The Watson transformation revisited

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Abstract

The Watson transformation is essentially the idea that a series can be converted into an integral in the complex plane via an "inverse residue" calculation. This idea has a long history of being used to convert expressions for the solutions of boundary value problems (BVPs) into alternative expressions from which it is easier to extract information about the solution. However, the technique is ad hoc, since given a series there are many different integrals whose residues are equal to that series. In this paper we show that, for the BVP on which the Watson transformation was first used, the optimal expression for the solution as an integral can be obtained in an algorithmic (as opposed to ad hoc) way using the Fokas transform method.

Keywords: transform method, Helmholtz equation, Watson transformation, Fokas method.

AMS subject classification: 35A22, 35C05, 35C15, 35J05, 35P10

1 Introduction

1.1 The background of the Watson transformation

For a very restrictive class of boundary value problems (BVPs), transform methods can be used to find explicit expressions for the solution. Even when one has such an expression, however, it may not be of any use. This situation was famously encountered for the BVPs of the Helmholtz equation posed in the exterior of a disk (in 2-d) or ball (in 3-d). In this paper we focus on the 2-d case, and thus the BVP we consider is the following.

Problem 1.1 (The Dirichlet problem for the Helmholtz equation in the exterior of a disk) *Let Ω be the exterior of a disk of radius a, i.e.*

$$\Omega = \left\{a < r < \infty,\ 0 \leq \theta < 2\pi\right\}. \tag{1.1}$$

Given $d \in C(\partial\Omega)$ and $k > 0$, find $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfying the PDE

$$\Delta u + k^2 u = 0 \quad \text{in } \Omega, \tag{1.2}$$

the Dirichlet boundary condition

$$u(a, \theta) = d(\theta), \quad 0 \leq \theta < 2\pi, \tag{1.3}$$

and the Sommerfeld radiation condition

$$\sqrt{r}\left(\frac{\partial u}{\partial r} - \mathrm{i}ku\right) \to 0 \quad \text{as } r \to \infty \tag{1.4}$$

(uniformly in θ).

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With solutions of the Helmholtz equation corresponding to solutions of the wave equation by multiplying by $\exp(-\mathrm{i}\omega t)$, the radiation condition (1.4) ensures that waves propagate away from the obstacle (in this case the disk), and the *wavenumber* k equals ω/c, where c is the wave speed.

For simplicity we only consider the homogeneous Helmholtz equation, but everything that follows also applies to the inhomogeneous Helmholtz equation, i.e. $\Delta u + k^2 u = -f$ for some prescribed function f (see [46, §4.3], [10]). In many scattering applications, the function d in the Dirichlet boundary condition is the incident wave restricted to the boundary (and therefore is a real-analytic function), and then u is the scattered field.

Seeking to find an explicit expression for the solution of Problem 1.1, we recall that the Laplacian is separable in 2-d polar coordinates, and thus either of the two separated ODEs (one in the r-variable and one in the θ-variable) can be used as the basis for a transform. The appropriate transform in the θ-variable for solving Problem 1.1 is the standard Fourier series (this fact can be determined by spectral analysis of the relevant ODE; see [24, §8.1.3], [48, p.154], [16, p.259], [39, §4.4, §5.7, §5.8], [15, §2.1]), and this transform yields the explicit expression for the solution

$$u(r,\theta) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{H_n^{(1)}(kr)}{H_n^{(1)}(ka)} \mathrm{e}^{\mathrm{i}n\theta} D(-\mathrm{i}n), \tag{1.5}$$

where $H_n^{(1)}$ denotes the Hankel function of the first kind, and

$$D(-\mathrm{i}n) = \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}n\phi} d(\phi)\, \mathrm{d}\phi, \quad n \in \mathbb{Z}.$$

On the face of it, we have found an explicit expression for the solution of Problem 1.1, so what more is to be said?

The answer is twofold: (i) we are interested in the large-k asymptotics of u (or, more precisely, the asymptotics as $ka \to \infty$), and (ii) it is extremely difficult to obtain these asymptotics from the explicit expression (1.5).

Regarding (i): in many applications one is interested in the large-k asymptotics of the solution of Problem 1.1, not only because it is the only BVP involving scattering by a bounded obstacle for which explicit solutions can be obtained, but also because in the geometrical theory of diffraction [25] it is the appropriate "canonical problem" for understanding scattering by obstacles with strictly positive curvature. Therefore, knowledge of the large-k asymptotics of Problem 1.1 has formed the basis of many investigations of scattering from general 2-d convex obstacles with strictly positive curvature [49], [4], [18], [50], [31], [5], [6, Chapter 13].

Regarding (ii): when $ka \gg 1$ the series (1.5) converges extremely slowly. Indeed, for a relative error of the order of one percent one needs to sum the first ka terms [9, §2.1], [38, §II] (and in the case when the sphere is the Earth and one considers radio waves, $ka = 8000$ [30, Page 118]).

In 1918, Watson overcame the difficultly of the slow convergence of the series (1.5) via the so-called *Watson transformation* [52]. This transformation converts the slowly converging series (1.5) into a different series that converges rapidly (with summing the first few terms of the new series sufficient for almost all applications). The Watson transformation consists of the following two steps:

(a) Convert the series into an integral using Cauchy's residue theorem. For example, if f is analytic in a neighbourhood of the real axis and has sufficient decay at infinity then

$$\sum_{n=-\infty}^{\infty} f(n) = -\int_C \frac{f(\nu)}{1 - \mathrm{e}^{2\pi \mathrm{i}\nu}} \mathrm{d}\nu, \tag{1.6}$$

where C is a contour that encloses the real ν-axis (in the positive sense) but not any of the singularities of $f(\nu)$ (see Figure 1).

(b) Deform the contour C to enclose the poles of $f(\nu)$ and then evaluate the integral in terms of the residues at these poles.

We now apply these two steps to the series solution (1.5).

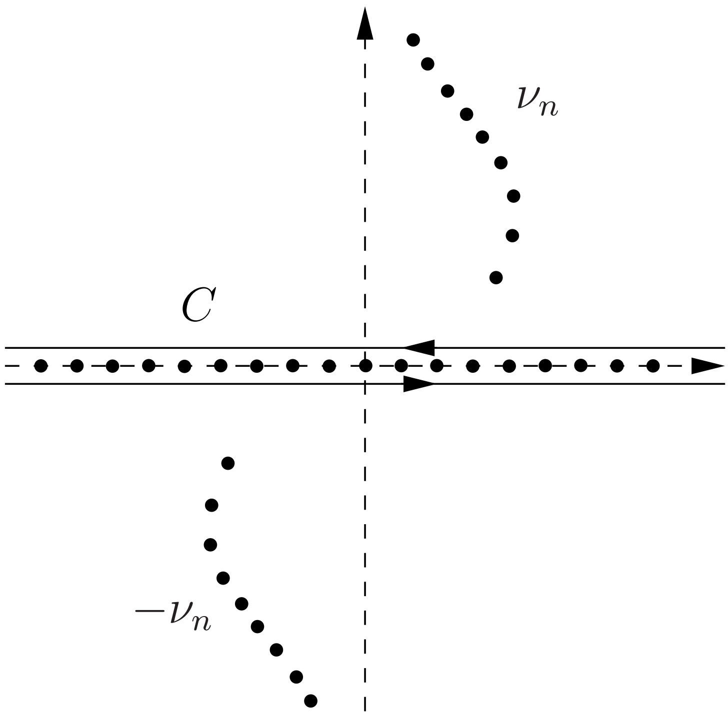

Figure 1: The poles and contours involved when the Watson transformation is applied to Problem 1.1.

Step (a). We have that

$$f(\nu) = \mathrm{e}^{\mathrm{i}\nu\theta} D(-\mathrm{i}\nu) \frac{H_\nu^{(1)}(kr)}{H_\nu^{(1)}(ka)}.$$

This function is analytic in the complex plane apart from poles at the zeros of $H_\nu^{(1)}(ka)$, which are in the first and third quadrants; see Theorem 3.1 below. We denote the zeros in the first quadrant by $\{\nu_n\}_{n=1}^{\infty}$; by the relation

$$H_{-\nu}^{(1)}(z) = \mathrm{e}^{\mathrm{i}\pi\nu} H_\nu^{(1)}(z), \tag{1.7}$$

the zeros in the third quadrant are $\{-\nu_n\}_{n=1}^{\infty}$. We apply (1.6) with the contour C such that

$$\int_C \mathrm{d}\nu = -\int_{-\infty+\mathrm{i}c}^{\infty+\mathrm{i}c} \mathrm{d}\nu + \int_{-\infty-\mathrm{i}c}^{\infty-\mathrm{i}c} \mathrm{d}\nu,$$

where the constant c is chosen so that $c < \Im(\nu_1)$; see Figure 1. We therefore obtain that

$$u(r,\theta) = \frac{1}{2\pi}\left(\int_{-\infty+\mathrm{i}c}^{\infty+\mathrm{i}c} - \int_{-\infty-\mathrm{i}c}^{\infty-\mathrm{i}c}\right) \frac{\mathrm{e}^{\mathrm{i}\nu\theta} D(-\mathrm{i}\nu)}{1-\mathrm{e}^{2\pi\mathrm{i}\nu}} \frac{H_\nu^{(1)}(kr)}{H_\nu^{(1)}(ka)} \mathrm{d}\nu. \tag{1.8}$$

Step (b). We now seek to evaluate the integral (1.8) as residues at $\{\pm\nu_n\}_{n=1}^{\infty}$. It is convenient to use the transformation $\nu \mapsto -\nu$ in the integral over $(-\infty - \mathrm{i}c, \infty - \mathrm{i}c)$ in (1.8) to convert the expression (1.8) into one integral on the contour $(-\infty + \mathrm{i}c, \infty + \mathrm{i}c)$ (i.e. an integral just above the real axis). Recalling the relation (1.7), we find that this procedure yields

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\infty+\mathrm{i}c}^{\infty+\mathrm{i}c} \left[\frac{\mathrm{e}^{\mathrm{i}\nu\theta} D(-\mathrm{i}\nu)}{1-\mathrm{e}^{2\pi\mathrm{i}\nu}} - \frac{\mathrm{e}^{-\mathrm{i}\nu\theta} D(\mathrm{i}\nu)}{1-\mathrm{e}^{-2\pi\mathrm{i}\nu}}\right] \frac{H_\nu^{(1)}(kr)}{H_\nu^{(1)}(ka)} \mathrm{d}\nu. \tag{1.9}$$

We now seek to close the contour of integration in (1.9) at infinity in the upper half-plane (UHP), and we therefore need to understand how the integrand of (1.9) behaves as $|\nu| \to \infty$ in this region. The behaviour of $H_\nu^{(1)}(kr)/H_\nu^{(1)}(ka)$ as $|\nu|\to\infty$ is subtle, and we return to this later. By integration by parts (assuming that d is sufficiently smooth), one can show that the second term in the square brackets in (1.9) is bounded as $\nu\to\infty$ in the UHP (see (4.4b) below), but the first term need not be. To deal with this latter fact, we use the equation

$$\frac{1}{1-\mathrm{e}^{2\pi\mathrm{i}\nu}} = 1 - \frac{1}{1-\mathrm{e}^{-2\pi\mathrm{i}\nu}} \tag{1.10}$$

in the first term in square brackets in (1.9), and obtain

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{H_\nu^{(1)}(kr)}{H_\nu^{(1)}(ka)}\mathrm{e}^{\mathrm{i}\nu\theta}D(-\mathrm{i}\nu)\mathrm{d}\nu - \frac{1}{2\pi}\int_{-\infty+\mathrm{i}c}^{\infty+\mathrm{i}c}\frac{H_\nu^{(1)}(kr)}{H_\nu^{(1)}(ka)}\left[\frac{\mathrm{e}^{\mathrm{i}\nu\theta}D(-\mathrm{i}\nu)+\mathrm{e}^{-\mathrm{i}\nu\theta}D(\mathrm{i}\nu)}{1-\mathrm{e}^{-2\pi\mathrm{i}\nu}}\right]\mathrm{d}\nu \tag{1.11}$$

(where have deformed the contour in first integral from $(-\infty+\mathrm{i}c,\infty+\mathrm{i}c)$ to real axis using analyticity and the fact that $H_\nu^{(1)}(kr)/H_\nu^{(1)}(ka)$ decays exponentially as $|\nu|\to\infty$ near the real axis). The term in square brackets in (1.11) is now bounded at infinity in the UHP. To deal with $\mathrm{e}^{\mathrm{i}\nu\theta}D(-\mathrm{i}\nu)$ in the first integral, we write $D(-\mathrm{i}\nu)$ as $D_L(-\mathrm{i}\nu)+D_R(-\mathrm{i}\nu)$, where

$$D_L(\pm\mathrm{i}\nu) = \int_0^\theta e^{\pm\mathrm{i}\nu\phi}d(\phi)\,\mathrm{d}\phi, \quad D_R(\pm\mathrm{i}\nu) = \int_\theta^{2\pi} e^{\pm\mathrm{i}\nu\phi}d(\phi)\mathrm{d}\phi,$$

let $\nu\mapsto-\nu$ in the term involving D_R (using (1.7)), and then deform the contours from $(-\infty,\infty)$ to $(-\infty+\mathrm{i}c,\infty+\mathrm{i}c)$. This yields the expression

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\infty+\mathrm{i}c}^{\infty+\mathrm{i}c}\frac{H_\nu^{(1)}(kr)}{H_\nu^{(1)}(ka)}\left[\mathrm{e}^{\mathrm{i}\nu\theta}D_L(-\mathrm{i}\nu)+\mathrm{e}^{-\mathrm{i}\nu\theta}D_R(\mathrm{i}\nu) - \frac{\mathrm{e}^{\mathrm{i}\nu\theta}D(-\mathrm{i}\nu)+\mathrm{e}^{-\mathrm{i}\nu\theta}D(\mathrm{i}\nu)}{1-\mathrm{e}^{-2\pi\mathrm{i}\nu}}\right]\mathrm{d}\nu. \tag{1.12}$$

Closing the contour of the integral in (1.12) at infinity in the UHP and evaluating the integral as residues on $\{\nu_n\}_{n=1}^\infty$ yields

$$u(r,\theta) = \mathrm{i}\sum_{n=1}^\infty\frac{H_{\nu_n}^{(1)}(kr)}{\dot{H}_{\nu_n}^{(1)}(ka)}\left[\mathrm{e}^{\mathrm{i}\nu_n\theta}D_L(-\mathrm{i}\nu_n)+\mathrm{e}^{-\mathrm{i}\nu_n\theta}D_R(\mathrm{i}\nu_n) - \frac{\mathrm{e}^{\mathrm{i}\nu_n\theta}D(-\mathrm{i}\nu_n)+\mathrm{e}^{-\mathrm{i}\nu_n\theta}D(\mathrm{i}\nu_n)}{1-\mathrm{e}^{-2\pi\mathrm{i}\nu_n}}\right], \tag{1.13}$$

where

$$\dot{H}_{\nu_n}^{(1)}(ka) = \frac{\mathrm{d}}{\mathrm{d}\nu}H_\nu^{(1)}(ka)\bigg|_{\nu=\nu_n},$$

and the Watson transformation procedure is now complete.

