Integrable Lagrangians and Picard modular forms

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Plan:

- Integrable Lagrangians $\int f(v_{x_1}, v_{x_2}, v_{x_3})\, dx_1 dx_2 dx_3$
- Integrability conditions
- Lagrangian density $f = v_{x_1} v_{x_2} v_{x_3}$
- Lagrangian densities $f = v_{x_1} v_{x_2} g(v_{x_3})$
- Lagrangian densities $f = v_{x_1} g(v_{x_2}, v_{x_3})$
- Generic Lagrangian densities $f(v_{x_1}, v_{x_2}, v_{x_3})$
- Concluding remarks

E.V. Ferapontov, K.R. Khusnutdinova and S.P. Tsarev, On a class of three-dimensional integrable Lagrangians, Comm. Math. Phys. **261**, N1 (2006) 225-243.

E.V. Ferapontov and A.V. Odesskii, Integrable Lagrangians and modular forms, J. Geom. Phys. **60**, no. 6-8 (2010) 896-906.

D. Zagier, On a U(3,1)-automorphic form of Ferapontov-Odesskii, talk in Utrecht on 17 April 2009.

Integrable Lagrangians $\int f(v_{x_1}, v_{x_2}, v_{x_3})\, dx_1 dx_2 dx_3$

Euler-Lagrange equation:

$$(f_{v_{x_1}})_{x_1} + (f_{v_{x_2}})_{x_2} + (f_{v_{x_3}})_{x_3} = 0.$$

Examples:

Dispersionless Kadomtsev-Petviashvili (dKP) equation

$$v_{x_1x_3} - v_{x_1}v_{x_1x_1} - v_{x_2x_2} = 0, \qquad f = v_{x_1}v_{x_2} - \frac{1}{3}v_{x_1}^3 - v_{x_2}^2.$$

Boyer-Finley (BF) equation

$$v_{x_1x_1} + v_{x_2x_2} - e^{v_{x_3}}v_{x_3x_3} = 0, \qquad f = v_{x_1}^2 + v_{x_2}^2 - 2e^{v_{x_3}}.$$

Three equivalent approaches to integrability

- The method of hydrodynamic reductions based on the requirement that the equation possesses infinitely many multi-phase solutions of special type.
- The method of dispersionless Lax pairs based on the representation of the equation as the compatibility condition of two Hamilton-Jacobi type equations.
- Integrability 'on solutions' based on the condition that the characteristic variety of the equation defines a conformal structure which is Einstein-Weyl on every solution.

All three approaches lead to the same set of integrability conditions for the Lagrangian density f.

Hydrodynamic reductions: example of dKP

First-order form of dKP equation $v_{x_1x_3}-v_{x_1}v_{x_1x_1}-v_{x_2x_2}=0$ (set $u=v_{x_1}$):

$$u_{x_3}-uu_{x_1}-w_{x_2}=0, \quad u_{x_2}-w_{x_1}=0.$$

Look for N-phase solutions: $u=u(R^1,...,R^n),\ w=w(R^1,...,R^n)$ where

$$R^i_{x_3}=\lambda^i(R)R^i_{x_1}, \quad R^i_{x_2}=\mu^i(R)R^i_{x_1}.$$

The substitution of u,w into the above first-order system implies

$$\partial_i w=\mu^i\partial_i u, \qquad \lambda^i=u+(\mu^i)^2,$$

as well as the following equations for $u(R)$ and $\mu^i(R)$ (Gibbons-Tsarev system):

$$\partial_j\mu^i=\frac{\partial_j u}{\mu^j-\mu^i}, \qquad \partial_i\partial_j u=2\frac{\partial_i u\partial_j u}{(\mu^j-\mu^i)^2}.$$

In involution! General solution depends on n arbitrary functions of one variable.

