# TORIC SELFDUAL EINSTEIN METRICS ON COMPACT ORBIFOLDS

DAVID M. J. CALDERBANK AND MICHAEL A. SINGER

ABSTRACT. We prove that any compact selfdual Einstein 4-orbifold of positive scalar curvature whose isometry group contains a 2-torus is, up to an orbifold covering, a quaternion Kähler quotient of $(k-1)$-dimensional quaternionic projective space by a $(k-2)$-torus for some $k \geqslant 2$. We also obtain a topological classification in terms of the intersection form of the 4-orbifold.

## INTRODUCTION

A selfdual Einstein (SDE) metric is a 4-dimensional Riemannian metric $g$ whose Weyl curvature $W$ is selfdual with respect to the Hodge star operator ($W = *W$), and whose Ricci tensor is proportional to the metric ($\mathrm{Ric} = \lambda g$). The only compact oriented 4-manifolds admitting SDE metrics of positive scalar curvature are $S^4$ and $\mathbb{C}P^2$, with the round metric and Fubini–Study metric respectively (cf. [3, Thm. 13.30]). However, if one considers 4-orbifolds, the class of examples is much wider. In [11], K. Galicki and H. B. Lawson constructed SDE 4-orbifolds by taking quaternion Kähler quotients of quaternionic projective spaces by tori, and this construction was systematically investigated by Boyer–Galicki–Mann–Rees [6].

These examples are all *toric*, i.e., the isometry group of the metric contains a 2-torus, hence they belong to the local classification by H. Pedersen and the first author of toric SDE metrics of nonzero scalar curvature [8], where it was shown that any such metric has an explicit local form determined by an eigenfunction of the Laplacian on the hyperbolic plane $\mathcal{H}^2$.

SDE 4-orbifolds have physical relevance as the simplest nontrivial target spaces for nonlinear sigma models in $N = 2$ supergravity (see [10]). They have also attracted interest recently because of the connection with M-theory and manifolds of $G_2$-holonomy [7, 12]: in particular the results in [8] have been exploited by L. Anguelova and C. Lazaroiu [1].

The first main theorem of this paper shows that the quaterion Kähler quotient is sufficient.

**Theorem A.** *Let $X$ be a compact selfdual Einstein 4-orbifold with positive scalar curvature, whose isometry group contains a 2-torus. Then, up to orbifold coverings, $X$ is isometric to a quaternion Kähler quotient of quaternionic projective space $\mathbb{H}P^{k-1}$, for some $k \geqslant 2$, by a $(k-2)$-dimensional subtorus of $\mathrm{Sp}(k)$. (We remark that the least such $k$ is $b_2(X) + 2$.)*

This result was already known when the 3-Sasakian 7-orbifold associated to $X$ is smooth, since toric 3-Sasakian 7-manifolds have been classified by R. Bielawski [4] using analytical techniques. Our methods are quite different, being elementary and entirely explicit.

Before outlining the proof, we recall that it was shown in [8] that the SDE metrics coming from (local) quaternion Kähler quotients of $\mathbb{H}P^{k-1}$ as above, are those for which the corresponding hyperbolic eigenfunction $F$ is a positive linear superposition of $k$ basic solutions which may be written

$$(1) \qquad F(\rho, \eta) = \sum_{i=1}^{k} \frac{\sqrt{a_i^2 \rho^2 + (a_i \eta - b_i)^2}}{\sqrt{\rho}},$$

where $a_i, b_i \in \mathbb{R}$, and $(\rho > 0, \eta)$ are half-space coordinates on $\mathcal{H}^2$. Our main task, therefore, is to show that the eigenfunction $F$ associated to $X$ in Theorem A is of the form (1).

*Proof.* We provide here the main line of the argument, relegating the detail to the body of the paper. First we observe that by Myers' Theorem (which extends easily to orbifolds [5]) the universal orbifold cover $\tilde{X}$ of $X$ is also a compact toric SDE 4-orbifold of positive scalar curvature, so we may assume $X = \tilde{X}$.

As we shall explain in section 1, compact simply connected 4-orbifolds $X$ with an effective action of a 2-torus $G = T^2$ may be classified by work of Orlik–Raymond [22] and Haefliger–Salem [13]. The orbit space $W = X/G$ is a polygonal disc whose edges $C_0, C_1, \ldots C_{k-1}, C_k = C_0$ (given in cyclic order) are labelled by *orbifold generators* $v_j = (m_j, n_j) \in \mathbb{Z}^2$, determined up to sign[1], with $m_j n_{j-1} - m_{j-1} n_j \neq 0$ for $j = 1, \ldots k$. The interior of $W$ is the image of the open subset $X_0$ of $X$ on which $G$ acts freely, the edges $C_j$ are the images of points with stabilizer $G(v_j) = \{(z_1, z_2) \in G : m_j z_1 + n_j z_2 = 0\}$ and cyclic orbifold structure groups of order $\gcd(m_j, n_j)$, and the corners are the images of the fixed points. A sign choice for $v_j$ is equivalent to an orientation of the corresponding circle orbits.

The classification result of [8], which we discuss in section 2, shows that the interior of $W$ is equipped with a hyperbolic metric $g_{\mathcal{H}^2}$ and hyperbolic eigenfunction $F$ (with $\Delta_{\mathcal{H}^2} F = \frac{3}{4} F$) such that the SDE metric $g_F$ on $X_0$ is given explicitly by the formula (2.1).

We shall show in section 3 that for the compactification of the metric $g_F$ on $X$, it is necessary that the edges of $W$ are at infinity with respect to the hyperbolic metric. It follows (by simple connectivity) that the interior of $W$ is identified with the entire hyperbolic plane. We also show in section 3 that for any half-space coordinates $(\rho > 0, \eta)$ on $\mathcal{H}^2$, the function $\sqrt{\rho} F(\rho, \eta)$ has a well-defined limit as $\rho \to 0$, which is a continuous piecewise linear function $f_0(\eta)$ of $\eta$ (whose corners are at the corners of $W$). The half-space coordinates can be chosen so that $f_0(\eta) = \pm(m_j \eta - n_j)$ on $C_j$.

As we shall discuss in section 2 (cf. [9]) a hyperbolic monopole $F$ on $\mathcal{H}^2$ is determined by its 'boundary value' $f_0$ via a 'Poisson formula'

$$F(\rho, \eta) = \frac{1}{2} \int \frac{f_0(y) \rho^{3/2} \, dy}{\left(\rho^2 + (\eta - y)^2\right)^{3/2}}.$$

Integrating twice by parts (in the sense of distributions), we then have

$$F(\rho, \eta) = \frac{1}{2} \int \frac{f_0''(y) \sqrt{\rho^2 + (\eta - y)^2} \, dy}{\sqrt{\rho}}.$$

In our case, $f_0$ is continuous and piecewise linear, so $f_0''$ is a linear combination of $k$ delta distributions and $F$ is therefore a linear combination of $k$ basic solutions, i.e., a $k$-pole solution in the sense of [8]. Since the SDE metric has positive scalar curvature, it follows from [8] that the determinant of a certain matrix $\Phi(\rho, \eta)$ associated to $F(\rho, \eta)$ (see section 2) is positive. Using [9] (see section 2 again) we find that

$$(2) \qquad \det \Phi(\rho, \eta) = \frac{1}{4} \iint \frac{f_0(y) f_0''(z) \rho \left(\rho^2 + (\eta - y)(\eta - z)\right)}{(\rho^2 + (\eta - y)^2)^{3/2} (\rho^2 + (\eta - z)^2)^{3/2}} \, dy \, dz.$$

Suppose now $\eta$ lies in the singular set of $f_0''$. Then as $\rho \to 0$, this integral is dominated by the contribution from $y$ near $\eta$ and the evaluation at $z = \eta$ given by the corresponding delta distribution in $f_0''$. It follows that $f_0$ is convex where it is positive and concave where it is negative. Thus, up to an irrelevant sign, $f_0(\eta)$ is positive and convex, hence $F(\rho, \eta)$ is of the form (1). According to [8], the metric $g$ is therefore locally isometric to a local quaternion Kähler quotient of $\mathbb{H}P^{k-1}$ by an explicitly defined $(k-2)$-dimensional abelian subgroup of $Sp(k)$, and one easily sees that the integrality conditions $v_j \in \mathbb{Z}^2$ imply that this subgroup is a torus. However, the quaternion Kähler quotient of $\mathbb{H}P^{k-1}$ by a $(k-2)$-torus is a compact 4-orbifold [6]. Therefore $X$ must be its universal orbifold cover. $\qquad \square$

---

[1]Thus $v_k = \pm v_0$ and it will be convenient later to take the sign to be negative.

