Department of Mechanical Engineering, University of Bath

ME10305 Mathematics 2

Solutions to Laplace Transforms Sheet 1

1. Find the Laplace Transforms of the following functions using the definition of the Laplace Transform (rather than by looking up the result in a table):

(a) $\mathcal{L}\big[e^{3t}\big] = \int_0^\infty e^{3t}e^{-st}dt = \int_0^\infty e^{-(s-3)t}dt = \dfrac{1}{3-s}\Big[e^{-(s-3)t}\Big]_0^\infty = \dfrac{1}{s-3}.$

Note that this result relies on the fact that $s > 3$; when $s \leq 3$ the result is infinite and we say that the Laplace Transform does not exist in this case.

(b) $\mathcal{L}\big[e^{-3t}\big] = \dfrac{1}{s+3}$ by analogy with (a). Here we require $s > -3$ for the transform to exist.

(c) $\mathcal{L}\big[\cos\omega t\big] = \int_0^\infty (\cos\omega t)e^{-st}dt = \dfrac{s}{s^2+\omega^2}$ using integration by parts. Here we require $s > 0$ for the Transform to exist.

(d) $\mathcal{L}\big[te^{-3t}\big] = \int_0^\infty te^{-(s+3)t}dt = \dfrac{1}{(s+3)^2}$ using integration by parts. Again $s > -3$ for the Transform to exist.

(e) $\mathcal{L}\big[t^3\big] = \dfrac{6}{s^4}.$

(f) $\mathcal{L}\big[e^{-at}\cos\omega t\big]$. Integration by parts will yield,

$$\mathcal{L}\big[e^{-at}\cos\omega t\big] = \frac{s+a}{(s+a)^2+\omega^2}.$$

Note that this result might also be obtained by applying the s-shift theorem [see lecture 3] to the answer to Q1c.

(g) $\mathcal{L}\big[f'''(t)\big] = -f''(0) - sf'(0) - s^2f(0) + s^3F(s)$, where $F(s) = \mathcal{L}\big[f(t)\big]$. Three integrations by parts for this one.

(h) $\mathcal{L}\big[f(t)\big] = (1-e^{-s})/s$. Should be an easy integral!

(i) $\mathcal{L}\big[\cosh\omega t\big] = \frac{1}{2}\Big(\dfrac{1}{s+\omega} + \dfrac{1}{s-\omega}\Big)$. This follows quickly from $\cosh\omega t = \frac{1}{2}(e^{\omega t} + e^{-\omega t})$.

(j) $\mathcal{L}\big[t^2e^{-t}\big]$ should come to $2/(s+1)^3$.

(k) Let $I = \mathcal{L}\big[t^{-1/2}\big] = \displaystyle\int_0^\infty t^{-1/2}e^{-st}dt$. Now let $x = (st)^{1/2}$. Hence $dx = (s^{1/2}/2t^{1/2})dt$, or, $(2/s^{1/2})dx = t^{-1/2}dt$, and therefore the integral becomes,

$$I = \int_0^\infty \frac{2}{s^{1/2}}e^{-x^2}dx = \frac{2}{s^{1/2}}\int_0^\infty e^{-x^2}dx = \frac{2}{s^{1/2}}\frac{\pi^{1/2}}{2} = \frac{\pi^{1/2}}{s^{1/2}}$$

on using the given result that

$$\int_0^\infty e^{-x^2}dx = \frac{\pi^{1/2}}{2}.$$

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2. Use the Laplace Transform to solve the following equations:

(a) $\dfrac{dy}{dt} + 4y = 6, \quad y(0) = 2.$

$\mathcal{L}\big[y'\big] = sY - y(0) = sY - 2$. Hence the ODE reduces to $(sY-2) + 4Y = 6/s$, from which Y may be shown to be

$$Y = \frac{2(s+3)}{s(s+4)} \equiv \frac{A}{s} + \frac{B}{s+4}.$$

Multiplying by $s(s+4)$ we obtain $2s+6 = A(s+4) + Bs$. When $s=0$ we see that $A = 3/2$, and when $s = -4$ we get $B = 1/2$. Hence

$$Y = \tfrac{1}{2}\Big[\frac{3}{s} + \frac{1}{s+4}\Big] \qquad \Rightarrow \qquad y = \tfrac{1}{2}[3 + e^{-4t}].$$

