TORIC CONTACT GEOMETRY IN ARBITRARY CODIMENSION

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ABSTRACT. We define toric contact manifolds in arbitrary codimension and give a description of such manifolds in terms of a kind of labelled polytope embedded into a grassmannian, analogous to the Delzant polytope of a toric symplectic manifold.

INTRODUCTION

Toric contact geometry lies at the interface of of two important themes in contemporary geometry. The first is the use of toric methods, beginning with toric varieties and toric symplectic manifolds, to provide large classes of geometric objects which can be understood in an essentially combinatorial way. The second is a growing interest in geometries where the tangent bundle is not completely reducible, but is instead equipped with a distribution (subbundle) or filtration. The simplest nontrivial example is a $(2m + 1)$-manifold N equipped with a corank one distribution $\mathcal{D} \leq TN$ which is locally the kernel of a *contact form*, i.e., a 1-form θ such that $\theta \wedge d\theta^{\wedge m}$ is a nonvanishing volume form. The restriction of $d\theta$ to $\mathcal{D}$ defines a symplectic form on $\mathcal{D}$, and thus $(N, \mathcal{D})$ may be viewed as an odd-dimensional analogue of a symplectic manifold called a *contact manifold*.

Any contact manifold has a canonical *symplectization*: the total space of the annihilator $\mathcal{D}^0 \leq T^*M$ of $\mathcal{D}$ inherits an exact 2-form $\Omega^{\mathcal{D}}$ from the tautological symplectic structure on T^*M, and $\Omega^{\mathcal{D}}$ is nondegenerate, hence symplectic, on the complement of the zero section. It follows that contact manifolds, like symplectic manifolds, have no local invariants, and it is natural to extend symplectic techniques to contact manifolds. In particular, a contact $(2m+1)$-manifold N is *toric* if its symplectization is, which means that N has an effective action of an $(m + 1)$-torus preserving the contact distribution $\mathcal{D}$, and such toric contact manifolds have been extensively studied.

Our interest here is in extending this theory to higher codimension, more precisely, to distributions $\mathcal{D}$ of corank $\ell > 1$ in TN. Such distributions arise naturally in CR geometry [12], for example on real submanifolds of codimension $\ell > 1$ in a complex manifold, and in subriemannian geometry [25]. However, it is not immediately clear how best to generalize the nondegeneracy condition, as the obstruction to integrability of $\mathcal{D}$ is now a 2-form on $\mathcal{D}$ with values in $TN/\mathcal{D}$, called the *Levi form*. We suggest that the most natural nondegeneracy requirement for a contact manifold $(N, \mathcal{D})$ in codimension ℓ is, as in codimension one, the local existence of a contact form, which is now a 1-form θ with $\mathcal{D} \leq \ker \theta$ and $d\theta|_{\mathcal{D}}$ nondegenerate. This means equivalently

that the Levi form of $\mathcal{D}$ has a nondegenerate component at every point, which is not as weak as it may seem, because nondegeneracy is a (fibrewise Zariski) open condition. Thus any contact manifold has a symplectization $U_{\mathcal{D}} \subseteq \mathcal{D}^0$ (with symplectic form induced from T^*N) which meets each fibre of $\mathcal{D}^0 \to N$ in a nonempty Zariski open subset. The complement of $U_{\mathcal{D}}$ in $\mathcal{D}^0$ is then, fibrewise, the cone over a projective variety (algebraically, a projective hypersurface of degree m, where $\mathcal{D}$ has rank $2m$) in $\mathrm{P}(\mathcal{D})$, which we call the *degeneracy variety* of $(N, \mathcal{D})$. In particular, for $\ell > 1$, a contact manifold $(N, \mathcal{D})$ has local invariants except in some low rank cases, and local classification would involve understanding the deformation theory of arbitrary projective hypersurfaces.

Despite these difficulties, *toric* contact geometry in higher codimension turns out to be surprisingly tractable. A contact manifold $(N, \mathcal{D})$ of dimension $2m + \ell$ and codimension ℓ has a symplectization $U_{\mathcal{D}}$ of dimension $2(m + \ell)$, hence is toric if it has an effective action of an $(m + \ell)$-torus preserving the contact distribution $\mathcal{D}$. In order to avoid technical difficulties appearing already in codimension 1, we also require that there is an ℓ-subtorus whose orbits are transverse to the contact distribution. In this setting, we find that despite the presence of local invariants, toric contact geometry is essentially encoded in a natural generalization of the momentum polytope of a toric symplectic manifold, namely a kind of polytope in the grassmannian of ℓ-dimensional subspaces of the dual of the Lie algebra of the torus (which is just the projectivization when $\ell = 1$). Thus the theory still has a strong combinatorial flavour, and the local invariants arise simply because polytopes in a grassmannian are more flexible than polytopes in an affine or projective space.

In more detail, the contents and main results of the paper are as follows. After defining contact manifolds and their symplectizations in §1.1 (and giving some simple examples, including products of odd dimensional spheres), we begin the study of contact actions in §1.2. Here we introduce as Condition 1, the transversality property that we use to avoid some technicalities in the theory. We also note that a local contact action satisfying this condition induces a family of 2-forms on $\mathcal{D}$ which we call *Levi structures*. In a companion paper [1], we use nondegenerate Levi structures, in the CR context, to construct Kähler metrics on quotients by transverse actions. While we do not pursue this angle in the present paper, our approach is in part motivated by it.

We begin our study of contact actions of a torus by introducing the local theory in §2.1. Here Levi structures are induced by *Levi pairs* $(\mathfrak{g}, \lambda)$, where $\mathfrak{g}$ is a subalgebra of the infinitesimal torus $\mathfrak{t}_N$ acting on the contact manifold N and $\lambda \in \mathfrak{g}^*$ picks out a component of the Levi form. We introduce fundamental methodology, Stratagem 1, for parametrizing such pairs $(\mathfrak{g}, \lambda)$ by maps to the quotients $\mathfrak{k} = \mathfrak{t}_N/\mathfrak{g}$ and $\mathfrak{h} = \mathfrak{t}_N/\ker \lambda$. This stratagem is closely linked to the affine geometry of momentum maps for hamiltonian torus actions, which we review in Appendix A. We thus see that a Levi pair provides an affine slice of the momentum map for the hamiltonian $\mathfrak{t}_N$ action on the symplectization $U_{\mathcal{D}}$ of N.