There are two important points to note about the series (1.13).

1. Around 1950, Sommerfeld [45, Appendix II of Chapter 5, Appendix to Chapter 6] showed that the series (1.13) can be obtained *directly* by constructing the transform associated with the *radial* ODE (using the algorithm described in, e.g., [24, §8.1.3], [48, p.154], [16, p.259], [39, §4.4,§5.7,§5.8], [15, §2.1])) and applying this transform to Problem 1.1 instead of the transform in the θ-variable. Indeed, spectral analysis of the radial ODE with the boundary condition $u(a,\theta)=0$ and the radiation condition (1.4) yields the (formal) completeness relation
$$\rho\,\delta(r-\rho) = -\pi\mathrm{i}\sum_{n=1}^\infty\frac{\nu_n H_{\nu_n}^{(1)}(k\rho)H_{\nu_n}^{(1)}(kr)J_{\nu_n}(ka)}{\dot{H}_{\nu_n}^{(1)}(ka)}, \tag{1.14}$$
see [35, p.299], [12, p.116] (these papers present the analogous completeness relation in the three dimensional case with Neumann boundary conditions, but the derviation of (1.14) is very similar). Applying the transform obtained from (1.14) to Problem 1.1 yields (1.13).

2. The series on the right-hand side of (1.13) does *not* converge for all θ. Indeed, if θ is such that $d(\theta)\neq 0$, then the series on the right-hand side of (1.13) diverges. Where did we go

wrong above in deriving (1.13)? The answer is that when $d(\theta) \neq 0$ and the contour in (1.12) is closed at infinity in the UHP, the integral at infinity is unbounded. This is due to the subtle behaviour of $H_\nu^{(1)}(kr)/H_\nu^{(1)}(ka)$ as $|\nu| \to \infty$; see §4. The divergence of the series (1.13) means that the transform associated with (1.14) is not complete. The relation (1.14) came from spectral analysis of the radial ODE

$$-r\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - k^2r^2u = \frac{\lambda}{r}u \tag{1.15}$$

on (a,∞), with the boundary condition $u(a,\theta) = 0$, and with the additional condition that the eigenfunctions satisfy the radiation condition (1.4). Although the differential operator (1.15) is formally self-adjoint, the presence of a complex coefficient in the radiation condition means that the appropriate constraints on the boundary conditions for self-adjointness are not satisfied, and thus this 1-d BVP is not self-adjoint. Therefore, the general results that ensure completeness of transforms associated with self-adjoint problems do not apply (see [15, §2.3] and the references therein). (Pflumm [42] was the first to note that the series in (1.13) diverges, and Cohen [10] proved that the transform pair obtained from (1.14) is *not* complete.)

Despite the fact that for some values of θ it diverges, the series on the right-hand side of (1.13) is still useful in the following three special cases:

1. When $d(\theta) = \delta(\theta-\theta_0)$, for some θ_0, the series (1.13) converges for all $\theta \neq \theta_0$. (Note that [49], [4], [50], [51], and [5] are interested in the analogue of this case for the Neumann problem, i.e. the boundary condition (1.3) is replaced by $(\partial u/\partial r)(a,\theta) = \delta(\theta-\theta_0)$, and the radial series for this problem also converges for all $\theta \neq \theta_0$.)

2. In some applications one is interested in the Green's function for Problem 1.1 (or the analogous Neumann problem), see, e.g., [18], [27, Part II], [28]. The analogue of (1.13) in this case converges when θ is not equal to the angular coordinate of the delta function on the right-hand side of the PDE (see [10, Theorem III]).

3. In the case of plane-wave scattering, d is given in terms of the incident field, and u is then the scattered field. By using certain identities involving Bessel functions [3, Equations 9.1.44 and 9.1.45], [22, Appendix A, Equation 1.23], one can obtain an expression for the total field as an integral similar to (1.12); see [6, Equation 13.1.5], [22, §8.3]. This integral can be evaluated as residues to obtain a radial series expansion, with this evaluation valid when (r,θ) is in the shadow of the obstacle (see [22, §8.3] for the proof of this in the case of an impedance boundary condition, and see also the discussion in [9, §I.2.13.6, Page 34]).

1.2 The main disadvantage of the Watson transformation

The main disadvantage of the Watson transformation is that *it is not clear a priori which "series-to-integral" formula (such as* (1.6)*) will allow you to easily obtain an integral in the complex plane that can be evaluated as residues at the poles of f.*

Indeed, for Problem 1.1 we used the formula (1.6) to convert the series solution (1.5) to the integral (1.8). However, there are many different ways in which we could have converted the series to an integral. For example, we could have used either

$$\sum_{n=-\infty}^{\infty} f(n) = \int_C \frac{f(\nu)}{1-\mathrm{e}^{-2\pi\mathrm{i}\nu}}\mathrm{d}\nu \tag{1.16}$$

or

$$\sum_{n=-\infty}^{\infty} f(n) = \frac{1}{2\pi\mathrm{i}}\int_C f(\nu)\frac{\pi\cos\pi\nu}{\sin\pi\nu}\mathrm{d}\nu \tag{1.17}$$

(with the latter expression appearing as [26, Equation 5.1]). Each of the three formulae (1.6), (1.16), (1.17) is valid if f is analytic in a neighbourhood of the real axis and has sufficient decay at infinity, and all three are equivalent (since each of the integrals is equal to the same series). The equation (1.16) follows from (1.6) by using (1.10), and writing the right-hand side of (1.17) as

$$\frac{1}{2}\int_C f(\nu)\frac{1+\mathrm{e}^{-2\pi\mathrm{i}\nu}}{1-\mathrm{e}^{-2\pi\mathrm{i}\nu}}\mathrm{d}\nu,$$

we see that (1.17) equals the average of (1.6) and (1.16).

If we use (1.16) to convert the series (1.5) into an integral instead of (1.6), we obtain an expression very similar to (1.8), and we can obtain the integral expression (1.12) is almost identical way to before. However, if we use (1.17) instead of (1.6), then we obtain the integral expression

$$u(r,\theta) = -\frac{1}{4\pi}\left(\int_{-\infty+\mathrm{i}c}^{\infty+\mathrm{i}c} - \int_{-\infty-\mathrm{i}c}^{\infty-\mathrm{i}c}\right)\mathrm{e}^{\mathrm{i}\nu\theta}D(-\mathrm{i}\nu)\frac{1+\mathrm{e}^{-2\pi\mathrm{i}\nu}}{1-\mathrm{e}^{2\pi\mathrm{i}\nu}}\frac{H_\nu^{(1)}(kr)}{H_\nu^{(1)}(ka)}\mathrm{d}\nu. \tag{1.18}$$

Mapping the integral below the real axis to an integral above the real axis using the transformation $\nu \mapsto -\nu$, we obtain

$$u(r,\theta) = -\frac{1}{4\pi}\int_{-\infty+\mathrm{i}c}^{\infty+\mathrm{i}c}\left[\mathrm{e}^{\mathrm{i}\nu\theta}D(-\mathrm{i}\nu)\frac{1+\mathrm{e}^{-2\pi\mathrm{i}\nu}}{1-\mathrm{e}^{2\pi\mathrm{i}\nu}} - \mathrm{e}^{-\mathrm{i}\nu\theta}D(\mathrm{i}\nu)\frac{1+\mathrm{e}^{2\pi\mathrm{i}\nu}}{1-\mathrm{e}^{-2\pi\mathrm{i}\nu}}\right]\frac{H_\nu^{(1)}(kr)}{H_\nu^{(1)}(ka)}\mathrm{d}\nu. \tag{1.19}$$

The second term in square brackets in (1.19) is bounded at infinity in the UHP, but it is not clear that the first term is. By using (1.10), one can eventually manipulate (1.19) into an expression similar to (1.11), and then use the decomposition $D(\pm\mathrm{i}\nu) = D_L(\pm\mathrm{i}\nu) + D_R(\pm\mathrm{i}\nu)$ and the transformation $\nu \mapsto -\nu$ to obtain an integral analogous to that in (1.12), i.e. one whose integrand is bounded in the UHP (apart from $H_\nu^{(1)}(kr)/H_\nu^{(1)}(ka)$). However, this procedure is much more involved than that in §1.1.

In summary, the amount of effort needed to obtain (1.12) (or an expression with similar properties) from (1.5) is highly dependent on which one of (1.6), (1.16), and (1.17) is used to convert the series into an integral, and it is not clear a priori which one of these formulae is the best.

The fact that the best "series-to-integral" formula to use depends on the particular f in question (and is hard to determine a priori) perhaps explains why the Watson transformation has *not* been applied uniformly to explicit expressions for solutions of BVPs arising from using transforms. Indeed, whereas the Watson transformation has been well-used in the context of scattering problems (where the BVPs involve elliptic PDEs), it has not been used for BVPs involving evolution PDEs. For example, if the heat equation, $u_t - u_{xx} = 0$, is posed for $0 < x < L$ and $0 < t < T$ with Robin boundary conditions at $x = 0$ and $x = L$, then the appropriate transform in the x-variable yields an expression for the solution as an infinite series over zeros of a transcendental equation (and thus evaluating the series requires finding these zeros); see, e.g., [43, §1.1.1-11], [11, Example 4.2.3], [19, §5.8]. The Watson transformation can be in principle be used to convert this series into an integral in the complex plane (with this integral then much easier to evaluate asymptotically or numerically than the series), however this appears not to have been done in the literature. (The integral expression for the solution of this BVP *has* been obtained using the Fokas transform method; see [14, §2.1], [15, §6.1].)

1.3 The main result of this paper

In hindsight, the "moral" of the Watson transformation is that, when a BVP can be solved by transforms, the best expression for the solution is an integral in the complex plane, which can then be deformed (and evaluated via residues if necessary) to yield either of the two expressions obtained by transforms (assuming these transforms are complete).

A new transform method was introduced in 1997 by Fokas [13] and further developed by Fokas and collaborators in the years since then (see the monograph [14] and the review paper [15]). This method arose from the theory of nonlinear integrable PDEs, but is also applicable to linear PDEs.

For linear BVPs that can be solved by classical transform methods, the Fokas method can be seen as realising the "moral" of the Watson transformation since it obtains an explicit expression for the solution as an integral in the complex plane that can be deformed to yield either of the two expressions (for a 2-d problem) obtained by transforms. Furthermore, the Fokas method can obtain explicit expressions for the solutions of certain BVPs for which there are no appropriate classical transforms (see [15, §6.2] for an overview of these problems). (The Fokas method has also been used to develop new numerical schemes for computing the solution of BVPs that cannot be solved explicitly, and to prove results about the existence, uniqueness, and regularity of certain BVPs; see the overview in [15, §6.3] and the references therein.)

The main goal of the present paper is to demonstrate that the expression (1.11) for the solution of Problem 1.1 can be obtained in an algorithmic way using the Fokas method, as opposed to the somewhat ad hoc method of the Watson transformation. (This fact was stated, but not proved, in [15, §5], and the present paper fills this gap.) We note that, although obtaining (1.11) via the Fokas method is more systematic than obtaining it via the Watson transformation, it is a longer and more involved calculation. However, the ideas underlying the Watson transformation (in particular the idea of deforming contours in the complex plane to obtain better representations of solutions to BVPs) are still being used by the research community, and we hope that this alternative algorithmic method of obtaining better representations of the solution of BVPs may prove useful in situations where it is still not clear what the best representation is or how to obtain it (we discuss this further in §5).

