ERROR CORRECTION CODING ①

THERE ARE 2^r POSSIBLE COMBINATIONS OF THE PARITY CHECKS, OF WHICH ONE COMBINATION MUST COINCIDE WITH NO ERROR.

A SINGLE ERROR CAN OCCUR IN ANY ONE OF THE 'n' POSITIONS OF THE CODEWORD SO;

$$2^r - 1 \geqslant n \quad , \quad 2^r \geqslant n+1$$

IF $2^r = n+1$, WE HAVE A PERFECT CODE

EXAMPLES

SINGLE-ERROR CORRECTING CODES (n, k)

$(7,4)$ $n=7,\ k=4,\ r=3$; $2^r = 8$, $n+1 = 8$ PERFECT

$(15,11)$ $n=15,\ k=11,\ r=4,$ $2^r = 16$, $n+1 = 16$ PERFECT

$(31,26)$ $n=31,\ k=26,\ r=5$ $2^r = 32$, $n+1 = 32$ PERFECT

$(6,3)$ $2^3 = 8 \neq 6+1$ IM-PERFECT

HIGHER-ORDER ERROR CORRECTION ②

DOUBLE, TRIPLE etc. (n, k, t) № CORRECTABLE ERRORS

THE NUMBER OF WAYS "i" ERRORS CAN OCCUR IN A WORD OF "n" BITS IS GIVEN BY THE BINOMIAL COEFFICIENT

$${}^nC_i$$

$${}^nC_i = \binom{n}{i} = \frac{n!}{i!\,(n-i)!} \quad ; \quad \binom{n}{0} \equiv 1$$

eg. $\binom{4}{1} = {}^4C_1 = \dfrac{4!}{1!\,(4-1)!} = \dfrac{4 \times \cancel{3!}}{\cancel{3!}} = \underline{\underline{4}}$

$\longleftarrow n \longrightarrow$

#1	X			
#2		X		
#3			X	
#4				X

eg: $\binom{4}{2} = {}^4C_2 = \dfrac{4!}{2!\,(4-2)!} = \dfrac{1}{2}\,\dfrac{4!}{2!} = \dfrac{4 \times 3 \times 2!}{2 \times 2!} = \underline{\underline{6}}$

$\longleftarrow n \longrightarrow$

#1	X	X		
#2	X		X	
#3	X			X
#4		X	X	
#5		X		X
#6			X	X

FOR A CODEWORD TO CORRECT UP TO "t" ERRORS, THE NUMBER OF POSSIBLE CHECK BIT PATTERNS MUST BE AT LEAST EQUAL TO THE NUMBER OF WAYS THAT "t" ERRORS CAN OCCUR.

HENCE;

$$2^r \geqslant \sum_{i=0}^{t} \binom{n}{i}$$

OR

$$2^r \geqslant \left[1 + \sum_{i=1}^{t} \binom{n}{i}\right]$$

EXAMPLES OF MULTIPLE ERROR CORRECTING CODES

$(15,8)$ - DOUBLE ERROR CORRECTION, $d_{min} = 5$

$(23,12)$ - TRIPLE ERROR CORRECTION, $d_{min} = 7$.

SINGLE ERROR CORRECTION CODING

AS AN EXAMPLE WE WILL CONSIDER A $(7,4)$ CODE
$n=7$, $k=4$, $r = n-k = 3$

THE CODEWORD DOESN'T HAVE TO BE SYSTEMATIC SO WE WON'T ASSUME IT IS TO START WITH.

LET THE CODEVECTOR BE

$$C = [c_1 \; c_2 \; c_3 \; c_4 \; c_5 \; c_6 \; c_7]$$

AT THE RECEIVER, WE CARRY OUT THE 3 PARITY CHECKS TO LOCATE THE ERROR

NOTE $2^3 = 8$, 7 ERROR POSITIONS, PLUS NO-ERROR

i.e.

P_1	P_2	P_3	ERROR POSITION
0	0	1	c_1
0	1	0	c_2
0	1	1	c_3
1	0	0	c_4
1	0	1	c_5
1	1	0	c_6
1	1	1	c_7

THE ALLOCATION COULD BE DIFFERENT, SO LONG AS EACH ERROR POSITION IS UNIQUELY IDENTIFIED. e.g. ROWS OF P_1, P_2, P_3 COULD BE SWAPPED

THUS, IF $P_1 = 1$, THEN THE ERROR LIES IN BITS C_4, C_5, C_6, C_7.