Dispersionless Lax pairs: example of dKP

The dKP equation $v_{x_1x_3} - v_{x_1}v_{x_1x_1} - v_{x_2x_2} = 0$ possesses dispersionless Lax representation (Zakharov):

$$S_{x_2} = \frac{1}{2}S_{x_1}^2 + v_{x_1}, \quad S_{x_3} = \frac{1}{3}S_{x_1}^3 + v_{x_1}S_{x_1} + v_{x_2}.$$

In parametric form:

$$S_{x_1} = p, \quad S_{x_2} = \frac{1}{2}p^2 + v_{x_1}, \quad S_{x_3} = \frac{1}{3}p^3 + v_{x_1}p + v_{x_2}.$$

Observation Integrability by the method of hydrodynamic reductions is equivalent to the existence of a dispersionless Lax representation (proved for broad classes of integrable models).

Integrability via Einstein-Weyl geometry: example of dKP

Einstein-Weyl geometry is a triple $(\mathbb{D},\ g,\ \omega)$ where $\mathbb{D}$ is a symmetric connection, g is a conformal structure and ω is a covector such that

$$\mathbb{D}_k g_{ij} = \omega_k g_{ij}, \quad R_{(ij)} = \Lambda g_{ij}.$$

Here $R_{(ij)}$ is the symmetrised Ricci tensor of $\mathbb{D}$ and Λ is some function (the first set of equations defines $\mathbb{D}$ uniquely, so it is sufficient to specify g and ω only).

Every solution of the dKP equation $v_{x_1x_3} - v_{x_1}v_{x_1x_1} - v_{x_2x_2} = 0$ carries Einstein-Weyl geometry (Dunajski, Mason, Tod):

$$g = 4dx_1dx_3 - dx_2^2 + 4v_{x_1}dx_3^2, \quad \omega = -4v_{x_1x_1}dx_3.$$

Observation Integrability by the method of hydrodynamic reductions is equivalent to the Einstein-Weyl property of the characteristic conformal structure of the equation (proved for broad classes of integrable models).

Integrability conditions

For a non-degenerate Lagrangian, the Euler-Lagrange equation is integrable (by either of the techniques mentioned above) if and only if the Lagrangian density f satisfies the relation

$$d^4f = d^3f\,\frac{dH}{H} + \frac{3}{H}\mathrm{det}(dM).$$

Here d^3f and d^4f are the symmetric differentials of f while the Hessian H and the 4×4 augmented Hessian matrix M are defined as

$$H = \det\begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{xy} & f_{yy} & f_{yz} \\ f_{xz} & f_{yz} & f_{zz} \end{pmatrix}, \quad M = \begin{pmatrix} 0 & f_x & f_y & f_z \\ f_x & f_{xx} & f_{xy} & f_{xz} \\ f_y & f_{xy} & f_{yy} & f_{yz} \\ f_z & f_{xz} & f_{yz} & f_{zz} \end{pmatrix}.$$

Here $(x, y, z) = (v_{x_1}, v_{x_2}, v_{x_3})$. The non-degeneracy condition is equivalent to $H \neq 0$. The system for f is in involution, and its solution space is 20-dimensional.

Lagrangian density $f = v_{x_1} v_{x_2} v_{x_3}$

Euler-Lagrange equation:

$$v_{x_3} v_{x_1 x_2} + v_{x_2} v_{x_1 x_3} + v_{x_1} v_{x_2 x_3} = 0.$$

Parametric Lax representation:

$$\frac{S_{x_1}}{v_{x_1}} = \zeta(p) + \frac{\wp'(p) + \lambda}{2\wp(p)}, \quad \frac{S_{x_2}}{v_{x_2}} = \zeta(p) + \frac{\wp'(p) - \lambda}{2\wp(p)}, \quad \frac{S_{x_3}}{v_{x_3}} = \zeta(p),$$

where $(\wp')^2 = 4\wp^3 + \lambda^2$ and $\zeta' = -\wp$ (Weierstrass $\wp$ and ζ functions). Note the algebraic identity

$$\left(\frac{S_{x_1}}{v_{x_1}} - \frac{S_{x_3}}{v_{x_3}}\right)\left(\frac{S_{x_3}}{v_{x_3}} - \frac{S_{x_2}}{v_{x_2}}\right)\left(\frac{S_{x_2}}{v_{x_2}} - \frac{S_{x_1}}{v_{x_1}}\right) = \lambda.$$

