EINSTEIN METRICS, HYPERCOMPLEX STRUCTURES AND THE TODA FIELD EQUATION

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Abstract. We obtain explicitly all solutions of the SU(∞) Toda field equation with the property that the associated Einstein-Weyl space admits a 2-sphere of divergence-free shear-free geodesic congruences. The solutions depend on an arbitrary holomorphic function and give rise to new hyperKähler and selfdual Einstein metrics with one dimensional isometry group. These metrics each admit a compatible hypercomplex structure with respect to which the symmetries are triholomorphic.

1. Introduction

LeBrun's $\mathcal{H}$-space construction [8] gives a method for constructing, at least in principle, a selfdual Einstein manifold M of negative scalar curvature with a prescribed conformal infinity given by an arbitrary real analytic conformal 3-manifold B. The Einstein metric is defined initially on a punctured collar neighbourhood of the conformal infinity, where it is uniquely determined by the conformal structure on B, but it often extends analytically to a larger manifold.

In practice, however, this "filling in" construction is difficult to carry out directly, except when B is conformally flat, when the Einstein metric is the hyperbolic metric. Conformally flat 3-manifolds may be characterized as locally admitting compatible Einstein metrics. A more general situation in which progress can be made is the case that B admits a compatible Einstein-Weyl structure. Hitchin [6] has shown that in this situation, the twistor space Z of M is the projectivized cotangent bundle of the minitwistor space $\mathcal{S}$ of B, and consequently that there is a conformal retraction of M onto B, i.e., a conformal submersion $M \to B$ inducing the identity map of B at infinity. If one has enough information about B or its minitwistor space, then one can hope to find M using this observation.

The first non-trivial examples of this construction were the Pedersen metrics on the unit ball in $\mathbb{R}^4$ [11], where the conformal infinity is a Berger 3-sphere with its standard Einstein-Weyl structure [7]. However, even in these examples, M is constructed indirectly and shown to be the desired 4-metric using the uniqueness clause of the LeBrun construction. In fact, after observing that the generator of the principal symmetry of the Berger sphere induces a Killing field on M, Pedersen applies the Jones-Tod construction [7] to see that the space of trajectories of this Killing field also carries an Einstein-Weyl structure, which he identifies as the standard Einstein metric of the round 3-sphere. This information, and a little inspired guesswork, is enough to find the Einstein metric on M explicitly, and it turns out that on one side of the conformal infinity, it extends to a complete metric on a ball.

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The following diagram summarizes the construction.

$$\begin{tikzcd} & M^4 \arrow[dl, shift right, "\xi"'] \arrow[dr, "K"] & \\ BS^3 \arrow[ur, hook, shift right] & & S^3 \end{tikzcd} \tag{1.0}$$

Here we have labelled the submersions over BS^3 (the Berger 3-sphere) and S^3 (the round 3-sphere) by vector fields tangent to the fibres: K is a Killing field and we take ξ to be a unit vector field. These submersions are well-defined if M^4 is the punctured ball: on the entire ball ξ has a point singularity, at which K vanishes.

A Weyl structure may be specified by giving a choice of representative g for the conformal metric and a 1-form ω which defines a connection on the line bundle L of scalars of weight one (see below). On the Berger spheres, these are given by

$$\begin{aligned} g &= d\theta^2 + \sin^2\theta d\phi^2 + a^2(d\psi + \cos\theta d\phi)^2 = \sigma_1^2 + \sigma_2^2 + a^2\sigma_3^2 \\ \omega &= b(d\psi + \cos\theta d\phi) = b\sigma_3, \end{aligned} \tag{1.1}$$

where a and b are constants with $b^2 = a^2(1-a^2)$, and the σ_i are the usual invariant 1-forms on SU(2). The principal symmetry is generated by $\partial/\partial\psi$.