$P_1 P_2 P_3 = 000$ - NO (OR UNDETECTED) ERROR.

HENCE;

$$P_1 = C_4' \oplus C_5' \oplus C_6' \oplus C_7'$$

$$P_2 = C_2' \oplus C_3' \oplus C_6' \oplus C_7'$$

$$P_3 = C_1' \oplus C_3' \oplus C_5' \oplus C_7'$$

WHERE C' IS THE RECEIVED CODE-WORD

ie.

$$C \rightarrow \boxed{Tx} \rightarrow \boxed{\text{CHANNEL}} \rightarrow \boxed{Rx} \rightarrow C'$$

AT THE TRANSMITTER, NO ERRORS WILL HAVE OCCURED SO $(P_1, P_2, P_3 = 0,0,0)$

$$\left.\begin{aligned} 0 &= C_4 \oplus C_5 \oplus C_6 \oplus C_7 \\ 0 &= C_2 \oplus C_3 \oplus C_6 \oplus C_7 \\ 0 &= C_1 \oplus C_3 \oplus C_5 \oplus C_7 \end{aligned}\right\} \text{THESE ARE THE PARITY CHECK EQUATIONS}$$

IF WE WRITE A "1" FOR EVERY POSITION CHECKED, AND A "0" OTHERWISE, WE OBTAIN THE PARITY CHECK MATRIX [H],

$$[H] = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

[AGAIN, WE COULD SWAP THE COLUMNS IF WE WISHED]

AND

$$[C_1\, C_2\, C_3\, C_4\, C_5\, C_6\, C_7] \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = [0\,0\,0]$$

THAT IS TO SAY;

$$C\,[H]^T = 0 \quad \text{FOR ALL VALID CODEWORDS}$$

IF C IS THE CODEVECTOR TRANSMITTED OVER A NOISY CHANNEL AND R IS THE NOISE CORRUPTED VECTOR AT THE RECEIVER;

$$R = C \oplus E,$$

WHERE E IS THE ERROR VECTOR OF THE FORM

$$E = [e_1\, e_2\, e_3\, e_4\, e_5\, e_6\, e_7]$$

WHICH INDICATES THE ERROR POSITION

AT THE RECEIVER, WE COMPUTE ...

$$S = R[H]^T = (C \oplus E)[H]^T$$

$$S = C[H]^T + E[H]^T.$$

RECALL THAT BY DEF^N. $C[H]^T \equiv 0$.

$$\Rightarrow \quad S = E[H]^T.$$

THE VECTOR S IS CALLED THE <u>ERROR SYNDROME</u>

IF AN ERROR OCCURS IN TRANSMISSION, S IS NON-ZERO AND IS RELATED TO THE ERROR VECTOR E. THE DECODER USES S TO DETECT AND CORRECT CODEWORDS

Eg. FOR OUR (7,4) CODE, A VALID CODEWORD IS 0110011. IF THIS IS RECEIVED WITHOUT ERROR $S = R[H]^T$ SHOULD BE ZERO

$$[0110011] \begin{bmatrix} 0&0&1 \\ 0&1&0 \\ 0&1&1 \\ 1&0&0 \\ 1&0&1 \\ 1&1&0 \\ 1&1&1 \end{bmatrix} = [1\oplus1, 1\oplus1\oplus1\oplus1, 1\oplus1] = [0\ 0\ 0]$$

NO ERROR

$$R \qquad [H]^T \qquad = S$$

SUPPOSE WE HAVE AN ERROR IN THE THIRD BIT THAT IS INSTEAD OF

$$R = [0110011]$$

WE HAVE

$$R = [0100011]$$

ie

$$C = 0110011$$
$$E = 0010000$$

$$R = C \oplus E = [0110011] \oplus [0010000]$$

$$[0100011] \begin{bmatrix} 0&0&1 \\ 0&1&0 \\ 0&1&1 \\ 1&0&0 \\ 1&0&1 \\ 1&1&0 \\ 1&1&1 \end{bmatrix} = [0\ 1\ 1]$$

Hence $S = [011]$, WHICH INDICATES THAT THERE IS AN ERROR IN THE THIRD BIT.

