University of Bath, Department of Mechanical Engineering

ME10305 Mathematics 2.

Matrices Sheet 3 — Eigenvalues, eigenvectors and solutions of ODE systems.

NOTE: that Q5 and Q6 are project-like questions involving eigenvectors and eigenvalues, and are beyond the remit of the unit.

Q1. Find the eigenvalues and eigenvectors of the following matrices;

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 3 & -1 \\ 0 & 4 & -1 \\ 1 & 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix},$$

$$F = \begin{pmatrix} b & a & 0 \\ c & b & a \\ 0 & c & b \end{pmatrix}, \quad G = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & 3 & 4 \end{pmatrix}.$$

Q2. Now we apply the eigenvalue theory to solving ODE systems. Solve:

(a) $$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{subject to} \quad x(0) = 1,\ y(0) = 0.$$

(b) $$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 & 6 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{subject to} \quad x(0) = 1,\ y(0) = 0.$$

(c) $$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 4 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{subject to} \quad x(0) = 4,\ y(0) = 0.$$

(d) $$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 3 & -1 \\ 0 & 4 & -1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{subject to} \quad x(0) = 2,\ y(0) = 2,\ z(0) = 3.$$

In cases (a) and (d) you will be able to use the results of part of Q1 to lighten your load!

Q3. Solve the following two systems of equations. Some of the work of Q2a may be used.

(a) $$\frac{d^2}{dt^2}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{subject to} \quad x(0) = 1,\ x'(0) = 0,\ y(0) = 0,\ y'(0) = 0.$$

(b) $$\frac{d^2}{dt^2}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{subject to} \quad x(0) = 1,\ x'(0) = 0,\ y(0) = 0,\ y'(0) = 0.$$

Q4. Solve the following systems of ODEs.

(a) $$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

(b) $$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\begin{pmatrix} 4 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

(c) $$\frac{d^2}{dt^2}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

If you have already found the eigenvalues and eigenvectors of J in Q1, then this should be very quick to answer.

Q5. One of the applications of matrix theory is in the area of Markov chains, which is about probabilities that are associated with sequences of events. This is an example adapted from the textbook by Glyn James.

It is stated that dry days follow dry days with a probability of 0.5 while wet days follow wet days with probability, 0.6. The notation, $P(D_n)$, is the probability that day n is dry, and clearly $P(W_n) = 1 - P(D_n)$ is that it is wet on the same day, where no other choices of weather are available! All of this may be written as follows,

$$\begin{pmatrix} P(D_{n+1}) \\ P(W_{n+1}) \end{pmatrix} = \begin{pmatrix} 0.5 & 0.4 \\ 0.5 & 0.6 \end{pmatrix}\begin{pmatrix} P(D_n) \\ P(W_n) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} P(D_0) \\ P(W_0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Here, the matrix is called a probability transition matrix and the initial condition (it is certainly dry!) is given. Although not necessary, see if you can understand how the matrix/vector equation has been constructed.

(a) Given the initial condition, find $P(D_1)$ and $P(W_1)$ using $n = 0$ in the above equation to find the prediction for the weather on day 1. Carry on like this to, say, day 4 to see if you can guess what the long-term trend is.

(b) Now find the eigenvalues and eigenvectors of the probability transition matrix, and see if you can relate these to your predicted long-term trend found in part (a).

(c) Given that $A\underline{v} = \lambda\underline{v}$ for eigenvectors and eigenvalues, the following is derived,

$$A^2\underline{v} = A(A\underline{v}) = A(\lambda\underline{v}) = \lambda\, A\underline{v} = \lambda^2\underline{v},$$

and so on for A^3, A^4 and so on. Now rewrite the original initial condition in the form,

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = A\underline{v}_1 + B\underline{v}_2,$$

i.e. find the constants A and B; here $\underline{v}_1$ and $\underline{v}_2$ are the eigenvectors. Now find out what happens as successive days fly by, but always keep the result in terms of a sum of multiples of the eigenvectors.

(d) The property you have just uncovered is a feature of probability transition matrices: one eigenvalue is equal to 1 and the corresponding eigenvector is the long-term trend. All of the other eigenvalues are smaller in magnitude. This happens because the elements in each column of the probability transition matrix adds to 1. Check all of these statements by finding the eigenvalues and eigenvectors of

$$\begin{pmatrix} a & b \\ 1-a & 1-b \end{pmatrix}.$$

(e) Finally we wish to design our probability so that we can have dry weekends and wet weekdays, so the long-term behaviour is that it will be dry 2/7ths of the time and wet 5/7ths of the time. Let $a = 0.5$ (as it was at the start) and find the value of b which will ensure this outcome.

Q6. This final question is lengthy and well above the standard required for the exam. However, it may be done if one is led carefully through it. The aim is to find the eigenvalues of tridiagonal matrices such as the following:

$$J_2 = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \quad J_3 = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix} \quad J_4 = \begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}.$$

You have already met these matrices in the problem sheet on determinants.

(i) Find the eigenvalues of J_2, J_3, J_4 and J_5 by using the standard techniques. I am not interested in finding the eigenvectors for now. Note that the factor $(\lambda+2)$ plays a vital role, and therefore *do not* expand your expressions for the determinants; keep them in terms of powers of $(\lambda + 2)$. You should be able to find simple analytical values for the eigenvalues, even for J_5 (for which three of the five eigenvalues are negative integers).

(ii) While undertaking part (i), you should have noticed how $\det(J_n)$ may be written in terms of $\det(J_{n-1})$ and $\det(J_{n-2})$ in the same manner as we found in Q6 of the determinants problem sheet. Show that,

$$\det(J_n) = -(2+\lambda)\det(J_{n-1}) - \det(J_{n-2}).$$

Now use your expressions for $\det(J_2)$ and $\det(J_3)$ obtained in part (i) to show that your expression for $\det(J_4)$ is correct. Likewise show that your expression for $\det(J_5)$ is correct. What is the polynomial which represents $\det(J_6) = 0$?

(iii) Now we will go into further detail with J_5 to find a general way of finding the eigenvalues for J_n and the eigenvectors. Assume that the eigenvector for J_5 has the following form,

$$\begin{pmatrix} -2-\lambda & 1 & 0 & 0 & 0 \\ 1 & -2-\lambda & 1 & 0 & 0 \\ 0 & 1 & -2-\lambda & 1 & 0 \\ 0 & 0 & 1 & -2-\lambda & 1 \\ 0 & 0 & 0 & 1 & -2-\lambda \end{pmatrix} \begin{pmatrix} \sin\alpha \\ \sin 2\alpha \\ \sin 3\alpha \\ \sin 4\alpha \\ \sin 5\alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

where α is currently unknown. Now write out the equation corresponding to, say, row 3 of this equation, rewrite this equation using multiple angle formulae (i.e. let $\sin 4\alpha = \sin(3\alpha + \alpha)$ and $\sin 2\alpha = \sin(3\alpha - \alpha)$), and hence show that,

$$\lambda = -2 + 2\cos\alpha.$$

Row 1 of the matrix equation is a special case as it has only two coefficients; check that it too gives the same expression for λ. Row 5 is also a special case, but this should yield $\sin 6\alpha = 0$. Now you are in a position to write down a simple formula for the eigenvalues, λ, for J_5 which should fit with your original calculations. What are the eigenvectors? What is the implication for J_n in general?

D.A.S.R. 08/01/2020