The selfdual Einstein metrics of [11] are

$$g_M = \frac{1}{(1-\rho^2)^2}\left[\frac{1+m^2\rho^2}{1+m^2\rho^4}d\rho^2 + \frac{1}{4}\rho^2\left((1+m^2\rho^2)(\sigma_1^2+\sigma_2^2) + \frac{1+m^2\rho^4}{1+m^2\rho^2}\sigma_3^2\right)\right].$$

The conformal structure extends to the conformal infinity at $\rho = 1$, which is the Berger sphere with $a^2 = 1/(1+m^2)$, so that $m^2 = b^2/a^4$.

Our aim in this paper is to generalize these metrics and the diagram (1.0) by replacing S^3 and BS^3 with other Einstein-Weyl spaces. To do this, we want to explain how the geometry of M^4 restricts the possible geometries of the Einstein-Weyl space M^4/K generalizing S^3. First of all, by the general theory of [13], since M^4 is an Einstein manifold with a Killing field K, the Einstein metric is conformal to a scalar-flat Kähler metric. Now an Einstein-Weyl space arising as the quotient of a scalar-flat Kähler metric by a Killing field is not arbitrary [9]: it admits a shear-free twist-free geodesic congruence [12]. Let us pause to define these terms.

Definition 1.1. A *Weyl space* is a conformal manifold (B^n, c) equipped with a torsion free connection D such that $D\mathsf{c} = 0$. (We view the conformal structure c as a metric on TB with values in the real line bundle L^2, where $L^{-n} = |\Lambda^n T^*B|$.) It is said to be *Einstein-Weyl* if the symmetric trace-free part of the Ricci tensor of this connection vanishes.

A *congruence* on an oriented three-dimensional Weyl space B^3 is (the foliation generated by) a weightless unit vector field $\chi \in C^\infty(B, L^{-1}\otimes TB)$, i.e., $\langle\chi,\chi\rangle = 1$, where the angle brackets denote the conformal metric. The congruence is *shear-free and geodesic* if

$$D\chi = \tau(id - \chi\otimes\chi) + \kappa *\chi$$

and τ, κ are called the *divergence* and *twist* of χ. They are sections of L^{-1}.

Our conventions mainly follow [2]. In particular we make free use of the isomorphism between $L^{w-k}\otimes\Lambda^k TB$ and $L^{w+k}\otimes\Lambda^k T^*B$ given by the conformal structure, and say that sections of these bundles have weight $w \in \mathbb{R}$. The Hodge star operator is the isomorphism $*\colon L^{w-k}\otimes\Lambda^k TB \to L^{w+n-k}\otimes\Lambda^{n-k}T^*B$ determined by the orientation form $*1 \in C^\infty(B, L^n\otimes\Lambda^n T^*B)$. Thus $*\chi$ may be viewed as a section of $\mathfrak{so}(TB)$, the bundle of skew endomorphisms, using the conformal structure.

Definition 1.2. We shall say an Einstein-Weyl 3-manifold is *Toda* if it admits a shear-free geodesic congruence with vanishing twist (a "Toda congruence"), and that it is *hyperCR* if it admits a shear-free geodesic congruence with vanishing divergence.

The reason for this terminology is that if B is Toda, there is a distinguished compatible metric, which we call the *LeBrun-Ward gauge*, such that the Weyl structure may be written

$$\begin{aligned} g &= e^u(dx^2 + dy^2) + dz^2 \\ \omega &= -u_z dz, \end{aligned} \tag{1.2}$$

where u is a solution of the Toda field equation $u_{xx} + u_{yy} + (e^u)_{zz} = 0$ (see [14]). Here ω is the connection 1-form of the covariant derivative on L given by the Weyl connection, relative to the trivialization of L given by the choice of representative metric. Hence $Dg = -2\omega \otimes g$. We used the same convention in (1.1).

On the other hand if B is hyperCR, then χ is not alone: in fact, if we view the orientation form $*1 \in \mathrm{C}^\infty(B, L^3 \otimes \Lambda^3 T^*B)$ as a section of $L \otimes T^*B \otimes \mathfrak{so}(TB)$ using the conformal structure, then $D - \kappa *1$ is a flat metric connection on $L^{-1} \otimes TB$ and the parallel weightless unit vector fields give a 2-sphere of divergence-free shear-free geodesic congruences. Each of these congruences defines a CR structure on B, hence the terms hyperCR and hyperCR structure are introduced by analogy with hypercomplex or hyperKähler. These Einstein-Weyl spaces were called *special* in [3] and [4]. It was shown there that monopoles over hyperCR Einstein-Weyl spaces define hypercomplex 4-manifolds by the construction of [7].

