Towards Atomic Graphs

4th International Workshop on Structures and Deduction

David R. Sherratt and Marco Solieri
d.r.sherratt@bath.ac.uk

September 9th, 2017

University of Bath

Table of contents

Why?

The λ-calculus

Sharing
Use of a single representation for multiple instances of some subterm

Duplication
Extraction of an object from a sharing to its context within some term

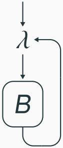
The atomic λ-calculus

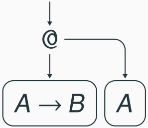
$$\frac{A \to (B \wedge C)}{(A \to B) \wedge (A \to C)}\ d$$

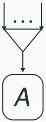
- A typed term calculus with explicit sharing
- Atomic duplication (on individual constructors)
- Achieves fully lazy sharing

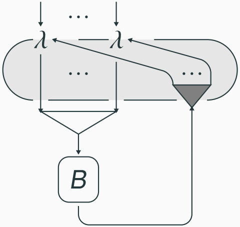
Graphical Intuition

$$\frac{B}{A \to (A \wedge B)}\,\lambda \qquad \frac{A \wedge (A \to B)}{B}\,@ \qquad \frac{A}{A \wedge \ldots \wedge A}\,\triangle \qquad \frac{A \to \dfrac{B}{(B \wedge \ldots \wedge B)}\,\triangle}{(A \to B) \wedge (A \to B)}\,d$$









Sharing Graphs

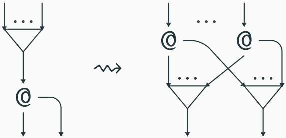
Optimal Reduction
To avoid useless repitition of work in the reduction of terms

- Introduced by Lévy (1970)
- Lamping defined an evaluator satisfying optimality requirements (1989/90)

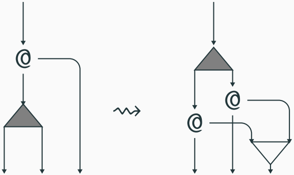
Similarities between Sharing Graphs and Atomic Graphs

- Graph reduction techniques
- Atomic Duplication

Differences to Sharing Graphs - Duplication of Application



Atomic Graphs

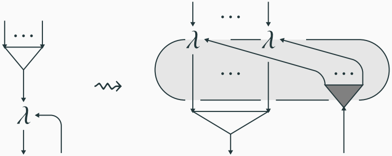


Sharing Graphs

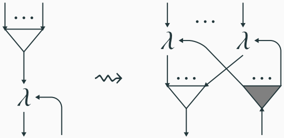
Differences to Sharing Graphs - Duplication of Abstraction

Co-sharing

Sharing Graphs	No notion of co-sharing
Atomic Graphs	A sharing with no rewrite rules



Atomic Graphs

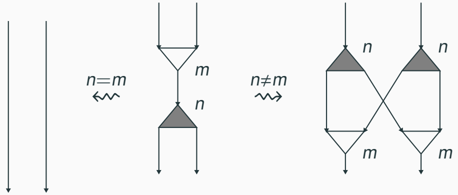


Sharing Graphs

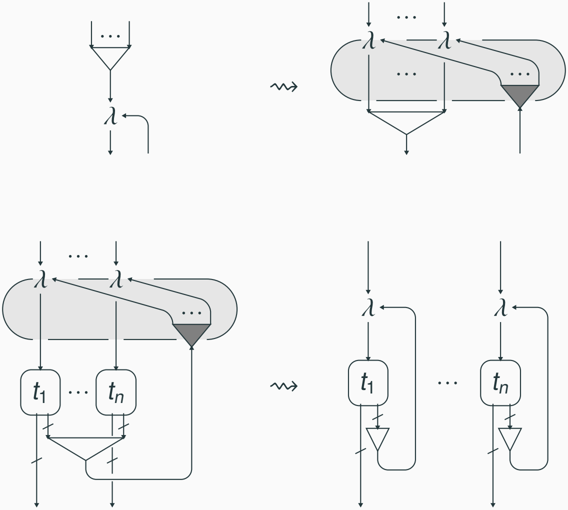
Differences to Sharing Graphs - Duplication of Abstraction

Co-sharing vs Sharing

How to decide if they duplicate or eliminate each other?