The central idea of the paper, the *grassmannian momentum map* of Definition 7, is a way to package these slices together in a single compact object (analogous to the polytope of a compact toric symplectic manifold or orbifold) in the grassmannian $\mathrm{Gr}_\ell(\mathfrak{t}_N^*)$ of ℓ-dimensional subspaces of $\mathfrak{t}_N^*$, which we call the *grassmannian image* of N.

When the $\mathfrak{t}_N$ action integrates to an effective contact action of a torus $\mathbb{T}_N$ on N, properties of the grassmannian image can be derived from fairly standard properties of the $\mathbb{T}_N$ action on N and $U_{\mathcal{D}}$. In §2.2, we apply the standard theory of symplectic

slices to obtain a local model for these actions. Then in §2.3, we obtain a connectedness and convexity result for the slices of $\mathfrak{t}_N$ action on $U_\mathcal{D}$ given by a Levi pair. This is essentially a special case of a convexity result for transverse symplectic foliations obtained independently by Ishida [17], and follows easily from the methods of Atiyah [2].

These preparations lead us to our main results on toric contact manifolds. In section §3.1, we show that the quotient of N by $\mathbb{T}_N$ is a simple polytope in the sense that it is at least a manifold with corners. We improve this in §3.2, where we show that any Levi pair $(\mathfrak{g}, \lambda)$ provides a diffeomorphism of the quotient $N/\mathbb{T}_N$ with a simple polytopes in the affine subspace of $\mathfrak{t}_N^*$ defined by $(\mathfrak{g}, \lambda)$. This shows that the grassmannian momentum map induces an embedding of $N/\mathbb{T}_N$ into $\mathrm{Gr}_\ell(\mathfrak{t}_N^*)$ whose image is polyhedral submanifold with corners as in Definition 14.

We would like to prove that toric contact manifolds are classified by their grassmannian image, with a suitable labelling of the codimension one faces. This is true locally by the local models obtained in §2.2 using symplectic slices. However, as in toric symplectic geometry [24] and toric contact geometry in codimension one [22], there remains a local-to-global question governed by the cohomology of a sheaf. Thus we have the following result.

Theorem. *The grassmannian image Ξ of a compact toric contact manifold of Reeb type $(N, \mathcal{D}, \mathbf{K})$ is Delzant and polyhedral of Reeb type in $\mathrm{Gr}_\ell(\mathfrak{t}_N^*)$, and there is a sheaf $\mathfrak{con}^\mathbb{T}(\mathcal{D})$ on Ξ such that toric contact manifolds of Reeb type with grassmannian image Ξ are parametrized up to isomorphism by $H^1(\Xi, \mathfrak{con}^\mathbb{T}(\mathcal{D}))$.*

The terminology in this theorem is defined in §3.3, where we prove that the grassmannian image has these properties, and establish the uniqueness, modulo $H^1(\Xi, \mathfrak{con}^\mathbb{T}(\mathcal{D}))$, of $(N, \mathcal{D}, \mathbf{K})$ given Ξ. Then in §3.4, we prove that any such Ξ arises in this way.

In contrast to the symplectic and codimension one cases, $\mathfrak{con}^\mathbb{T}(\mathcal{D})$ is the sheaf of solutions of a linear partial differential equation (for infinitesimal contactomorphisms), which is overdetermined and typically not involutive. We show that $H^1(\Xi, \mathfrak{con}^\mathbb{T}(\mathcal{D}))$ vanishes in special cases, such as when $(N, \mathcal{D}, \mathbf{K})$ is a product of codimension one toric contact manifolds. However, it remains an open question whether it vanishes in general.

1. CONTACT GEOMETRY IN ARBITRARY CODIMENSION

1.1. **Levi nondegenerate distributions and symplectization.** Let N be smooth manifold of real dimension $2m+\ell$ equipped with a rank $2m$ vector distribution $\mathcal{D} \leq TN$. Let $\mathcal{D}^0$ be the annihilator of $\mathcal{D}$, which is a rank ℓ subbundle of T^*N. We thus have dual short exact sequences of vector bundles

$$0 \to \mathcal{D} \to TN \xrightarrow{q_\mathcal{D}} TN/\mathcal{D} \to 0$$

$$0 \to \mathcal{D}^0 \to T^*N \to \mathcal{D}^* \to 0,$$

where the transpose of the quotient $q_\mathcal{D} \colon TN \to TN/\mathcal{D}$ identifies $\mathcal{D}^0$ canonically with $(TN/\mathcal{D})^*$: thus we identify an element or section α of $(TN/\mathcal{D})^*$ with $\alpha \circ q_\mathcal{D}$ in $\mathcal{D}^0$.

Definition 1. The *Levi form* $L_\mathcal{D} \colon \wedge^2 \mathcal{D} \to TN/\mathcal{D}$ of $\mathcal{D} \leq TN$ is defined, using sections $X, Y \in \Gamma(\mathcal{D})$, by the tensorial expression

$$(1) \qquad\qquad L_\mathcal{D}(X, Y) = -q_\mathcal{D}([X, Y]).$$

The *nondegeneracy locus* of $\mathcal{D}$ is the open subset

$$U_\mathcal{D} = \{\alpha \in \mathcal{D}^0 \cong (TN/\mathcal{D})^* \mid \alpha \circ L_\mathcal{D} \text{ is nondegenerate}\}$$

of $\mathcal{D}^0$. We say $\mathcal{D}$ is *Levi nondegenerate*, and $(N, \mathcal{D})$ is *contact of rank m and codimension ℓ*, if $U_\mathcal{D}$ has nonempty intersection with each fibre of $\mathcal{D}^0$ over N. A (local) section of $U_\mathcal{D}$ is called a (local) *contact form* on N. We often describe $U_\mathcal{D}$ by its complement, which is (the bundle of cones over) the *degeneracy variety*

$$V_\mathcal{D} = \{[\alpha] \in \mathrm{P}(\mathcal{D}^0) \mid \alpha \circ L_\mathcal{D} \text{ is degenerate}\},$$

viewed as a bundle of submanifolds of the fibrewise projectivization $\mathrm{P}(\mathcal{D}^0)$ of $\mathcal{D}^0$.