THE SYNDROME S FOR AN ERROR IN THE THIRD BIT OF R IS THE THIRD ROW OF $[H]^T$, ie. THE 3rd COLUMN OF $[H]$

THE SYNDROME IS A BINARY REPRESENTATION OF THE LOCATION OF THE ERROR IN POSITIONS 1-7.

CODE GENERATION

THE PARITY CHECK EQUATIONS FOR OUR (7,4) CODE ARE REPEATED HERE AS;

$$0 = C_4 \oplus C_5 \oplus C_6 \oplus C_7$$

$$0 = C_2 \oplus C_3 \oplus C_6 \oplus C_7$$

$$0 = C_1 \oplus C_3 \oplus C_5 \oplus C_7$$

THESE PARITY CHECK BITS HAVE BEEN OBTAINED FROM A LINEAR COMBINATION OF MESSAGE BITS

PARITY BITS b_1, b_2, b_3

MESSAGE BITS m_1, m_2, m_3, m_4.

THE SIMPLEST WAY TO GENERATE THE CODE SINCE C_1, C_2 AND C_4 APPEAR ONLY ONCE IN ONLY ONE OF THE PARITY CHECK EQUATIONS IS TO LET THESE BE THE PARITY CHECK BITS

ie $b_3 = C_4$, $b_2 = C_2$, $b_1 = C_1$

HENCE;

$$b_3 = C_5 \oplus C_6 \oplus C_7$$

$$b_2 = C_3 \oplus C_6 \oplus C_7$$

$$b_1 = C_3 \oplus C_5 \oplus C_7$$

HENCE WE ALLOCATE THE MESSAGE BITS AS FOLLOWS;

$$b_3 = m_2 \oplus m_3 \oplus m_4$$

$$b_2 = m_1 \oplus m_3 \oplus m_4$$

$$b_1 = m_1 \oplus m_2 \oplus m_4$$

HENCE, THE CODE-WORD IS;

$$\begin{array}{ccccccc} b_1 & b_2 & & b_3 & & & \\ \downarrow & \downarrow & & \downarrow & & & \\ C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 \\ & & \uparrow & & \uparrow & \uparrow & \uparrow \\ & & m_1 & & m_2 & m_3 & m_4 \end{array}$$

ie $[b_1\ b_2\ m_1\ b_3\ m_2\ m_3\ m_4]$

THIS CODE IS NOT SYSTEMATIC.

$C_1 \ldots\ldots C_7$	SYNDROME	#		NOTE
0 0 0 0 0 0 0	0 0 0	0	b_1	ERRORS IN
1 0 0 0 0 0 0	0 0 1	1	b_1	CHECK
0 1 0 0 0 0 0	0 1 0	2	b_2	POSITIONS
0 0 1 0 0 0 0	0 1 1	3	m_1	HAVE SINGLE
0 0 0 1 0 0 0	1 0 0	4	b_3	BIT SYNDROMES
0 0 0 0 1 0 0	1 0 1	5	m_2	
0 0 0 0 0 1 0	1 1 0	6	m_3	
0 0 0 0 0 0 1	1 1 1	7	m_4	

WE CAN CORRECT THE CODEWORD DIRECTLY FROM THE SYNDROME

SYSTEMATIC LINEAR BLOCK CODES

WE REQUIRE A MORE STRUCTURED WAY OF GENERATING A CODE THAT IS EASIER TO HANDLE MATHEMATICALLY...

FOR A (7,4) CODE (SYSTEMATIC) WE HAVE;

$$\begin{matrix} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 \\ m_1 & m_2 & m_3 & m_4 & b_1 & b_2 & b_3 \end{matrix}$$

HENCE, THE PARITY CHECK EQUATIONS (BOTTOM p5) BECOME;

$$0 = m_4 \oplus b_1 \oplus b_2 \oplus b_3 \qquad -1$$
$$0 = m_2 \oplus m_3 \oplus b_2 \oplus b_3 \qquad -2$$
$$0 = m_1 \oplus m_3 \oplus b_1 \oplus b_3 \qquad -3$$

SOLVING FOR b_1, b_2, b_3 (BY SUBSTITUTION AND RE-ARRANGEMENT) FOR EXAMPLE FROM 2 WE HAVE $m_2 \oplus m_3 = b_2 \oplus b_3$, SUBSTITUTING THIS INTO 1 WE HAVE $b_1 = m_2 \oplus m_3 \oplus m_4$

WE OBTAIN;

$$0 = m_2 \oplus m_3 \oplus m_4 \oplus b_1$$
$$0 = m_1 \oplus m_3 \oplus m_4 \oplus b_2$$
$$0 = m_1 \oplus m_2 \oplus m_4 \oplus b_3$$

IN MATRIX FORM WE HAVE

$$[H_s] = \left[\begin{array}{cccc:ccc} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

AS BEFORE, $C[H_s]^T = 0$ FOR ALL VALID CODEWORDS.