Outline of the paper. The details of the Fokas method applied to Problem 1.1 are given in §2 below. This procedure requires certain results about the asymptotics of Bessel and Hankel functions and in particular the asymptotics of ν_n (the zeros of $H_\nu^{(1)}(ka)$), and we give these in §3. These asymptotic results are also precisely the results needed to prove that when the contour of the integral in (1.12) is closed at infinity in the UHP, the integral at infinity is unbounded if $d(\theta) \neq 0$ and zero if $d(\theta) = 0$, and we give this proof in §4. We do this, not only for completeness, but also because, although this result is relied upon by many authors (e.g. [49], [28], [18], [50], [31]), the only complete proofs the author is aware of are in [17, §3] (for the Helmholtz equation in the exterior of a prolate spheroid in 3-d), [38, §III] (for the sphere), and [36, Pages 114-115 and 119-120] (for the disk), and these proofs are perhaps not presented in their simplest possible forms. Finally, we end the paper in §5 by discussing its contents both from the point of view of spectral theory and in light of current uses of Watson-type transformations.

2 The Fokas method applied to Problem 1.1

2.1 Outline of the Fokas method applied to linear BVPs

The Fokas method for linear BVPs consists of the following three steps (which we implement for Problem 1.1 in §2.2 below):

1. Rewrite the PDE as a one-parameter family of equations in divergence form. Integrating this divergence form over the boundary of the domain gives the *global relation*, which is an algebraic equation coupling the transforms of all boundary values. (The terminology emphasises that this relation contains global, as opposed to local, information about the boundary values.)

2. Derive an *integral representation* of the solution involving *transforms* of all boundary values. This equation is analogous to Green's integral representation, but rather than being formulated in the physical space, it is formulated in the spectral (Fourier) space.

3. Eliminate from the integral representation the transforms of the unknown boundary values and thus obtain an expression for the solution in terms of the known boundary data. This step involves algebraic manipulation of both the global relation and equations obtained from it via certain transformations in the complex Fourier plane, as well as deforming the contours in the integral representation and using Cauchy's theorem.

Furthermore, Step 3 can be broken down into the following three substeps:

(a) Use the global relation to express some of the transforms of the unknown boundary values that appear in the integral representation in terms of the smallest possible subset of other unknown transforms (if there exist different possibilities, use the one that yields the smallest number of unknowns in each equation).

(b) Identify the domains in the complex ν-plane where the integrands of the integral representation are bounded and analytic, and also identify the location of any singularities. At this substep some unknown transforms can be eliminated directly from the integral representation using analyticity and Cauchy's theorem.

(c) Deform contours and use the global relation again so that the contribution from the remaining unknown boundary values vanish by analyticity (using Cauchy's theorem).

For certain BVPs the solution can be obtained without using all three of (a)-(c), however Problem 1.1 requires all three substeps.

2.2 The Fokas method applied to Problem 1.1

Our goal in this section is to prove the following theorem.

Theorem 2.1 *Applying the steps of the Fokas method outlined in §2.1 to the BVP of Problem 1.1 yields the explicit expression for the solution* (1.11)*, where* $D(\pm \mathrm{i}\nu)$ *are defined by*

$$D(\pm \mathrm{i}\nu) = \int_0^{2\pi} \mathrm{e}^{\pm \mathrm{i}\nu\phi}\, d(\phi)\, \mathrm{d}\phi, \quad \nu \in \mathbb{C},$$

and c *in the limits of the second integral is a constant such that* $0 < c < \Im \nu_1$*.*

Before we begin, we note that we need to consider the θ-variable as non-periodic (i.e. we consider $0 < \theta < 2\pi$ with $\theta = 0$ and $\theta = 2\pi$ not automatically identified as the same line). This is not just a feature of applying the Fokas method, and is encountered classically. Indeed, with u defined by the angular series solution (1.5), $u(r, \theta + 2\pi n) = u(r, \theta)$ when $n \in \mathbb{Z}$, but with u defined by the radial series solution (1.13), $u(r, \theta + 2\pi n) \neq u(r, \theta)$ (this can be seen from the series solution itself, but also from that fact that for the terms in brackets in (1.12) to be bounded at infinity in the UHP we need $0 < \theta < 2\pi$). Since we know that the restriction $0 < \theta < 2\pi$ is necessary to obtain the radial series solution, we impose it from the beginning. (We note, however, that the Fokas method imposes this requirement independently in Step 2.)

Therefore, we let $\widetilde{\Omega}$ be the domain exterior to a disc of radius a centered at the origin with a cut along the line $\theta = 0$, i.e.

$$\widetilde{\Omega} = \{a < r < \infty, \quad 0 < \theta < 2\pi\},$$

and we consider the problem of finding $\widetilde{u}$ such that $\Delta\widetilde{u} + k^2\widetilde{u} = 0$ in $\widetilde{\Omega}$, $\widetilde{u}(a, \theta) = d(\theta)$ for $0 < \theta < 2\pi$, $\widetilde{u}$ satisfies the radiation condition (1.4), and $\widetilde{u}$ satisfies the following boundary conditions on the cut

$$\widetilde{u}(r, 2\pi) = \widetilde{u}(r, 0), \quad \widetilde{u}_\theta(r, 2\pi) = \widetilde{u}_\theta(r, 0). \tag{2.1}$$

The conditions (2.1) (along with the fact that $\Delta\widetilde{u} + k^2\widetilde{u} = 0$) imply that the periodic extension of $\widetilde{u}(r, \cdot)$ is in $C^2(\mathbb{T})$, and then by uniqueness this extension is the solution of Problem 1.1, u. Since $\widetilde{u}$ is the restriction of u to $0 < \theta < 2\pi$, in what follows we denote the solution of the BVP in $\widetilde{\Omega}$ as u.

We now implement Steps 1 to 3 of §2.1. Steps 1 and 2 of this procedure were outlined in [15, §5]; for completeness we give the main details here too.

2.2.1 Step 1

The Helmholtz equation is formally self-adjoint, and the divergence form of (1.2) is

$$\nabla \cdot \big(v\nabla u - u\nabla v\big) = 0. \tag{2.2}$$

Integrating (2.2) over $\widetilde{\Omega}$ and using the divergence theorem yields the global relation

$$0 = \int_{\partial\widetilde{\Omega}} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) \mathrm{d}S, \tag{2.3}$$

where v is a one-parameter family of solutions to (1.2) such that the integral at infinity vanishes; this is the case if v also satisfies the radiation condition (1.4).

We can obtain a one-parameter family of adjoint solutions v via separation of variables. Since the BVP is separable in polar coordinates, a one-parameter family is given by

$$v(r,\theta) = \mathrm{e}^{\mathrm{i}\nu\theta} H_\nu^{(1)}(kr), \quad \nu \in \mathbb{C}, \tag{2.4}$$

where the solution in the radial variable, $H_\nu^{(1)}(kr)$, is chosen by the requirement that it satisfies the radiation condition (1.4).

Substituting v given by (2.4) into (2.3), writing the resulting equation in polar coordinates, and recalling that $u(r,\theta)$ satisfies (2.1) on the cut along $\theta = 0$, we find the following global relation

$$-aH_\nu^{(1)}(ka)N(\mathrm{i}\nu) + akH_\nu^{(1)'}(ka)D(\mathrm{i}\nu) + \left(1 - \mathrm{e}^{2\pi\mathrm{i}\nu}\right)\Big(\mathrm{i}\nu D_0(\nu) - N_0(\nu)\Big) = 0, \tag{2.5}$$

where

$$D_0(\nu) = \int_a^\infty H_\nu^{(1)}(k\rho)\, u(\rho,0)\frac{\mathrm{d}\rho}{\rho}, \quad N_0(\nu) = \int_a^\infty H_\nu^{(1)}(k\rho)\, u_\theta(\rho,0)\frac{\mathrm{d}\rho}{\rho}, \quad \nu \in \mathbb{C}, \tag{2.6}$$

and

$$D(\mathrm{i}\nu) = \int_0^{2\pi} \mathrm{e}^{\mathrm{i}\nu\phi}\, u(a,\phi)\mathrm{d}\phi, \quad N(\mathrm{i}\nu) = \int_0^{2\pi} \mathrm{e}^{\mathrm{i}\nu\phi}\, u_r(a,\phi)\mathrm{d}\phi, \quad \nu \in \mathbb{C}. \tag{2.7}$$

The notation indicates that D and D_0 are transforms of the Dirichlet boundary values, and N and N_0 are transforms of the Neumann boundary values, with D and N the boundary values on the surface of the disk, and D_0 and N_0 the boundary values on the cut.

2.2.2 Step 2

In general, the integral representation of the Fokas method can be obtained in the following three different ways: (i) spectral analysis of the differential form behind the global relation [14, Part III], (ii) applying the global relation in a subdomain [14, Chapters 1 and 2], or (iii) via Green's integral representation [14, §11.4], [47]. For Problem 1.1 it is easiest to use the third method.

Green's integral representation for the solution of (1.2) in $\widetilde{\Omega}$ is

$$u(\boldsymbol{x}) = \int_{\partial\widetilde{\Omega}} \left(E(\boldsymbol{\xi},\boldsymbol{x})\frac{\partial u}{\partial n}(\boldsymbol{\xi}) - u(\boldsymbol{\xi})\frac{\partial E}{\partial n}(\boldsymbol{\xi},\boldsymbol{x})\right)\mathrm{d}S(\boldsymbol{\xi}), \quad \boldsymbol{x} \in \widetilde{\Omega}, \tag{2.8}$$

where $E(\boldsymbol{\xi},\boldsymbol{x})$ is the fundamental solution (or free-space Green's function) satisfying

$$\left(\Delta_{\boldsymbol{\xi}} + k^2\right)E(\boldsymbol{\xi},\boldsymbol{x}) = -\delta(\boldsymbol{\xi} - \boldsymbol{x}), \quad \boldsymbol{\xi} \in \mathbb{R}^2. \tag{2.9}$$

For the Helmholtz equation in two dimensions with the radiation condition (1.4) the fundamental solution is given by

$$E(\boldsymbol{\xi},\boldsymbol{x}) = \frac{\mathrm{i}}{4}H_0^{(1)}\left(k|\boldsymbol{\xi} - \boldsymbol{x}|\right), \tag{2.10}$$

however we do not use this fact directly, and instead start from the PDE defining E (2.9).

The integral representation of the Fokas method can be obtained by substituting two different expressions for the fundamental solution into Green's integral representation (2.8). These two expressions are obtained by solving (2.9) by the two appropriate classical transforms (associated with each separated ODE) in the chosen coordinate system. One slight subtlety is that since we are considering the θ variable as non-periodic, we must consider a different fundamental solution to (2.10).

To emphasise that our fundamental solution is *not* the usual one, we denote it by E_s (following the notation in [48, p. 270] where Sommerfeld's use of the non-periodic fundamental solution in a different context is discussed). The non-periodic fundamental solution E_s satisfies equation (2.9) and the radiation condition (1.4) in the domain defined in polar co-ordinates (ρ,ϕ) by $-\infty < \theta < \infty$ and $0 < r < \infty$. The two expressions for E_s, obtained by the appropriate transforms, are given in the following theorem.

Theorem 2.2 (Expansions of E_s in angular and radial eigenfunctions [47, Proposition 3.1]) *The non-periodic fundamental solution E_s for the Helmholtz equation can be expressed as expansions in terms of either the angular or radial eigenfunctions. The* angular expansion *is*

$$E_s(\rho,\phi;r,\theta)=\frac{\mathrm{i}}{4}\left(\int_0^\infty H_\nu^{(1)}(kr_>)J_\nu(kr_<)\mathrm{e}^{\mathrm{i}\nu(\theta-\phi)}\mathrm{d}\nu+\int_0^\infty H_\nu^{(1)}(kr_>)J_\nu(kr_<)\mathrm{e}^{-\mathrm{i}\nu(\theta-\phi)}\mathrm{d}\nu\right), \tag{2.11}$$

where $r_>=\max(r,\rho)$, $r_<=\min(r,\rho)$ and $0<\theta,\phi<2\pi$. The radial expansion *is*

$$E_s(\rho,\phi;r,\theta)=\lim_{\varepsilon\to0}\frac{\mathrm{i}}{4}\left(\int_0^{\mathrm{i}\infty}\mathrm{e}^{\varepsilon\nu^2}H_\nu^{(1)}(kr_1)J_\nu(kr_2)\mathrm{e}^{\mathrm{i}\nu|\theta-\phi|}\mathrm{d}\nu+\int_0^{-\mathrm{i}\infty}\mathrm{e}^{\varepsilon\nu^2}H_\nu^{(1)}(kr_1)J_\nu(kr_2)\mathrm{e}^{-\mathrm{i}\nu|\theta-\phi|}\mathrm{d}\nu\right), \tag{2.12}$$

where either $r_1=r$ and $r_2=\rho$ or vice versa.

Equations (2.11) and (2.12) are obtained in [47, Proposition 3.1] using the two completeness relations

$$\delta(\theta-\phi)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\nu(\theta-\phi)}\mathrm{d}\nu \tag{2.13}$$

and

$$\rho\,\delta(r-\rho)=\lim_{\varepsilon\to0}\frac{1}{2}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\mathrm{e}^{\varepsilon\nu^2}H_\nu^{(1)}(kr_1)J_\nu(kr_2)\,\nu\,\mathrm{d}\nu, \tag{2.14}$$

where either $r_1=r$ and $r_2=\rho$ or visa versa. The first completeness relation (2.13) is associated with the angular ODE under non-periodicity, and corresponds to the Fourier transform, whereas equation (2.14) is associated with the radial ODE under the radiation condition (1.4), and corresponds to the Kontorovich-Lebedev transform. (This is the point at which the requirement that θ is non-periodic arises when implementing the Fokas method, since there does not exist an expansion of the fundamental solution in radial eigenfunctions if θ is periodic; see [47, Remark 3.4].)