Our normalization of the Levi form is chosen so that for any section α of $\mathcal{D}^0$, the restriction of $d\alpha$ to $\wedge^2 \mathcal{D} \leq \wedge^2 TN$ is $\alpha \circ L_\mathcal{D}$. A Levi nondegenerate distribution $\mathcal{D}$ has local frames $\alpha_1, \ldots \alpha_\ell$ for $\mathcal{D}^0$ in $U_\mathcal{D}$, and hence each fibre of the degeneracy variety $V_\mathcal{D}$ is the projective hypersurface of degree m where $(\sum_{i=1}^\ell t_i \alpha_i) \circ L_\mathcal{D}$ degenerates (i.e., the determinant, a homogeneous degree m polynomial in the ℓ variables $t_1, \ldots t_\ell$, vanishes).

Examples 1. (i) The projectivization $\mathrm{P}(W) = W^\times / \mathbb{R}^\times \cong \mathbb{R}P^{2m+1}$ of a symplectic vector space (W, ω) (with $\dim W = 2m + 2$) is contact of codimension one, where $\mathcal{D}_\xi = \mathrm{Hom}(\xi, \xi^\perp / \xi) \leq \mathrm{Hom}(\xi, W/\xi) = T_\xi \mathrm{P}(W)$ and $\xi^\perp$ is the orthogonal space to ξ with respect to ω; thus $U_\mathcal{D} = \mathcal{D}^0 \setminus 0$ and the degeneracy variety $V_\mathcal{D}$ is empty. The odd dimensional sphere $\mathbb{S}^{2m+1} \cong W^\times / \mathbb{R}^+$ (the space of rays in W) is also contact, being a double cover of $\mathrm{P}(W)$.

(ii) If $(N_i, \mathcal{D}_i)$ are contact manifolds, with codimensions ℓ_i, for $i \in \{1, \ldots n\}$, then so is $(\prod_{i=1}^n N_i, \mathcal{D}_1 \oplus \cdots \oplus \mathcal{D}_n)$, with codimension $\ell = \ell_1 + \cdots + \ell_n$ and $U_\mathcal{D} = \prod_{i=1}^n U_{\mathcal{D}_i}$. In particular, the product of $n = \ell$ codimension one spheres $\mathbb{S}^{2m_1+1} \times \cdots \times \mathbb{S}^{2m_\ell+1}$ is a contact manifold with codimension ℓ, where each fibre of $U_\mathcal{D}$ is the disjoint union of the 2^ℓ open ℓ-quadrants (simple cones) spanned by $\pm\eta_1, \ldots, \pm\eta_\ell$. Each fibre of the degeneracy variety $V_\mathcal{D}$ is thus an union of ℓ-hyperplanes (i.e., the facets of an $(\ell - 1)$-simplex) with multiplicities m_i.

(iii) Another special case, studied in [6, 11], is that of *ℓ-contact manifolds*. In the terminology of this paper, these are contact manifolds $(N, \mathcal{D})$ of rank m and codimension ℓ for which the Levi form $L_\mathcal{D} \colon \wedge^2 \mathcal{D} \to TN/\mathcal{D}$ has rank one (but this component is nondegenerate) at every point. Equivalently, each fibre of the degeneracy variety $V_\mathcal{D}$ is a single hyperplane of multiplicity m.

(iv) The case that the degeneracy variety $V_\mathcal{D}$ is empty (i.e., in each fibre, it has no real points) has been studied in subriemannian geometry as a natural generalization of the codimension one case [25]. This condition means equivalently that for any $z \in N$ and any nonzero $X \in T_z N$, $L_\mathcal{D}(X, \cdot) \colon \mathcal{D}_z \to T_z N / \mathcal{D}_z$ is surjective. A distribution with this property is called *fat*.

Definition 2. A diffeomorphism Ψ between contact manifolds $(N, \mathcal{D})$ and $(N', \mathcal{D}')$ is called a *contactomorphism* if $\Psi_*(\mathcal{D}) = \Psi^* \mathcal{D}' \leq \Psi^* TN'$. We denote by $\mathfrak{con}(N, \mathcal{D})$ the Lie algebra of *infinitesimal contactomorphisms* of $(N, \mathcal{D})$, i.e., the space of vector fields X on N such that

$$\mathcal{L}_X \Gamma(\mathcal{D}) \subseteq \Gamma(\mathcal{D}).$$

We now show that $U_\mathcal{D} \subseteq \mathcal{D}^0 \leq T^*N$ (or any connected component of $U_\mathcal{D}$) provides a "symplectization" of N, to which contactomorphisms lift as hamiltonian vector fields. For this recall that the *tautological 1-form* τ on T^*N is defined by $\tau_\alpha = \alpha \circ p_*$, where $\alpha \in T^*N$ and p_* is the differential of the projection $p \colon T^*N \to N$. The latter fits into an exact sequence

$$(2) \qquad\qquad 0 \to p^* T^*N \to T(T^*N) \xrightarrow{p_*} p^* TN \to 0,$$

where (the image of) p^*T^*N is the vertical bundle of T^*N. The *canonical symplectic form* on T^*N is then $\Omega = d\tau$. For any local section α of T^*N, $\alpha^*\tau = \alpha$ and hence $\alpha^*\Omega = \alpha^*d\tau = d(\alpha^*\tau) = d\alpha$. Since τ vanishes on p^*T^*N, so does Ω, and the induced pairing between p^*T^*N and p^*TN the natural one: for any 1-form α and any lift $\tilde{X}$ of a vector field X on N, $\Omega(p^*\alpha, \tilde{X}) = d(\tau(\tilde{X}))(p^*\alpha) = \alpha(X)$, since $\tau(\tilde{X})\colon T^*N \to \mathbb{R}$ is evaluation at X, which is linear on the fibres of p.

Any vector field X on N admits a *natural lift* to T^*N, whose local flow is the natural lift of the local flow of X. For further use, we recall the following well-known facts.