The round metric on S^3, as an Einstein-Weyl structure, is both Toda and hyperCR. More precisely, it admits a Toda congruence on the complement of any pair of antipodal points, and also two hyperCR structures. The Toda congruence is given by the geodesics joining the antipodal points, while the congruences of the two hyperCR structures are the left and right invariant congruences respectively.

Returning now to the Pedersen metric over S^3, we see that not only is it conformally scalar-flat Kähler, but also, since S^3 is hyperCR, it admits two compatible hypercomplex structures [4, 10]. We shall see later that these hypercomplex structures are also induced by a hyperCR structure on the Berger 3-sphere at infinity.

Therefore, in order to generalize the Pedersen metrics and diagram (1.0), one approach is to look for Einstein-Weyl spaces, generalizing S^3, which are both Toda and hyperCR. Our first result is that all such spaces can be found.

Theorem 1.3. *For any holomorphic function h on an open subset of S^2, the Einstein-Weyl space given by*

$$\begin{aligned} g &= (z+h)(z+\overline{h})g_{S^2} + dz^2 \\ \omega &= -\frac{2z + h + \overline{h}}{(z+h)(z+\overline{h})}dz, \end{aligned}$$

where g_{S^2} is the spherical metric, is both Toda and hyperCR. Furthermore any Toda Einstein-Weyl space admitting a hyperCR structure arises in this way, with the exception of the Toda solutions given by a parallel congruence on flat space.

We prove this theorem in section 2. Then, in sections 3 and 4, we consider monopoles over these hyperCR Toda spaces. In particular, over each such space, we find an Einstein metric with symmetry and with a conformal infinity given by another Einstein-Weyl space from a class generalizing the Berger spheres. This will give the desired generalization of diagram (1.0).

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2. The Toda solutions

In order to prove Theorem 1.3 we must find all solutions of the Toda field equation admitting a hyperCR structure. As very few solutions of the Toda equation are known, this is an interesting exercise in its own right. The condition that an Einstein-Weyl space is hyperCR is equivalent [4] to the existence of a section κ of L^{-1} with

$$\kappa^2 = \frac{1}{6} scal^D \tag{2.1}$$

$$D\kappa = -\frac{1}{2} {*F^D}, \tag{2.2}$$

where F^D is the curvature of the Weyl connection on L. Our aim is to impose this condition on the Toda field equation. We start with equation (2.2) which can be written in a gauge as $d\kappa - \omega\kappa = -\frac{1}{2}{*d\omega}$. In the LeBrun-Ward gauge, this becomes

$$\kappa_x dx + \kappa_y dy + (\kappa_z + u_z\kappa)dz = -\frac{1}{2}u_{yz}dx + \frac{1}{2}u_{xz}dy.$$

We deduce from this the equations $\kappa_x = -\frac{1}{2}u_{yz}$ and $\kappa_y = \frac{1}{2}u_{xz}$, which have an integrability condition:

$$u_{xxz} = u_{xzx} = 2\kappa_{yx} = 2\kappa_{xy} = -u_{yzy} = -u_{yyz}.$$

Therefore $0 = (u_{xx} + u_{yy})_z = -(e^u)_{zzz}$ and so we may write

$$e^u = e^{f(x,y)}\big(az^2 + b(x,y)z + c(x,y)\big).$$

The Toda field equation with this Ansatz can be solved explicitly as follows. We compute:

$$\begin{aligned} u_{xx} + u_{yy} &= \frac{\big((b_{xx} + b_{yy})z + c_{xx} + c_{yy}\big)(az^2 + bz + c) - (b_x z + c_x)^2 - (b_y z + c_y)^2}{(az^2 + bz + c)^2} \\ &\quad + f_{xx} + f_{yy} \\ (e^u)_{zz} &= 2ae^f, \end{aligned}$$