The proof of this theorem shows explicitly how the isotropy data of a toric SDE orbifold $X$ give rise to the hyperbolic eigenfunction defining the SDE metric on $X$ and hence to its realization as quaternion Kähler quotient. However, it is not yet clear which isotropy data give a toric orbifold admitting an SDE metric. To understand this, we consider the inverse construction. Suppose therefore that $X$ is a quaternion Kähler quotient of $\mathbb{H}P^{k-1}$ by a $(k-2)$-dimensional subtorus $H$ of the standard maximal torus $T^k = \mathbb{R}^k/2\pi\mathbb{Z}^k$ in $\mathrm{Sp}(k)$. Then the isometry group of $X$ contains the quotient torus $T^k/H$. If we choose an identification of $T^k/H$ with with $\mathbb{R}^2/2\pi\mathbb{Z}^2$, then $H = \mathfrak{h}/2\pi\Lambda$ is determined by a map from $\mathbb{Z}^k \to \mathbb{Z}^2$ with kernel $\Lambda$, or equivalently by $(a_i, b_i) \in \mathbb{Z}^2$ for $i = 1, \ldots k$ (the images of the standard basis elements of $\mathbb{Z}^k$). Any two $(a_i, b_i)$ span $\mathbb{Z}^2 \otimes_\mathbb{Z} \mathbb{Q}$: otherwise $X$ is a quaternion Kähler quotient of $\mathbb{H}P^{j-1}$ for some $j < k$. Using the choice of basis of $\mathbb{Z}^2$, we can suppose $a_i \neq 0$ for all $i$, and then using the ordering and signs of the standard basis of $\mathbb{Z}^k$, we can assume that $a_i > 0$ and that the sequence $(y_i := b_i/a_i)$ is increasing. We set $y_0 = -\infty$ and $y_{k+1} = +\infty$.

Now, by [8], the hyperbolic eigenfunction generating the SDE metric is given by (1) and therefore the boundary value of $\sqrt{\rho}F(\rho, \eta)$ is the continuous piecewise linear convex function

$$(3) \qquad f_0(\eta) = \sum_{i=1}^k |a_i \eta - b_i|,$$

whose value on the edge $(y_i, y_{i+1})$ is $m_i \eta - n_i$, where

$$v_j = (m_j, n_j) = \sum_{i=1}^j (a_i, b_i) - \sum_{i=j+1}^k (a_i, b_i)$$
$$2(a_i, b_i) = (m_i, n_i) - (m_{i-1}, n_{i-1})$$

and we set $(m_0, n_0) = -(m_k, n_k)$. Up to orbifold covering, $X$ is the compact toric orbifold with stabilizers and orbifold structure groups determined by $\pm v_j$ in that cyclic order.

In terms of the $v_j$, the conditions $a_i > 0$ and $(y_i)$ strictly increasing mean that:

(a) the sequence $m_j$ is strictly increasing;

(b) the sequence $(n_j - n_{j-1})/(m_j - m_{j-1})$ is strictly increasing.

(Relative to a given basis of $\mathbb{Z}^2$, condition (a) determines the cyclic permutation of the $v_j$, since it forces $m_0 = -m_k < m_j < m_k$ for $0 < j < k$, so $|m_0| = |m_k|$ is the largest $|m_j|$. It also fixes the signs, apart from that of the smallest $|m_j|$, but this is fixed by condition (b).)

Conversely, given isotropy data for a toric orbifold $X$, if we can choose the signs and the cyclic permutation so that (a)–(b) hold, we can define the corresponding $(a_i, b_i)$ and $f_0(\eta)$ inducing these data (up to a factor of 2). $X$ will then admit a toric SDE metric since, up to orbifold covering, it is a quaternion Kähler quotient as above.

We now give a topological interpretation of these conditions, which leads to a classification result (cf. [2]). Before stating it, we define $\Delta_{i,j} = m_i n_j - m_j n_i$ and introduce the term *exceptional surface* to refer to the inverse image in a toric orbifold $X$ of an edge of $X/G$.

**Theorem B.** *Let $X$ be a compact, simply connected, oriented toric 4-orbifold, with $k = b_2(X) + 2$. Then the following are equivalent.*

(i) *$X$ admits a selfdual Einstein metric of positive scalar curvature such that the exceptional surfaces are totally geodesic.*

(ii) *The intersection form of $X$ is positive definite and for any exceptional surface $S$, the rational homology class $[\bar{S}]$ has self-intersection number $e = [\bar{S}] \cdot [\bar{S}] < \chi_{\mathrm{orb}}(\bar{S})$, where $\chi_{\mathrm{orb}}(\bar{S})$ is the orbifold Euler characteristic of the closure $\bar{S}$ of $S$.*

(iii) *If $S_1, S_2, \ldots S_k$ are the exceptional surfaces oriented so that the orbifold generators $v_1, v_2 \ldots v_k$ satisfy $\Delta_{0,j} \geqslant 0$ where $v_0 = -v_k$, then the $v_j$ are in cyclic order ($\Delta_{j-1,j} > 0$) and*

$$(4) \qquad \Delta_{j-1,j+1} < \Delta_{j-1,j} + \Delta_{j,j+1} \qquad \text{for all} \quad 0 < j < k.$$

*If (i)–(iii) hold, then $X$ admits a toric selfdual Einstein metric, unique up to homothety and pullback by an equivariant diffeomorphism.*

*Proof.* (i)$\Rightarrow$(ii). Since the metric is selfdual with positive scalar curvature, the intersection form must be positive definite (by Hodge theory for orbifolds and a Bochner argument— cf. [17]). Now we observe that $e < \chi_{\mathrm{orb}}(\bar{S})$ follows from a positive scalar curvature analogue of [9, Theorem F]. Indeed, the proof in [9, Section 6] generalizes to orbifolds, and reversing the sign of the scalar curvature there, we see that any compact, connected, totally geodesic 2-suborbifold of an SDE orbifold of positive scalar curvature must satisfy $\Sigma \cdot \Sigma < \chi_{\mathrm{orb}}(\Sigma)$.

(ii)$\Rightarrow$(iii). The positivity of the intersection form is equivalent to the fact that $[v_j] \in \mathbb{R}P^1$ are in cyclic order (see section 1), which proves the first part. Because of this, we have $\Delta_{j-1,j} > 0$ for $1 \leqslant j \leqslant k$. Now since $[\bar{S}_j] \cdot [\bar{S}_j] < \chi_{\mathrm{orb}}(\bar{S}_j)$, (4) follows from the following formulae (see section 1 for the first, the second is standard for an orbifold 2-sphere):

$$[\bar{S}_j] \cdot [\bar{S}_j] = \frac{\Delta_{j-1,j+1}}{\Delta_{j-1,j}\Delta_{j,j+1}}, \qquad \chi_{\mathrm{orb}}(\bar{S}_j) = \frac{\Delta_{j-1,j} + \Delta_{j,j+1}}{\Delta_{j-1,j}\Delta_{j,j+1}}.$$

(iii)$\Rightarrow$(i). Under the conditions in (iii) we are still free to cyclicly permute the $v_j$ by changing signs and relabelling. We use this freedom to ensure that $m_0 = -m_k < m_j < m_k$ for all $0 < j < k$. We want to show that conditions (a)–(b) above hold since we already know that under these conditions $X$ admits a toric SDE metric, and the exceptional surfaces, as fixed point sets of a subgroup of the isometry group, are totally geodesic.

To establish (a)–(b) we note that (4) may be rewritten as

(5)             $$(n_{j+1} - n_j)(m_j - m_{j-1}) > (m_{j+1} - m_j)(n_j - n_{j-1}).$$

Since $\Delta_{j-1,j} > 0$, this says that $(m_{j+1}, n_{j+1})$ lies on the side containing the origin of the line $L_j$ joining $(m_{j-1}, n_{j-1})$ to $(m_j, n_j)$. We use induction to show that $(m_j)$ is increasing. Clearly $m_1 > m_0$, so suppose $m_j > m_{j-1}$. Then dividing (5) by $m_j - m_{j-1}$, we see that $(m_{j+1}, n_{j+1})$ is above $L_j$, hence so is the origin. If $m_j \leqslant 0$, then this, together with the fact that $\Delta_{j,j+1} > 0$, shows that $m_{j+1} > m_j$. Hence the sequence $(m_j : m_j \leqslant 0)$ is increasing. A similar induction starting from the fact that $m_{k-1} < m_k$ shows that the sequence $(m_j : m_j \geqslant 0)$ is increasing. Thus (a) holds, and dividing (5) by $(m_j - m_{j-1})(m_{j+1} - m_j) > 0$, we obtain (b).