Both equations (2.14) and (2.12) contain the regularising factor $\exp(\varepsilon\nu^2)$, which illustrates the technical complications that arise due to the non-self-adjointness of the Helmholtz equation in unbounded domains. Performing the classical transform method algorithm described in [15, §2.1] on the radial ODE with the radiation condition, one arrives at (2.14) *without* the regularising factor $\exp(\varepsilon\nu^2)$. However, D. S. Jones showed in [21] that without this factor the transform is *not* complete (the transform even fails for the simple function $\exp(-a\rho)$, $a>0$), and he established, via a proof of completeness, the version (2.14). Nevertheless, many of the classical solutions of BVPs obtained via (2.14) without the term $\exp(\varepsilon\nu^2)$ are still correct because the contour is deformed (albeit illegally) and the resulting expression converges (the term $\exp(\varepsilon\nu^2)$ rigorously justifies the contour deformation, after which ε can be set to zero); see [21, §7].

The integral representation of the Fokas method for Problem 1.1 is

$$\begin{aligned}u(r,\theta)=\lim_{\varepsilon\to0}\frac{\mathrm{i}}{4}\Bigg[&\int_0^{\mathrm{i}\infty}\mathrm{e}^{\varepsilon\nu^2}J_\nu(kr)\left[-\left(\mathrm{e}^{\mathrm{i}\nu\theta}+\mathrm{e}^{\mathrm{i}\nu(2\pi-\theta)}\right)\mathrm{i}\nu D_0(\nu)-\left(\mathrm{e}^{\mathrm{i}\nu\theta}-\mathrm{e}^{\mathrm{i}\nu(2\pi-\theta)}\right)N_0(\nu)\right]\mathrm{d}\nu\\&+\int_0^{-\mathrm{i}\infty}\mathrm{e}^{\varepsilon\nu^2}J_\nu(kr)\left[\left(\mathrm{e}^{-\mathrm{i}\nu\theta}+\mathrm{e}^{-\mathrm{i}\nu(2\pi-\theta)}\right)\mathrm{i}\nu D_0(\nu)-\left(\mathrm{e}^{-\mathrm{i}\nu\theta}-\mathrm{e}^{-\mathrm{i}\nu(2\pi-\theta)}\right)N_0(\nu)\right]\mathrm{d}\nu\Bigg]\\&-\frac{\mathrm{i}a}{4}\Bigg[\int_0^\infty\mathrm{e}^{\mathrm{i}\nu\theta}H_\nu^{(1)}(kr)\Big(J_\nu(ka)N(-\mathrm{i}\nu)-kJ'_\nu(ka)D(-\mathrm{i}\nu)\Big)\mathrm{d}\nu\\&+\int_0^\infty\mathrm{e}^{-\mathrm{i}\nu\theta}H_\nu^{(1)}(kr)\Big(J_\nu(ka)N(\mathrm{i}\nu)-kJ'_\nu(ka)D(\mathrm{i}\nu)\Big)\mathrm{d}\nu\Bigg].\end{aligned} \tag{2.15}$$

To obtain (2.15) we write Green's integral representation (2.8) in polar coordinates, and then substitute the expansions (2.11) and (2.12) in the resulting expression; on the boundaries where ϕ is fixed and ρ varies we use the radial expansion (2.12), whereas on the boundaries where ρ is fixed and ϕ varies we use the angular expansion (2.11). Changing the order of the physical integrals (in ρ or ϕ) and the spectral integrals (in ν) we find (2.15), provided we choose $r_1=\rho$ and $r_2=r$ in the radial representation (2.12) (the alternative choice $r_1=r$, $r_2=\rho$ still leads to an integral representation, but the transforms of the boundary values are not the same transforms that appear in the global relation (2.5)).

2.2.3 Step 3

Our task is to eliminate the transforms of the unknown boundary values $D_0(\nu)$, $N_0(\nu)$, and $N(\pm\mathrm{i}\nu)$ from the integral representation (2.15), using the global relation (2.5), and obtain the expression (1.11). Recall that the description of Step 3 says that we should do this using "both the global relation and equations obtained from it via certain transformations in the complex Fourier plane". In this situation the appropriate transformation is $\nu \mapsto -\nu$; this is a consequence of the symmetry (1.7) and the fact that the integral representation (2.15) involves $D(-\mathrm{i}\nu)$ and $N(-\mathrm{i}\nu)$ whereas the global relation (2.5) involves only $D(\mathrm{i}\nu)$ and $N(\mathrm{i}\nu)$. Letting $\nu \mapsto -\nu$ in (2.5) and using (1.7) results in the two equations

$$- aH_\nu^{(1)}(ka)N(\pm\mathrm{i}\nu) + akH_\nu^{(1)'}(ka)D(\pm\mathrm{i}\nu) + \left(1 - \mathrm{e}^{\pm 2\pi\mathrm{i}\nu}\right)\Big(\pm \mathrm{i}\nu D_0(\nu) - N_0(\nu)\Big) = 0. \tag{2.16}$$

(Alternatively, one can obtain the second equation by noting that $\mathrm{e}^{-\mathrm{i}\nu\theta}H_\nu^{(1)}(kr)$, $\nu \in \mathbb{C}$, is also a one-parameter family of solutions of the adjoint equation that satisfy the radiation condition.)

We now use the three substeps (a)-(c) described in §2.1 to eliminate the unknown transforms D_0, N_0, and N from the representation (2.15).

Before we proceed, we note that the following identity holds for any function $L(\nu)$ (provided the integrals exist):

$$\begin{aligned}&\int_0^{\mathrm{i}\infty} J_\nu(kr)H_\nu^{(1)}(ka)L(\nu)\,\mathrm{d}\nu + \int_0^{-\mathrm{i}\infty} J_\nu(kr)H_\nu^{(1)}(ka)L(-\nu)\,\mathrm{d}\nu \\ &= \int_0^{\mathrm{i}\infty} J_\nu(ka)H_\nu^{(1)}(kr)L(\nu)\,\mathrm{d}\nu + \int_0^{-\mathrm{i}\infty} J_\nu(ka)H_\nu^{(1)}(kr)L(-\nu)\,\mathrm{d}\nu.\end{aligned} \tag{2.17}$$

This identity can be derived by expanding $H_\nu^{(1)}$ as a linear combination of J_ν and $J_{-\nu}$ (using its definition (3.9)), and then by letting $\nu \mapsto -\nu$ in the term involving $J_{-\nu}$. (This identity shows reciprocity in r and ρ in the expression (2.12), and a similar identity is used when solving problems using the Kontorovich-Lebedev transform [21, §5], [22, §9.19].)

In the course of the proof, we need various results about (i) the asymptotics of $J_\nu(z)$ and $H_\nu^{(1)}(z)$ as $|\nu| \to \infty$ with $z \in \mathbb{R}$ fixed, (ii) the asymptotics of ν_n (the zeros of $H_\nu^{(1)}(ka)$ in the first quadrant) as $n \to \infty$. These results are collected in §3, and we refer to them as necessary during the course of the proof.

Substep (a). The two global relations (2.16) involve four unknown functions: $N(\mathrm{i}\nu), N(-\mathrm{i}\nu), N_0(\nu)$, and $D_0(\nu)$. Therefore, these two equations can express any one unknown in terms of two others. Here we use (2.16) to express $N(\pm\mathrm{i}\nu)$ in terms of $N_0(\nu)$ and $D_0(\nu)$ (the details for any different choice, e.g. expressing $N_0(\nu)$ and $D_0(\nu)$ in terms of $N(\pm\mathrm{i}\nu)$, are very similar to those below).

Solving (2.5) for $N(\pm\mathrm{i}\nu)$, we obtain that

$$aN(\pm\mathrm{i}\nu) = \frac{1}{H_\nu^{(1)}(ka)}\Big(akH_\nu^{(1)'}(ka)D(\pm\mathrm{i}\nu) + \left(1 - \mathrm{e}^{\pm 2\pi\mathrm{i}\nu}\right)\left(\pm \mathrm{i}\nu D_0(\nu) - N_0(\nu)\right)\Big). \tag{2.18}$$

Substituting these expressions into (2.15), we find that the unknown parts of the integrals over $(0,\infty)$ in (2.15) are

$$\begin{aligned}-\frac{\mathrm{i}}{4}\Bigg(&\int_0^\infty \mathrm{e}^{\mathrm{i}\nu\theta}\frac{H_\nu^{(1)}(kr)J_\nu(ka)}{H_\nu^{(1)}(ka)}\left(1 - \mathrm{e}^{-2\pi\mathrm{i}\nu}\right)\left(- \mathrm{i}\nu D_0(\nu) - N_0(\nu)\right)\mathrm{d}\nu \\ &+ \int_0^\infty \mathrm{e}^{-\mathrm{i}\nu\theta}\frac{H_\nu^{(1)}(kr)J_\nu(ka)}{H_\nu^{(1)}(ka)}\left(1 - \mathrm{e}^{2\pi\mathrm{i}\nu}\right)\left(\mathrm{i}\nu D_0(\nu) - N_0(\nu)\right)\mathrm{d}\nu\Bigg);\end{aligned} \tag{2.19}$$

for brevity of presentation, we focus only on the unknown terms in (2.15) and do not display the known terms involving $D(\pm\mathrm{i}\nu)$.

Since u is a solution of (1.2) with the outgoing radiation condition (1.4), we expect the r dependence of the solution to be of the form $H_\nu^{(1)}(kr)$. Indeed, the integrals over $(0,\infty)$ in (2.15) are of this form, but the integrals over $(0, \mathrm{i}\infty)$ are not. To rectify this situation we multiply and divide by $H_\nu^{(1)}(ka)$ in the first two terms of (2.15) and use the identity (2.17). Multiplying and dividing by $H_\nu^{(1)}(ka)$, and noting that there are no zeros of $H_\nu^{(1)}(ka)$ on the contour, we find that the first two terms of (2.15) become

$$\lim_{\varepsilon\to 0}\frac{\mathrm{i}}{4}\left(\int_0^{\mathrm{i}\infty}\mathrm{e}^{\varepsilon\nu^2}J_\nu(kr)H_\nu^{(1)}(ka)M(\nu)\mathrm{d}\nu+\int_0^{-\mathrm{i}\infty}\mathrm{e}^{\varepsilon\nu^2}J_\nu(kr)H_\nu^{(1)}(ka)M(-\nu)\mathrm{d}\nu\right), \tag{2.20}$$

where

$$M(\nu)=-\left(\mathrm{e}^{\mathrm{i}\nu\theta}+\mathrm{e}^{\mathrm{i}\nu(2\pi-\theta)}\right)\mathrm{i}\nu\frac{D_0(\nu)}{H_\nu^{(1)}(ka)}-\left(\mathrm{e}^{\mathrm{i}\nu\theta}-\mathrm{e}^{\mathrm{i}\nu(2\pi-\theta)}\right)\frac{N_0(\nu)}{H_\nu^{(1)}(ka)}.$$

The identity (2.17) with $L(\nu)=\mathrm{e}^{\varepsilon\nu^2}M(\nu)$ then implies that the arguments of $H_\nu^{(1)}$ and J_ν in (2.20) can be interchanged. Combining the resulting terms with (2.19), we find that the unknowns in the integral representation (2.15) are given by

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{\mathrm{i}}{4}\Bigg(&\int_{A'B'D}I(\varepsilon,\nu)\,H_\nu^{(1)}(kr)J_\nu(ka)\left[\mathrm{e}^{\mathrm{i}\nu\theta}\frac{-\mathrm{i}\nu D_0(\nu)-N_0(\nu)}{H_\nu^{(1)}(ka)}+\mathrm{e}^{\mathrm{i}\nu(2\pi-\theta)}\frac{-\mathrm{i}\nu D_0(\nu)+N_0(\nu)}{H_\nu^{(1)}(ka)}\right]\mathrm{d}\nu\\&+\int_{A'B'E}I(\varepsilon,\nu)H_\nu^{(1)}(kr)J_\nu(ka)\left[\mathrm{e}^{-\mathrm{i}\nu\theta}\frac{\mathrm{i}\nu D_0(\nu)-N_0(\nu)}{H_\nu^{(1)}(ka)}+\mathrm{e}^{-\mathrm{i}\nu(2\pi-\theta)}\frac{\mathrm{i}\nu D_0(\nu)+N_0(\nu)}{H_\nu^{(1)}(ka)}\right]\mathrm{d}\nu\Bigg),\end{aligned} \tag{2.21}$$

where the contours of integration are shown in Figure 2 (and A', D, and E are points at infinity), and

$$I(\varepsilon,\nu):=\begin{cases}\mathrm{e}^{\varepsilon\nu^2} & \text{if } |\Im\nu|>1,\\ 1 & \text{otherwise.}\end{cases}$$

The reason we introduce $I(\varepsilon,\nu)$ is so that we can write integrals over the contours $A'B'D$ and $A'B'E$ that have the regularising factor $\mathrm{e}^{\varepsilon\nu^2}$ on $\mathrm{i}\mathbb{R}^+$ but not on $\mathbb{R}$. In the next substep we will be concerned with the analyticity properties of the integrands in (2.21), however we observe immediately that $I(\varepsilon,\nu)$ is not an analytic function of ν. This will not be a problem, since our procedure will always be to deform an integral on $(\mathrm{i},\mathrm{i}\infty)$, say, to one where the integral converges absolutely when $\varepsilon=0$; ε can then be set to zero, and then the factor $I(\varepsilon,\nu)$ can be replaced by one (we go through this in more detail below).

