Department of Mechanical Engineering, University of Bath

ME10305 Mathematics 2

Laplace Transforms Sheet 1

It is normal in questions on Laplace Transforms to have ready access to the LTs of functions like sinusoids, exponentials and powers.

1. Find the Laplace Transforms of the following functions using the definition of the Laplace Transform (rather than by looking up the result in a table):

(a) e^{3t} (b) e^{-3t} (c) $\cos\omega t$ (d) te^{-3t} (e) t^3 (f) $t\cos\omega t$ (g) $f'''(t)$

(h) The unit pulse: $f(t) = 1$ for $t < 1$, $f(t) = 0$ otherwise (i) $\cosh\omega t$ (j) $t^2 e^{-t}$

(k) $t^{-1/2}$ [Hint: set $x = (st)^{1/2}$ to transform the integral and use the result $\int_0^\infty e^{-x^2}\,dx = \sqrt{\pi}/2$.]

2. Use the Laplace Transform to solve the following equations:

$$\text{(a)} \quad \frac{dy}{dt} + 4y = 6, \qquad y(0) = 2.$$

$$\text{(b)} \quad \frac{d^2y}{dt^2} + 16y = 0, \qquad y(0) = 0, \quad \frac{dy}{dt}(0) = 1.$$

$$\text{(c)} \quad \frac{d^2y}{dt^2} + 4y = 29e^{-5t}, \qquad y(0) = 0, \quad \frac{dy}{dt}(0) = -3.$$

$$\text{(d)} \quad y''' + y'' + 4y' + 4y = 0, \qquad y(0) = 0, \quad y'(0) = 3, \quad y''(0) = -5.$$

[You may also practice on any of the linear constant coefficient equations from the ODEs section of the unit, but note that there may be some awkwardnesses due to the fact that (i) the questions weren't designed for nice LT solutions, (ii) many don't have initial conditions specified, (iii) some of the results derived in the 3rd and 4th Laplace Transform lectures may be of considerable use.]

3. Find the Laplace Transform of $z(t) = \int_0^t y(\tau)\,d\tau$. [Hint: recall that $z'(t) = y(t)$ here.]

4. Find the solution of the ODE, $y'' + 2y' + y = 2e^{-t}$, subject to $y(0) = y'(0) = 0$. [Hint: you may need to consult the solution to Q1j.]

5. Factorise the denominator of the following fractions into complex factors, and use partial fractions to find their Inverse Laplace Transforms:

(a) $\dfrac{1}{s^2 + b^2}$ (b) $\dfrac{s}{s^2 + b^2}$ (c) $\dfrac{1}{s^2 + 2cs + c^2 + d^2}$ (d) $\dfrac{s + c}{s^2 + 2cs + c^2 + d^2}$.

[**Note:** that I won't expect such complex factorisation in the exam.]

These results may be used to solve the following equations:

(e) $y'' + 4y' + 5y = 0, \quad y(0) = 0, \quad y'(0) = 1.$

(f) $y'' + 2y' + 2y = e^{-t}, \quad y(0) = 0, \quad y'(0) = 0.$

6. Write down the values of the following integrals.

$$\int_{-\infty}^{\infty} \delta(t) e^{2t}\, dt, \qquad \int_{-\infty}^{\infty} \delta(t-1) e^{-t^2}\, dt, \qquad \int_{-\infty}^{\infty} \delta(t-2) \sin \pi t\, dt, \qquad \int_{0}^{\infty} \delta(t+2) t^3\, dt.$$

7. Find the Laplace Transforms of the following functions:

$$\text{(a)}\ \ e^{e^t}\delta(t-1), \qquad \text{(b)}\ \ \sum_{n=0}^{\infty} \delta(t-n) = \delta(t) + \delta(t-1) + \delta(t-2) + \delta(t-3) + \ldots.$$

[Look out for the geometric series....]

8. Use the Laplace Transform to solve the following equations:

$$\text{(a)}\quad \frac{dy}{dt} + 3y = \delta(t), \qquad y(0) = 1.$$

$$\text{(b)}\quad \frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \delta(t), \qquad y(0) = 1, \quad \frac{dy}{dt}(0) = b, \qquad \text{where } b \text{ is a constant.}$$

$$\text{(c)}\quad \frac{d^3y}{dt^3} - \frac{dy}{dt} = 3\delta(t), \qquad y(0) = 1, \quad \frac{dy}{dt}(0) = 0, \quad y''(0) = -1.$$

9. Laplace Transforms are perfectly set up to solve Initial Value Problems, but let us try it out on a Boundary Value Problem. The aim, then, is to solve $y'' + y = 0$, subject to $y(0) = 1$ and $y(\frac{1}{2}\pi) = 1$. At the outset, let $y'(0) = c$ and carry out the analysis using this unknown constant. Eventually you will have the opportunity to find c.