Proposition 1. *The natural lift of any vector field X on N to TN is the unique lift $\tilde{X}$ with $\mathcal{L}_{\tilde{X}}\tau = 0$; it is hamiltonian with respect to Ω, with momentum map $\alpha \mapsto \alpha(X)$, and is given by $\tilde{X}_\alpha = \alpha_*(X) - p^*\mathcal{L}_X\alpha$ for any extension of $\alpha \in T^*N$ to a local section.*

Proof. The natural lift of any diffeomorphism of N to T^*N preserves τ (by its tautological construction), and hence the natural lift of X preserves τ. On the other hand for any vector field $\tilde{X}$ on T^*N preserving τ, $0 = \mathcal{L}_{\tilde{X}}\tau = \iota_{\tilde{X}}\Omega + d(\tau(\tilde{X}))$, so $\tilde{X}$ has hamiltonian $\tau(\tilde{X})$. If $\tilde{X}$ is a lift of X then $\tau(\tilde{X})(\alpha) = \alpha(X)$, which determines $\tilde{X}$ uniquely from X. For the final formula, observe that $\alpha^*(\iota_{\tilde{X}}\Omega) = -d(\alpha(X)) = \iota_X d\alpha - \mathcal{L}_X\alpha = \alpha^*(\iota_{\alpha_*(X)}\Omega - \iota_{p^*\mathcal{L}_X\alpha}\Omega)$, since $\alpha^*(\iota_{\alpha_*(X)}\Omega) = \iota_X(\alpha^*\Omega) = \iota_X d\alpha$; from this the result follows by nondegeneracy of Ω. $\qquad\square$

Let $\tau^{\mathcal{D}}$ be the pullback of τ to $\mathcal{D}^0 \leq T^*N$, so that $\Omega^{\mathcal{D}} := d\tau^{\mathcal{D}}$ is the pullback of Ω. Henceforth, p will denote the bundle projection of $\mathcal{D}^0$ rather than T^*N above, and we now have an exact sequence

$$(3) \qquad\qquad 0 \to p^*\mathcal{D}^0 \to T\mathcal{D}^0 \xrightarrow{p_*} p^*TN \to 0.$$

Proposition 2. *For a contact manifold $(N, \mathcal{D})$, $U_{\mathcal{D}}$ is the open subset of $\mathcal{D}^0$ on which $\Omega^{\mathcal{D}}$ is nondegenerate, hence a symplectic form. Also, for any contact form $\alpha\colon N \to U_{\mathcal{D}}$, $TN = \mathcal{D} \oplus \mathcal{K}^\alpha$, where $\mathcal{K}^\alpha$ is the distribution orthogonal to $\mathcal{D}$ via $d\alpha$. The natural lift of any $X \in \mathfrak{con}(N, \mathcal{D})$ to T^*N is tangent to $\mathcal{D}^0$, and its restriction defines an isomorphism between $\mathfrak{con}(N, \mathcal{D})$ and the space of projectable vector fields $\tilde{X}$ on $\mathcal{D}^0$ with $\mathcal{L}_{\tilde{X}}\tau^{\mathcal{D}} = 0$.*

Proof. The restriction of $\Omega_\alpha^{\mathcal{D}}$ to $V_\alpha(\mathcal{D}^0) \times T_zN$ descends to the canonical nondegenerate pairing between $V_\alpha(\mathcal{D}^0) \cong \mathcal{D}_z^0$ and $T_zN/\mathcal{D}_z$. Hence Ω_α is nondegenerate if and only if $\alpha^*\Omega^{\mathcal{D}} = d\alpha$ is nondegenerate on $\mathcal{D}$ for any extension of α to a local section. But $d\alpha|_{\mathcal{D}} = \alpha \circ L_{\mathcal{D}}$, which proves the first part. The second part is immediate from the nondegeneracy of $d\alpha$ on $\mathcal{D}$.

For the last part, observe that the local flow of $X \in \mathfrak{con}(N, \mathcal{D})$ preserves $\mathcal{D}$, hence its natural lift to T^*N preserves $\mathcal{D}^0$, i.e., the induced vector field is tangent to $\mathcal{D}^0$. The restriction $\tilde{X}$ to $\mathcal{D}^0$ is clearly a lift of X, hence projectable, and $\mathcal{L}_{\tilde{X}}\tau^{\mathcal{D}} = 0$ since $\tau^{\mathcal{D}}$ is the pullback of τ. Conversely if a lift $\tilde{X}$ of X to $\mathcal{D}$ satisfies $\mathcal{L}_{\tilde{X}}\tau^{\mathcal{D}} = 0$ then for any section Y of $\mathcal{D}$, any lift $\tilde{Y}$ of Y to $\mathcal{D}^0$, and any $\alpha \in \mathcal{D}^0$, $0 = (\mathcal{L}_{\tilde{X}}\tau^{\mathcal{D}})_\alpha(\tilde{Y}) = -\tau_\alpha^{\mathcal{D}}([\tilde{X}, \tilde{Y}]) = \alpha(p_*[\tilde{X}, \tilde{Y}]) = \alpha([p_*\tilde{X}, p_*\tilde{Y}]) = \alpha(\mathcal{L}_X Y)$. Hence $\mathcal{L}_X Y$ is a section of $\mathcal{D}$, i.e., $X \in \mathfrak{con}(N, \mathcal{D})$. Furthermore, if $f_X\colon \mathcal{D}^0 \to \mathbb{R}$ is defined by $f_X(\alpha) = \alpha(X)$ then $df_X + \iota_{\tilde{X}}\Omega^{\mathcal{D}} = 0$, which determines $\tilde{X}$ uniquely from X on $U_{\mathcal{D}}$, hence on $\mathcal{D}^0$, since $U_{\mathcal{D}}$ is dense in $\mathcal{D}^0$. $\qquad\square$

We refer to $\mathcal{K}^\alpha$ as the *Reeb distribution* of α.

If we identify $\mathcal{D}^0$ with $(TN/\mathcal{D})^*$, then the function $f_X\colon \mathcal{D}^0 \to \mathbb{R}$ with $f_X(\alpha) = \alpha(X)$ is the fibrewise linear form defined by $q_{\mathcal{D}}(X) \in TN/\mathcal{D}$. Since f_X is a hamiltonian for the lift $\tilde{X}$ on $U_{\mathcal{D}}$, it follows that $\tilde{X}$ is in fact uniquely determined by $q_{\mathcal{D}}(X)$.