Differences to Sharing Graphs - Duplication of Abstraction



Duplication of Abstraction - Proof Transformations

$$
\dfrac{A \to \begin{array}{c} \dfrac{B}{A \wedge B}\,\lambda \\ \Downarrow \\ C \end{array}}{(A \to C) \wedge (A \to C)}\,\triangle
\quad \rightsquigarrow \quad
\dfrac{A \to \begin{array}{c} \dfrac{B}{A \wedge B}\,\lambda \\ \Downarrow \\ \dfrac{C}{C \wedge C}\,\triangle \end{array}}{(A \to C) \wedge (A \to C)}\,d
$$

$$
\dfrac{A \to \begin{array}{c} \dfrac{}{\dfrac{A}{A \quad A}\,\triangle}\,\lambda \\ \Downarrow \wedge \Downarrow \\ C \quad C \end{array}}{(A \to C) \wedge (A \to C)}\,d
\quad \rightsquigarrow \quad
A \to \begin{array}{c} \dfrac{}{A}\,\lambda \\ \Downarrow \\ C \end{array}
\wedge
A \to \begin{array}{c} \dfrac{}{A}\,\lambda \\ \Downarrow \\ C \end{array}
$$

What?

The Aim

Atomic Graphs

Formulation of the atomic λ-calculus as a graph rewriting system that is

- Local
- Sound
- Complete

with respect to reduction in the atomic λ-calculus.

Sharing Graphs

Better understand the relationship between Sharing Graphs and the atomic λ-calculus

Soundness and Completeness Conjecture

$$[\![t]\!] : \Lambda_a \to \mathfrak{G}_a$$

$$(\!|t|\!) : \mathfrak{G}_a \to \Lambda_a$$

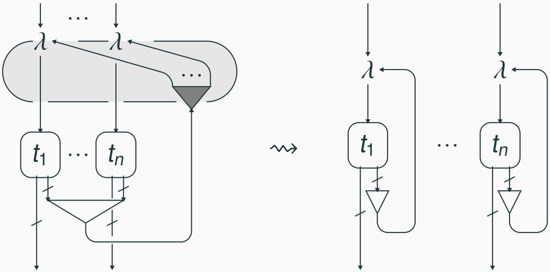
Conjecture

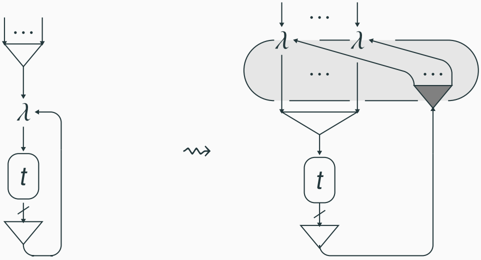
Given $t \in \Lambda_a$:

1. if $[\![t]\!] \rightsquigarrow G$ for some $G \in \mathfrak{G}_a$, then $t \rightsquigarrow^* (\!|G|\!)$;
2. if $t \rightsquigarrow t'$ for some $t \in \Lambda_a$, then $[\![t]\!] \rightsquigarrow^* [\![t']\!]$.

How?

Which sharing meets the co-sharing?