which must sum to zero. We multiply through by $(az^2 + bz + c)^2$ and equate coefficients of the resulting quartic in z. The leading term is a Liouville equation:

$$f_{xx} + f_{yy} + 2ae^f = 0.$$

The general solution of this Liouville equation may be written

$$e^{f(x,y)} = \frac{4|F'(x+iy)|^2}{(1 + a|F(x+iy)|^2)^2}$$

in terms of an arbitrary nonconstant F, holomorphic in $x+iy$. The other coefficients now give the following equations:

$$a(b_{xx}+b_{yy})=0 \tag{2.3}$$

$$a(c_{xx}+c_{yy})+b(b_{xx}+b_{yy})=b_x^2+b_y^2 \tag{2.4}$$

$$b(c_{xx}+c_{yy})+c(b_{xx}+b_{yy})=2(b_xc_x+b_yc_y) \tag{2.5}$$

$$c(c_{xx}+c_{yy})=c_x^2+c_y^2. \tag{2.6}$$

If $a=0$ then equations (2.4) and (2.6) are solved by letting $b=B|e^{\phi}|^2$, $c=C|e^{\psi}|^2$ with B,C constant and ϕ,ψ holomorphic. Equation (2.5) now gives $B=0$, $C=0$ or $|\phi'-\psi'|=0$ and so the functional dependence of b and c can be absorbed into f and we have a separable solution for e^u.

If a is not zero, then equations (2.3) and (2.6) give $b=a(h+\overline{h})$, $c=C|e^{\psi}|$, with C constant and h,ψ holomorphic. Equation (2.4) now gives $aC|\psi'|^2|e^{\psi}|^2=a^2|h'|^2$ and so C/a is nonnegative. If $C=0$ then h is constant and we have a separable solution; otherwise, without loss of generality, we may take $C=a$ and $h=e^{\psi}+\mu$ where μ is a real constant. Finally, equation (2.5) reduces to $\mu|\psi'|^2=0$ and so either $\mu=0$ or ψ is constant, the latter case again giving a separable solution.

The separable solutions are all known and the Einstein-Weyl structures are all given by 3-metrics of constant curvature [13]: in our case the curvature must be nonnegative in order to satisfy (2.1), and these solutions, generating the metrics of $\mathbb{R}^3$ and S^3, are the ones we are trying to generalize. The new solutions of the Toda equation are:

$$e^u=\frac{4a(z+h)(z+\overline{h})|F'|^2}{(1+a|F|^2)^2},$$

and positivity forces $a>0$. We readily verify that equation (2.2) is now satisfied with

$$\kappa=\frac{i(h-\overline{h})}{2(z+h)(z+\overline{h})}.$$

Furthermore, a computation shows that $\frac{1}{6}scal^D=\frac{1}{2}u_{zz}+\frac{1}{4}u_z^2$, from which it follows that (2.1) is also satisfied, and so the Einstein-Weyl space is indeed hyperCR. Since F cannot be constant, we may use $\sqrt{a}F$ as a holomorphic coordinate in place of $x+iy$ and we easily obtain Theorem 1.3.

3. Scalar-flat Kähler metrics

In this section we study abelian monopoles over the hyperCR Toda spaces. On any Toda space B, these are defined to be solutions of the equation

$$w_{xx}+w_{yy}+(e^u w)_{zz}=0.$$

LeBrun [9] shows that each solution of this equation generates a scalar-flat Kähler metric with a Killing field, given explicitly by

$$g_M=w\,e^u(dx^2+dy^2)+w\,dz^2+w^{-1}(dt+\theta)^2,$$

where θ is a 1-form on B with $*(dw-\omega w)=d\theta$.

Consequently we can construct a large family of scalar-flat Kähler metrics from the Einstein-Weyl spaces of section 2. Since the Einstein-Weyl spaces are hyperCR, these scalar-flat Kähler metrics admit compatible hypercomplex structures with respect to which $\partial/\partial t$ is triholomorphic. For most choices of h, the hyperCR Toda spaces have no continuous symmetries, and so these scalar-flat Kähler spaces will generically have only a one-dimensional symmetry group, generated by $\partial/\partial t$.