Lagrangian densities $f = v_{x_1} v_{x_2} g(v_{x_3})$

Euler-Lagrange equation:

$$(v_{x_2} g(v_{x_3}))_{x_1} + (v_{x_1} g(v_{x_3}))_{x_2} + (v_{x_1} v_{x_2} g'(v_{x_3}))_{x_3} = 0.$$

Integrability condition for $g(z)$:

$$g''''(g^2 g'' - 2g(g')^2) - 9(g')^2(g'')^2 + 2gg'g''g''' + 8(g')^3 g''' - g^2(g''')^2 = 0.$$

$GL(2, \mathbb{R})$-invariance:

$$\tilde{z} = \frac{\alpha z + \beta}{\gamma z + \delta}, \qquad \tilde{g} = (\gamma z + \delta) g.$$

This invariance allows one to linearise the integrability condition for $g(z)$.

Auxiliary hypergeometric equation

Consider the auxiliary hypergeometric equation

$$u(1-u)h_{uu} + (1-2u)h_u - \frac{2}{9}h = 0,$$

parameters $(1/3, 2/3, 1)$. The geometry behind this equation is a 1-parameter family of genus 2 trigonal curves

$$r^3 = q(q-1)(q-u)^2$$

supplied with the holomorphic differential $\omega = dq/r$. The corresponding periods, $h = \int_a^b \omega$ where $a, b \in \{0, 1, \infty, u\}$, form a 2-dimensional vector space and satisfy the above (Picard-Fuchs) hypergeometric equation.

Generic solution $g(z)$

The generic solution $g(z)$ can be represented in either of the 3 equivalent forms:

1. Parametric form:

$$z = \frac{h_1(u)}{h_2(u)}, \quad g = h_2(u)$$

   where h_i are 2 linearly independent solutions of the hypergeometric equation. $GL(2,\mathbb{R})$-invariance corresponds to the freedom in the choice of basis h_i.

2. Theta representation:

$$g(z) = \sum_{(k,l)\in\mathbb{Z}^2} e^{2\pi i(k^2+kl+l^2)z} = 1+6q+6q^3+6q^4+12q^4+\ldots, \quad q = e^{2\pi i z}.$$

3. Power series:

$$g(z) = \sum_{k\geq 0} B_k^2 \frac{z^{6k+1}}{(6k+1)!}$$

   where B_k are certain integers.

Lagrangian densities $f = v_{x_1} g(v_{x_2}, v_{x_3})$

Euler-Lagrange equation

$$(g)_{x_1} + \left(v_{x_1} g_{v_{x_2}}\right)_{x_2} + \left(v_{x_1} g_{v_{x_3}}\right)_{x_3} = 0.$$

Integrability conditions lead to an involutive system of 5 PDEs for $g(y, z)$ which are invariant under the 10-dimensional symmetry group:

$$\tilde{y} = \frac{l_1(y, z)}{l(y, z)}, \quad \tilde{z} = \frac{l_2(y, z)}{l(y, z)}, \quad \tilde{g} = \alpha g + \beta,$$

where l, l_1, l_2 are arbitrary (inhomogeneous) linear forms. This invariance allows one to linearise the integrability conditions for $g(y, z)$.