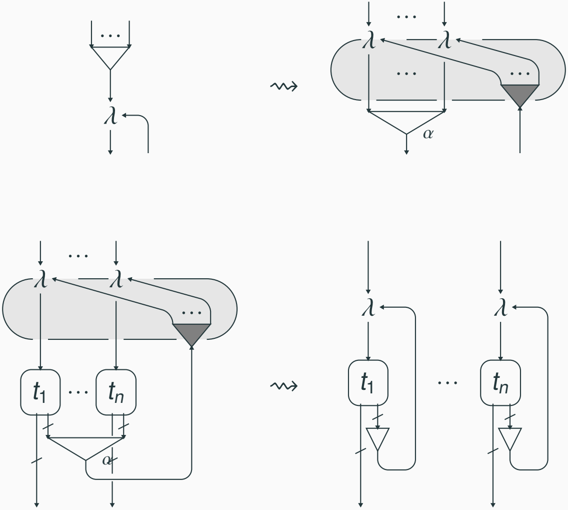


When sharing meets the co-sharing - Proof Theory

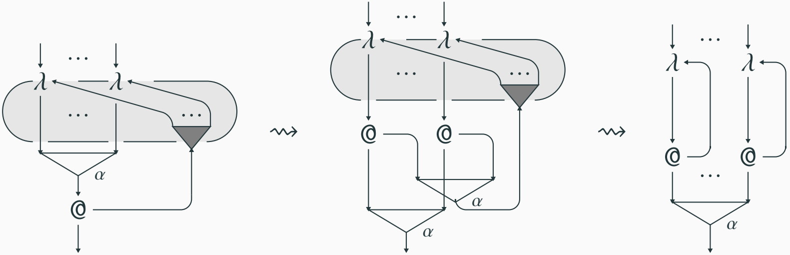
$$
\cfrac{\overline{A \to \begin{array}{c} \cfrac{A}{A \quad A}\,\triangle \\ \Downarrow \;\wedge\; \Downarrow \\ C \quad C \end{array}}\;\lambda}{(A \to C) \wedge (A \to C)}\; d
\quad\rightsquigarrow\quad
\overline{A \to \begin{array}{c} A \\ \Downarrow \\ C \end{array}}\;\lambda
\;\wedge\;
\overline{A \to \begin{array}{c} A \\ \Downarrow \\ C \end{array}}\;\lambda
$$

$$
\cfrac{\cfrac{B}{A \to \begin{array}{c} \cfrac{A}{A \wedge A}\,\triangle \;\wedge\; B \\ \Downarrow \\ C \end{array}}\;\lambda}{(A \to C) \wedge (A \to C)}\;\triangle
\quad\rightsquigarrow\quad
\cfrac{\cfrac{B}{A \to \begin{array}{c} \cfrac{A}{A \wedge A}\,\triangle \;\wedge\; B \\ \Downarrow \\ \cfrac{C}{C \wedge C}\,\triangle \end{array}}\;\lambda}{(A \to C) \wedge (A \to C)}\; d
$$