Remark 1. The degeneracy variety is a local contact invariant of a contact manifold $(N, \mathcal{D})$. For example $\mathbb{S}^1 \times \mathbb{S}^5$ and $\mathbb{S}^3 \times \mathbb{S}^3$ cannot be even locally contactomorphic because the degeneracy variety of the latter has two points in each fibre, whereas the former has only one (with multiplicity two). Also the lift $\tilde{X}$ to $\mathcal{D}^0$, for any $X \in \mathfrak{con}(N, \mathcal{D})$ must preserve (the cone over) the degeneracy variety. Thus although $\mathfrak{con}(N, \mathcal{D})$ may be identified with a linear subspace of $\Gamma(TN/\mathcal{D})$, this linear subspace is typically small.

1.2. **Local contact actions and transversality.** Effective actions of Lie groups by contactomorphisms on $(N, \mathcal{D})$ may be described locally as follows.

Definition 3. A (*local, effective*) *contact action* of a Lie algebra $\mathfrak{g}$ on a contact manifold $(N, \mathcal{D})$ is a Lie algebra monomorphism $\mathbf{K} \colon \mathfrak{g} \to \mathfrak{con}(N, \mathcal{D})$. For $v \in \mathfrak{g}$, we write K_v for the induced vector field $\mathbf{K}(v)$, and we define $\kappa^{\mathfrak{g}} \colon N \times \mathfrak{g} \to TN$ by $\kappa^{\mathfrak{g}}(z, v) = K_{v,z}$. Let $\mathcal{K}^{\mathfrak{g}} \le TN$ be the image of $\kappa^{\mathfrak{g}}$, i.e., $\mathcal{K}^{\mathfrak{g}}_z := \mathrm{span}\{K_{v,z} \mid v \in \mathfrak{g}\}$.

Observation 1. *Let* $\mathbf{K} \colon \mathfrak{g} \to \mathfrak{con}(N, \mathcal{D})$ *be a local contact action of* $\mathfrak{g}$ *on* $(N, \mathcal{D})$ *and define* $\mu_{\mathfrak{g}} \colon \mathcal{D}^0 \to \mathfrak{g}^*$ *by* $\langle \mu_{\mathfrak{g}}(\alpha), v \rangle = \alpha(K_v)$ *for* $\alpha \in \mathcal{D}^0$ *and* $v \in \mathfrak{g}$. *Then the lift of* $\mathbf{K}$ *to* T^*N *preserves* $\mathcal{D}^0$, *and the induced local action* $\tilde{\mathbf{K}}$ *is hamiltonian on* $U_{\mathcal{D}}$ *with momentum map* $\mu_{\mathfrak{g}}|_{U_{\mathcal{D}}}$; *in particular* $\langle d\mu_{\mathfrak{g}}(\tilde{K}_v), w \rangle = -\langle \mu_{\mathfrak{g}}, [v, w]_{\mathfrak{g}} \rangle$ *for all* $v, w \in \mathfrak{g}$.

This is immediate from the discussion at the end of §1.1.

Example 2. Let G be a Lie group of dimension ℓ with Lie algebra $\mathfrak{g}$ and let $\pi \colon N \to M$ be a principal G-bundle with connection $\eta \colon TN \to \mathfrak{g}$, where $\dim M = 2m$. Then $\mathcal{D} := \ker \eta$ is a rank $2m$ distribution on N, G acts on $(N, \mathcal{D})$ by contactomorphisms, and η induces a bundle isomorphism of $TN/\mathcal{D}$ with $N \times \mathfrak{g}$. In this trivialization, the Levi form of $\mathcal{D}$ is $d\eta + \frac{1}{2}[\eta \wedge \eta]_{\mathfrak{g}}$, the pullback to N of the curvature F^{η} of η.

Suppose $\mathbf{K} \colon \mathfrak{g} \to \mathfrak{con}(N, \mathcal{D})$ is a local contact action where $(N, \mathcal{D})$ is contact of codimension $\ell = \dim \mathfrak{g}$. Abstracting the local geometry of Example 2, we say the action of $\mathfrak{g}$ is *transversal* if the following condition holds.

Condition 1. At every point of N, $\mathcal{D} + \mathcal{K}^{\mathfrak{g}} = TN$. Equivalently:
 (i) $\mathrm{rank}\, \mathcal{K}^{\mathfrak{g}} = \ell$ everywhere on N;
 (ii) $\mathcal{D} \cap \mathcal{K}^{\mathfrak{g}}$ is the zero section of TN (and thus $TN = \mathcal{D} \oplus \mathcal{K}^{\mathfrak{g}}$).

The composite $q_{\mathcal{D}} \circ \kappa^{\mathfrak{g}} \colon N \times \mathfrak{g} \to \mathcal{K}^{\mathfrak{g}} \to TN/\mathcal{D}$ is a bundle isomorphism and so there is a canonically defined 1-form $\eta^{\mathfrak{g}} \colon TN \to \mathfrak{g}$, characterized by

$$\ker \eta^{\mathfrak{g}} = \mathcal{D} \quad \text{and} \quad \forall v \in \mathfrak{g}, \quad \eta^{\mathfrak{g}}(K_v) = v.$$

We also denote by $\eta^{\mathfrak{g}}$ the induced map from $TN/\mathcal{D}$ to $\mathfrak{g}$. We can now verify the usual properties of a connection 1-form.

Lemma 1. *Let* $\eta = \eta^{\mathfrak{g}}$ *denote the* $\mathfrak{g}$-*valued 1-form of a contact action on* N *satisfying Condition 1. Then for any* $v \in \mathfrak{g}$, $\mathcal{L}_{K_v} \eta + [v, \eta]_{\mathfrak{g}} = 0$, *and* $d\eta + \frac{1}{2}[\eta \wedge \eta]_{\mathfrak{g}} = \eta \circ L_{\mathcal{D}}$, *where* $L_{\mathcal{D}}$ *is extended by zero from* $\mathcal{D}$ *to* $TN = \mathcal{D} \oplus \mathcal{K}^{\mathfrak{g}}$.

