University of Bath, Department of Mechanical Engineering

ME10305 Mathematics 2.

Matrices Sheet 1 — Multiplication.

The aim of this problem sheet is to get used to performing matrix multiplication and to know what *compatibility with respect to multiplication* means. There are also some properties which involve matrix transposes which are useful to know, and also the concepts of commutivity and distributivity.

Q1. The matrices, A, B, C and D, are defined as follows,

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 3 & 5 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

Classify all these matrices in terms of numbers of rows and columns. Now make a list of which pairs may be multiplied together (i.e. are *compatible* with respect to multiplication) — for example, both AB and BA belong to this list. Now find all the permissible products of two matrices.

Q2. Having now determined AB, where A and B are as given in Q1, write down $(AB)^T$, the transpose of AB. Now calculate B^TA^T. Is $(AB)^T = B^TA^T$? Is it obvious whether this last answer is true in general?

Q3. The matrix A is defined by

$$A = \begin{pmatrix} 2 & 1 & 2 \\ 0 & 3 & -3 \\ 1 & 2 & -1 \end{pmatrix}.$$

Find A^T. Now form the sums $A + A^T$ and $A - A^T$. What do you notice about these new matrices? Find the products AA^T and A^TA. What do you conclude from these results?

Q4. We have seen that matrix multiplication, where the matrices are compatible, yields $AB \neq BA$ in general, i.e. matrix multiplication is non-commutative. But I would like you to show that matrix multiplication is associative, that is, $A(BC) = (AB)C$, where the term in brackets is computed first. Check one specific case:

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Now check the general case for 2×2 matrices:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \qquad C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}.$$

Try to think of a way of generalising this result to any three square matrices, and then to any set of compatible matrices.

Q5. Not really part of the syllabus, but I needed something to fill the gap at the bottom of this page! If you have two tridiagonal matrices which are compatible with respect to multiplication and are subsequently multiplied together, then is there is a general statement that can be made about the pattern of the components in that product?

Q6. Rotation matrices are important for many applications and are especially so in robotics. I am not going to teach this formally, but I would like to introduce them and to play around with them a little.

We may define the following three rotation matrices:

$$R_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} \qquad \text{(Rotation by an angle } \alpha \text{ about the } x\text{-axis.)}$$

$$R_y(\beta) = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \qquad \text{(Rotation by an angle } \beta \text{ about the } y\text{-axis.)}$$

$$R_z(\gamma) = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{(Rotation by an angle } \gamma \text{ about the } z\text{-axis.)}$$

Therefore if the position vector of a point is $\underline{r}$, and if that point is rotated by an angle, α, about the x-axis, then the new location of the point is given by the matrix/vector product, $R_x(\alpha)\,\underline{r}$. If this new point is subsequently rotated by γ about the z-axis, then its new location is given by, $R_z(\gamma)\,R_x(\alpha)\,\underline{r}$.

Thus a rotation about the x-axis followed by a rotation about the z-axis is $R_z(\gamma)\,R_x(\alpha)$, where the rotation matrices only appear to have been written down in the wrong order!

(i) Perhaps it is not surprising that the inverse matrix of $R_x(\alpha)$ is $R_x(-\alpha)$, given what this notation means. Check that $R_x(\alpha)\,R_x(-\alpha) = I$, the 3×3 identity matrix.

(ii) Find both $R_z(\gamma)\,R_x(\alpha)$ and $R_x(\alpha)\,R_z(\gamma)$. Are they equal? What is the implication of this general result? What about when $\alpha = \gamma = \frac{1}{4}\pi$? What about when $\alpha = \gamma = \frac{1}{2}\pi$?

(iii) If you really have time spare, then you could try the following. A point suffers the grave indignity of the following sequence of rotations: $R_x(\alpha)$ then $R_z(\gamma)$ then $R_x(-\alpha)$ then $R_z(-\gamma)$. This expresses a possibly naive thought that an arbitrarily chosen point will return to where it started after this sequence; do you think that it will? If not, what are the correct third and fourth rotations to cause the point to return?

Q7. A question for interest, perhaps. Fermat's last theorem is well-known: when a, b, c and n take positive integer values, the equation $a^n + b^n = c^n$ has solutions only when $n = 2$. However, a Michael Penn youtube video alerted me to the fact that this theorem doesn't apply when a, b and c are matrices! So here's a straightforward exercise in matrix multiplication to check if Prof. Penn is correct:

$$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}^3 + \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}^3 = \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}^3.$$

D.A.S.R. 25/02/2022