The proof of (iii)$\Rightarrow$(i) establishes the existence of a toric SDE metric, which is unique as stated by the proof of Theorem A and the classification of simply connected toric 4-orbifolds. $\qquad\square$

*Remarks.* The second condition in (ii) means equivalently that for any negative toric complex structure on the complement of any fixed point in $X$, $K_X^{-1}$ is nef. Indeed for a toric complex structure on the complement of $x \in X$, $K_X^{-1}$ being nef is equivalent, by the adjunction formula, to $[\bar{S}] \cdot [\bar{S}] < \chi_{\mathrm{orb}}(\bar{S})$ for all $\bar{S}$ that do not meet $x$. Now we note that by [16] (see also [8]), $X$ admits toric scalar-flat Kähler metrics on the complement of any fixed point.

Note that $X$ does not admit a global toric complex structure of either orientation unless it is a weighted projective space (or an orbifold quotient thereof). This can be seen by observing that toric complex orbifolds are symplectic and so the sequence $[v_1], [v_2], \ldots [v_k] \in \mathbb{R}P^1$ must have winding number two (since the $v_j$ are the normals to the faces of a convex polytope in $\mathfrak{g}^*$). It follows easily that the signature is $\pm(k-4)$, which equals $\mp(k-2)$ iff $k = 3$.

Theorem B shows that not every compact, simply connected toric 4-orbifold admits an SDE metric. On the other hand, as was noted in [6], there are SDE toric 4-orbifolds with arbitrarily large second Betti number. It is straightforward to construct such 4-orbifolds and compute their rational homology using the methods presented here (the integer pairs $(a_i, b_i)$, $i = 1, \ldots k$, used in formula (1) are constrained only by pairwise linear independence).

*Examples.* If $k = b_2(X) + 2 = 2$ then $X$ is necessarily isometric to an orbifold quotient of $S^4$ with the round metric: after an $\mathrm{SL}(2, \mathbb{Z})$ transformation we may take $v_0 = (-m, 0)$, $v_1 = (0, -n)$ and $v_2 = -v_0$ as orbifold data, $S^4$ itself being given by $mn = \pm 1$.

When $k = 3$, $X$ is a weighted projective space [11]. For instance, orbifold data for $\mathbb{C}P^2$ can be taken to be $(-2, -1)$, $(-1, -1)$, $(1, 0)$, $(2, 1)$.

An example with $k = 4$ and only one orbifold singularity is given by the data $(-2, -3)$, $(-1, -1)$, $(0, -1)$, $(1, 0)$, $(2, 3)$. More generally by taking the $n_j$ sufficiently negative, one can construct infinitely many examples with $b_2(X) + 2 = k$ for any $k \leqslant 2|m_0|$. The graph of $z = f_0(y)$ is a union of line segments with integer slopes in the region $\{(y, z) : z \geqslant |m_0 y - n_0|\}$ as shown in Figure 1.
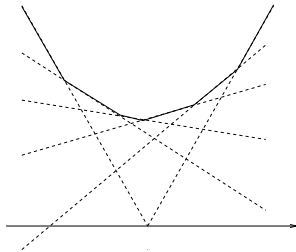


FIGURE 1. Graph of a typical boundary value for a compact SDE orbifold.

The body of the paper is really a series of appendices. In section 1, we review the classification of compact toric 4-orbifolds, following [22, 13]. In fact, we present the classification of simply connected compact $n$-orbifolds with a cohomogeneity two torus action, since this is the most natural context and the fundamental paper of Haefliger and Salem [13] rather understates the power of their theory in proving results such as this.

In section 2, we review the material from [8, 9] that we use. Then, in section 3, we present the main technical arguments that we skipped in the proof of Theorem A.

We assume throughout that the reader is familiar with the theory of orbifolds.

## 1. TORUS ACTIONS ON ORBIFOLDS

In this section we summarize the description of compact orbifolds with torus actions due to Orlik–Raymond [22] and Haefliger–Salem [13] (see also [14, 18]).

1.1. **Lie groups and tori acting on orbifolds.** Let $X$ be an oriented $n$-orbifold with a smooth effective action of a compact Lie group $G$. Fix $x \in X$ with stabilizer $H \leqslant G$ and orbifold structure group $\gamma$. Let $\phi \colon \tilde{U} \to \tilde{U}/\gamma = U$ be an $H$-invariant uniformizing chart about $x \in U$ and let $\tilde{H}$ be the group of diffeomorphisms of $\tilde{U}$ which project to diffeomorphisms induced by elements of $H$. Thus $\gamma$ is a normal subgroup of $\tilde{H}$ and $H = \tilde{H}/\gamma$.

Elements of the Lie algebra $\mathfrak{g}$ of $G$ induce $\gamma$-invariant vector fields on $\tilde{U}$ and the integral submanifold through $\phi^{-1}(x)$ is $\phi^{-1}(G \cdot x \cap U)$. Let $W = \tilde{T}_x X / \tilde{T}_x(G \cdot x)$ be the quotient of the uniformized tangent spaces to $X$ and $G \cdot x$ at $x$. Since $\tilde{H}$ preserves $\tilde{T}_x(G \cdot x)$, it acts linearly on $W$ and this induces an action of $H$ on $W/\gamma$. Hence by the differentiable slice theorem:

> there is a $G$-invariant neighbourhood of the orbit $G \cdot x$ that is $G$-equivariantly diffeomorphic to $G \times_H (B/\gamma)$, where $B$ is a $\tilde{H}$-invariant ball in $W$.

Now suppose that $G = \mathfrak{g}/2\pi\Lambda$ is an $m$-torus $(m \leqslant n)$, where $\Lambda$ is a lattice in $\mathfrak{g}$. Then we can improve on the above as follows. Let $U$ now be a $G$-invariant tubular neighbourhood of $G \cdot x$ with orbifold fundamental group $\Gamma$. Since $\pi_2(G \cdot x) = 0$, the universal orbifold cover $\pi \colon \hat{U} \to \hat{U}/\Gamma = U$ is smooth [13]. Let $\hat{G}$ be the group of diffeomorphisms of $\hat{U}$ that project

to diffeomorphisms of $U$ induced by elements of $G$, and let $\hat{H}$ be the stabilizer of a point $\hat{x}$ in $\pi^{-1}(x)$, so that $\Gamma$ is normal in $\hat{G}$ and $G = \hat{G}/\Gamma$. Then by the differentiable slice theorem:

> there is a $G$-invariant neighbourhood of the orbit $G \cdot x$ that is $G$-equivariantly diffeomorphic to $(\hat{G} \times_{\hat{H}} B)/\Gamma$, where $B$ is a $\hat{H}$-invariant ball in $W$.

Observe that $\hat{H} \cap \Gamma = \gamma$, so that $(\hat{G} \times_{\hat{H}} B)/\Gamma = (\hat{G}/\Gamma) \times_{\hat{H}/\gamma} (B/\gamma) = G \times_H (B/\gamma)$ as before.

Since $\hat{U}$ is 1-connected, $\hat{G}/\hat{H}$ is the universal cover of $G/H$, namely $\mathfrak{g}/\mathfrak{h}$. Thus $\hat{G}/\hat{G}_0 = \hat{H}/\hat{H}_0$ is a finite group $D$, where $\hat{G}_0 = \mathfrak{g}/2\pi\Lambda = G$ and $\hat{H}_0 = \mathfrak{h}/2\pi\Lambda_0$ denote the identity components, $\Lambda_0$ being a subgroup of $\Lambda$. Since $\hat{H}$ is the (unique) maximal compact subgroup of $\hat{G}$ we have the following.