Proof. For any $v \in \mathfrak{g}$, K_v preserves $\mathcal{D}$ and $\mathcal{K}^{\mathfrak{g}}$, hence also $\kappa^{\mathfrak{g}} \circ \eta \colon TN \to \mathcal{K}^{\mathfrak{g}}$. Thus

$$0 = \mathcal{L}_v(\mathbf{K} \circ \eta)(X) = (\mathcal{L}_v \mathbf{K})(\eta(X)) + \mathbf{K}(\mathcal{L}_{K_v} \eta)(X).$$

Since $\mathbf{K}$ is a morphism of Lie algebras, $(\mathcal{L}_{K_v} \mathbf{K})(w) = [K_v, K_w] = \mathbf{K}([v, w]_{\mathfrak{g}})$, so $(\mathcal{L}_{K_v} \eta)(X) + [v, \eta(X)]_{\mathfrak{g}} = 0$ by the injectivity of $\mathbf{K}$. Now for $X, Y \in \Gamma(\mathcal{D})$ and $v, w \in \mathfrak{g}$, $d\eta(X + K_v, Y + K_w) = -\eta([X + K_v, Y + K_w]) = -\eta([X, Y]) - [\eta(K_v), \eta(K_w)]_{\mathfrak{g}} = \eta(L_{\mathcal{D}}(X + K_v, Y + K_w)) - [\eta(X + K_v), \eta(Y + K_w)]_{\mathfrak{g}}$. $\qquad\square$

Since $\mathbf{K}\colon \mathfrak{g} \to \mathfrak{con}(N,\mathcal{D})$ is a Lie algebra morphism, $\mathcal{K}^{\mathfrak{g}}$ is an integrable distribution, equal to $\mathcal{K}^{\alpha}$ for any contact form α such that $\alpha(K_v)$ is constant for all $v \in \mathfrak{g}$. For any $\lambda \in \mathfrak{g}^*$, define $\eta^{\lambda} = \eta^{\mathfrak{g},\lambda}\colon N \to \mathcal{D}^0$ by $\eta_z^{\lambda}(X) = \langle \eta_z(X), \lambda \rangle$, so that $\eta^{\lambda}(K_v) = \lambda(v)$ and $d\eta^{\lambda}|_{\mathcal{D}} = \langle d\eta|_{\mathcal{D}}, \lambda \rangle = \eta^{\lambda} \circ L_{\mathcal{D}}$ is the λ-component of the Levi form of $\mathcal{D}$.

Definition 4. For a contact action $\mathbf{K}\colon \mathfrak{g} \to \mathfrak{con}(N,\mathcal{D})$ satisfying Condition 1, we refer to $\mathcal{K}^{\mathfrak{g}}$ and its integral submanifolds as the associated *Reeb distribution* and *Reeb foliation* transverse to $\mathcal{D}$. For $\lambda \in [\mathfrak{g},\mathfrak{g}]^0 \le \mathfrak{g}^*$, we call $(\mathcal{D}, d\eta^{\lambda}|_{\mathcal{D}})$ the induced *Levi structure*; it is said to be *nondegenerate* $d\eta^{\lambda}|_{\mathcal{D}}$ is, i.e., η^{λ} is a contact form ($U_{\mathcal{D}}$-valued).

2. CONTACT TORUS ACTIONS

2.1. **Abelian local contact actions.** Suppose that $\mathbf{K}\colon \mathfrak{t}_N \to \mathfrak{con}(N,\mathcal{D})$ is a local contact action of an abelian Lie algebra $\mathfrak{t}_N$ on a contact manifold $(N,\mathcal{D})$. Let

$$\kappa := \kappa^{\mathfrak{t}_N}\colon N \times \mathfrak{t}_N \to TN, \qquad \text{with} \qquad \kappa(z,v) = K_{v,z}, \quad \text{and}$$

$$\mu := \mu_{\mathfrak{t}_N}\colon \mathcal{D}^0 \to \mathfrak{t}_N^*, \qquad \text{with} \qquad \langle \mu(\alpha), v \rangle = \alpha(K_v),$$

so that $(p,\mu)\colon \mathcal{D}^0 \to N \times \mathfrak{t}_N^*$ is the pointwise transpose of $q_{\mathcal{D}} \circ \kappa\colon N \times \mathfrak{t}_N \to TN/\mathcal{D}$. By Observation 1, $\mathbf{K}$ lifts to a hamiltonian action on $U_{\mathcal{D}}$ with momentum map $\mu|_{U_{\mathcal{D}}}$. The following is the main tool in our analysis of abelian local contact actions.

Definition 5. An ℓ-dimensional subalgebra $\iota_{\mathfrak{g}}\colon \mathfrak{g} \hookrightarrow \mathfrak{t}_N$ and an element $\lambda \in \mathfrak{g}^* \setminus 0$ together form a *Levi pair* $(\mathfrak{g},\lambda)$ for $\mathbf{K}$ if:

• $\mathfrak{g}$ acts transversally on N via $\mathbf{K}$, i.e., $\mathcal{K}^{\mathfrak{g}} := \mathrm{span}\{K_{v,z} \mid v \in \mathfrak{g}\}$ satisfies Condition 1.

Let $\eta\colon TN \to \mathfrak{g}$ be the connection 1-form of $\mathfrak{g}$; we say $(\mathfrak{g},\lambda)$ is *nondegenerate* if:

• $\eta^{\lambda} = \langle \eta, \lambda \rangle$ is a contact form, i.e., $(\mathcal{D}, d\eta^{\lambda}|_{\mathcal{D}})$ is a nondegenerate Levi structure.

Thus $(p,\mu_{\mathfrak{g}})\colon \mathcal{D}^0 \to N \times \mathfrak{g}^*$, with $\mu_{\mathfrak{g}} := \iota_{\mathfrak{g}}^{\top} \circ \mu$, is an isomorphism. We say $(N,\mathcal{D},\mathbf{K})$ is *Reeb type* if it admits a nondegenerate Levi pair.