(b) $\dfrac{d^2y}{dt^2} + 16y = 0, \quad y(0) = 0, \quad \dfrac{dy}{dt} = 1.$

$\mathcal{L}\big[y''\big] = s^2Y - y'(0) - sy(0) = s^2Y - 1$. The equation reduces to

$$(s^2Y - 1) + 16Y = 0 \qquad \Rightarrow \qquad Y = \frac{1}{s^2+16} = \tfrac{1}{4}\Big[\frac{4}{s^2+4^2}\Big] \qquad \Rightarrow \qquad y = \tfrac{1}{4}\sin 4t$$

using tables.

(c) $\dfrac{d^2y}{dt^2} + 4y = 29e^{-5t}, \qquad y(0) = 0, \quad \dfrac{dy}{dt}(0) = -3.$

The equation becomes,

$$s^2Y + 3 + 4Y = \frac{29}{s+5} \qquad \Rightarrow \qquad Y = \frac{29}{(s+5)(s^2+4)} - \frac{3}{s^2+4}.$$

After some partial fractions work, this simplifies to,

$$Y = \frac{1}{s+5} + \frac{2}{s^2+4} - \frac{s}{s^2+4}.$$

After using standard results for the LTs of exponentials and sinusoids, we obtain,

$$y = e^{-5t} + \sin 2t - \cos 2t.$$

(d) $y''' + y'' + 4y' + 4y = 0, \qquad y(0) = 0, \quad y'(0) = 3, \quad y''(0) = -5.$

Using the standard results for the LTs of the various derivatives, we get,

$$\Big[s^3Y - y''(0) - sy'(0) - s^2y(0)\Big] + \Big[s^2Y - y'(0) - sy(0)\Big] + 4\Big[sY - y(0)\Big] + 4Y = 0.$$

Using the initial conditions and simplifying, we get,

$$(s^3 + s^2 + 4s + 4)Y = 3s - 2.$$

The cubic multiplying Y may be factorised and therefore we have,

$$Y = \frac{3s-2}{(s^2+4)(s+1)}.$$

Standard partial fractions of the form,

$$\frac{3s-2}{(s^2+4)(s+1)} = \frac{As+B}{s^2+4} + \frac{C}{s+1},$$

yields $A = 1$, $B = 2$ and $C = -1$. Therefore,

$$\begin{aligned} Y &= \frac{s+2}{s^2+4} - \frac{1}{s+1}, \\ &= \frac{s}{s^2+4} + \frac{2}{s^2+4} - \frac{1}{s+1}. \end{aligned}$$

All three of these terms have standard inverse LTs. Therefore we have,

$$y = \cos 2t + \sin 2t - e^{-t}.$$

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3. Find the Laplace Transform of $z(t) = \int_0^t y(\tau)\,d\tau$. [Hint: recall that $z'(t) = y(t)$ here.]

We'll integrate by parts once only and differentiate the z-term:

$$\mathcal{L}\Big[\int_0^t y(\tau)\,d\tau\Big]\,dt = \mathcal{L}\big[z(t)\big] = \int_0^\infty z(t)e^{-st}\,dt$$

$$= \Big[z\Big]\Big[\frac{e^{-st}}{-s}\Big]_0^\infty - \int_0^\infty \Big[z'\Big]\Big[\frac{e^{-st}}{-s}\Big]\,dt$$

$$= 0 + \frac{1}{s}\int_0^\infty ye^{-st}\,dt = \frac{Y}{s}$$

In the above, note that $z(0) = 0$, given its definition.

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4. Find the solution of the ODE, $y'' + 2y' + y = 2e^{-t}$, subject to $y(0) = y'(0) = 0$. [Hint: you may need to consult the solution to Q1j.]

The Laplace Transform of the ODE yields,

$$(s^2 + 2s + 1)Y = \frac{2}{s+1} \qquad \Longrightarrow \qquad Y = \frac{2}{(s+1)^3}.$$

Question 1j has the solution, $\mathcal{L}\big[t^2 e^{-1}\big] = 2/(s+1)^3$, therefore the present solution is $y = t^2 e^{-t}$.