1.2. **Proposition.** [13] *Let $G = \mathfrak{g}/2\pi\Lambda$ be an $m$-torus acting effectively on an oriented $n$-manifold $X$ and let $G \cdot x$ be an orbit with $k$-dimensional stabilizer $H$. Then there is*
- *a rank $k$ sublattice $\Lambda_0$ of $\Lambda$,*
- *a finite group $D$ with a central extension*

$$0 \to \Lambda/\Lambda_0 \to \Gamma \to D \to 1,$$

- *and a faithful representation $\hat{H} \to SO(n - m + k)$, where $\hat{H}$ is the maximal compact subgroup of the pushout extension $\hat{G} = \Gamma \times_{\Lambda/\Lambda_0} \mathfrak{g}/2\pi\Lambda_0$,*

*such that a $G$-invariant tubular neighbourhood $U$ of $G \cdot x$ is $G$-equivariantly diffeomorphic to $(\hat{G} \times_{\hat{H}} B)/\Gamma$ for a ball $B \subset \mathbb{R}^{n-m+k}$. These data classify tubular neighbourhoods of orbits up to $G$-equivariant diffeomorphism.*

This result is easy to apply when $k = n - m$ or $k = n - m - 1$, when $\hat{H}_0 = \mathfrak{h}/2\pi\Lambda$ is a maximal torus in $SO(2(n - m))$ or $SO(2(n - m - 1) + 1)$. Then $\hat{H} = \hat{H}_0$ (since $\hat{H}$ is in the centralizer of $\hat{H}_0$), so $D = 1$ and $\Gamma = \Lambda/\Lambda_0$. Hence a tubular neighbourhood $U$ of such an orbit is classified by a subgroup $\Lambda_0$ of $\Lambda$ such that $\hat{H} = \mathfrak{h}/2\pi\Lambda_0$.

- When $k = n - m$, $U/G$ is homeomorphic to $[0, 1)^{n-m}$ and $\Lambda_0 = \bigoplus_{j=1}^{n-m} \Lambda_0^j$, where $\Lambda_0^j$ are linearly independent rank one sublattices of $\Lambda$ such that $(\Lambda_0^j \otimes_{\mathbb{Z}} \mathbb{R})/2\pi\Lambda_0^j$ is the stabilizer the orbits over the $j$th face of $U/G$.
- When $k = n - m - 1$, $U/G$ is homeomorphic to $[0, 1)^{n-m-1} \times (-1, 1)$ and $\Lambda_0 = \bigoplus_{j=1}^{n-m-1} \Lambda_0^j$, with $\Lambda_0^j$ as before.

To obtain a global classification, one must patch together such local tubes. This is conveniently encoded by the Čech cohomology of $W = X/G$ with values in $\Lambda$.

1.3. **Proposition.** [13] *Suppose $W = \bigcup_i W_i$ is a union of open sets and $(X_i, \pi_i \colon X_i \to W_i)$ are $G$-orbifolds with orbit maps $\pi_i$. Then there is a $G$-orbifold $(X, \pi \colon X \to W)$ with $\pi^{-1}(W_i)$ $G$-equivariantly diffeomorphic to $X_i$ if and only if a Čech cohomology class in $H^3(W, \Lambda)$ associated to $\{(X_i, \pi_i)\}$ vanishes. If this is the case then the set of such $G$-orbifolds $(X, \pi)$ is an affine space modelled on $H^2(W, \Lambda)$.*

1.4. **Cohomogeneity two torus actions on orbifolds.** Let us now specialize to the case $\dim W = 2$ (i.e., $n = m + 2$). The union $X_0$ of the $m$-dimensional orbits is the dense open subset on which the action of $G$ is locally free, hence $W_0 = X_0/G$ is a 2-orbifold. The remaining orbits have dimension $m-1$ or $m-2$, i.e., stabilizers of dimension $k = n-m-1 = 1$ or $k = n-m = 2$. Hence we can obtain a global classification in this case. (Similar arguments give a global classification when $\dim W = 1$.)

1.5. **Theorem.** [22, 13]

(i) *Let $X$ be a compact connected oriented $(m + 2)$-orbifold with a smooth effective action of an $m$-torus $G = \mathfrak{g}/2\pi\Lambda$. Then $W = X/G$ is a compact connected oriented 2-orbifold with boundary and corners, equipped with a labelling $\Lambda_0^j$ of the edges of $W$ (the connected*

*components of the smooth part of the boundary) by rank 1 sublattices of $\Lambda$ such that at each corner of $W$ the corresponding two lattices are linearly independent.*

(ii) *For any such data on $W$, there is a $G$-orbifold $X$ inducing these data, and if $H^2(W, \Lambda) = 0$ then $X$ is uniquely determined up to $G$-equivariant diffeomorphism.*

(iii) *The orbifold fundamental group of $X$ is determined by the long exact sequence:*

$$\pi_2^{orb}(W_0) \to \Lambda / \sum_j \Lambda_0^j \to \pi_1^{orb}(X) \to \pi_1^{orb}(W_0) \to 1.$$

*In particular $X$ is simply connected if and only if either $W_0$ is a smooth open disc and the lattices $\Lambda_0^j$ generate $\Lambda$, or $W_0 = W$ is a simply connected orbifold 2-sphere (so that $\pi_2^{orb}(W) = \mathbb{Z}$) and $\pi_2^{orb}(W_0) \to \Lambda$ is an isomorphism (so $m = 1$).*

1.6. **Compact toric 4-orbifolds.** We now apply the preceding result when $m = 2$ and $X$ is a simply connected 4-orbifold. Then $W$ is a smooth polygonal disc with rank 1 sublattices $\Lambda_0^j \subset \Lambda \cong \mathbb{Z}^2$ $(j = 1, \dots k)$ labelling the edges $C_j$ of $W$, which we order cyclically. $\Lambda_0^j$ is determined by one of its generators $v_j = (m_j, n_j) \in \mathbb{Z}^2$, which is unique up to a sign. The corner conditions mean that $v_{j-1}, v_j$ are linearly independent, or equivalently $\Delta_{j-1,j} \neq 0$, for $j = 1, \dots k$ (where $v_0 = -v_k$ and $\Delta_{i,j} := m_i n_j - m_j n_i$). The simple connectivity of $X$ means that $\{v_j : j = 1, \dots k\}$ spans $\mathbb{Z}^2$. Since $H^2(W, \mathbb{Z}^2) = 0$, $X$ is uniquely determined by these data. (The classification for manifolds is more subtle, since then we must have $\Delta_{j-1,j} = \pm 1$, i.e., adjacent pairs of labels form a $\mathbb{Z}$-basis, which leads to combinatorial problems.)

We end by discussing the rational homology of such a toric 4-orbifold $X$. Since $X$ is oriented and simply connected, this amounts to describing $H_2(X, \mathbb{Q})$ and its intersection form. We have already remarked that $b_2(X) = k - 2$ (and this is easy to establish by a spectral sequence argument): in fact the closures $\bar{S}_j$ of the exceptional surfaces $S_j = \pi^{-1}(C_j)$, once oriented, define rational homology classes generating $H_2(X, \mathbb{Q})$. Obviously the only nontrivial intersections are the self-intersections and the intersections of adjacent $\bar{S}_j$. For the latter, we note that in the orbifold uniformizing chart of order $|\Delta_{j,j+1}|$ about $\bar{S}_j \cap \bar{S}_{j+1}$, the intersection number is $\pm 1$ and hence $[\bar{S}_j] \cdot [\bar{S}_{j+1}] = \pm 1/\Delta_{j,j+1}$. Similarly, by considering the link of $S_j$ (which is an orbifold lens space), we find that $[\bar{S}_j] \cdot [\bar{S}_j] = \pm \Delta_{j-1,j+1}/(\Delta_{j-1,j}\Delta_{j,j+1})$. In fact our orientation conventions give

$$(1.1) \quad [\bar{S}_j] \cdot [\bar{S}_j] = \Delta_{j-1,j+1}/(\Delta_{j-1,j}\Delta_{j,j+1}), \qquad [\bar{S}_j] \cdot [\bar{S}_{j+1}] = [\bar{S}_{j+1}] \cdot [\bar{S}_j] = -1/\Delta_{j,j+1}.$$

We notice that $\sum_{j=1}^{k} m_j [\bar{S}_j]$ and $\sum_{j=1}^{k} n_j [\bar{S}_j]$ have trivial intersection with any $[\bar{S}_i]$. Since the latter classes span the rational homology, and the intersection form is nondegenerate, we have $\sum_{j=1}^{k} m_j [\bar{S}_j] = 0 = \sum_{j=1}^{k} n_j [\bar{S}_j]$. Since $b_2(X) = k - 2$, these span the relations amongst the rational classes $[\bar{S}_j]$.