Auxiliary hypergeometric system

Consider the auxiliary (Appell) hypergeometric system

$$h_{u_1 u_2} = \frac{1}{3}\frac{h_{u_1} - h_{u_2}}{u_1 - u_2},$$

$$h_{u_1 u_1} = -\frac{h}{9u_1(u_1-1)} + \frac{h_{u_2}}{3(u_1-u_2)}\frac{u_2(u_2-1)}{u_1(u_1-1)} - \frac{h_{u_1}}{3}\left(\frac{1}{u_1-u_2} + \frac{2}{u_1} + \frac{2}{u_1-1}\right),$$

$$h_{u_2 u_2} = -\frac{h}{9u_2(u_2-1)} + \frac{h_{u_1}}{3(u_2-u_1)}\frac{u_1(u_1-1)}{u_2(u_2-1)} - \frac{h_{u_2}}{3}\left(\frac{1}{u_2-u_1} + \frac{2}{u_2} + \frac{2}{u_2-1}\right).$$

The geometry behind this system is the family of genus 3 Picard trigonal curves

$$r^3 = q(q-1)(q-u_1)(q-u_2)$$

supplied with the holomorphic differential $\omega = dq/r$. The corresponding periods, $h = \int_a^b \omega$ where $a, b \in \{0, 1, \infty, u_1, u_2\}$, form a 3-dimensional vector space and satisfy the above (Picard-Fuchs) hypergeometric system.

Generic solution $g(y,z)$

The generic solution $g(y,z)$ can be represented in either of the 3 equivalent forms:

1. Parametric form:

$$y = \frac{h_1(u_1,u_2)}{h_3(u_1,u_2)}, \quad z = \frac{h_2(u_1,u_2)}{h_3(u_1,u_2)}, \quad g = F(s), \; s = \frac{u_1(u_2-1)}{u_2(u_1-1)}$$

   where h_i are 3 linearly independent solutions of the hypergeometric system and $F' = [s(s-1)]^{-2/3}$.

2. Theta representation:

$$g(y,z) = y + \sum_{(k,l)\in\mathbb{Z}^2\backslash 0} \frac{\tilde{\theta}((k+\epsilon l)y)}{k+\epsilon l} e^{2\pi i(k^2+kl+l^2)z}, \quad \epsilon = e^{\pi i/3}.$$

3. Power series:

$$g(y,z) = \sum_{j,k\geq 0} B_j B_k B_{j+k} \frac{y^{6j+1}}{(6j+1)!} \frac{z^{6k+1}}{(6k+1)!}.$$

Relation to Picard modular forms

The period map

$$y = \frac{h_1(u_1, u_2)}{h_3(u_1, u_2)}, \quad z = \frac{h_2(u_1, u_2)}{h_3(u_1, u_2)},$$

was inverted by Picard (1883):

$$u_1 = \frac{\varphi_1(y, z)}{\varphi_0(y, z)}, \; u_2 = \frac{\varphi_2(y, z)}{\varphi_0(y, z)},$$

where $\varphi_\nu(y, z) \in M_3(\Gamma[\sqrt{-3}])$ are single-valued modular forms on a 2-dimensional complex ball $2\mathrm{Re}y + |z|^2 < 0$ with respect to the Picard modular group $\Gamma[\sqrt{-3}] = \{g \in U(2, 1; \mathbb{Z}[\rho]) : g \equiv 1 (\mathrm{mod} \sqrt{-3})\}, \; \rho = e^{2\pi i/3}$.

Picard modular forms were extensively studied by Holzapfel, Feustel, Finis, Shiga, Cléry and van der Geer.

Picard modular forms via theta functions

Explicitly, $\varphi_\nu(y,z) = \theta_\nu^3$ where theta functions $\theta_\nu(y,z)$ are defined as

$$\theta_\nu(y,z) = \sum_{\xi \in \mathbb{Z}[\rho]} \rho^{-\nu(\xi+\bar{\xi})} Y(\xi y) e^{\frac{2\pi z}{\sqrt{3}}\xi\bar{\xi}},$$

here

$$Y(u) = \frac{1}{k} e^{\frac{\pi u^2}{\sqrt{3}}} \vartheta\begin{bmatrix} 1/6 \\ 1/6 \end{bmatrix}(u, -\rho^2), \quad k = \vartheta\begin{bmatrix} 1/6 \\ 1/6 \end{bmatrix}(0, -\rho^2),$$