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5. Factorise the denominator of the following fractions into complex factors, and use partial fractions to find their Inverse Laplace Transforms:

(a) $\dfrac{1}{s^2+b^2} = \dfrac{1}{(s+bj)(s-bj)} = \dfrac{1}{2bj}\Big[\dfrac{1}{s-bj} - \dfrac{1}{s+bj}\Big].$

Hence

$$\mathcal{L}^{-1}\Big[\frac{1}{s^2+b^2}\Big] = \frac{1}{2bj}\mathcal{L}^{-1}\Big[\frac{1}{s-bj} - \frac{1}{s+bj}\Big] = \frac{1}{2bj}[e^{bjt} - e^{-bjt}]$$
$$= \frac{1}{2bj}\Big[(\cos bt + j\sin bt) - (\cos bt - j\sin bt)\Big] = \frac{\sin bt}{b}.$$

(b) $\dfrac{s}{s^2+b^2} = \dfrac{1}{2}\Big[\dfrac{1}{s-bj} + \dfrac{1}{s+bj}\Big].$

Hence

$$\mathcal{L}^{-1}\Big[\frac{s}{s^2+b^2}\Big] = \tfrac{1}{2}\Big[e^{bjt} + e^{-bjt}\Big] = \cos bt.$$

(c) $\dfrac{1}{s^2+2cs+c^2+d^2} = \dfrac{1}{2dj}\Big[\dfrac{1}{s+c-dj} - \dfrac{1}{s+c+dj}\Big].$

Note that this exercise is very similar to part (a). The solution is $(1/d)e^{-ct}\sin dt$.

(d) $\dfrac{s+c}{s^2+2cs+c^2+d^2} = \dfrac{1}{2}\Big[\dfrac{1}{s+c-dj} + \dfrac{1}{s+c+dj}\Big].$ The solution is $e^{-ct}\cos dt$.

Use some of these results to solve the following equations:

(e) $y'' + 4y' + 5y = 0, \quad y(0) = 0, \quad y'(0) = 1.$

Here we have

$$\mathcal{L}\big[y''\big] = s^2Y - sy(0) - y'(0) = s^2Y - 1, \qquad \mathcal{L}\big[y'\big] = sY - y(0) = sY.$$

The equation transforms to $(s^2 + 4s + 5)Y = 1$ and hence $Y = 1/(s^2 + 4s + 5)$. Now we can use the result of part (c) with $c = 2$ and $d = 1$: $y = e^{-2t}\sin t$.

(f) $y'' + 2y' + 2y = e^{-t}, \quad y(0) = 0, \quad y'(0) = 0.$

Here we find that $\mathcal{L}\big[y'\big] = sY$ and $\mathcal{L}\big[y''\big] = s^2Y$. The equation transforms to $(s^2 + 2s + 2)Y = 1/(s+1)$, from which we obtain

$$Y = \frac{1}{(s+1)(s^2+2s+2)} = \frac{1}{s+1} - \frac{s+1}{s^2+2s+2} \qquad \Rightarrow \qquad y = (1-\cos t)e^{-t}$$

after using partial fractions and part (d) with $c = d = 1$.

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6. Write down the values of the following integrals.

In all cases we apply the result that the integral of $g(t)\delta(t-a)$ is $g(a)$, unless a lies outside of the range of integration, in which case the result is 0.

$$\int_{-\infty}^{\infty} \delta(t)e^{2t}\,dt = 1.$$

$$\int_{-\infty}^{\infty} \delta(t-1)e^{-t^2}\,dt = e^{-1}.$$

$$\int_{-\infty}^{\infty} \delta(t-2)\sin\pi t\,dt = \sin 2\pi t = 0.$$

$$\int_{0}^{\infty} \delta(t+2)t^3\,dt = 0.$$

The final answer is zero because the impulse is at $t = -2$ which outside the range of integration. So the impulse function is always zero within that range.

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7. Find the Laplace Transforms of the following functions:

(a) $\mathcal{L}\big[e^{e^t}\delta(t-1)\big] = \displaystyle\int_0^{\infty} e^{e^t}\delta(t-1)e^{-st}dt = e^{e^1}e^{-s} = e^{e-s}$ using the result for the integrals of delta functions.

