Department of Mechanical Engineering, University of Bath

ME10305 Mathematics 2

Laplace Transforms Sheet 2

It is normal in questions on Laplace Transforms to have ready access to the LTs of functions like sinusoids, exponentials and powers.

10. First sketch the following functions, and then Find their Laplace Transforms:

(a) $H(t-a)t^3$ (b) $\sum_{n=0}^{\infty}(-1)^n H(t-n) = H(t) - H(t-1) + H(t-2) - H(t-3) + \ldots$ (c) $H(a-t)$,

[In one case it may be possible to simplify the final answer...]

11. Use the s-shift theorem to find the Inverse Laplace Transform of:

(a) $\dfrac{1}{s+a}$ (b) $\dfrac{2}{(s+a)^3}$ (c) $\dfrac{b}{(s+a)^2+b^2}$ (d) $\dfrac{s}{(s+a)^2+b^2}$.

12. [This is an exam-style question.] Find the Laplace Transforms of both $\cos bt$ and $\sin bt$. Then use the s-shift theorem to write down the Laplace Transforms of $e^{-at}\cos bt$ and $e^{-at}\sin bt$. Hence solve the ODE,

$$y'' + 6y' + 25y = 0$$

subject to $y(0) = 1$ and $y'(0) = 5$.

13. Use the t-Shift Theorem to find the Inverse Laplace Transform of:

(a) $\dfrac{e^{-as}}{s^3}$ (b) $\dfrac{e^{-as}}{s+b}$ (c) $\dfrac{e^{-as}}{s^2+b^2}$ (d) $\dfrac{e^{-as}}{(s+c)^2+b^2}$.

14. Use the convolution theorem to find the Inverse Laplace Transform of:

(a) $\dfrac{1}{(s+a)^2}$ (b) $\dfrac{1}{(s+a)(s^2+b^2)}$ (c) $\dfrac{1}{(s^2+a^2)^2}$ (d) $e^{-as} \times \dfrac{1}{s^2}$.

15. Use the convolution theorem to find the solutions to the following equations:

(a) $y' + 3y = e^{-2t}, \quad y(0) = 0;$

(b) $y'' + 5y' + 6y = 0, \quad y(0) = 0, \quad y'(0) = 1;$

(c) $y'' + y = e^{-t}, \quad y(0) = 0, \quad y'(0) = 0.$

16. Solve the system of equations,

$$x' = 2x - 4y + \delta(t),$$

$$y' = 3x - 5y,$$

subject to the initial conditions, $x(0) = y(0) = 0$.

17. Solve the system of equations,

$$x'' + 2x - 2y = \delta(t),$$

$$y'' - x + 3y = 0,$$

subject to the initial conditions, $x(0) = x'(0) = y(0) = y'(0) = 0$. When the final solution has been obtained, determine which of the four initial conditions has been violated, but can you guess this in advance?

18. [This is a project-like question which combines quite a large number of mathematical results.]

The overall aim for this question is to solve the ODE,

$$y' + y = \text{Ш}(t) = \sum_{n=-\infty}^{\infty} \delta(t-n).$$

The unusual symbol, Ш, which I cannot typeset properly(!), is known as the Shah function, and the symbol itself is the Cyrillic character, sha, which mimics the shape of the function. In various contexts it is also known as (i) the Dirac comb, (ii) more picturesquely as the bed of nails function, and (iii) more functionally as an impulse train.

The solution is a periodic function which has a period of 1, but this can't be found simply using the Laplace Transform because that is an integral from $t = 0$ onwards, whereas the Shah function consists of impulses at all integer values of t, both positive and negative. Therefore we will do this by determining the eventual 'steady periodic state' that is achieved when t becomes large. Enjoy the ride!

(a) Find the Laplace Transform of e^{-t}, and then apply the t-shift theorem to find the inverse Laplace Transform of $e^{-ns}/(s+1)$.

(b) Use the Laplace Transform on the ODE,

$$y' + y = \sum_{n=0}^{\infty} \delta(t-n), \qquad y(0) = 0,$$

to find an expression for $Y(s)$, the transform of $y(t)$. Do not simplify this expression for Y by, say, summing the geometric series!

(c) Now use the result of part (a) to write down an expression for $y(t)$ in terms of a sum of terms involving unit step functions.

(d) The sum we have obtained for $y(t)$ is infinite in length; do make sure that you're happy with this idea! Now we let $t = n + \epsilon$ in your expression for y, where n is the first positive integer below t, and where $0 \le \epsilon < 1$. You should then be able to factor $e^{-\epsilon}$ out of the resulting mess(!), and then be able to sum the resulting geometric series to obtain a compact formula for y.

(e) Now find $\lim_{n\to\infty} y$. This will give the long-term formula for $y(t)$, but written in terms of ϵ — this will be a valid formula for $y(t)$ between two neighbouring integer values of t. In the ultimate steady periodic state, what are the maximum and minimum values of y?

(f) See if you can guess what $y(t)$ looks like.

(g) An easier way to solve the main problem is to concentrate on the interval of time, $0 \le t < 1$, and to solve for

$$y' + y = \delta(t), \qquad y(0) = c,$$

using Laplace Transforms. The value of c may be found by insisting that $y(1) = y(0)$.