For Theorem B we need a formula for the signature of $X$ (i.e., of the intersection form on $H_2(X, \mathbb{Q})$) which was given by Joyce [16] in the manifold case and by Hattori–Masuda [14] in general. For this formula, choose an arbitrary vector $v = (m, n) \in \mathbb{Z}^2$ which is not a multiple of any $v_j$ and define $\Delta_j = mn_j - nm_j$. Then

$$\sigma(X) = \sum_{j=1}^{k} sign(\Delta_{j-1}\Delta_{j-1,j}\Delta_j).$$

This is evidently independent of the sign choices for the $v_j$, and it is independent of the choice of $v$, since if we move $v$ so that one $\Delta_j$ changes sign then only two terms in the above sum change sign, but they have the opposite sign. Let us choose the signs of the $v_j$ so that $\Delta_j > 0$ for $i = 1, \dots k$ (so that the $v_j$ lie in a half-space bounded by the span of $v$); then $\Delta_0 < 0$. It follows that $|\sigma(X)| = k - 2 = b_2(X)$ iff $\Delta_{j-1,j}$ all have the same sign for $j = 1, \dots k$, i.e., iff $[v_1], \dots [v_k]$ are in cyclic order in $\mathbb{R}P^1$. Thus $\{\pm v_j : j = 1, \dots k\}$ are the normals to a compact convex polytope in $\mathfrak{g}^*$ symmetric under $v \mapsto -v$, as in Anguelova–Lazaroiu [2].

## 2. Toric selfdual Einstein metrics

In this section we give a brief account of the relevant results of [8] and [9].

2.1. **Local classification.** Let $\mathcal{H}^2$ denote the hyperbolic plane, which we regard as the positive definite sheet of the hyperboloid $\{a \in S^2\mathbb{R}^2 : \det a = 1\}$ in the space $S^2\mathbb{R}^2$ of symmetric $2 \times 2$ matrices, with the induced metric ($-\det$ is the quadratic form of a Minkowski metric on this vector space).

Let $G$ denote the standard 2-torus $\mathbb{R}^2/2\pi\mathbb{Z}^2$, with linear coordinates $z = (z_1, z_2)$. Consider the metric $g_F$ constructed on (open subsets of) $\mathcal{H}^2 \times G$ through the formula

$$(2.1) \qquad g_F = \frac{|\det \Phi|}{F^2}\left(g_{\mathcal{H}^2} + dz\,\Phi^{-1}A\Phi^{-1}dz^t\right)$$

where $F$ is an eigenfunction of the hyperbolic Laplacian,

$$(2.2) \qquad \Delta_{\mathcal{H}^2}F = \frac{3}{4}F,$$

$dz = (dz_1, dz_2)$, $A$ is the tautological $S^2\mathbb{R}^2$-valued function on $\mathcal{H}^2$ (with $A_a = a$), and $\Phi = \frac{1}{2}FA - dF$ (with $dF_a \in T_a^*\mathcal{H}^2 \cong a^\perp \subset S^2\mathbb{R}^2$). We note that our sign convention for the Laplacian is $\Delta_{\mathcal{H}^2} = \delta d$, where $\delta = -d^*$ is minus the formal adjoint of $d$ with respect to $g_{\mathcal{H}^2}$.

2.2. **Theorem.** [8] *$g_F$ is an selfdual Einstein metric whose scalar curvature scalar curvature has the opposite sign to the quantity $\det \Phi = \frac{1}{4}F^2 - |dF|^2$. (The metric has singularities where $F = 0$ or $\det \Phi = 0$.) Furthermore, any SDE metric with nonzero scalar curvature and a 2-torus in its isometry group is obtained locally from this construction.*

A more explicit form of the metric can be obtained by introducing half-space coordinates $(\rho(a) > 0, \eta(a))$ on $\mathcal{H}^2$. The standard basis of $\mathbb{R}^2$ leads to a preferred choice

$$A(\rho, \eta) = \frac{1}{\rho}\begin{bmatrix} 1 & \eta \\ \eta & \rho^2 + \eta^2 \end{bmatrix}.$$

However—and this will be crucial later—all these local formulae are $SL_2(\mathbb{R})$-invariant, and other half-space coordinates are given by other unimodular bases of $\mathbb{R}^2$. Identifying $\mathbb{R}^2$ with the Lie algebra of $G$, it follows that we can work with any oriented $\mathbb{Z}$-basis for the lattice $\mathbb{Z}^2$: the above formulae will transform naturally under $SL_2(\mathbb{Z})$.

One easily computes $\det A = 1$ and $\det dA = (d\rho^2 + d\eta^2)/\rho^2$, so that $(\rho, \eta)$ so defined are half-space coordinates. It is convenient to set

$$(2.3) \qquad f(\rho, \eta) = \sqrt{\rho}F(\rho, \eta), \quad v_1 = (f_\rho, \eta f_\rho - \rho f_\eta), \quad v_2 = (f_\eta, \rho f_\rho + \eta f_\eta - f),$$

so that $\Phi = \lambda_1 \otimes v_1 + \lambda_2 \otimes v_2$, where $\lambda_1 = (\sqrt{\rho}, 0)$ and $\lambda_2 = (\eta/\sqrt{\rho}, 1/\sqrt{\rho})$ form an orthonormal frame ($A = \lambda_1^2 + \lambda_2^2$). A straightforward computation then gives

$$(2.4) \qquad g_F = \frac{\rho|\varepsilon(v_1, v_2)|}{f^2}\left(\frac{d\rho^2 + d\eta^2}{\rho^2} + \frac{\varepsilon(v_1, dz)^2 + \varepsilon(v_2, dz)^2}{\varepsilon(v_1, v_2)^2}\right),$$

where $\varepsilon(\cdot, \cdot)$ denotes the standard symplectic form on $\mathbb{R}^2$.

2.3. **Boundary behaviour.** The replacement of $F$ by $f$ is quite natural because one can prove that if $F$ satisfies (2.2) in a set of the form $\{0 < \rho < a\} \times \{b < \eta < c\}$ and is of power law growth in $\rho$, then $f$ must have an asymptotic expansion of the form

$$(2.5) \qquad f(\rho, \eta) = \sqrt{\rho}F(\rho, \eta) \sim \left(f_0(\eta) + f_1(\eta)\rho^2 + \cdots\right) + \left(-\tfrac{1}{2}f_0''(\eta)\rho^2 + \cdots\right)\log\rho$$

where $f_0$ and $f_1$ are in general distributions. The coefficients of the higher powers of $\rho$ are also distributions, uniquely determined by $f_0$ and $f_1$; only even powers of $\rho$ can occur. Such boundary regularity results are discussed in a much more general setting in [20, 21].

In addition to this local regularity, we shall need the following uniqueness result.

**2.4. Proposition.** *If $F$ satisfies (2.2) globally on $\mathcal{H}^2$ and $f_0 = 0$ on the boundary $\mathbb{R}P^1$ of $\mathcal{H}^2$, then $F = 0$.*

Some care is needed in interpreting this result in half-space coordinates. At first sight, the function $F = \rho^{3/2}$ appears to be a counter-example. However, if one changes to coordinates

$$(2.6) \qquad \tilde{\rho} = \rho/(\rho^2 + \eta^2), \quad \tilde{\eta} = -\eta/(\rho^2 + \eta^2),$$

then $F = \tilde{\rho}^{3/2}(\tilde{\rho}^2 + \tilde{\eta}^2)^{-3/2}$ and $\sqrt{\rho}F \to 2\delta(\tilde{\eta})$ as $\tilde{\rho} \to 0$, so that $f_0$ does not vanish at $\infty$ in the original coordinates.

More generally, if we define

$$\tilde{f}_0(\tilde{\eta}) = \lim_{\tilde{\rho} \to 0} \sqrt{\tilde{\rho}} F(\rho, \eta)$$

with $(\rho, \eta)$ and $(\tilde{\rho}, \tilde{\eta})$ related as in (2.6), then we see that

$$(2.7) \qquad \tilde{f}_0(-1/\eta) = |\eta| f_0(\eta)$$

Thus $f_0$ is the restriction of a distributional section of the line-bundle $\mathcal{O}(1) \otimes L$ over $\mathbb{R}P^1$, where $L$ is the Möbius bundle and $\mathcal{O}(1)$ is the dual of the tautological line bundle. Sections of this bundle can also be viewed as functions $\hat{f}_0$ on $\mathbb{R}^2 \setminus \{0\}$ satisfying $\hat{f}_0(\lambda v) = |\lambda| \hat{f}_0(v)$.

An elementary way to prove Proposition 2.4 is via the maximum principle.