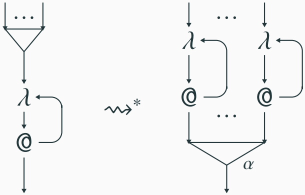
Some Bookkeeping - Active Sharing

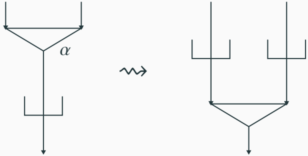


Some Bookkeeping - Wild Active Sharings

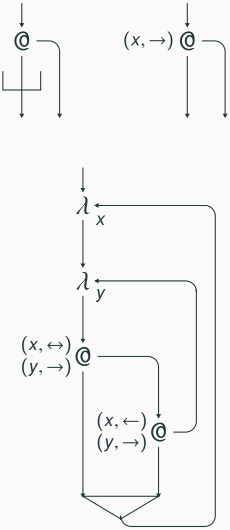


More Bookkeeping - End of Scope



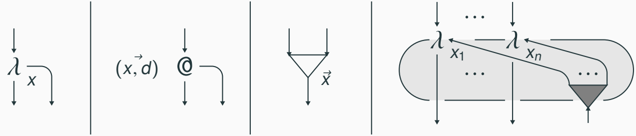


More Bookkeeping - Director Annotations



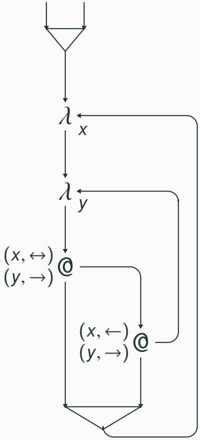
More Bookkeeping - Director Annotations

$\mathfrak{G}_a \quad :=$

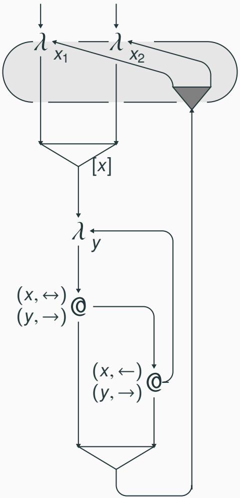


$$d \quad := \quad \leftarrow \mid \rightarrow \mid \leftrightarrow$$

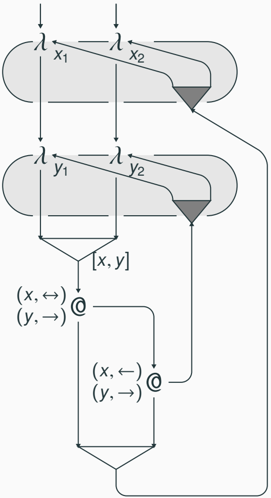
More Bookkeeping - Duplication



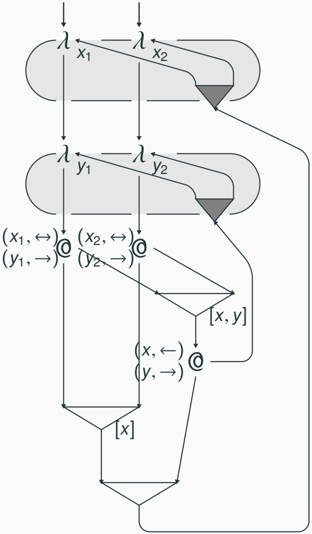
More Bookkeeping - Duplication



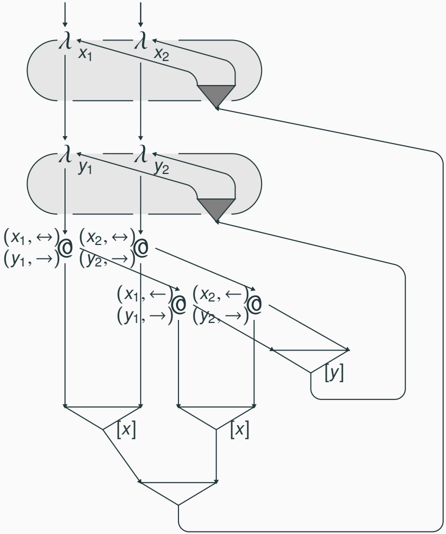
More Bookkeeping - Duplication



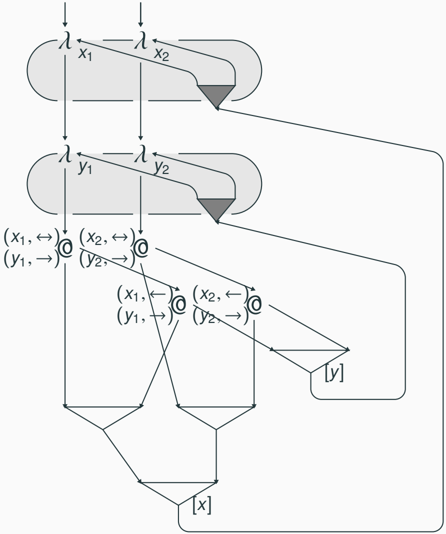
More Bookkeeping - Duplication



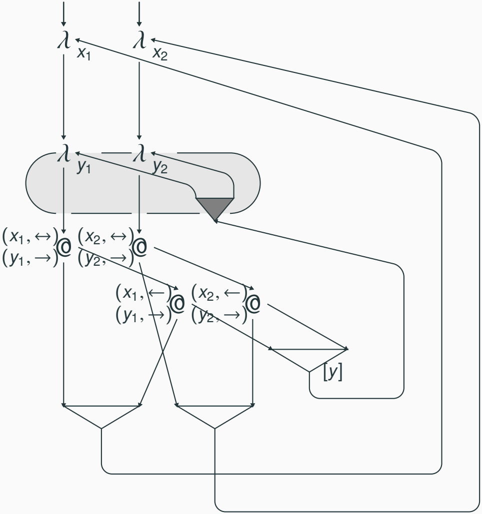
More Bookkeeping - Duplication



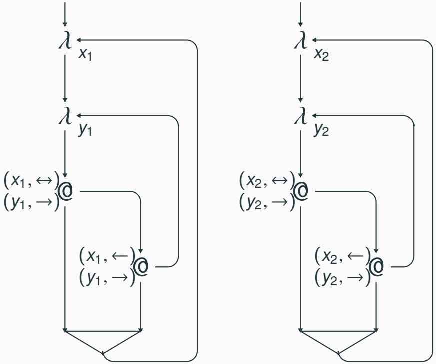
More Bookkeeping - Duplication



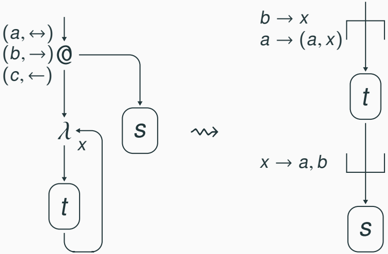
More Bookkeeping - Duplication

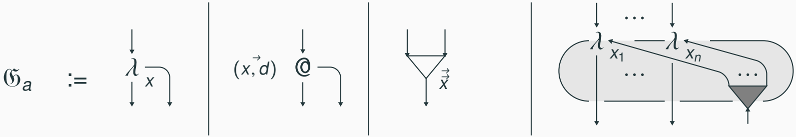


More Bookkeeping - Duplication



Beta Reduction





$$d \quad := \quad \leftarrow \mid \rightarrow \mid \leftrightarrow$$

Proof Theoretic Interpretation

$$\frac{}{A \to A}\ i \qquad \frac{\Gamma, A; \Delta \vdash B}{\Gamma; \Delta \vdash A \to B}\ \lambda$$

$$\frac{\Gamma; \Delta \vdash A \to B \qquad \Gamma'; \Delta' \vdash A}{\Gamma \cup \Gamma'; \Delta \cup \Delta' \vdash B}\ @ \qquad \frac{\Gamma; \Delta \vdash A}{\Gamma; \Delta \vdash A \qquad \Gamma; \Delta \vdash A}\ \triangle$$

