Example 1

Which one of these two calculations do you find easier to understand?

CALCULATION 1

$$\int \sin^4 x\,dx \qquad u = \sin^3 x \quad \frac{dv}{dx} = \sin x$$

$$\frac{du}{dx} = 3\sin^2 x\cos x \quad v = -\cos x.$$

$$= -\sin^3 x\cos x - \int -3\sin^2 x\cos^2 x\,dx$$

$$= -\sin^3 x\cos x + 3\int \sin^2 x\cos^2 x\,dx$$

$$= -\sin^3 x\cos x + 3\int \sin^2 x(1-\sin^2 x)\,dx$$

$$= -\sin^3 x\cos x + 3\int \sin^2 x\,dx - 3\int \sin^4 x\,dx$$

$$\cos(2x) = 1 - 2\sin^2 x$$
$$\sin^2 x = \tfrac{1}{2}(1-\cos(2x))$$
$$\int \sin^2 x\,dx = \tfrac{1}{2}\int 1-\cos(2x)\,dx = \tfrac{1}{2}x - \tfrac{1}{4}\sin(2x)$$

$$-\sin^3 x\cos x + 3\int \sin^2 x\,dx - 3\int \sin^4 x\,dx$$

$$= -\sin^3 x\cos x + 3\left(\tfrac{1}{2}x - \tfrac{1}{4}\sin(2x)\right) - 3\int \sin^4 x\,dx$$

$$= -\sin^3 x\cos x + \tfrac{3}{2}x - \tfrac{3}{2}\sin x\cos x - 3\int \sin^4 x\,dx$$

$$4\int \sin^4 x\,dx = -\sin^3 x\cos x + \tfrac{3}{2}x - \tfrac{3}{2}\sin x\cos x$$

$$\boxed{\tfrac{3}{8}x - \tfrac{1}{4}\sin^3 x\cos x - \tfrac{3}{8}\sin x\cos x + c}$$

CALCULATION 2

Let $I = \int \sin^4 x\,dx$.

Let $u = \sin^3 x$ so $\frac{du}{dx} = 3\sin^2 x\cos x$ and $\frac{dv}{dx} = \sin x$ so $v = -\cos x$.

Then integrating by parts gives

$$I = -\sin^3 x\cos x - \int -3\sin^2 x\cos^2 x\,dx$$

$$= -\sin^3 x\cos x + 3\int \sin^2 x\cos^2 x\,dx$$

$$= -\sin^3 x\cos x + 3\int \sin^2 x(1-\sin^2 x)\,dx$$

$$= -\sin^3 x\cos x + 3\int \sin^2 x\,dx - 3\int \sin^4 x\,dx$$

$$= -\sin^3 x\cos x + 3\int \sin^2 x\,dx - 3I$$

So $4I = -\sin^3 x\cos x + 3\int \sin^2 x\,dx$ $(*)$

To find $\int \sin^2 x\,dx$, note that

$$\cos(2x) = 1 - 2\sin^2 x \quad \text{so} \quad \sin^2 x = \tfrac{1}{2}(1-\cos(2x)).$$

So, from $(*)$,

$$4I = -\sin^3 x\cos x + 3\int \sin^2 x\,dx$$

$$= -\sin^3 x\cos x + \tfrac{3}{2}\int 1-\cos(2x)\,dx$$

$$= -\sin^3 x\cos x + \tfrac{3}{2}x - \tfrac{3}{4}\sin(2x) + c.$$

$$= -\sin^3 x\cos x + \tfrac{3}{2}x - \tfrac{3}{2}\sin x\cos x + c.$$

So $I = \frac{3}{8}x - \frac{1}{4}\sin^3 x\cos x - \frac{3}{8}\sin x\cos x + c.$

(taken from "How to study for a mathematics degree" by Lara Alcock)