Here ϑ-functions with characteristics are defined as

$$\vartheta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](z,\tau) = \sum_{n \in \mathbb{Z}} e^{\pi i \tau (n+a)^2 + 2\pi i (n+a)(z+b)}.$$

Differential dg via Picard modular forms

There is a simple expression of the differential dg is terms of φ_ν:

$$dg = \frac{\varphi_1\varphi_2(\varphi_2 - \varphi_1)d\varphi_0 + \varphi_0\varphi_2(\varphi_0 - \varphi_2)d\varphi_1 + \varphi_0\varphi_1(\varphi_1 - \varphi_0)d\varphi_2}{\zeta^2}$$

where $\zeta \in S_6(\Gamma[\sqrt{-3}], \det)$ is a modular form defined as

$$\zeta^3 = \varphi_0\varphi_1\varphi_2(\varphi_1 - \varphi_0)(\varphi_2 - \varphi_0)(\varphi_2 - \varphi_1).$$

Up to a constant factor, the differential dg coincides with the Eisenstein series $E_{1,1}$ which was introduced in:

H. Shiga, On the representation of the Picard modular function by θ constants. I, II. Publ. Res. Inst. Math. Sci. **24**, no. 3 (1988) 311-360.

F. Cléry, G. van der Geer, Generators for modules of vector-valued Picard modular forms. Nagoya Math. J. **212** (2013) 19-57.

Generic Lagrangian densities $f(v_{x_1}, v_{x_2}, v_{x_3})$

Euler-Lagrange equation:

$$(f_{v_{x_1}})_{x_1} + (f_{v_{x_2}})_{x_2} + (f_{v_{x_3}})_{x_3} = 0.$$

Integrability conditions lead to a system of 15 PDEs for $f(x, y, z)$ which are invariant under 20-dimensional symmetry group:

$$\tilde{x} = \frac{l_1(x,y,z)}{l(x,y,z)}, \ \ \tilde{y} = \frac{l_2(x,y,z)}{l(x,y,z)}, \ \ \tilde{z} = \frac{l_3(x,y,z)}{l(x,y,z)}, \ \ \tilde{f} = \frac{f}{l(x,y,z)},$$

as well as obvious symmetries of the form

$$\tilde{f} = \epsilon f + \alpha x + \beta y + \gamma z + \delta,$$

where l, l_1, l_2, l_3 are arbitrary (inhomogeneous) linear forms. This invariance allows one to linearise the integrability conditions for $f(x, y, z)$.

Auxiliary hypergeometric system

Consider the auxiliary (Appell) hypergeometric system

$$h_{u_i u_j} = \frac{1}{3}\frac{h_{u_i} - h_{u_j}}{u_i - u_j},$$

$$h_{u_i u_i} = -\frac{2}{9}\frac{h}{u_i(u_i-1)} - \frac{1}{3u_i(u_i-1)}\sum_{j\neq i}^{3}\frac{u_j(u_j-1)}{u_j-u_i}h_{u_j} + \\ -\frac{1}{3}\Big(\sum_{j\neq i}^{3}\frac{1}{u_i-u_j} + \frac{2}{u_i} + \frac{2}{u_i-1}\Big).$$

The geometry behind this system is the family of genus 4 Picard trigonal curves

$$r^3 = q(q-1)(q-u_1)(q-u_2)(q-u_3)$$

supplied with the holomorphic differential $\omega = dq/r$. The corresponding periods, $h = \int_a^b \omega$ where $a, b \in \{0, 1, \infty, u_1, u_2, u_3\}$, form a 4-dimensional vector space and satisfy the above (Picard-Fuchs) hypergeometric system.