Our aim is to show that (2.21) is equal to an expression that involves only the known transforms $D(\pm\mathrm{i}\nu)$.

Substep (b) We now look at where the integrands in (2.21) are analytic, and we also determine their asymptotics as $|\nu|\to\infty$.

For fixed $z\in\mathbb{R}^+$, $H_\nu^{(1)}(z)$ and $J_\nu(z)$ are entire functions, and thus so are $D_0(\nu)$ and $N_0(\nu)$. We need the asymptotics as $|\nu|\to\infty$ of the product $H_\nu^{(1)}(kr)J_\nu(ka)$ and the fractions $D_0(\nu)/H_\nu^{(1)}(ka)$ and $N_0(\nu)/H_\nu^{(1)}(ka)$. We introduce the following three regions (which are needed to describe the asymptotics of $H_\nu^{(1)}(z)$): Region 3 is the 2nd quadrant of the complex ν-plane, the union of Regions 1 and 2 is the 1st quadrant, and the boundary between Regions 1 and 2 is the curve h_1 defined by

$$\Re\left(\nu\log\left(\frac{2\nu}{\mathrm{e}z}\right)\right)=0,$$

see Figure 3 (note that the curve h_1 is asymptotically independent of the value of z). The angular width of Region 2 is $\mathrm{ord}(1/\log|\nu|)$ as $|\nu|\to\infty$ (and thus the arguments of points on the curve h_1 tend to zero as $|\nu|\to\infty$). The asymptotics of $H_\nu^{(1)}(z)$ and $J_\nu(z)$ as $|\nu|\to\infty$ for fixed $z\in\mathbb{R}$ are given below in Theorem 3.1 and imply that, as $|\nu|\to\infty$,

$$H_\nu^{(1)}(x)J_\nu(y)\sim\frac{1}{\mathrm{i}\pi\nu}\left(\frac{y}{x}\right)^\nu\quad\text{in both Region 1 and the 4th quadrant,} \tag{2.22a}$$

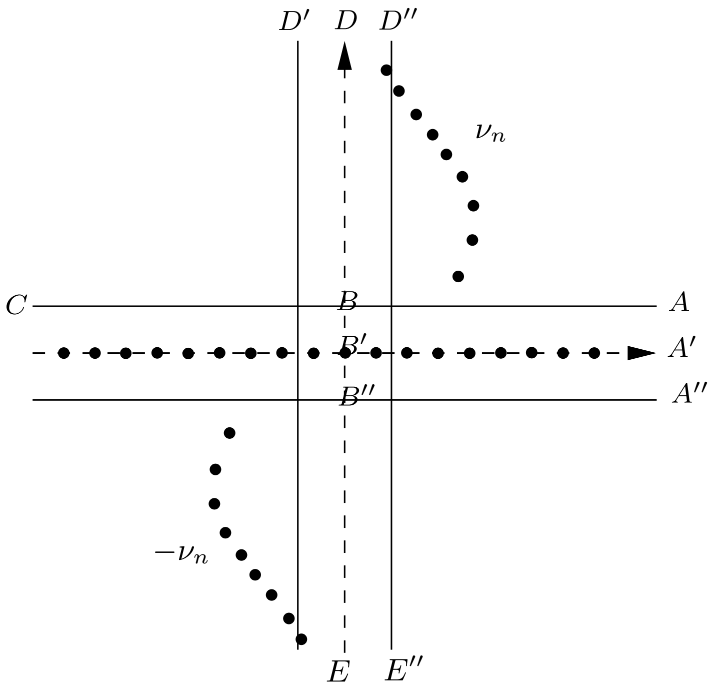

Figure 2: The poles of the integrands of (1.11) and the contours of integration in the complex ν-plane.

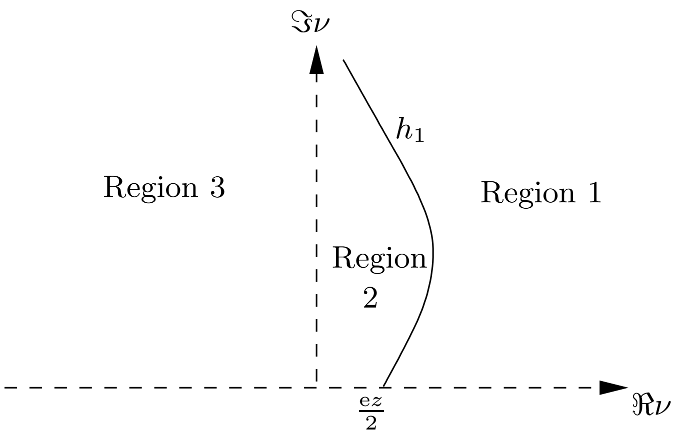

Figure 3: Regions 1, 2, and 3 and the curve h_1 used to describe the asymptotics of $H_\nu^{(1)}(z)$ as $|\nu| \to \infty$ in Theorem 3.1. (In a similar way to Figure 1, we plot the curve h_1 in a way that emphasises that the argument of a point on the curve tends to $\pi/2$ as $|\nu| \to \infty$.)

$$H_\nu^{(1)}(x)J_\nu(y) \sim \frac{1}{\pi\nu}(yx)^\nu \left(\frac{\mathrm{e}}{2\nu}\right)^\nu \quad \text{in Regions 2 and 3,} \tag{2.22b}$$

(note that the asymptotics of $H_\nu^{(1)}(x)$ in the 4th quadrant follow from the asymptotics in Region 3 and the relation (1.7)).

Turning to $D_0(\nu)/H_\nu^{(1)}(ka)$ and $N_0(\nu)/H_\nu^{(1)}(ka)$, we see that both these fractions have poles at the zeros of $H_\nu^{(1)}(ka)$, which occur in the first and third quadrants of the complex ν-plane (with the zeros in the first quadrant denoted by ν_n and the zeros in the third quadrant then equal to $-\nu_n$ by (1.7)). The asymptotics of ν_n as $n \to \infty$ are given in Theorem 3.1; for our purposes we only need to know that $\arg \nu_n \to \pi/2$ as $n \to \infty$. Both $D_0(\nu)/H_\nu^{(1)}(ka)$ and $N_0(\nu)/H_\nu^{(1)}(ka)$ involve $H_\nu^{(1)}(k\rho)/H_\nu^{(1)}(ka)$ for $\rho \geq a$. By Lemma 3.7 below, this fraction $\sim (a/\rho)^\nu$ in Region 1 and the 4th quadrant, and is $\mathcal{O}((r/a)^\nu)$ in Region 2 and a neighbourhood of the curve h_1 (away from ν_n). The key points, therefore, are that $D_0(\nu)/H_\nu^{(1)}(ka)$ and $N_0(\nu)/H_\nu^{(1)}(ka)$ are bounded in Region 1 and the 4th quadrant, but are exponentially large in Region 2 and have poles on h_1.

Returning to (2.21) with these asymptotics, we see that the integral on $A'B'E$ in (2.21) is zero. Indeed, by Cauchy's theorem we can deform the contour from $A'B'E$ to $A'B'B''E''$, where $\Im B'' < -1$ (to ensure that $I(\varepsilon,\nu) = \mathrm{e}^{\varepsilon\nu^2}$ on the deformed part), and where $-\pi/2 < \arg E'' < -\pi/4$ (so that the term $\mathrm{e}^{\varepsilon\nu^2}$ still decays). By the asymptotics (2.22a), the resulting integral converges absolutely even when $\varepsilon = 0$, and then by the dominated convergence theorem ε can be set to zero (and thus $I(\varepsilon,\nu)$ can be replaced by one). By closing the contour at infinity in the fourth quadrant and using Cauchy's theorem, we see that this integral equals zero (the contribution at infinity is zero since the integrand decays exponentially due to (2.22a)).

In contrast, we cannot deform the contour of the integral over $A'B'D$ off $\mathrm{i}\mathbb{R}$ because of the poles of $H_\nu^{(1)}(ka)$. Therefore, we are left with the only unknown in the integral representation equal to the first integral in (2.21).

Substep (c). Our instructions are now to "deform contours and use the global relation again so that the contribution from the remaining unknown boundary values vanish by analyticity (using Cauchy's theorem)". The unknown terms involve $-\mathrm{i}\nu D_0(\nu) \pm N_0(\nu)$, and the global relation (2.16) gives these quantities in terms of $N(\pm\mathrm{i}\nu)$ and $D(\pm\mathrm{i}\nu)$ as follows,

$$-\mathrm{i}\nu D_0(\nu) - N_0(\nu) = \frac{a\left(H_\nu^{(1)}(ka)N(-\mathrm{i}\nu) - kH_\nu^{(1)'}(ka)D(-\mathrm{i}\nu)\right)}{1-\mathrm{e}^{-2\pi\mathrm{i}\nu}} \tag{2.23a}$$

and

$$-\mathrm{i}\nu D_0(\nu) + N_0(\nu) = \frac{-a\left(H_\nu^{(1)}(ka)N(\mathrm{i}\nu) - kH_\nu^{(1)'}(ka)D(\mathrm{i}\nu)\right)}{1-\mathrm{e}^{2\pi\mathrm{i}\nu}}. \tag{2.23b}$$

Since the denominators of the right-hand sides of these last two equations involve $1-\mathrm{e}^{\pm 2\pi\mathrm{i}\nu}$, which has zeros on $\mathbb{R}$, if we want to use (2.23) in the first integral in (2.21) we need to deform the contour of this term off $\mathbb{R}$. We therefore deform the contour of the first integral in (2.21) from $A'B'D$ to ABD, where $0 < \Im B < \min(\Im\nu_1, 1)$; note that requiring $\Im B < 1$ ensures that $I(\varepsilon,\nu) = 1$ in the region where the integral is deformed (and thus the deformation is justified by Cauchy's theorem).

Substituting (2.23) into the integral over ABD, we find that the terms involving $N(\pm\mathrm{i}\nu)$ vanish because of analyticity. Indeed, when the contour ABD is closed at infinity in the first quadrant there are no poles of the integrand inside the contour (the term $H_\nu^{(1)}(ka)$ in the denominator is cancelled by the same term appearing in the numerator), the contour does not enclose any of the zeros of $1-\mathrm{e}^{\pm 2\pi\mathrm{i}\nu}$ (which are on the real axis), and the contribution from the integral at infinity is zero since the asymptotics

$$\frac{\mathrm{e}^{\mathrm{i}\nu\theta}N(-\mathrm{i}\nu)}{1-\mathrm{e}^{-2\pi\mathrm{i}\nu}} = \frac{\mathrm{e}^{\mathrm{i}\nu\theta}}{\mathrm{i}\nu}\left(u_\theta(a,2\pi) - u_\theta(a,0)\mathrm{e}^{2\pi\mathrm{i}\nu}\right)\left(1+\mathcal{O}\left(\frac{1}{\nu}\right)\right), \quad \text{and} \tag{2.24a}$$

$$\frac{\mathrm{e}^{-\mathrm{i}\nu\theta}N(\mathrm{i}\nu)}{1-\mathrm{e}^{-2\pi\mathrm{i}\nu}} = \frac{\mathrm{e}^{\mathrm{i}\nu(2\pi-\theta)}}{\mathrm{i}\nu}\left(\mathrm{e}^{2\pi\mathrm{i}\nu}u_\theta(a,2\pi) - u_\theta(a,0)\right)\left(1+\mathcal{O}\left(\frac{1}{\nu}\right)\right) \tag{2.24b}$$

(obtained by integration by parts) along with (2.22) imply that if $0 \le \theta < 2\pi$ then the relevant integrand decays exponentially as $|\nu| \to \infty$ with $\Im\nu > 0$. Before applying Cauchy's theorem, we must remove the factor $I(\varepsilon, \nu)$ in a similar way to above; i.e., we first deform the contour at infinity from D to D'' (where $\pi/4 < \arg D'' < \pi/2$), and then set $\varepsilon = 0$. The integrand of the resulting integral is analytic, and then Cauchy's theorem can be applied.