**2.5. Proposition.** *Suppose that $F$ is defined in $\mathcal{H}^2$ and satisfies*

$$\Delta F = \alpha(\alpha + 1)F, \text{ where } \alpha \in \mathbb{R}.$$

*If the boundary value $f_0(\eta) = \lim_{\rho \to 0} \rho^\alpha F(\rho, \eta)$ vanishes for all $\eta$ in $\mathbb{R}P^1$, then $F = 0$.*

*Proof.* We pass to the Poincaré model of $\mathcal{H}^2$: the unit disc with coordinates $(x, y)$, $r^2 = x^2 + y^2 < 1$. Define

$$u = \frac{1 - r^2}{2} \text{ so } \Delta = u^2(\partial_x^2 + \partial_y^2).$$

If $f = u^\alpha F$, then we have

$$\Delta F = \Delta(u^{-\alpha} f) = (\Delta u^{-\alpha})f + 2\nabla u^{-\alpha} \cdot \nabla f + u^{-\alpha} \Delta f.$$

Rearranging this,

$$\Delta f + 2u^\alpha \nabla u^{-\alpha} \cdot \nabla f = \alpha(\alpha + 1)f - u^\alpha(\Delta u^{-\alpha})f.$$

By an easy computation,

$$u^\alpha \Delta u^{-\alpha} = \alpha(\alpha + 1) - 2\alpha^2 u.$$

Hence

$$\Delta f + 2u^\alpha \nabla u^{-\alpha} \cdot \nabla f = 2\alpha^2 u f.$$

Since $u \geqslant 0$ in $\mathcal{H}^2$, it follows from the maximum principle that $f$ cannot have either a positive interior maximum nor a negative interior minimum. Hence if $f \to 0$ at the boundary, then $F = 0$. It is clear that the boundary value of $f$ differs from $f_0$, defined using half-space coordinates, by multiplication by a positive function. The proof is complete. $\square$

When $\alpha = 1/2$, we recover Propostion 2.4.

Note that if we replace the eigenvalue $\alpha(\alpha + 1)$ by $\beta(\beta + 1)$, then we obtain

$$\Delta f + 2u^\alpha \nabla u^{-\alpha} \cdot \nabla f = [(\beta - \alpha)(1 + \alpha + \beta) + 2\alpha^2 u]f.$$

Since $u \to 0$ at the boundary, we cannot apply the maximum principle if $(\beta - \alpha)(1 + \alpha + \beta) < 0$; in particular it is not applicable for $\alpha > \beta = 1/2$.

2.6. **The Poisson formula.** A Poisson formula reconstructs the eigenfunction $F$ from its boundary value $f_0$; in half-space coordinates,

$$(2.8) \qquad F(\rho, \eta) = \frac{1}{2} \int \frac{f_0(y)\rho^{3/2}\, dy}{\left(\rho^2 + (\eta - y)^2\right)^{3/2}}.$$

It is straightforward to check (e.g., by making the change of variables $y = \eta + \rho x$) that

$$f_0(\eta) = \lim_{\rho \to 0} \sqrt{\rho} F(\rho, \eta).$$

Again, despite appearances, equation (2.8) is really $SL_2(\mathbb{R})$-equivariant. Indeed, the kernel

$$\frac{\rho}{\rho^2 + (\eta - y)^2}|dy|$$

is $SL_2(\mathbb{R})$ invariant for the diagonal action of $SL_2(\mathbb{R})$ on $\mathcal{H}^2 \times \mathbb{R}P^1$ and (2.8) is the $(3/2)$-power of this kernel applied to $f_0(y)|dy|^{-1/2}$, which is also $SL_2(\mathbb{R})$ invariant.

We have seen that the map $\mathcal{P}\colon f_0 \mapsto F$ given by (2.8) is injective. In fact, its image (operating on $\mathcal{D}'(\mathbb{R}P^1)$) is the space of solutions of (2.2) that grow at most exponentially with geodesic distance from a point [19]. We shall not need this; the interested reader is referred to Theorem 4.24 of the Introduction in Helgason's book [15].

We end by remarking that [9, §5.1] gives an integral formula for the determinant of the matrix $\Phi = \frac{1}{2}FA - dF$:

$$\det \Phi(\rho, \eta) = -\frac{1}{8} \iint \frac{(y - z)(\mu(y)\nu(z) - \mu(z)\nu(y))\rho^3}{\left(\rho^2 + (\eta - y)^2\right)^{3/2}\left(\rho^2 + (\eta - z)^2\right)^{3/2}}\, dy\, dz,$$

where $\mu(y) = f_0'(y)$ and $\nu(y) = yf_0'(y) - f_0(y)$. Substituting for $\mu$ and $\nu$, we see that

$$\det \Phi(\rho, \eta) = -\frac{1}{4} \iint \frac{(y - z)f_0(y)f_0'(z)\rho^3}{\left(\rho^2 + (\eta - y)^2\right)^{3/2}\left(\rho^2 + (\eta - z)^2\right)^{3/2}}\, dy\, dz.$$

Integrating by parts with respect to $z$ (differentiating $f_0'(z)$), we obtain formula (2).

## 3. Boundary behaviour of $F$

Let $X$ be a compact simply connected 4-orbifold with an effective action of a 2-torus $G$, equipped with a $G$-invariant SDE metric of positive scalar curvature. We have seen that $X/G$ is a polygonal disc and the interior of $X/G$ is equipped with a hyperbolic metric and a hyperbolic eigenfunction $F$. In this section we prove:

- the edges of the polygon are at infinity with respect to the hyperbolic metric;
- there are half-space coordinates $(\rho, \eta)$ such that if $f_0(\eta) = \lim_{\rho \to 0} \sqrt{\rho} F(\rho, \eta)$ then on each edge $C_j$, labelled by $\pm(m_j, n_j)$, $f_0(\eta)$ is equal, up to sign, to the linear function $m_j \eta - n_j$;
- $f_0(\eta)$ is continuous at the vertices of the polygon.

As we have remarked in section 2, for the second of these facts we can make a unimodular change of basis and suppose that $(m_j, n_j) = (0, \ell_j)$, where $\ell_j = \gcd(m_j, n_j)$ is the order of the orbifold structure group of points in the corresponding special orbits. We then show that for the half-space coordinates corresponding to such a unimodular basis, $f_0(\eta) = \pm\ell_j$ on $C_j$.

At the corner $\bar{C}_j \cap \bar{C}_{j+1}$ corresponding to a fixed point $x$, we would like to use the primitive vectors $(m_j, n_j)/\ell_j$ and $(m_{j+1}, n_{j+1})/\ell_{j+1}$ as a basis for $\mathbb{Z}^2$; unfortunately they only form a basis for a sublattice of index $(m_j n_{j+1} - m_{j+1}n_j)/(\ell_j \ell_{j+1})$. However, to prove the continuity of $f_0$ at the corner, we may as well pass to the orbifold covering of a neighbourhood of $x$ defined by this sublattice. Hence there is no loss in supposing that this index is 1.

3.1. **Exceptional surfaces.** Let $C$ be an edge of the polygon $W$ and $S$ its inverse image in $X$ (whose closure is an orbifold 2-sphere). We let $2\pi/\ell$ be the cone angle of $S$ in $X$ so that points in $S$ have orbifold structure group $\mathbb{Z}_\ell$.

Near any point of $S$ we can introduce *Fermi coordinates*. For $x$ near $S$, we write $r(x)$ for the distance from $x$ to $S$ and introduce an angular coordinate $\theta$ (of period $2\pi$) such that $d\theta$ vanishes on the radial geodesics and evaluates to 1 on the generator $K$ of the action of the stabilizer of $S$ (note that $\theta$ is far from unique). The metric then takes the form

$$(3.1) \qquad g = dr^2 + r^2 d\theta^2/\ell^2 + h_1 + r h_2 + r^2 h_3$$

where $h_1$ is the 'first fundamental form' (restriction of the metric to $S$), $h_2$ is the second fundamental form, and $h_3$ is a form on $TX$ bilinear in $rd\theta$ and $TS$. Since $S$ is a fixed point set of the isometry group generated by $K$, $h_2 = 0$, which we shall use in the next subsection, but not here. (Also, by Gauss's lemma, $h_3$ does not contain terms in $dr$, but we shall not need this precision.)