In order to obtain explicit metrics, we still have a linear differential equation, the monopole equation, to solve. Fortunately, there are some interesting solutions available, given to us for free by the geometry. These solutions may be viewed as arising from LeBrun's observation [9] that the monopole equation above is the linearized Toda equation, and so monopoles can be found by linearizing a family of solutions of the Toda equation. In particular, the affine change $(x, y, z) \mapsto (ax, ay, az - b)$ induces a symmetry of the Toda equation. Linearizing a family of solutions generated by this gauge transformation shows that for any $a, b \in \mathbb{R}$, $a(1-\frac{1}{2}zu_z)+\frac{1}{2}bu_z$ defines a monopole on *any* Toda Einstein-Weyl space [1, 5, 9, 13].

For our explicit solutions, these monopoles may also be obtained by linearizing with respect to affine changes of the holomorphic function h. Ian Strachan (private communication) has pointed out that by linearizing the solutions with respect to arbitrary holomorphic changes of h, one sees, more generally, that

$$\mathrm{w} = \frac{f}{2(z+h)} + \frac{\overline{f}}{2(z+\overline{h})}$$

is a monopole for any holomorphic function f ($f = ah + b$ being a special case). To compute θ note that $*(d\mathrm{w} - \omega\mathrm{w}) = d(\mathrm{v}\, dz) + \frac{1}{2}(f + \overline{f})vol_{S^2}$ where

$$\mathrm{v} = \frac{if}{2(z+h)} - \frac{i\overline{f}}{2(z+\overline{h})}.$$

Hence one can write $dt + \theta = \beta + \mathrm{v}\, dz$, where β is a 1-form independent of z such that $d\beta = \frac{1}{2}(f + \overline{f})vol_{S^2}$, so that the scalar-flat Kähler metric is:

$$g_M = \mathrm{w}\,(z+h)(z+\overline{h})g_{S^2} + \mathrm{w}dz^2 + \mathrm{w}^{-1}(\beta + \mathrm{v}\, dz)^2. \tag{3.1}$$

For definiteness, one could take

$$\beta = dt + \frac{i}{1+\zeta\overline{\zeta}}\left(\frac{f\, d\zeta}{\zeta} - \frac{\overline{f}\, d\overline{\zeta}}{\overline{\zeta}}\right)$$

where ζ is a holomorphic coordinate on S^2 with $vol_{S^2} = 2i\, d\zeta \wedge d\overline{\zeta}/(1+\zeta\overline{\zeta})^2$.

These scalar-flat Kähler metrics will not be Einstein or conformally Einstein in general. However, they do have the property that the lift of $\partial/\partial z$ given by $\beta(\partial/\partial z) = 0$ defines a conformal submersion. To see this, write the conformal structure on M as $\mathsf{c} = \varepsilon_0^2 + \cdots + \varepsilon_3^2$, where ε_0 and ε_3 are the weightless unit 1-forms corresponding to $\mathrm{w}dz$ and $\beta + \mathrm{v}\, dz$. Let ξ be the weightless unit 1-form dual to $\partial/\partial z$, so that

$$\xi = \frac{\mathrm{w}\varepsilon_0 + \mathrm{v}\varepsilon_3}{\sqrt{\mathrm{w}^2 + \mathrm{v}^2}}.$$

Now $\varepsilon_0^2 + \varepsilon_3^2 - \xi^2 = \eta^2$, where

$$\eta = \frac{\mathrm{v}\varepsilon_0 - \mathrm{w}\varepsilon_3}{\sqrt{\mathrm{w}^2 + \mathrm{v}^2}} = \frac{\mathrm{w}\beta}{\sqrt{\mathrm{w}^2 + \mathrm{v}^2}}.$$

Hence $\mathsf{c} - \xi^2$ may be represented by the metric

$$(\mathrm{w}^2 + \mathrm{v}^2)|z+h|^2 g_{S^2} + \beta^2 = |f|^2 g_{S^2} + \beta^2,$$

which is independent of z, so that ξ is a conformal submersion over this metric.

