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Mathematics 2 ME10305

Problem Sheet — Fourier Series

Note: For the purposes of exam revision, questions 2, 4 and 5 are the important ones. Question 1 has a general importance while questions 3 and 6 provide some good background knowledge.

Q1. Which of the following functions are even, odd or neither about $x = 0$? Of those which are periodic, find the fundamental period.

(i) $\sin t$ (ii) $\sin^2 t$ (iii) $\sqrt{1-t^2}$ $(-1 \le t \le 1)$ (iv) te^{-t} (v) e^{-t^2}

(vi) te^{-t^2} (vii) $\sin t + \sin 3t$ (viii) $\sin t \sin 3t$ (ix) $\sin t \sin \sqrt{2}t$

(x) $f(t) = t + 1$ for $-1 < t < 1$, $f(t) = f(t+2)$

The final two functions have one definition in part of the period and another in the remaining part.

(xi) $f(t) = t$ for $0 \le t \le 1$, $f(t) = 2 - t$ for $1 \le t \le 2$, $f(t) = f(t+2)$

(xii) $f(t) = 1$ for $0 < t \le 1$, $f(t) = 2 - t$ for $1 \le t < 2$, $f(t) = f(t+2)$

Q2. Find the Fourier Series representations of the following functions, bearing in mind that quicker results may be obtained when symmetries are accounted for. In all cases, (i) sketch the function, (ii) try to predict in advance how fast the Fourier coefficients decay by checking the continuity of each function **before** attempting to find the Fourier Series.

(a) $f(t) = t^2$ $-\pi \le t \le \pi$ with $f(t) = f(t + 2\pi)$.

(b) $f(t) = t - t^2$ $0 \le t \le 1$ with $f(t) = f(t+1)$.

(c) $f(t) = \pi^2 t - t^3$ $-\pi \le t \le \pi$ with $f(t) = f(t + 2\pi)$.

(d) $f(t) = t - t^3$ $-1 \le t \le 1$ with $f(t) = f(t+2)$.

(e) $f(t) = \cos \alpha t$ $-1 \le t \le 1$ with $f(t) = f(t+2)$.

(f) $f(t) = \cosh \alpha t$ $-1 \le t \le 1$ with $f(t) = f(t+2)$.

(g) $f(t) = 1$ for $0 < t < 1$, $f(t) = -1$ for $1 < t < 2$, with $f(t) = f(t+2)$.

(h) $f(t) = 3t^5 - 10t^3 + 7t$ for $-1 \le t \le 1$ with $f(t) = f(t+2)$.

(i) $f(t) = |\sin t|$.

(j) $f(t) = t^2$ for $0 < t < 1$ with $f(t) = f(t+1)$.

Q3. The aim of this question is two-fold, to derive the formulae for the Fourier coefficients and to prove Parseval's theorem. For simplicity we will consider functions of period 2π. **(This question is over and above what would be expected in an exam question, but it is included to show where the formulae for the Fourier coefficients come from.)**

If m and n are nonzero integers, first show that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos nt \cos mt \, dt = \begin{cases} 0 & \text{when } n \ne m, \\ 1 & \text{when } n = m. \end{cases}$$

Do the same for the integral of the product of two sines. Finally, show that the integral of $\sin nt \cos mt$ over the same range is zero.

Hence use these results and the standard definition of the Fourier series,

$$f(t) = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty}(A_n \cos nt + B_n \sin nt),$$

to find expressions for the Fourier coefficients.

For a function of period 2π, Parseval's theorem is

$$\frac{1}{\pi}\int_{-\pi}^{\pi}[f(t)]^2 dt = \tfrac{1}{2}A_0^2 + \sum_{n=1}^{\infty}[A_n^2 + B_n^2];$$

prove this using the results you have already derived. This result is related to the energy content of a periodic signal.

Q4. If $g(t) = t^2$ in the range $-\pi \le t \le \pi$, and $g(t)$ has a period equal to 2π, find its Fourier series. Hence find the Particular Integral of the ordinary differential equation,

$$\frac{dy}{dt} + cy = g(t).$$

Q5. Consider the response of the following undamped mass/spring system to a rectified sine wave signal:

$$\frac{d^2y}{dt^2} + K^2 y = |\sin t|.$$

By sketching the signal confirm that its period is π and determine its Fourier series. Hence find the response $y(t)$. For which values of K is there resonance?

Q6. Find the Fourier series of the response to the following damped system to the rectified sine wave:

$$y'' + cy' + K^2 y = |\sin t|.$$

For which values of K is there resonance?