Let $(\mathfrak{g},\lambda)$ be a Levi pair. For any $v \in \mathfrak{t}_N$, $(d\eta^{\lambda})(K_v,\cdot) = -d(\eta^{\lambda}(K_v))$. We may thus view $\eta^{\lambda}(K_v) = \langle \mu(\eta^{\lambda}), v \rangle$ as the "horizontal momentum" of K_v with respect to the Levi structure $(\mathcal{D}, d\eta^{\lambda}|_{\mathcal{D}})$. Observe that if $v \in \mathfrak{g}$, $\eta_z^{\lambda}(K_v) = \langle v, \lambda \rangle$, which vanishes for $v \in \ker \lambda \le \mathfrak{g}$. Hence $z \mapsto \mu(\eta_z^{\lambda}) \in \mathfrak{t}_N^*$ takes values in $(\ker \lambda)^0 \cong (\mathfrak{t}_N/\ker \lambda)^*$.

Stratagem 1. For any pair $(\mathfrak{g},\lambda)$ with $\mathfrak{g} \le \mathfrak{t}_N$ and $\lambda \in \mathfrak{g}^* \setminus 0$, the quotient $\mathfrak{t}_N/\ker \lambda$ is an extension by $\mathbb{R}$ of the quotient $\mathfrak{t}_N/\mathfrak{g}$. To allow $(\mathfrak{g},\lambda)$ to vary, it is convenient to fix this extension $\mathfrak{h} \to \mathfrak{k}$ (where $\mathfrak{h}$ and $\mathfrak{k}$ are abelian Lie algebras of dimensions $n - \ell + 1$ and $n - \ell$ for $n = \dim \mathfrak{t}_N$); then the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{g} & \overset{\iota_{\mathfrak{g}}}{\longrightarrow} & \mathfrak{t}_N & \overset{\mathbf{u}}{\longrightarrow} & \mathfrak{k} & \longrightarrow & 0 \\
& & {\scriptstyle \lambda}\downarrow & & {\scriptstyle \mathbf{L}}\downarrow & & \| & & \\
0 & \longrightarrow & \mathbb{R} & \overset{\varepsilon}{\longrightarrow} & \mathfrak{h} & \overset{d}{\longrightarrow} & \mathfrak{k} & \longrightarrow & 0.
\end{array}
$$

of short exact sequences associates pairs $(\mathfrak{g},\lambda)$, with $\mathfrak{t}_N/\ker \lambda \cong \mathfrak{h}$, to surjective linear maps $\mathbf{L}\colon \mathfrak{t}_N \to \mathfrak{h}$ (thus $\mathfrak{g}$ is the kernel of $\mathbf{u} := d \circ \mathbf{L}$, and λ is induced by $\mathbf{L}|_{\mathfrak{g}}$).

Let $\mathcal{A} \subseteq \mathfrak{h}^*$ be the affine subspace $(\varepsilon^{\top})^{-1}(1)$ modelled on $\mathfrak{k}^*$; then $\mathfrak{h}$ may be identified with the affine linear functions $\ell\colon \mathcal{A} \to \mathbb{R}$, whence $d\ell \in \mathfrak{k}$ is the linear part of $\ell \in \mathfrak{h}$.

If $(\mathfrak{g},\lambda)$, defined by $\mathbf{L}\colon \mathfrak{t}_N \to \mathfrak{h}$, is a Levi pair, then the map $\mu^{\lambda}\colon N \to \mathcal{A} \subseteq \mathfrak{h}^*$, determined uniquely by the formula

$$(4) \qquad\qquad \langle \mu^{\lambda}(z), \mathbf{L}(v) \rangle = \eta_z^{\lambda}(K_v)$$

for all $z \in N$ and $v \in \mathfrak{t}_N$, will be called the *horizontal (natural) momentum map* of $(\mathcal{D}, d\eta^\lambda|_{\mathcal{D}})$ (cf. Appendix A). Equivalently the diagram

$$
\begin{array}{ccc}
N & \xrightarrow{\ \eta^\lambda\ } & U_{\mathcal{D}} \\
{\scriptstyle \mu^\lambda}\downarrow & & \downarrow{\scriptstyle \mu} \\
\mathfrak{h}^* & \xrightarrow{\ \mathbf{L}^\top\ } & \mathfrak{t}_N^*
\end{array}
$$

commutes, i.e., $\mathbf{L}^\top \circ \mu^\lambda = \mu \circ \eta^\lambda$.

Lemma 2. *Let $(N, \mathcal{D}, \mathbf{K})$ be a contact manifold with an abelian local contact action. Then the following are equivalent.*

(i) *$(N, \mathcal{D}, \mathbf{K})$ admits a nondegenerate Levi pair in a neighbourhood of any point.*

(ii) *$(p, \mu) \colon \mathcal{D}^0 \to N \times \mathfrak{t}_N^*$ is injective.*

(iii) *$q_{\mathcal{D}} \circ \kappa \colon N \times \mathfrak{t}_N \to TN/\mathcal{D}$ is surjective.*

Proof. The last two conditions are equivalent because (p, μ) is the transpose of $q_{\mathcal{D}} \circ \kappa$. Over any open neighbourhood where there is a nondegenerate Levi pair, $(p, \mu_{\mathfrak{g}})$ is an isomorphism and hence (p, μ) is injective. Conversely, if $q_{\mathcal{D}} \circ \kappa$ surjects, a complement to the kernel at a point gives a locally transversal subalgebra $\iota_{\mathfrak{g}} \colon \mathfrak{g} \hookrightarrow \mathfrak{t}_N$, hence also a Levi pair: $U_{\mathcal{D}}$ is open with nonempty fibres, so we can find $\lambda \in \mathfrak{g}^*$ such that η^λ is locally a contact form. $\qquad\square$

Remark 2. If N is compact, $\{\lambda \in \mathfrak{g}^* \setminus 0 \mid \eta^\lambda \in \Gamma(U_{\mathcal{D}})\}$ is an open cone $\mathcal{C}_{\mathfrak{g}} \subseteq \mathfrak{g}^*$.

Definition 6. $(N, \mathcal{D}, \mathbf{K})$ is *locally Reeb type* if the conditions of Lemma 2 hold.