In our case, we also know that the metric is a toric SDE metric of positive scalar curvature and so is given explicitly by

$$(3.2) \qquad g_F = \frac{d\rho^2 + d\eta^2}{k^2 f^2} + \frac{k^2}{f^2}(d\psi_1, d\psi_2) P^t P \begin{pmatrix} d\psi_1 \\ d\psi_2 \end{pmatrix}$$

where

$$P = \begin{pmatrix} \rho f_\eta - \eta f_\rho & f_\rho \\ f - \rho f_\rho - \eta f_\eta & f_\eta \end{pmatrix} \qquad \text{and} \qquad k = \frac{\sqrt{\rho}}{\sqrt{f f_\rho - \rho(f_\rho^2 + f_\eta^2)}} = \frac{\sqrt{\rho}}{\sqrt{-\det P}}.$$

Here the angular coordinates $(\psi_1, \psi_2): X_0 \to \mathbb{R}^2/2\pi\mathbb{Z}^2$ are canonically defined up to a change of $\mathbb{Z}$-basis, and the metric is invariant under such changes provided we make the corresponding change of half-space coordinates $(\rho, \eta)$ (see section 2). We use this freedom to let $d\psi_1$ vanish on $K$, so that we can take $\theta = \psi_2$. There is still the freedom to add a multiple of $\psi_1$ to $\psi_2$ and $\theta$, and we use this (in a rather mild way) to ensure that the half-space coordinates $(\rho, \eta)$ are bounded near $S$. Note that *a priori* the coordinates $(\rho, \eta)$ are only independent in a punctured neighbourhood of $S$.

We complete the coordinates $r, \psi_1, \psi_2 = \theta$ by a coordinate $y$ so that to leading order in $r$, as a bilinear form in $dr, dy, d\psi_1, rd\psi_2$, the metric (3.1) is given by

$$(3.3) \qquad dy^2 + dr^2 + a^2 d\psi_1^2 + r^2 d\psi_2^2/\ell^2$$

where $a(y) > 0$. The equality of the angular parts of the metrics (3.2) and (3.3) now reduces to the following equation:

$$\frac{k^2}{f^2} P^t P = \begin{pmatrix} a^2 & 0 \\ 0 & r^2/\ell^2 \end{pmatrix} + \begin{pmatrix} O(r) & O(r^2) \\ O(r^2) & O(r^3) \end{pmatrix}.$$

It follows that

$$\frac{k^2}{f^2} \begin{pmatrix} a^{-1} & 0 \\ 0 & \ell/r \end{pmatrix} P^t P \begin{pmatrix} a^{-1} & 0 \\ 0 & \ell/r \end{pmatrix} = I + O(r).$$

This matrix is symmetric and so, by binomial series expansion (for $r$ sufficiently small), it has an inverse square root which is also symmetric and of the form $I + O(r)$. Multiplying on both sides by this inverse square root, we deduce that there is an orthogonal matrix with determinant $-1$

$$\begin{pmatrix} c & s \\ s & -c \end{pmatrix}, \quad c^2 + s^2 = 1$$

such that

$$\frac{k}{f} P \begin{pmatrix} a^{-1} & 0 \\ 0 & \ell/r \end{pmatrix} = \begin{pmatrix} c & s \\ s & -c \end{pmatrix} + O(r),$$

or in other words,

$$(3.4) \qquad \frac{k}{f}\begin{pmatrix} \rho f_\eta - \eta f_\rho & f_\rho \\ f - \rho f_\rho - \eta f_\eta & f_\eta \end{pmatrix} = \begin{pmatrix} ca & sr/\ell \\ sa & -cr/\ell \end{pmatrix} + \begin{pmatrix} O(r) & O(r^2) \\ O(r) & O(r^2) \end{pmatrix}.$$

This equation contains all the information we need. Taking determinants, we obtain

$$(3.5) \qquad \frac{\rho}{f^2} = \frac{k^2}{f^2}\det P = \frac{ar}{\ell} + O(r^2).$$

Also, if we use the $(1,2)$ and $(2,2)$ components to eliminate $kf_\rho/f$ and $kf_\eta/f$ from the $(1,1)$ and $(2,1)$ components of $(3.4)$, we have

$$(3.6) \qquad -\rho cr/\ell - \eta sr/\ell = ca + O(r)$$

$$(3.7) \qquad k - \rho sr/\ell + \eta cr/\ell = sa + O(r).$$

In particular $c$ is $O(r)$ and so $s = \pm 1 + O(r^2)$. Thus $k \to \pm a$ as $r \to 0$, which is nonzero.

**3.2. Proposition.** *$f$ is bounded away from zero and infinity as $r \to 0$ and $\rho$ is a defining function for the exceptional surface $S$ with $(\rho, \eta)$ independent in a neighbourhood of $S$.*

*Proof.* From the non-angular part of the metric $g_F$, we deduce that $d\rho \wedge d\eta / f^2 = k^2 dr \wedge dy + O(r)$. Since $d\rho \wedge d\eta$ is well defined as $r \to 0$ and $k^2 dr \wedge dy$ does not vanish at $r = 0$, $f$ must be bounded. Now by $(3.5)$,

$$(3.8) \qquad \rho = af^2\ell^{-1}r + O(r^2)$$

and so $\rho = O(r)$. Now by the boundary regularity results discussed in §2, $f$ as a function of $\rho$ must be asymptotically $O(1)$ or $O(\rho^2)$ and it follows easily from $(3.8)$ that only the first case is possible. Thus $f$ is bounded away from zero and infinity and therefore $\rho$ has precisely order $r$ and $d\rho \wedge d\eta$ does not vanish as $\rho \to 0$. $\qquad \square$

It is now straightforward to see that $f$ has the required boundary behaviour.

**3.3. Proposition.** *$f(\rho, \eta) = \pm\ell + O(\rho^2)$.*

*Proof.* From $(3.4)$, we have $(\log f)_\rho = (\log f)_\eta = O(\rho)$ and so $f(\rho, \eta) = A + O(\rho^2)$ for some constant $A$. Now $(3.8)$ gives $\rho = aA^2\ell^{-1}r + O(r^2)$, so that $d\rho = aA^2\ell^{-1}dr + O(r)$ and $d\rho^2 = a^2A^4\ell^{-2}dr^2 + O(r)$. By comparing $(3.2)$ and $(3.3)$ we deduce that $a^2A^4\ell^{-2} = k^2f^2 + O(r) = a^2A^2 + O(r)$ so that $A^2/\ell^2 = 1$ (since both are constant). $\qquad \square$

We end this section by remarking that if we had interchanged the roles of $\psi_1$ and $\psi_2$ in the above argument (corresponding to an inversion of half-space coordinates), then we would have, in place of $(3.4)$,

$$(3.9) \qquad \frac{k}{f}\begin{pmatrix} \rho f_\eta - \eta f_\rho & f_\rho \\ f - \rho f_\rho - \eta f_\eta & f_\eta \end{pmatrix} = \begin{pmatrix} cr/\ell & sa \\ sr/\ell & -ca \end{pmatrix} + \begin{pmatrix} O(r^2) & O(r) \\ O(r^2) & O(r) \end{pmatrix}$$

and hence

$$(3.10) \qquad -\rho ca - \eta sa = cr/\ell + O(r)$$

$$(3.11) \qquad k - \rho sa + \eta ca = sr/\ell + O(r).$$

Assuming (as we may) that $\rho \to 0$ and $\eta$ is bounded away from zero as $r \to 0$, the argument goes through in a similar way. This time, $s = O(r)$, hence $c = \pm 1 + O(r^2)$ and $k \to \mp\eta a$ as $r \to 0$. Therefore $(3.9)$ gives $(\log f)_\rho = O(\rho)$ but $(\log f)_\eta = 1/\eta + O(\rho)$, so $f(\rho, \eta) = A\eta + O(\rho^2)$ for some constant $A$, and we compute as before that $A = \pm\ell$.

Thus $f_0(\eta) = \pm\ell\eta$, as we would expect from the coordinate invariance of our formulae.