$$\frac{\Gamma, A; \Delta \vdash B}{\Gamma; A, \Delta \vdash B}\ A \to A \qquad \frac{\Gamma; A, \Delta' \vdash B}{\Gamma, A; \Delta' \vdash B}\ A \to A$$

$$\frac{\Gamma; \Delta, \Delta' \vdash B}{\Gamma; \Delta'' \vdash B}\ \Delta'' \to \Delta, \Delta' \qquad \frac{\Gamma; \Delta'' \vdash C}{\Gamma; \Delta, \Delta' \vdash B}\ \Delta, \Delta' \to \Delta''$$

Proof Theoretic Interpretation

$$
\dfrac{\dfrac{\begin{array}{c}\dfrac{}{A \vdash A}\,i \\ \Downarrow \\ \Gamma, A; \Delta \vdash B\end{array}}{\Gamma; \Delta \vdash A \to B}\,\lambda \qquad \dfrac{\pi}{\Gamma'; \Delta' \vdash A}}{\Gamma \cup \Gamma'; \Delta \cup \Delta', A \vdash B}\,@
\quad \rightsquigarrow \quad
\dfrac{\begin{array}{c}\dfrac{\begin{array}{c}\pi \\ \Gamma'; \Delta' \vdash B\end{array}}{; A \vdash A}\; A \to A^* \\ \Downarrow' \\ \Gamma; \Delta, A \vdash B\end{array}}{\Gamma \cup \Gamma'; \Delta \cup \Delta', A \vdash B}\; A \to A^*
$$

Conclusion.

Conclusion

Why?

Explore an efficient non-optimal implementation of the λ-calculus

What?

Atomic graphs: a graph rewriting system based on the atomic λ-calculus

How?

Use of director annotations to guide active sharings through the graph

Use of gates to maintain annotations after β-reduction

Yet can we prove they are sound and complete?
Is this mechanism equally as complex as sharing graphs?
Is it necessary? Can we make the mechanism simpler?