(b) $\mathcal{L}\big[\delta(t) + \delta(t-1) + \delta(t-2) + \delta(t-3) + \ldots\big]$.

As $\mathcal{L}\big[\delta(t-n)\big] = e^{-ns}$, this transform is $e^0 + e^{-s} + e^{-2s} + e^{-3s} + \ldots$.

Another way of writing this is $\displaystyle\sum_{n=0}^{\infty} e^{-ns}$, which is a geometrical series and may be summed to get $1/(1-e^{-s})$.

The following summation of unit impulses, $\displaystyle\sum_{n=-\infty}^{\infty} \delta(t-n)$, (noting the lower limit) is known as the Shah function ($\text{Ш}(t)$) or, more descriptively, as the Dirac comb. It is used in signal processing and sampling.

8. Use the Laplace Transform to solve the following equations:

(a) $\dfrac{dy}{dt} + 3y = \delta(t), \qquad y(0) = 1.$

The Laplace Transform of the full ODE is

$$(s+3)Y - 1 = 1,$$

and hence,

$$Y = \frac{2}{s+3} \qquad \Longrightarrow \qquad y = 2e^{-3t}.$$

Clearly the initial displacement given by the solution is not what was set as the initial condition, but this is due to the impulse.

(b) $\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} + 2y = \delta(t), \qquad y(0) = 1, \quad \dfrac{dy}{dt}(0) = b$ where b is a constant.

After taking the Laplace Transform the ODE becomes,

$$s^2Y - b - s + 3sY - 3 + 2Y = 1.$$

Hence,

$$(s^2 + 3s + 2)Y = s + 4 + a.$$

After rearrangement and then partial fractions we obtain,

$$Y = \frac{s+4+b}{(s+1)(s+2)} = \frac{3+b}{s+1} - \frac{2+a}{s+2},$$

upon using partial fractions. So the final solution is,

$$y = (3+b)e^{-t} - (2+b)e^{-2t}.$$

Although this solution yields $y(0) = 0$, as desired, it is easily shown that $y'(0) = 1 + b$. So a unit momentum has been added.

(c) $\dfrac{d^3y}{dt^3} - \dfrac{dy}{dt} = 3\delta(t), \qquad y(0) = 1, \quad \dfrac{dy}{dt}(0) = 0, \quad y''(0) = -1.$

After the taking of Laplace Transforms we get,

$$s^3Y - sY + 2 - s^2 = 3 \qquad \Longrightarrow \qquad Y = \frac{s^2+1}{s^3 - s} \qquad \Longrightarrow \qquad Y = \frac{1}{s+1} + \frac{1}{s-1} - \frac{1}{s}.$$

Hence the final solution is,

$$y = e^t + e^{-t} - 1.$$

If we check the initial conditions, then we see that $y(0) = 1$ and $y'(0) = 0$, as required. However, $y''(0) = 2$, according to the solution, whereas we imposed $y''(0) = -1$ at the start. We need to bear in mind that the $3\delta(t)$ forcing term for this 3rd order ODE increases the value of $y''(0)$ immediately, and this is what we have seen.

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9. Laplace Transforms are perfectly set up to solve Initial Value Problems, but let us try it out on a Boundary Value Problem. The aim, then, is to solve $y''+y=0$, subject to $y(0)=1$ and $y(\frac{1}{2}\pi)=1$. At the outset, let $y'(0)=c$ and carry out the analysis using this unnown constant. Eventually you will have the opportunity to find c.

Setting $y'(0)=c$, the Laplace Transform of the ODE yields,

$$(s^2+1)Y-s-c=0.$$

Hence,

$$Y=\frac{s+c}{s^2+1}=\frac{s}{s^2+1}+c\frac{1}{s^2+1}.$$

Hence,

$$y=\cos t+c\sin t.$$

Now we are in a position to satisfy the second Boundary Condition, $y(\frac{1}{2}\pi)=1$; hence $c=1$. The final solution is,

$$y=\cos t+\sin t.$$

Comment: So clearly it is possible to solve BVPs using Laplace Transforms. I am not sure how tricky this will become if we have to carry more than one unknown initial condition.