NOTE, WE COULD HAVE OBTAINED $[H_s]$ BY SHUFFLING THE COLUMNS OF $[H]$

IF THE OPERATION $R[H_s]^T = S$ IS USED THEN A SINGLE ERROR WILL PRODUCE A SYNDROME WHOSE ROW NUMBER IN THE $[H_s]^T$ MATRIX IS THE SAME AS THE ERROR POSITION

E.g. FOR AN INPUT MESSAGE OF 1110, FIND THE CODEWORD AND SHOW THAT THE SYNDROME IDENTIFIES AN ERROR IN POSITION 5 OF THE CODE WORD

FROM THE [H]-MATRIX, CONSTRUCT THE PARITY EQUATIONS, i.e.

$$\left[\begin{array}{cccc:ccc} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \qquad \begin{aligned} b_1 &= m_2 \oplus m_3 \oplus m_4 \\ b_2 &= m_1 \oplus m_3 \oplus m_4 \\ b_3 &= m_1 \oplus m_2 \oplus m_4 \end{aligned}$$

THEREFORE, THE CODEWORD FOR $\overset{m_1 \dots m_4}{1110}$ IS

$$b_1 = 1 \oplus 1 \oplus 0 = 0$$
$$b_2 = 1 \oplus 1 \oplus 0 = 0$$
$$b_3 = 1 \oplus 1 \oplus 0 = 0$$

HENCE THE COMPLETE CODE WORD INCLUDING PARITY BITS IS

$$\overbrace{1110}^{m_1 \dots m_4}\ \underbrace{000}_{b_1 \dots b_3}$$

WITH AN ERROR IN THE 5^{th} BIT THE CODE WORD BECOMES 1110100.

THE SYNDROME $S = R[H_s]^T$.

$$[1110100]\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [1 \oplus 1 \oplus 1 \quad 1 \oplus 1 \quad 1 \oplus 1]$$
$$[100]$$

100, i.e. THE FIFTH ROW OF $[H_s]^T$

NOTE THERE IS ONLY ONE "1" IN THE SYNDROME THEREFORE THE ERROR IS IN THE PARITY BITS, SO THE MESSAGE IS OK

SYSTEMATIC CODE GENERATION

THE PARITY CHECK MATRIX WILL BE OF THE FORM:

$$[H_s] = \left[\begin{array}{ccccc|ccccc} h_{11} & h_{12} & h_{13} & \cdots & h_{1k} & 1 & 0 & 0 & \cdots & 0 \\ h_{21} & h_{22} & h_{23} & \cdots & h_{2k} & 0 & 1 & 0 & \cdots & 0 \\ h_{31} & h_{32} & h_{33} & \cdots & h_{3k} & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{r1} & h_{r2} & h_{r3} & \cdots & h_{rk} & 0 & 0 & 0 & \cdots & 1 \end{array}\right]$$

PARITY SECTION — IDENTITY MATRIX

FOR AN (n,k) CODE WITH CODEWORDS

$$C = [C_1 C_2 \dots C_k \dots C_n]$$
$$[m_1 m_2 \dots m_k b_1 \dots b_r]$$

$$k + r = n$$

So;

$$C_1 = m_1 = 1m_1 \oplus \emptyset m_2 \oplus \emptyset m_3 \dots \emptyset m_k$$
$$C_2 = m_2 = 0m_1 \oplus 1m_2 \oplus \emptyset m_3 \dots \emptyset m_k$$
$$\vdots$$
$$C_k = m_k = \emptyset m_1 \oplus \emptyset m_2 \oplus \emptyset m_3 \dots 1m_k$$
$$C_{k+1} = b_1 = h_{11}m_1 \oplus h_{12}m_2 \oplus h_{13}m_3 \dots h_{1k}m_k$$
$$C_{k+2} = b_2 = h_{21}m_1 \oplus h_{22}m_2 \oplus h_{23}m_3 \dots h_{2k}m_k$$
$$\vdots$$
$$C_n = C_{k+r} = b_r = h_{r1}m_1 \oplus h_{r2}m_2 \oplus h_{r3}m_3 \dots h_{rk}m_k$$