In summary, we have eliminated all the unknown transforms from the integral representation. Collecting all the remaining terms together, we have that

$$\begin{aligned} u(r,\theta) = &-\frac{\mathrm{i}ka}{4}\left(\int_0^\infty \mathrm{e}^{\mathrm{i}\nu\theta} D(-\mathrm{i}\nu)\frac{H_\nu^{(1)}(kr)}{H_\nu^{(1)}(ka)}\left[J_\nu(ka)H_\nu^{(1)'}(ka) - J_\nu'(ka)H_\nu^{(1)}(ka)\right]\mathrm{d}\nu\right.\\ &\left.\int_0^\infty \mathrm{e}^{-\mathrm{i}\nu\theta} D(\mathrm{i}\nu)\frac{H_\nu^{(1)}(kr)}{H_\nu^{(1)}(ka)}\left[J_\nu(ka)H_\nu^{(1)'}(ka) - J_\nu'(ka)H_\nu^{(1)}(ka)\right]\mathrm{d}\nu\right)\\ &-\lim_{\varepsilon\to 0}\frac{\mathrm{i}ka}{4}\int_{ABD} I(\varepsilon,\nu)\frac{H_\nu^{(1)}(kr)}{H_\nu^{(1)}(ka)}J_\nu(ka)H_\nu^{(1)'}(ka)\left[\frac{\mathrm{e}^{\mathrm{i}\nu\theta}D(-\mathrm{i}\nu)+\mathrm{e}^{-\mathrm{i}\nu\theta}D(\mathrm{i}\nu)}{1-\mathrm{e}^{-2\pi\mathrm{i}\nu}}\right]\mathrm{d}\nu. \end{aligned} \tag{2.25}$$

This equation can be simplified to (1.11) as follows. Using the Wronskian relation

$$J_\nu(ka)H_\nu^{(1)'}(ka) - J_\nu'(ka)H_\nu^{(1)}(ka) = \frac{2\mathrm{i}}{\pi ka} \tag{2.26}$$

in the integrals on $(0,\infty)$, along with the transformation $\nu \mapsto -\nu$, we find that these integrals are equal to the integral on $(-\infty,\infty)$ in (1.11). The relation (2.26) also implies that the integral over ABD in (2.25) equals

$$\begin{aligned} &-\lim_{\varepsilon\to 0}\frac{\mathrm{i}ka}{4}\int_{ABD} I(\varepsilon,\nu)H_\nu^{(1)}(kr)J_\nu'(ka)\left[\frac{\mathrm{e}^{\mathrm{i}\nu\theta}D(-\mathrm{i}\nu)+\mathrm{e}^{-\mathrm{i}\nu\theta}D(\mathrm{i}\nu)}{1-\mathrm{e}^{-2\pi\mathrm{i}\nu}}\right]\mathrm{d}\nu\\ &+\lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_{ABD} I(\varepsilon,\nu)\frac{H_\nu^{(1)}(kr)}{H_\nu^{(1)}(ka)}\left[\frac{\mathrm{e}^{\mathrm{i}\nu\theta}D(-\mathrm{i}\nu)+\mathrm{e}^{-\mathrm{i}\nu\theta}D(\mathrm{i}\nu)}{1-\mathrm{e}^{-2\pi\mathrm{i}\nu}}\right]\mathrm{d}\nu. \end{aligned} \tag{2.27}$$

If we deform the contour of the first integral in (2.27) from ABD to ABD'' (staying on $\mathrm{i}\mathbb{R}$ until $\Im\nu > 1$), then we can set ε to zero, and then the resulting integral vanishes when the contour is closed at infinity (the contribution at infinity is zero due to the asymptotics (2.24) with N replaced by D and asymptotics similar to (2.22)). If we deform the contour of the second integral in (2.27) from ABD to ABD', then we can set ε to zero, and the contour can then be deformed to ABC (again using asymptotics similar to (2.24) and (2.22) to show that the contribution at infinity is zero). This term then equals the second term in (1.11) and we are done.

3 The asymptotics of $H_\nu^{(1)}(z)$ and its zeros as $|\nu| \to \infty$

The asymptotics of the Hankel function $H_\nu^{(1)}(z)$ when $|\nu| \to \infty$ and z is fixed can be extracted from the classic text of Watson [53, §8.6, Page 262] (with a particularly accessible account of these results given in [38, Appendix A]). Instead of just quoting these results, we give a short proof from first principles.

Theorem 3.1 (Asymptotic behaviour of $H_\nu^{(1)}(z)$ as $\nu \to \infty$, $\Im\nu > 0$, with $z \in \mathbb{R}^+$ fixed) *Let $z \in \mathbb{R}^+$ be fixed. Then*

$$H_\nu^{(1)}(z) \sim -2\mathrm{e}^{\mathrm{i}\pi/4}\sqrt{\frac{2}{\pi\nu}}\sinh\left(\nu\log\left(\frac{2\nu}{\mathrm{e}z}\right)+\frac{\mathrm{i}\pi}{4}\right) \qquad \text{as } |\nu| \to \infty \text{ with } 0 < \arg\nu < \pi. \tag{3.1}$$

Let the curve h_1 be defined by

$$\Re\left(\nu\log\left(\frac{2\nu}{\mathrm{e}z}\right)\right) = 0, \qquad \Im\nu > 0. \tag{3.2}$$

This curve divides the 1st quadrant of the complex ν-plane into two regions, see Figure 3; call these Regions 1 and 2. The angular width of Region 2 is $\operatorname{ord}(1/\log|\nu|)$ *as* $|\nu| \to \infty$ *(recall that* $a = \operatorname{ord}(b)$ *if* $a = \mathcal{O}(b)$ *and* $b = \mathcal{O}(a)$*). Let Region 3 be the 2nd quadrant of the complex ν-plane.*

The asymptotics (3.1) *then imply that, as* $|\nu| \to \infty$,

$$
\begin{aligned}
H_\nu^{(1)}(z) &\sim \sqrt{\frac{2}{\pi\nu}}\left(\frac{\mathrm{e}z}{2\nu}\right)^\nu, \qquad \text{for } \nu \text{ in Regions 2 and 3,}\\
&\sim \frac{1}{\mathrm{i}}\sqrt{\frac{2}{\pi\nu}}\left(\frac{2\nu}{\mathrm{e}z}\right)^\nu, \qquad \text{for } \nu \text{ in Region 1.}
\end{aligned} \tag{3.3}
$$

Furthermore, $H_\nu^{(1)}(z)$ *has zeros (as a function of ν) in the first quadrant and these lie on* h_1 *as* $|\nu| \to \infty$*. Denote these zeros by* ν_n *and let* $\nu_n = \rho_n \mathrm{e}^{\mathrm{i}\phi_n}$*. Then, as* $n \to \infty$,

$$
\rho_n = \frac{\pi\left(n - \frac{1}{4}\right)}{\log\left(\frac{2\pi\left(n-\frac{1}{4}\right)}{\mathrm{e}z}\right)}\left(1 + \mathcal{O}\left(\frac{\log\log n}{\log n}\right)\right) \tag{3.4}
$$

and

$$
\phi_n = \frac{\pi}{2}\left(1 - \frac{1}{\log\left(\frac{2\pi\left(n-\frac{1}{4}\right)}{\mathrm{e}z}\right)}\left(1 + \mathcal{O}\left(\frac{\log\log n}{\log n}\right)\right)\right). \tag{3.5}
$$

Remark 3.2 *The curve* h_1 *is often defined as*

$$
\Re\left(\sqrt{\nu^2 - z^2} - \nu\log\left(\frac{\nu + \sqrt{\nu^2 - z^2}}{z}\right)\right) = 0, \quad \Im\nu > 0; \tag{3.6}
$$

see, e.g., [38, Appendix A]. When $|\nu| \gg z$ *the equation* (3.6) *becomes*

$$
\Re\left(\nu - \nu\log\left(\frac{2\nu}{z}\right)\right) = 0, \quad \Im\nu > 0,
$$

which is (3.2)*. The more general definition of* h_1 (3.6) *appears in the asymptotics of* $H_\nu^{(1)}(z)$ *for other parameter regimes (in particular the limit* $\nu \to \infty$ *and* $z \to \infty$ *with* $|\nu| \sim z$*).*

Remark 3.3 *The asymptotics* (3.3) *are often quoted as*

$$
H_\nu^{(1)}(z) \sim \frac{1}{\mathrm{i}}\sqrt{\frac{2}{\pi\nu}}\left(\frac{2\nu}{z\mathrm{e}}\right)^\nu \qquad \text{as } |\nu| \to \infty \text{ with } -\pi/2 \le \arg\nu \le \pi/2 - \delta, \qquad \text{for all } \delta > 0, \tag{3.7}
$$

see, e.g., [22, Chapter 8, Equation 3.10], meaning that given any $\delta > 0$ *there exists an* R *such that the asymptotics in* (3.7) *hold for* $|\nu| > R$*. This last statement is correct, however it "hides" the reciprocal behaviour in a neighbourhood of the imaginary axis for a fixed* $|\nu|$*.*

Proof of Theorem 3.1. The power series definition of $J_\nu(z)$ (see, e.g., [22, §A.2], [3, Equation 9.1.10]) implies that

$$
J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)}\left(1 + \mathcal{O}\left(\frac{1}{\nu}\right)\right) \quad \text{as } |\nu| \to \infty,
$$

where $\Gamma(\cdot)$ is the Gamma function.

Using the method of steepest descent on the integral formula

$$
\Gamma(\nu+1) = -\frac{1}{2\mathrm{i}\sin\pi\nu}\int_{\mathcal{H}} \mathrm{e}^{t+\nu\log t}\,\mathrm{d}t,
$$

where $\mathcal{H}$ is the Hankel contour [1, Equation 6.7.19], one can show that

$$
\Gamma(\nu+1) \sim \nu^\nu \mathrm{e}^{-\nu}\sqrt{2\pi\nu} \quad \text{and} \quad \Gamma(-\nu+1) \sim \frac{1}{\sin\pi\nu}\sqrt{\frac{\pi\nu}{2}}\nu^{-\nu}\mathrm{e}^{\nu}
$$

as $|\nu| \to \infty$ with $-\pi < \arg \nu < \pi$ [8, §6.6, Example 9] Therefore,

$$J_\nu(z) \sim \left(\frac{z}{2}\right)^\nu e^\nu \nu^{-\nu} \frac{1}{\sqrt{2\pi\nu}} \quad \text{and} \quad J_{-\nu}(z) \sim \sin \pi\nu \sqrt{\frac{2}{\pi\nu}} \left(\frac{z}{2}\right)^{-\nu} \nu^\nu e^{-\nu} \tag{3.8}$$

as $|\nu| \to \infty$ with $-\pi < \arg \nu < \pi$.

Using (3.8) in the definition of $H_\nu^{(1)}(z)$ [3, Equation 9.1.3]

$$H_\nu^{(1)}(z) := \frac{1}{i \sin \pi\nu} \Big(J_{-\nu}(z) - J_\nu(z) e^{-i\pi\nu} \Big), \tag{3.9}$$

yields

$$H_\nu^{(1)}(z) \sim \frac{1}{i} \sqrt{\frac{2}{\pi\nu}} \left(\frac{z}{2}\right)^{-\nu} \frac{\nu^\nu}{e^\nu} - \frac{1}{i\sqrt{2\pi\nu}} \left(\frac{z}{2}\right)^\nu \frac{e^\nu}{\nu^\nu} \frac{e^{i\pi\nu}}{\sin \pi\nu} \tag{3.10}$$

as $|\nu| \to \infty$ with $-\pi < \arg \nu < \pi$. When $0 < \arg \nu < \pi$,

$$\frac{e^{-i\pi\nu}}{\sin \pi\nu} \sim -2i$$

as $|\nu| \to \infty$, and using this in (3.10) we find (3.1).

By looking at the argument of the sinh function in (3.1) we see that the first exponential dominates the second in Region 1, whereas the second exponential dominates the first in Regions 2 and 3. The proof that the width of Region 2 is $\mathcal{O}(1/\log|\nu|)$ is very similar to the proof of the asymptotics of ϕ_n below.

Letting $\nu = \rho e^{i\phi}$, we see that the argument of the sinh term in (3.1) is zero when

$$\rho \left(\cos \phi \log \left(\frac{2\rho}{ez} \right) - \phi \sin \phi \right) = 0, \tag{3.11}$$

(the real part) and

$$\rho \left(\sin \phi \log \left(\frac{2\rho}{ez} \right) + \phi \cos \phi \right) = \left(n - \frac{1}{4} \right) \pi \tag{3.12}$$

(the imaginary part). The equation (3.11) shows that as ρ tends to infinity, $\cos \phi$ must tend to zero and thus ϕ must tend to $\pi/2$. The equation (3.12) then gives us that

$$\rho_n \log \left(\frac{2\rho_n}{ez} \right) \sim \left(n - \frac{1}{4} \right) \pi \quad \text{as } n \to \infty, \tag{3.13}$$

and this relation may be solved by iteration (see, e.g., [20, §1.1 and 1.5]) to give (3.4). Expanding ϕ_n about $\pi/2$ in (3.11) then yields

$$\phi_n \sim \frac{\pi}{2} \left(1 - \frac{1}{\log\left(\frac{2\rho_n}{ez}\right)} \right) \quad \text{as } n \to \infty. \tag{3.14}$$

Using the asymptotics for ρ_n (3.4) in (3.14), we obtain (3.5). ■

Remark 3.4 *The asymptotics of ϕ_n are sometimes stated in the literature as*

$$\phi_n = \frac{\pi}{2} \left(1 - \frac{1}{\log\left(\frac{2\pi\left(n-\frac{1}{4}\right)}{z}\right)} \left(1 + \mathcal{O}\left(\frac{\log \log n}{\log n} \right) \right) \right) \tag{3.15}$$

(see, e.g., [10, §5], [28, §5]), i.e. with no factor of e *in the denominator of the logarithm compared to (3.5). Nevertheless, the asymptotics (3.15) are consistent with (3.5) since*

$$\frac{1}{\log(n/e)} = \frac{1}{\log n} \left(1 + \mathcal{O}\left(\frac{1}{\log n} \right) \right).$$

Remark 3.5 *A key feature of the zeros of $H^{(1)}_\nu(z)$ is that they get closer together as $\nu \to \infty$. Indeed, the asymptotics* (3.4) *show that*

$$\rho_{n+1} - \rho_n = \mathcal{O}\left(\frac{1}{\log n}\right) \qquad \text{as } n \to \infty.$$

To evaluate the integral in (1.12) as residues at the poles ν_n, one needs to define a sequence of contours that pass between these poles.

