University of Bath, Department of Mechanical Engineering

ME10305 Mathematics 2.

Matrices Sheet 2 — Determinants, Cramer's rule and Gaussian Elimination.

Q1. Find the determinant of the following matrices. Which matrices are singular (i.e. have a zero determinant)? Use the row and column manipulation technique described in the lecture. For (d) and (g) attempt the evaluation of the determinant in more than one way just to practice the skill.

$$\text{(a)}\ \begin{pmatrix} 6 & 2 \\ 8 & 3 \end{pmatrix} \qquad \text{(b)}\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{(c)}\ \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \qquad \text{(d)}\ \begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & -3 \\ 1 & 2 & -1 \end{pmatrix}$$

$$\text{(e)}\ \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \qquad \text{(f)}\ \begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix} \qquad \text{(g)}\ \begin{pmatrix} 2 & 1 & 1 & 1 \\ 0 & 3 & -3 & 1 \\ 1 & 2 & -1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \qquad \text{(h)}\ \begin{pmatrix} b & a & a & a \\ a & b & a & a \\ a & a & b & a \\ a & a & a & b \end{pmatrix}$$

Q2. The matrix J_n is an $n \times n$ matrix where the diagonal entries have the value -2, the superdiagonal and subdiagonal entries the value 1, and 0 elsewhere. For example, J_1, J_2 and J_5 are

$$J_1 = (-2) \qquad J_2 = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \qquad J_5 = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}.$$

Such matrices arise in the numerical solution of second order ordinary differential equations.

Assume that $|J_1| = -2$, and then evaluate $|J_2|$, $|J_3|$, $|J_4|$ and $|J_5|$ directly from the matrix definitions. This should show you how to derive the recurrence relation,

$$|J_n| = -2|J_{n-1}| - |J_{n-2}|.$$

Finally, what is the explicit value of $|J_n|$?

Q3. Use Cramer's rule to solve the following systems of equations.

$$\text{(a)}\ \begin{aligned} 2x + 5y &= -1 \\ -3x + 2y &= 2 \end{aligned} \qquad \text{(b)}\ \begin{aligned} 2x_1 + 3x_2 - 2x_3 &= 1 \\ 6x_1 - 2x_2 - x_3 &= 2 \\ x_1 - x_2 + x_3 &= 2 \end{aligned} \qquad \text{(c)}\ \begin{aligned} x_1 + 3x_2 - x_3 &= 3 \\ x_2 - 7x_3 &= 2 \\ 2x_1 - 5x_3 &= 1 \end{aligned}$$

Q4. Use Gaussian Elimination to solve the following systems of equations.

(a) $$\begin{aligned} 2x + 5y &= -1 \\ -3x + 2y &= 2 \end{aligned}$$

(b) $$\begin{aligned} 2x_1 + 3x_2 - 2x_3 &= 1 \\ 6x_1 - 2x_2 - x_3 &= 2 \\ x_1 - x_2 + x_3 &= 2 \end{aligned}$$

(c) $$\begin{aligned} x_1 + 3x_2 - x_3 &= 3 \\ x_2 - 7x_3 &= 2 \\ 2x_1 - 5x_3 &= 1 \end{aligned}$$

Note that the above three systems of equations are identical to those in which were solved in Q3 using Cramer's rule.

(d) $$\begin{pmatrix} 1 & 2 & -1 & 1 \\ 1 & 1 & -2 & 6 \\ 3 & 0 & 1 & 1 \\ -2 & 1 & -3 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 2 \\ -1 \end{pmatrix}$$

(e) $$\begin{pmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \\ 7 \end{pmatrix}$$

Q5. The matrix given in Q1e is singular, by which is meant that it has a zero determinant, and therefore it either has no solution or an infinite number of them. The aim of this question is to see how Gaussian Elimination copes with such a situation.

Try to solve the matrix/vector equation

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

using Gaussian Elimination to see the manner in which the procedure fails when the matrix is singular. However, it is possible to write down solutions for this case. I haven't covered this in the lectures and therefore you'll need to work out how to do it.

Now try to find the solution when the right hand side vector is $(-2, 1, 5)^T$. Can you explain why two separate equations involving the same matrix has solutions in one case but not in another? (Hint: use $(a, b, c)^T$ as the right hand side as a third case.)

Q6. The aim here is to find the inverse of some matrices using Gaussian Elimination starting with the identity matrix as part of the augmented matrix scheme. Other general properties of inverses will arise along the way. Treat this question as practice in Gaussian elimination; the computation of inverses takes too long in the examination context (with the possible exception of a tridiagonal 3×3 matrix). Find the inverses of the following matrices.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 3 & -1 \\ 0 & 4 & -1 \\ 1 & 1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix},$$

$$D = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \\ 1/4 & 1/5 & 1/6 \end{pmatrix}$$

You may check your answer either by forming the product $M^{-1}M$ or the product MM^{-1} or by consulting the web page: `https://matrix.reshish.com/inverse.php`.

What conclusion can you draw about the inverses of matrices which are symmetric, anti-symmetric or tridiagonal?

D.A.S.R. 18/02/2021