IN MATRIX FORM THIS IS;

$$[C_1 C_2 \ldots C_n] = [M_1 M_2 \ldots M_k]\left[\begin{array}{ccccc|cccc} 1 & 0 & 0 & \cdots & 0 & h_{11} & h_{21} & \cdots & h_{r1} \\ 0 & 1 & 0 & \cdots & 0 & h_{12} & h_{22} & \cdots & h_{r2} \\ 0 & 0 & 1 & \cdots & 0 & h_{13} & h_{23} & \cdots & h_{r3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & h_{1k} & h_{2k} & & h_{rk} \end{array}\right]$$

THAT IS $\quad C = M[G]$

$[G]$ IS THE <u>GENERATOR MATRIX</u> OF THE CODE

$$[G] = [I_k \vdots P]_{k \times n}$$

$[G]$ HAS BEEN DERIVED FROM THE PARITY CHECK MATRIX $[H]$

$$[H] = [P^T \vdots I_R]_{r \times n}$$

$$[H]^T = \left[\begin{array}{c} P \\ \hline I_R \end{array}\right]_{n \times r}$$

FOR EXAMPLE; FOR THE (7,4) CODE WE HAVE BEEN CONSIDERING;

$$[H] = \left[\begin{array}{cccc|ccc} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] = [P^T \vdots I_3]$$

HENCE $[G] = [I_k \vdots P] = [I_4 \vdots P]$

$$[G] = \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$$

<u>NOTE</u> ALL THE REMAINING CODEWORDS CAN BE OBTAINED BY MOD-2 ADDING THESE VECTORS

e.g. $M = 0000 \qquad C = 0000\ 000$ BY DEF^N.
$ = 0001 \qquad C = 0001\ 111$
$ = 0010 \qquad C = 0010\ 110$
$ = 0011 \qquad C = 0011\ 001$ MOD-2 ADD THESE

etc.

CODE GENERATION

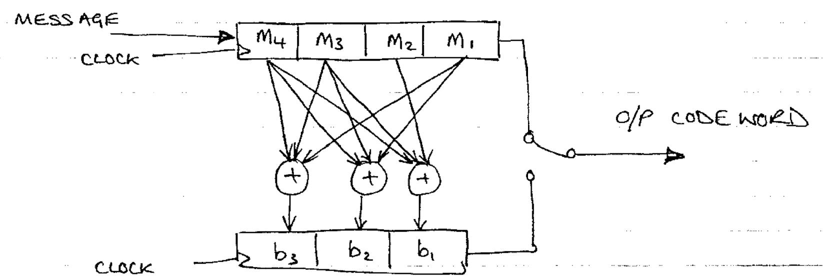
FOR THIS EXAMPLE, A CODEWORD IS;

$$C_1 \ldots\ldots\ldots\ldots C_7$$
$$m_1\ m_2\ m_3\ m_4\ b_1\ b_2\ b_3$$

$$(C_5)\ b_1 = m_2 \oplus m_3 \oplus m_4$$
$$(C_6)\ b_2 = m_1 \oplus m_3 \oplus m_4$$
$$(C_7)\ b_3 = m_1 \oplus m_3 \oplus m_4$$

THIS CODE CAN BE IMPLEMENTED USING TWO SHIFT REGISTERS, AND 'r' MOD-2 ADDERS (EX-OR GATE). SHIFT REGISTERS ARE "k" AND "r" BITS LONG.

THE INFORMATION (MESSAGE) BITS ARE SHIFTED INTO "k"-BIT SHIFT REGISTER. THE "r" PARITY CHECK BITS ARE COMPUTED AND TEMPORARILY STORED IN A SHIFT REGISTER. THE MESSAGE BITS ARE THEN CLOCKED OUT FOLLOWED BY THE PARITY BITS.



SYSTEMATIC LINEAR BLOCK CODER