Definition 3.6 (The contour $\mathcal{C}(N)$) *For $N \in \mathbb{Z}^+$, let the contour $\mathcal{C}(N)$ be defined by*

$$\mathcal{C}(N) = \Big\{\nu : \, |\nu| = \widetilde{\rho}_N, \quad 0 \le \arg\nu \le \pi\Big\}$$

where

$$\widetilde{\rho}_N \log\left(\frac{2\widetilde{\rho}_N}{\mathrm{e}ka}\right) = \left(N + \frac{1}{4}\right)\pi. \tag{3.16}$$

In a similar way to how the asymptotics (3.4) followed from (3.13), the definition (3.16) implies that

$$\widetilde{\rho}_N = \frac{\pi\left(N+\frac{1}{4}\right)}{\log\left(\frac{2\pi\left(N+\frac{1}{4}\right)}{\mathrm{e}ka}\right)}\left(1 + \mathcal{O}\left(\frac{\log\log N}{\log N}\right)\right) \tag{3.17}$$

as $N \to \infty$ (and so, comparing (3.17) to (3.4), we see that $\widetilde{\rho}_N$ lies "halfway" between ρ_N and ρ_{N+1}). Therefore, as $N \to \infty$, $H^{(1)}_\nu(ka) \neq 0$ on $\mathcal{C}(N)$, and the area enclosed by $\mathcal{C}(N)$ and the real axis contains the first N zeros of $H^{(1)}_\nu(ka)$.

We have chosen a sequence of semicircular contours between the ν_n, and this is also what Nussenzveig [38, Equation 4.11] and Goodrich and Kazarinoff [17, Page 172] did. Another option is to define a sequence of composite curves, consisting of (i) an arc in the neighbourhood of the zeros, whose definition is motivated by the Hankel-function asymptotics, and (ii) a large semicircle in the rest of the upper half-plane; this is what Pflumm [42, Pages 6-15] and Naylor [36, Pages 115-116] did.

Lemma 3.7 (Behaviour of the ratio of Hankel functions on $\mathcal{C}(N)$) *Let the curve h_1 be defined by* (3.2) *with $z = ka$. Let $\widetilde{\rho}_N$ be defined by* (3.16)*. Then, as $N \to \infty$ with $|\nu| = \widetilde{\rho}_N$, we have that $H^{(1)}_\nu(ka) \neq 0$ and*

$$\frac{H^{(1)}_\nu(kr)}{H^{(1)}_\nu(ka)} \sim \begin{cases} \left(\dfrac{r}{a}\right)^\nu & \text{in Regions 2 and 3,} \\[2ex] \operatorname{ord}\left(\left(\dfrac{r}{a}\right)^\nu\right) & \text{in a neighbourhood of the curve } h_1, \\[2ex] \left(\dfrac{a}{r}\right)^\nu & \text{in Region 1.} \end{cases}$$

Proof. The asymptotics (3.1) imply that

$$\frac{H^{(1)}_\nu(kr)}{H^{(1)}_\nu(ka)} \sim \frac{\sinh\left(\nu\log\left(\frac{2\nu}{\mathrm{e}kr}\right) + \mathrm{i}\frac{\pi}{4}\right)}{\sinh\left(\nu\log\left(\frac{2\nu}{\mathrm{e}ka}\right) + \mathrm{i}\frac{\pi}{4}\right)}$$

as $|\nu| \to \infty$ with $0 < \arg\nu < \pi$. Using

$$\nu\log\left(\frac{2\nu}{\mathrm{e}kr}\right) = \nu\log\left(\frac{2\nu}{\mathrm{e}ka}\right) + \nu\log\left(\frac{a}{r}\right)$$

and the formula $\sinh(A+B) = \sinh A\cosh B + \cosh A\sinh B$, we find that, with $|\nu| = \widetilde{\rho}_N$,

$$\frac{H^{(1)}_\nu(kr)}{H^{(1)}_\nu(ka)} = \cosh\left(\nu\log\left(\frac{a}{r}\right)\right) + \frac{\cosh A}{\sinh A}\sinh\left(\nu\log\left(\frac{a}{r}\right)\right) \tag{3.18}$$

where

$$A := \nu \log\left(\frac{2\nu}{\mathrm{e}ka}\right) + \mathrm{i}\frac{\pi}{4} = \cos\phi\left(N+\frac{1}{4}\right)\pi - \widetilde{\rho}_N\phi\sin\phi + \mathrm{i}\left[\sin\phi\left(N+\frac{1}{4}\right)\pi + \frac{\pi}{4} + \widetilde{\rho}_N\phi\cos\phi\right].$$

In Region 1, $\Re A \to \infty$ as $N \to \infty$ (since in this region $\cos\phi\,(N+1/4)\,\pi > \widetilde{\rho}_N\phi\sin\phi$), thus $\cosh A/\sinh A \to 1$, and then (3.18) implies that $H_\nu^{(1)}(kr)/H_\nu^{(1)}(ka) \sim (a/r)^\nu$. In Regions 2 and 3, $\Re A \to -\infty$ (since in this region $\widetilde{\rho}_N\phi\sin\phi < \cos\phi\,(N+1/4)\,\pi$), thus $\cosh A/\sinh A \to -1$, and then (3.18) implies that $H_\nu^{(1)}(kr)/H_\nu^{(1)}(ka) \sim (r/a)^\nu$.

The curve h_1 is defined by $\Re A = 0$, and almost identical arguments to those that give the asymptotics of ϕ_n (3.5) show that on h_1

$$\phi = \frac{\pi}{2}\left(1 - \frac{1}{\log\left(\frac{2\pi\left(N+\frac{1}{4}\right)}{\mathrm{e}ka}\right)}\left(1+\mathcal{O}\left(\frac{\log\log N}{\log N}\right)\right)\right). \tag{3.19}$$

These asymptotics imply that, on h_1, $A \sim \mathrm{i}(N+1/2)\pi$, and thus $\cosh A \sim 0$, $\sinh A \sim \mathrm{i}(-1)^N$, and

$$\frac{H_\nu^{(1)}(kr)}{H_\nu^{(1)}(ka)} \sim \cosh\left(\nu\log\left(\frac{a}{r}\right)\right) = \mathrm{ord}\left(\left(\frac{r}{a}\right)^\nu\right).$$

■

4 Convergence of the radial series (1.13)

We now show how the asymptotics in §3 can be used to determine the conditions under which the integral in (1.12) can be evaluated as residues at ν_n, and hence the conditions under which the radial series (1.13) converges.

Recall from Definition 3.6 that the contour $\mathcal{C}(N)$ is a large semicircle in the upper half-plane that (asymptotically) encloses the first N zeros of $H_\nu^{(1)}(ka)$. Evaluating the integrals in (1.11) as residues on ν_n yields

$$u(r,\theta) = \lim_{N\to\infty}\left(\mathrm{i}\sum_{n=1}^{N} a_n(r,\theta) - \frac{1}{2\pi}\int_{\mathcal{C}(N)} \frac{H_\nu^{(1)}(kr)}{H_\nu^{(1)}(ka)}\left[\mathrm{e}^{\mathrm{i}\nu\theta}D_L(-\mathrm{i}\nu) + \mathrm{e}^{-\mathrm{i}\nu\theta}D_R(\mathrm{i}\nu) - \frac{\mathrm{e}^{\mathrm{i}\nu\theta}D(-\mathrm{i}\nu)+\mathrm{e}^{-\mathrm{i}\nu\theta}D(\mathrm{i}\nu)}{1-\mathrm{e}^{-2\pi\mathrm{i}\nu}}\right]\mathrm{d}\nu\right), \tag{4.1}$$

where

$$a_n(r,\theta) := \frac{H_{\nu_n}^{(1)}(kr)}{\dot{H}_{\nu_n}^{(1)}(ka)}\left[\mathrm{e}^{\mathrm{i}\nu_n\theta}D_L(-\mathrm{i}\nu_n) + \mathrm{e}^{-\mathrm{i}\nu_n\theta}D_R(\mathrm{i}\nu_n) - \frac{\mathrm{e}^{\mathrm{i}\nu_n\theta}D(-\mathrm{i}\nu_n)+\mathrm{e}^{-\mathrm{i}\nu_n\theta}D(\mathrm{i}\nu_n)}{1-\mathrm{e}^{-2\pi\mathrm{i}\nu_n}}\right]. \tag{4.2}$$

Theorem 4.1 (Convergence of the radial series (1.13) via evaluating the integral (1.12) as residues)

(a) *If the function $d(\phi)$ is zero in a neighbourhood of θ, i.e. there exists a constant $c_1 > 0$ such that $d(\phi) = 0$ for all $|\phi - \theta| \le c_1$, then*

$$\int_{\mathcal{C}(N)} \frac{H_\nu^{(1)}(kr)}{H_\nu^{(1)}(ka)}\left[\mathrm{e}^{\mathrm{i}\nu\theta}D_L(-\mathrm{i}\nu) + \mathrm{e}^{-\mathrm{i}\nu\theta}D_R(\mathrm{i}\nu) - \frac{\mathrm{e}^{\mathrm{i}\nu\theta}D(-\mathrm{i}\nu)+\mathrm{e}^{-\mathrm{i}\nu\theta}D(\mathrm{i}\nu)}{1-\mathrm{e}^{-2\pi\mathrm{i}\nu}}\right]\mathrm{d}\nu \to 0 \tag{4.3}$$

as $N\to\infty$, and so (4.1) implies that

$$u(r,\theta) = \mathrm{i}\sum_{n=1}^{\infty} a_n(r,\theta),$$

which is (1.13).

(b) *Conversely, if $d(\theta) \neq 0$ then $|\int_{\mathcal{C}(N)}| \to \infty$ as $N \to \infty$ and therefore the radial series (1.13) for $u(r,\theta)$ does not converge.*

Remark 4.2 *Note that we have not dealt with the case that $d(\phi) = 0$ when $\phi = \theta$, but d is not zero in a neighbourhood of θ (e.g. if d has a simple zero at θ). However, as discussed in §1.1, the most interesting cases physically are (i) d is a delta function, and (ii) d is the restriction on $\partial\Omega$ of a plane wave. In (i), d is zero except at a single point, and in (ii) $d(\phi) \neq 0$ for all ϕ.*

Proof of Theorem 4.1. Lemma 3.7 gives us the asymptotics on $\mathcal{C}(N)$ of the Hankel functions appearing in the integrand of (1.12), and so we now need to obtain the asymptotics for the terms involving D, D_L, and D_R. The key point in the proof is that the rate of decay of the terms in the integral involving D_L and D_R depends on whether or not θ is in the support of the function d, and the faster rate of decay (for θ not in the support) is needed for the integrand to go to zero near the imaginary axis.

We assume that $d \in C^2$, and then two integration by parts give us the following asymptotics.

1. If $0 \leq \theta < 2\pi$ then

$$\frac{\mathrm{e}^{\mathrm{i}\nu\theta} D(-\mathrm{i}\nu)}{1 - \mathrm{e}^{-2\pi\mathrm{i}\nu}} = \frac{\mathrm{e}^{\mathrm{i}\nu\theta}}{\mathrm{i}\nu}\left(d(2\pi) + \mathcal{O}\left(\frac{1}{\nu}\right)\right) \tag{4.4a}$$

$$\frac{\mathrm{e}^{-\mathrm{i}\nu\theta} D(\mathrm{i}\nu)}{1 - \mathrm{e}^{-2\pi\mathrm{i}\nu}} = -\frac{\mathrm{e}^{\mathrm{i}\nu(2\pi-\theta)}}{\mathrm{i}\nu}\left(d(0) + \mathcal{O}\left(\frac{1}{\nu}\right)\right) \tag{4.4b}$$

as $|\nu| \to \infty$ with $0 < \arg\nu < \pi$ (compare to (2.24)).

2. If there exists a $c_1 > 0$ such that $d(\phi) = 0$ for all $|\phi - \theta| < c_1$, then (with $0 \leq \theta < 2\pi$)

$$\mathrm{e}^{\mathrm{i}\nu\theta} D_L(-\mathrm{i}\nu) = -\frac{\mathrm{e}^{\mathrm{i}\nu c_1}}{\mathrm{i}\nu}\left(d(\theta - c_1) + \mathcal{O}\left(\frac{1}{\nu}\right)\right) + \frac{\mathrm{e}^{\mathrm{i}\nu\theta}}{\mathrm{i}\nu}\left(d(0) + \mathcal{O}\left(\frac{1}{\nu}\right)\right), \tag{4.5a}$$

$$\mathrm{e}^{-\mathrm{i}\nu\theta} D_R(\mathrm{i}\nu) = \frac{\mathrm{e}^{\mathrm{i}\nu(2\pi-\theta)}}{\mathrm{i}\nu}\left(d(2\pi) + \mathcal{O}\left(\frac{1}{\nu}\right)\right) - \frac{\mathrm{e}^{\mathrm{i}\nu c_1}}{\mathrm{i}\nu}\left(d(\theta + c_1) + \mathcal{O}\left(\frac{1}{\nu}\right)\right), \tag{4.5b}$$

as $|\nu| \to \infty$ with $0 < \arg\nu < \pi$. Furthermore, if $d(\theta) \neq 0$ then both $\mathrm{e}^{\mathrm{i}\nu\theta} D_L(-\mathrm{i}\nu)$ and $\mathrm{e}^{-\mathrm{i}\nu\theta} D_R(\mathrm{i}\nu)$ are $\mathrm{ord}(1/\nu)$.