The conformal structures $|f|^2 g_{S^2} + \beta^2$, depending on an arbitrary holomorphic function f, arise elsewhere, namely in the classification of Einstein-Weyl spaces admitting a "geodesic symmetry" (a conformal vector field preserving the Weyl connection whose trajectories are geodesics of the Weyl connection).

Theorem 3.1. [2] *The three dimensional Einstein-Weyl spaces with geodesic symmetry are either flat with translational symmetry or are given locally by:*

$$\begin{aligned} g &= |H|^{-2} g_{S^2} + \beta^2 \\ \omega &= \tfrac{i}{2}(H - \overline{H})\beta \\ d\beta &= \tfrac{1}{2}(H + \overline{H})|H|^{-2} vol_{S^2} \end{aligned}$$

where H is any nonvanishing holomorphic function on an open subset of S^2. The geodesic symmetry K is dual to β and the congruence $K/|K|$ has divergence $\tau = \frac{i}{2}(H - \overline{H})\mu_g^{-1}$ and twist $\kappa = \frac{1}{4}(H + \overline{H})\mu_g^{-1}$. Furthermore, these spaces are all hyperCR, the flat connection on $L^{-1} \otimes TB$ being $D + \kappa *1$.

Replacing H by $1/f$ we see that the scalar flat Kähler metrics of this section fibre over the Einstein-Weyl spaces with geodesic symmetry. In the next section we shall fill in these Einstein-Weyl spaces with Einstein metrics.

4. Selfdual Einstein metrics

In the previous section we noted in passing, that when $f = ah + b$, the monopoles $w = \frac{1}{2}\big(f/(z+h) + \overline{f}/(z+\overline{h})\big)$ may be identified with the geometrically significant monopoles $a(1 - \frac{1}{2}zu_z) + \frac{1}{2}bu_z$ which are canonically defined on any Toda Einstein-Weyl space. There is also a special monopole defined on any hyperCR space [3, 4], namely $w = \kappa$, as one easily sees from equation (2.2). We note that for the hyperCR Toda spaces, this monopole is obtained by setting $f = i$.

The significance of these monopoles is that they all give rise to Einstein metrics.

4.1. **Scalar-flat Kähler metrics which are conformally hyperKähler.**

By [3, 4], the hypercomplex structure we obtain from the κ monopole is conformally hyperKähler, and the symmetry $\partial/\partial t$ is a triholomorphic homothetic vector field of the hyperKähler metric. Hence when $f = i$, the scalar-flat Kähler metric (3.1) is conformally Ricci-flat. The Einstein-Weyl space with geodesic symmetry obtained from the conformal submersion ξ is $\mathbb{R}^3$ (with a radial symmetry).

4.2. **HyperKähler metrics with compatible hypercomplex structures.**

By [1, 5, 9], on any Toda space, the scalar-flat Kähler metric corresponding to the monopole u_z is in fact Ricci-flat and therefore hyperKähler: the symmetry $\partial/\partial t$ is a Killing field of the hyperKähler metric, but is not triholomorphic unless the Toda space is $\mathbb{R}^3$ (with a translational congruence). However, in the case of a hyperCR Toda space, this Ricci-flat metric admits another compatible hypercomplex structure with respect to which the symmetry *is* triholomorphic, and so we have nontrivial examples of selfdual spaces with two compatible hypercomplex structures. In summary, when $f = 1$, the scalar-flat Kähler metric (3.1) is hyperKähler with a Killing field and an additional hypercomplex structure. The Einstein-Weyl space with geodesic symmetry obtained from the conformal submersion ξ is S^3 (with symmetry given by a Hopf fibration).