Inhomogeneous hypergeometric extension

We will also need the inhomogeneous hypergeometric system

$$h_{u_i u_j} = \frac{1}{3}\frac{h_{u_i} - h_{u_j}}{u_i - u_j} + \epsilon_{ijk}\frac{u_k(u_k - 1)(u_i - u_j)}{U^{2/3}},$$

$$h_{u_i u_i} = -\frac{2}{9}\frac{h}{u_i(u_i - 1)} - \frac{1}{3u_i(u_i - 1)}\sum_{j\neq i}^{3}\frac{u_j(u_j - 1)}{u_j - u_i}h_{u_j} + \\ -\frac{1}{3}\Big(\sum_{j\neq i}^{3}\frac{1}{u_i - u_j} + \frac{2}{u_i} + \frac{2}{u_i - 1}\Big)h_{u_i},$$

where ϵ_{ijk} is the totally antisymmetric tensor and

$$U = u_1 u_2 u_3 (u_1 - 1)(u_2 - 1)(u_3 - 1)(u_1 - u_2)(u_2 - u_3)(u_3 - u_1).$$

The inhomogeneous system for h is in involution.

Generic solution $f(x, y, z)$

The generic solution $g(y, z)$ can be represented in either of the 3 equivalent forms:

1. Parametric form:

$$x = \frac{h_1}{h_4}, \quad y = \frac{h_2}{h_4}, \quad z = \frac{h_3}{h_4}, \quad f = \frac{F}{h_4}$$

   where $h_i(u_1, u_2, u_3)$ are 4 independent solutions of the hypergeometric system and $F(u_1, u_2, u_3)$ is a solution of the inhomogeneous system.

2. Theta representation:

$$f(x, y, z) = xy + \sum_{(k,l)\in\mathbb{Z}^2\setminus 0} \frac{\tilde{\theta}((k+\epsilon l)x)\tilde{\theta}((k+\epsilon l)y)}{(k+\epsilon l)^2} e^{2\pi i(k^2+kl+l^2)z}.$$

3. Power series:

$$f(x, y, z) = \sum_{i,j,k\geq 0} B_i B_j B_k B_{i+j+k} \frac{x^{6i+1}}{(6i+1)!} \frac{y^{6j+1}}{(6j+1)!} \frac{z^{6k+1}}{(6k+1)!}.$$

Theta functions $\tilde{\theta}$ and integers B_n

We define

$$\tilde{\theta}(z) = \frac{i}{\theta'(0)} e^{\pi \frac{z^2}{\sqrt{3}} - \pi i \frac{\epsilon}{4}} \vartheta(\epsilon, z), \quad \theta'(0) = -\frac{i}{(2\pi)^2} e^{\frac{\pi\sqrt{3}}{8}} 3^{3/8} \Gamma(1/3)^{9/2}$$

where

$$\vartheta(\tau, z) = \sum_{n \in \mathbb{Z}} e^{\pi i [(n+1/2)^2 \tau + 2(n+1/2)(z+1/2)]}$$

is the classical Jacobi theta function. The Taylor expansion of $\tilde{\theta}$ about $z = 0$ is

$$\tilde{\theta}(z) = \sum_{n \geqslant 0} b_{6n+1}\, z^{6n+1} = z + b_7\, z^7 + \dots$$

where

$$b_{6n+1} = \frac{(-3)^n \Gamma(1/3)^{18n}\, B_n}{(2\pi)^{6n} (6n+1)!}.$$

Here B_n are integers.

Concluding remarks

- There exists a whole variety of integrable Lagrangians whose densities f are polynomial or can be expressed in terms of elementary functions. It would be interesting to clarify how these (and similar) examples can be recovered as degenerations of the 'master-Lagrangian', and to describe singular orbits of lower dimension. This should be related to understanding degenerations/compactifications of the moduli space of Picard curves.

- Although our parametric, theta and power series representations for integrable densities possess straightforward generalisations to dimensions higher than 3, the relation to integrable Lagrangians will be lost: one can show that in higher dimensions every integrable first-order Lagrangian density $f(v_x)$ is necessarily of the form $f = \frac{Q(v_x)}{l(v_x)}$ where Q and l are arbitrary quadratic and linear functions of the first-order derivatives. Thus, the occurence of modular forms is the essentially 3-dimensional phenomenon.