These asymptotics show that the final term in square brackets in (4.3) (i.e. the fraction) decays exponentially regardless of whether or not θ is in the support of d. However, the first two terms in square brackets are $\mathrm{ord}(1/\nu)$ when $d(\theta) \neq 0$ and $\mathrm{ord}(\exp(\mathrm{i}\nu c_1)/\nu)$ if d is zero in a neighbourhood of θ (noting that $c_1 < \theta$).

We split the integral over $AGC(N)$ as follows

$$\int_{\mathcal{C}(N)} = \int_{\substack{\mathcal{C}(N)\\ \text{Region 1}}} + \int_{\substack{\mathcal{C}(N)\\ \text{nbhd of } h_1}} + \int_{\substack{\mathcal{C}(N)\\ \text{Region 2}}} + \int_{\substack{\mathcal{C}(N)\\ \text{Region 3}}} .$$

Since the ratio of Hankel functions decays exponentially in Regions 1 and 3 (by Lemma 3.7), the integrals in Regions 1 and 3 tend to zero as $N \to \infty$ regardless of whether or not θ is in the support of d (there is a loss of decay on the imaginary axis, which is the boundary of Region 3, but this is exactly the situation where Jordan's lemma [1, Lemma 4.2.2] shows the integral still goes to zero.)

In Region 2 and the neighbourhood of h_1 the Hankel functions grow exponentially. Indeed, when

$$\phi = \frac{\pi}{2}\left(1 - \mathrm{ord}\left(\frac{1}{\log N}\right)\right) \tag{4.6}$$

on the contour $\mathcal{C}(N)$,

$$\left(\frac{r}{a}\right)^{\nu} = \mathrm{ord}\left(\exp\left(\widetilde{\rho}_N \cos\phi \log\left(\frac{r}{a}\right)\right)\right) = \mathrm{ord}\left(\exp\left(c_2 \frac{N}{(\log N)^2}\right)\right),$$

for some $c_2 > 0$ (where we have used the asymptotics of $\widetilde{\rho}_n$ (3.17)).

The asymptotic behaviour of the Hankel functions in Region 2 is the same as in the neighbourhood of h_1, but Region 2 has a larger angular width than the neighbourhood. Therefore, we only need to consider the integral over Region 2, since the integral over the neighbourhood of h_1 is of higher order. Furthermore, the presence of $(r/a)^\nu$ in the asymptotics of the integrand means that the leading order behaviour of the integral over Region 2 comes from when (4.6) holds. Therefore, when estimating this integral we only need to estimate the integral over the region (4.6).

If $d(\theta) \neq 0$ then the terms in square brackets in (4.3) are $\mathrm{ord}(1/\nu)$ and

$$\left| \int_{\substack{\mathcal{C}(N) \\ \text{Region } 2}} \right| = \mathrm{ord}\left(\frac{\exp\left(c_2 \frac{N}{(\log N)^2}\right)}{\log N} \right), \tag{4.7}$$

which tends to infinity as $N \to \infty$ (thus establishing Part (b) of the Theorem).

If d is zero in a neighbourhood of θ, then the terms in square brackets in (4.3) are $\mathrm{ord}(\exp(\mathrm{i}\nu c_3)/\nu)$, where $c_3 = \min(\theta, 2\pi - \theta, c_1)$ and c_1 is as in (4.5). Therefore, in this case

$$\left| \int_{\substack{\mathcal{C}(N) \\ \text{Region } 2}} \right| = \mathrm{ord}\left(\frac{\exp\left(c_2 \frac{N}{(\log N)^2} - c_3 \frac{N}{\log N}\right)}{\log N} \right) = \mathrm{ord}\left(\frac{\exp\left(-c_3 \frac{N}{\log N}\right)}{\log N} \right), \tag{4.8}$$

which tends to zero as $N \to \infty$ (thus establishing Part (a) of the Theorem). ∎

4.1 Investigating convergence of the series (1.13) directly

In Theorem 4.1 we determined whether or not the series (1.13) converges by investigating the behaviour of the integral on $\mathcal{C}(N)$ as $N \to \infty$. Alternatively, one can investigate the convergence of the series directly. This was presented in a clear way by Cohen [10, Theorem IV] for the BVP

$$\begin{aligned} \Delta u + k^2 u &= -f \quad \text{in } \Omega, \\ u(a, \theta) &= 0, \quad \text{for } 0 \leq \theta < 2\pi, \end{aligned}$$

where f is a given function with compact support in Ω, and u also satisfies the radiation condition (1.4). The proof for Problem 1.1 is very similar, and we include it for completeness.

Theorem 4.3 (Convergence of the radial series (1.13) via the asymptotics of $a_n(r,\theta)$) *With $a_n(r,\theta)$ defined by (4.2),*

1. *if d is zero in a neighbourhood of θ then $a_n(r,\theta) \to 0$ exponentially quickly as $n \to \infty$, and thus the series $\sum_{n=1}^\infty a_n$ converges;*
2. *if $d(\theta) \neq 0$ then $a_n(r,\theta) \to \infty$ as $n \to \infty$.*

Proof. Using the asymptotics of the Hankel function (3.1) and the asymptotics of the ν_n (3.4) and (3.5), we find that

$$\frac{H^{(1)}_{\nu_n}(kr)}{\dot{H}^{(1)}_{\nu_n}(ka)} \sim -\frac{1}{2}\left(\frac{r}{a}\right)^{\nu_n} \frac{1}{\log\left(\frac{2\nu_n}{eka}\right)} \tag{4.9}$$

([10, Equations 22 and 24]).

If d is zero in a neighbourhood of θ, then the asymptotics (4.4) and (4.5) imply that

$$\left[\mathrm{e}^{-\mathrm{i}\nu_n\theta} D_R(\mathrm{i}\nu_n) + \mathrm{e}^{\mathrm{i}\nu_n\theta} D_L(-\mathrm{i}\nu_n) - \frac{\mathrm{e}^{\mathrm{i}\nu_n\theta} D(-\mathrm{i}\nu_n) + \mathrm{e}^{-\mathrm{i}\nu_n\theta} D(\mathrm{i}\nu_n)}{1 - \mathrm{e}^{-2\pi\mathrm{i}\nu_n}} \right] = \mathrm{ord}\left(\frac{\mathrm{e}^{\mathrm{i}\nu_n c_3}}{\nu_n} \right), \tag{4.10}$$

with $c_3 = \min(\theta, 2\pi - \theta, c_1)$. Combining (4.9) and (4.10) and using Cohen's notation that $\nu_n = \alpha_n + \mathrm{i}\beta_n$, we have that

$$a_n(r,\theta) = \mathrm{ord}\left(\frac{\mathrm{e}^{\,\mathrm{i}\nu_n c_3 + \nu_n \log(r/a)}}{\nu_n \log\left(\frac{2\nu_n}{eka}\right)} \right) = \mathrm{ord}\left(\frac{\mathrm{e}^{-\beta_n\left(c_3 - \frac{\alpha_n}{\beta_n}\log(r/a)\right)}}{n} \right)$$

(where we have used the fact that $\nu_n \log(2\nu_n/(\mathrm{e}ka)) = (n-1/4)\pi\mathrm{i}$).

The asymptotics of ρ_n and ϕ_n, (3.4) and (3.5) respectively, imply that

$$\alpha_n = \mathrm{ord}\left(\frac{n}{(\log n)^2}\right), \quad \beta_n = \mathrm{ord}\left(\frac{n}{\log n}\right),$$

and so $\alpha_n/\beta_n \to 0$ as $n \to \infty$. Therefore, if d is zero in a neighbourhood of θ then

$$a_n(r,\theta) = \mathrm{ord}\left(\frac{\mathrm{e}^{-c_3\beta_n}}{n}\right),$$

which tends to zero as $n \to \infty$ (compare to (4.8)).

If $d(\theta) \neq 0$ then the left hand side of (4.10) is only $\mathrm{ord}(\nu_n^{-1})$. Therefore, in this case

$$a_n(r,\theta) = \mathrm{ord}\left(\frac{\mathrm{e}^{\alpha_n \log(r/a)}}{n}\right),$$

which tends to infinity as $n \to \infty$ (compare to (4.7)). ∎

5 Concluding remarks

We conclude by discussing the results of this paper, first in the context of spectral theory, and then in the context of current uses of Watson-type transformations.

Implications for spectral theory. Given a separable BVP in 2-d, the Watson transformation implies that the transform associated with one of the separated ODEs (plus boundary conditions) can be obtained from the transform associated with the other separated ODE (plus boundary conditions). For example, for Problem 1.1, the transform associated with the ODE in the θ-variable is the usual Fourier series, and in §1.1 we started from this and then used the Watson transformation to obtain the expression for the solution obtained by the transform associated with the r-variable (although we saw that this second transform is not complete).

The Fokas method implies that given the two transforms associated with the two separated ODEs *in the whole space* (i.e. without boundary conditions), one can obtain the transforms associated with either of the ODEs plus boundary conditions. For example, given the completeness relations (2.13) and (2.14) for the radial and angular ODEs with no boundary conditions, we can obtain the expression (1.11) for the solution of Problem 1.1, from which either the angular or the radial series ((1.5) or (1.13)) can be obtained. (We obtained the radial series from (1.11) in §1.1 and §4, and the details of how to obtain the angular series are in [46, §4.3.1.2].) Actually, one only needs the completeness relation for *one* of the ODEs in the whole space. Indeed, the completeness relation (2.13) gave us the expression for the fundamental solution (2.11), and the completeness relation (2.14) gave us the expression (2.12); however, either one of (2.11) and (2.12) can be obtained from the other by deforming contours, and thus we only need *one* of the completeness relations (2.13) and (2.14).

In other words, using the Fokas method, one can obtain the completeness relations for operators *with* boundary conditions (if these completeness relations exist) from the completeness relations for the operators *without* boundary conditions. This fact has recently been exploited in the context of third- and higher-order differential operators by Pelloni and Smith [41], [44]. In this situation, completeness relations exists for the operators without boundary conditions, and these authors (i) find conditions under which completeness relations exist for the operators with boundary conditions [41, Theorem 2.6], and (ii) show that completeness relations do not exist for the operators with certain boundary conditions [41, Theorem 3.1].

As mentioned in §1.1, BVPs for the Helmholtz equation in unbounded domains are not self-adjoint because of the radiation conditon (1.4). For separable Helmholtz BVPs, this non-self-adjointness causes the transforms associated with the radial ODE to *not* be complete. Indeed, the usual form of the Kontorovich–Ledebev transform, the transform associated with the radial ODE in the whole space, is not complete (as discussed in §2.2.2), and the transform (1.14) associated with the radial ODE on $a < r < \infty$ with a zero Dirichlet boundary condition at $r = a$ is not complete (this is proved in [10, §3] using similar arguments to those in §4.1). With the Fokas method, one only needs to overcome the problem of non-completeness for the radial ODE in the whole space (via Jones' regularised version of the Kontorovich–Ledebev transform (2.14)), and then one can solve any separable Helmholtz BVP in polar coordinates using the expansions of the fundamental solution (2.11) and (2.12) (along with the other steps of the Fokas method).

Current uses of Watson-type transformations. In this paper, we have only discussed how transform methods can be used to find explicit expressions for the solutions of certain BVPs. (By "explicit expression" we mean an integral or a series involving known functions, and we consider special functions, such as Hankel and Bessel functions, as "known", although this distinction is rather simplistic.)

Transform methods can also be used to obtain expressions for the solutions of more complicated BVPs in terms of integrals or series that involve more complicated mathematical objects. For example, the *Wiener-Hopf method* expresses the solutions of certain BVPs as integrals whose integrands involve the solution of a Wiener-Hopf problem (or, more generally, a *Riemann–Hilbert* problem), which is equivalent to a singular integral equation; see [37], [1, Chapter 7]. Similarly, the *Sommerfeld–Malyuzhinets technique* expresses the solutions of BVPs for the Helmholtz equation in wedge and cone geometries as integrals whose integrands involve the solution of a *functional-difference equation*; see e.g. [7], [40]. In general, the Wiener–Hopf problems and functional-difference equations cannot be solved explicitly, but one can often obtain from them the asymptotics of the solution of the BVP in relevant parameter limits (see, e.g., [37, Chapter 5], [7]), and they can also be solved numerically (see, e.g., [2], [29] and the references therein).

For many BVPs tackled by this more general use of transform methods, it is often not clear what the best representation of the solution is, and the idea of deforming contours in the complex plane (central to the Watson transformation) is often used; see, e.g., [33, §5], [32]. The author hopes that the ideas in this paper, and the Fokas method in general, will eventually be able to shed new light on these problems, however further development of the method is required. In particular, the Fokas method was used to find explicit expressions for the solutions of BVPs for evolution equations in three dimensions (two spatial dimensions and one time dimension) in [23] and [34]; however, to the authors' knowledge, the method has not yet been used to find an explicit expression for the solution of a BVP for an elliptic PDE in three dimensions (although this is, in principle, possible).

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