3.4. **Corner behaviour.** We now show that $f_0$ is continuous at a corner $\bar{C}_1 \cap \bar{C}_2$, for edges $C_1, C_2$ corresponding to exceptional surfaces $S_1, S_2$. To do this, we argue as in §3.1, but with the metric expanded about the point $\bar{S}_1 \cap \bar{S}_2$. It is now natural to introduce coordinates $r_1$ and $r_2$, the distance functions from $S_1$ and $S_2$ respectively. Since $S_1$ and $S_2$ are totally geodesic, the restriction of $r_2$ to $S_1$ is also the distance function from $\bar{S}_1 \cap \bar{S}_2$ and similarly for $r_1$ and $S_2$. Therefore, after adapting the basis of $\mathbb{R}^2/2\pi\mathbb{Z}^2$, we can suppose that the metric (3.2), to leading order in $r_1 r_2$ (as a bilinear form in $dr_1, dr_2, r_1 d\psi_1, r_2 d\psi_2$), is equal to

$$dr_1^2 + dr_2^2 + r_1^2 d\psi_1^2/\ell_1^2 + \ell_2^2 r_2^2 d\psi_2^2 \ell_2^2$$

for some positive integers $\ell_1$ and $\ell_2$. If we carry through the calculations of §3.1 with this metric, but now take into account that $S_1$ and $S_2$ are totally geodesic (since we now need better control over the error terms), then we certainly have

$$(3.12) \qquad \frac{k}{f}\begin{pmatrix} \rho f_\eta - \eta f_\rho & f_\rho \\ f - \rho f_\rho - \eta f_\eta & f_\eta \end{pmatrix} = \begin{pmatrix} cr_1/\ell_1 & sr_2/\ell_2 \\ sr_1/\ell_1 & -cr_2/\ell_2 \end{pmatrix} + O(r_1^2 r_2^2).$$

Hence

$$(3.13) \qquad \rho cr_2/\ell_2 + \eta sr_2/\ell_2 + cr_1/\ell_1 = O(r_1^2 r_2^2)$$

$$(3.14) \qquad k - \rho sr_2/\ell_2 + \eta cr_2/\ell_2 - sr_1/\ell_1 = O(r_1^2 r_2^2),$$

and so away from $r_1 = r_2 = 0$ we have

$$s = \pm\frac{\rho r_2/\ell_2 + r_1/\ell_1}{\sqrt{(\rho r_2/\ell_2 + r_1/\ell_1)^2 + (\eta r_2/\ell_2)^2}} + O(r_1^2 r_2^2)$$

$$c = \mp\frac{\eta r_2/\ell_2}{\sqrt{(\rho r_2/\ell_2 + r_1/\ell_1)^2 + (\eta r_2/\ell_2)^2}} + O(r_1^2 r_2^2)$$

$$k = \pm\sqrt{(\rho r_2/\ell_2 + r_1/\ell_1)^2 + (\eta r_2/\ell_2)^2} + O(r_1^2 r_2^2).$$

(We must be careful as $s$ and $c$ are not continuous at $r_1 = r_2 = 0$.) We deduce from these formulae that

$$(3.15) \qquad \begin{aligned} (\log f)_\rho &= \frac{\rho r_2^2/\ell_2^2 + r_1 r_2/(\ell_1 \ell_2)}{(\rho r_2/\ell_2 + r_1/\ell_1)^2 + (\eta r_2/\ell_2)^2} + O(r_1^2 r_2^2) \\ (\log f)_\eta &= \frac{\eta r_2^2/\ell_2^2}{(\rho r_2/\ell_2 + r_1/\ell_1)^2 + (\eta r_2/\ell_2)^2} + O(r_1^2 r_2^2). \end{aligned}$$

Let the corner correspond to $\eta = \eta_0$. We shall investigate the behaviour of $f$ in polar coordinates centred at $(0, \eta_0)$, by introducing

$$\eta - \eta_0 = R\cos\Theta, \quad \rho = R\sin\Theta.$$

Then

$$\partial_R \log f = (\cos\Theta \, \partial_\eta + \sin\Theta \, \partial_\rho) \log f$$

and this is uniformly bounded for any fixed $\Theta$ by (3.15). Hence $f(R, \Theta)$ has a limit $f(0, \Theta)$ as $R \to 0$, for each fixed $\Theta \in (0, \pi)$. It also has a limit as $\Theta \to 0$ or $\pi$ for each fixed $R > 0$, namely the boundary value $f_0(\eta_0 \pm R)$. Thus $f$ is bounded in $[0, \varepsilon] \times [-\pi, \pi]$ for some $\varepsilon > 0$.

Now we note that

$$\partial_\Theta f = f \partial_\Theta \log f = fR(-\sin\Theta \, \partial_\eta + \cos\Theta \, \partial_\rho) \log f$$

and the right hand side is $O(R)$, uniformly in $\Theta$ by (3.15). Hence by integration, we have $|f(R, \Theta_1) - f(R, \Theta_2)| = O(R)$ for any $\Theta_1, \Theta_2 \in (0, \pi)$. We deduce that

$$|f_0(\eta_0 + R) - f_0(\eta_0 - R)| = O(R)$$

and hence finally, taking $R \to 0$, that $f_0$ is continuous at the corner.

## References

[1] L. Anguelova and C. I. Lazaroiu, *M-theory on toric $G_2$ cones and its type II reduction*, J. High Energy Phys. (2002) 038, 41pp.

[2] L. Anguelova and C. I. Lazaroiu, *M-theory compactifications on certain "toric" cones of $G_2$ holonomy*, J. High Energy Phys. (2003) 066, 72pp.

[3] A. L. Besse, *Einstein Manifolds*, Ergeb. Math. Grenzgeb. **10**, Springer, Berlin (1987).

[4] R. Bielawski, *Complete hyper-Kähler $4n$-manifolds with a local tri-Hamiltonian $\mathbf{R}^n$-action*, Math. Ann. **314** (1999) 505–528.

[5] J. E. Borzellino, *Orbifolds of maximal diameter*, Indiana Univ. Math. J. **42** (1993) 37–53.

[6] C. P. Boyer, K. Galicki, B. M. Mann and E. G. Rees, *Compact 3-Sasakian 7-manifolds with arbitrary second Betti number*, Invent. Math. **131** (1998) 321–344.

[7] R. L. Bryant and S. M. Salamon, *On the construction of some complete metrics with exceptional holonomy*, Duke Math. J. **58** (1989) 829–850.

[8] D. M. J. Calderbank and H. Pedersen, *Selfdual Einstein metrics with torus symmetry*, J. Diff. Geom. **60** (2002) 485–521.

[9] D. M. J. Calderbank and M. Singer, *Einstein metrics and complex singularities*, Invent. Math. **156** (2004) 405–443.

[10] K. Galicki, *New matter couplings in $N = 2$ supergravity*, Nuclear Phys. **B 289** (1987) 573–588.

[11] K. Galicki and H. B. Lawson, Jr., *Quaternionic reduction and quaternionic orbifolds*, Math. Ann. **282** (1988) 1–21.

[12] G. W. Gibbons, D. N. Page and C. N. Pope, *Einstein metrics on $S^3$, $\mathbf{R}^3$ and $\mathbf{R}^4$ bundles*, Comm. Math. Phys. **127** (1990) 529–553.

[13] A. Haefliger and É. Salem, *Actions of tori on orbifolds*, Ann. Global Anal. Geom. **9** (1991) 37–59.

[14] A. Hattori and M. Masuda, *Theory of multi-fans*, Osaka J. Math. **40** (2003) 1–68.

[15] S. Helgason, *Groups and geometric analysis*, Mathematical Surveys and Monographs **83**, American Mathematical Society, Providence (2000), Integral geometry, invariant differential operators, and spherical functions, Corrected reprint of the 1984 original.

[16] D. D. Joyce, *Explicit construction of self-dual 4-manifolds*, Duke Math. J. **77** (1995) 519–552.

[17] C. LeBrun, *On the topology of self-dual 4-manifolds*, Proc. Amer. Math. Soc. **98** (1986) 637–640.

[18] E. Lerman and S. Tolman, *Hamiltonian torus actions on symplectic orbifolds and toric varieties*, Trans. Amer. Math. Soc. **349** (1997) 4201–4230.

[19] J. B. Lewis, *Eigenfunctions on symmetric spaces with distribution-valued boundary forms*, J. Funct. Anal. **29** (1978) 287–307.

[20] R. Mazzeo, *The Hodge cohomology of a conformally compact metric*, J. Diff. Geom. **28** (1988) 309–339.

[21] R. R. Mazzeo and R. B. Melrose, *Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature*, J. Funct. Anal. **75** (1987) 260–310.

[22] P. Orlik and F. Raymond, *Actions of the torus on 4-manifolds* I, Trans. Amer. Math. Soc. **152** (1970) 531–559.

School of Mathematics, University of Edinburgh, King's Buildings, Mayfield Road, Edinburgh EH9 3JZ, Scotland.
  *E-mail address*: davidmjc@maths.ed.ac.uk

  *E-mail address*: michael@maths.ed.ac.uk