4.3. **Selfdual Einstein metrics with negative scalar curvature.**

These are the most interesting examples for us as they generalize diagram (1.0). The article [13] implies that, for any $a, b \in \mathbb{R}$, the scalar-flat Kähler metric generated by the monopole $a(1 - \frac{1}{2}zu_z) + \frac{1}{2}bu_z$ on any Toda space is conformal to an Einstein metric with scalar curvature $-3a$, via the conformal factor $1/(az-b)^2$. For $a = 0$ these are the hyperKähler metrics discussed in 4.2 above, and for $a \neq 0$ we can set $b = 0$ by translating the z coordinate. For our explicit solutions, this corresponds to adding a real constant to h. Thus when $f = h$, i.e.,

$$w = \frac{\frac{1}{2}(h+\overline{h})z + h\overline{h}}{(z+h)(z+\overline{h})},$$

the scalar-flat Kähler metric (3.1) is conformal to an Einstein metric with negative scalar curvature, given explicitly by

$$\frac{1}{z^2}\left[\frac{\frac{1}{2}(h+\overline{h})z + h\overline{h}}{(z+h)(z+\overline{h})}dz^2 + \bigl(\tfrac{1}{2}(h+\overline{h})z + h\overline{h}\bigr)g_{S^2} + \frac{(z+h)(z+\overline{h})}{\frac{1}{2}(h+\overline{h})z + h\overline{h}}(dt+\theta)^2\right].$$

This has a conformal infinity at $z = 0$ with conformal metric $|h|^2 g_{S^2} + \beta^2$, where $dt + \theta = \beta - \kappa z\, dz$ and so $d\beta = \frac{1}{2}(h+\overline{h}) vol_{S^2}$.

Hence we see that we have found the selfdual Einstein metrics M filling in every Einstein-Weyl space admitting a geodesic symmetry.

We recover the Berger spheres by taking h to be constant. The form of the Einstein metrics we have found is easily related to the Pedersen family by putting $h = 1 + im$ and setting $z = (1-\rho^2)/\rho^2$.

5. Additional remarks

We have shown that applying the LeBrun construction to an Einstein-Weyl space with geodesic symmetry gives an Einstein metric with compatible hypercomplex structure fibering over the general hyperCR Toda space.

The Einstein-Weyl spaces with geodesic symmetry are all hyperCR and this gives an explanation for the hypercomplex structure coming with our Einstein metrics. The twistor point of view gives a particularly quick way to see this: a hyperCR structure on an Einstein-Weyl space B corresponds to a holomorphic map from its minitwistor space $\mathcal{S}$ to $\mathbb{C}P^1$. Composing this with the projection from $PT^*\mathcal{S}$ to $\mathcal{S}$, we see that the twistor space Z of M has a holomorphic map to $\mathbb{C}P^1$. Thus applying the LeBrun construction to a hyperCR Einstein-Weyl space always gives a hypercomplex Einstein space.

There are many more hyperCR spaces than the spaces with geodesic symmetry arising here, but it will be much harder to fill them in explicitly, since the Einstein metric may no longer have a symmetry, so that it is harder to find indirectly. It is perhaps easier to ask how other Einstein-Weyl spaces with symmetry fill in. Indeed, it may be that there are other hyperCR spaces with symmetry where the symmetry is *not* geodesic, in which case we would obtain selfdual Einstein metrics with a hypercomplex structure and a symmetry which is not triholomorphic. However, even in this case, it is not clear how to solve the Toda equation.

In summary, we note that we can generalize and augment diagram (1.0) as follows:

$$\begin{tikzcd}
& M^4 \arrow[dl, shift right, "\xi"'] \arrow[dr, "K"] & \\
B^3 \arrow[ur, hook, shift right] \arrow[dr, "K"'] & & \tilde{B}^3 \arrow[dl, "\partial/\partial z"] \\
& S^2 &
\end{tikzcd}$$

For the lower part of the diagram, we use the fact that the Killing field K on M^4 descends to the geodesic symmetry of B^3, and $\partial/\partial z$ is a conformal submersion on the hyperCR Toda space $\tilde{B}^3$, since it is shear-free. The surface over which B^3 and $\tilde{B}^3$ both fibre comes with a natural spherical metric and so it would seem that it is the geometry of S^2 which lies behind our constructions.

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