For a vector space $\mathfrak{t}$ of dimension $n \geq \ell$, we denote by $\mathrm{Gr}_\ell(\mathfrak{t}^*)$ the grassmannian of ℓ-dimensional subspaces of $\mathfrak{t}^*$. This is a manifold of dimension $\ell(n - \ell)$ with distinguished *affine charts* defined as follows. Let $\iota_{\mathfrak{g}} \colon \mathfrak{g} \hookrightarrow \mathfrak{t}$ be an ℓ-dimensional subspace of $\mathfrak{t}$ and let $\mathrm{Gr}_\ell^{\mathfrak{g}}(\mathfrak{t}^*) = \{\xi \in \mathrm{Gr}_\ell(\mathfrak{t}^*) \mid \iota_{\mathfrak{g}}^\top(\xi) = \mathfrak{g}^*\}$ be set of subspaces $\xi \leq \mathfrak{t}^*$ complementary to the kernel of $\iota_{\mathfrak{g}}^\top \colon \mathfrak{t}^* \to \mathfrak{g}^*$. Then $\mathrm{Gr}_\ell^{\mathfrak{g}}(\mathfrak{t}^*)$ is an open subset of $\mathrm{Gr}_\ell(\mathfrak{t}^*)$, and is isomorphic to the affine space $\{\theta \in \mathrm{Hom}(\mathfrak{g}^*, \mathfrak{t}^*) \mid \iota_{\mathfrak{g}}^\top \circ \theta = Id_{\mathfrak{g}^*}\}$ via the map sending θ to $\mathrm{im}\,\theta$. The inverse sends $\xi \in \mathrm{Gr}_\ell^{\mathfrak{g}}(\mathfrak{t}^*)$ to $\psi_{\mathfrak{g}}(\xi) \colon \mathfrak{g}^* \to \mathfrak{t}^*$ where

(5)   for $\lambda \in \mathfrak{g}^*$, $\langle \psi_{\mathfrak{g}}(\xi), \lambda \rangle \in \mathfrak{t}^*$ is the unique point in $(\iota_{\mathfrak{g}}^\top)^{-1}(\lambda) \cap \xi$.

Definition 7. Let $\mathcal{K} = \mathcal{K}^{\mathfrak{t}_N} = \mathrm{im}\,\kappa$, $\mathcal{E} := \mathrm{im}(p, \mu) \leq N \times \mathfrak{t}_N^*$, and $\Theta := \mathcal{E}^0 = \ker(q_{\mathcal{D}} \circ \kappa) = \kappa^{-1}(\mathcal{D}) \leq N \times \mathfrak{t}_N$. If $(N, \mathcal{D}, \mathbf{K})$ is locally Reeb type, $\mathcal{K} \cap \mathcal{D}$ has codimension ℓ in $\mathcal{K}$ and $\mathcal{E}$ is a rank ℓ subbundle of $N \times \mathfrak{t}_N^*$ (with $\mathcal{E}^* \cong (N \times \mathfrak{t}_N)/\Theta \cong TN/\mathcal{D}$). Thus $\mathcal{E}$ defines a smooth map $N \to \mathrm{Gr}_\ell(\mathfrak{t}_N^*); z \mapsto \mathcal{E}_z$ which we call the *grassmannian momentum map* of $(N, \mathcal{D}, \mathbf{K})$, and also denote by $\mathcal{E}$.

Remark 3. If $(\mathfrak{g}, \lambda)$ is a nondegenerate Levi pair (over an open subset of N) then $\mathcal{E}$ takes values in the affine chart $\mathrm{Gr}_\ell^{\mathfrak{g}}(\mathfrak{t}_N^*)$ (on that open subset) and $\langle \psi_{\mathfrak{g}} \circ \mathcal{E}, \lambda \rangle = \mu \circ \eta^\lambda = \mathbf{L}^\top \circ \mu^\lambda$.

Example 3. Let us revisit Examples 1.

(i) The codimension one case is particularly straightforward as $\mathrm{Gr}_1(\mathfrak{t}_N^*)$ is just the projectivization of $\mathfrak{t}_N^*$ and the grassmannian momentum map $\mathcal{E} \colon N \to \mathrm{P}(\mathfrak{t}_N^*)$ is just the projectivization of the momentum map μ on $U_{\mathcal{D}}$, i.e., $\mathcal{E}_{p(\alpha)} = [\mu(\alpha)]$. In particular, if $N = \mathbb{S}^{2m+1} = W^\times/\mathbb{R}^+$ and $\mathfrak{t}_N$ is the Lie algebra of the diagonal $(m + 1)$-torus with respect to a basis of W, then $\mathrm{im}\,\mu$ is the union of the standard quadrant in $\mathfrak{t}_N^*$ and its negation. Hence $\mathrm{im}\,\mathcal{E}$ is the projectivization of this quadrant, the standard simplex.

(ii) Let $N = \mathbb{S}^{2m_1+1} \times \cdots \times \mathbb{S}^{2m_\ell+1}$ be a product of ℓ contact spheres $\mathbb{S}^{2m_i+1} = W_i^\times/\mathbb{R}^+$, with the product local contact action of an abelian Lie algebra $\mathfrak{t}_N = \bigoplus_{i=1}^\ell \mathfrak{t}_i$, where $\mathfrak{t}_i$ has dimension $m_i + 1$ and acts diagonally on W_i as in (i). Then, via the natural embedding of $\prod_{i=1}^\ell \mathrm{P}(\mathfrak{t}_i^*)$ into $\mathrm{Gr}_\ell(\mathfrak{t}_N^*)$, sending $(U_1, \ldots U_\ell)$ to $U_1 \oplus \cdots \oplus U_\ell$, the image of $\mathcal{E}$ is a product of simplices in $\mathrm{Gr}_\ell(\mathfrak{t}_N^*)$.

Proposition 3. *Let* $\mathbf{K}\colon \mathfrak{t}_N \to \mathfrak{con}(N,\mathcal{D})$ *be an abelian local contact action. If* $\mathbf{K}$ *is locally Reeb type, the lift* $\tilde{\mathbf{K}}$ *of* $\mathbf{K}$ *to* $\mathcal{D}^0 \to N$ *is transverse to the fibres, i.e., if* $K_{v,z} = 0$ *(for* $v \in \mathfrak{t}_N, z \in N$*) then* $\tilde{K}_v$ *is identically zero along* $\mathcal{D}_z^0 = p^{-1}(z)$.

Proof. Since $(p, \mu)\colon \mathcal{D}^0 \to N \times \mathfrak{t}_N^*$ is a monomorphism of vector bundles, it is an immersion. However, for any $v \in \mathfrak{t}_N$, $d\mu(\tilde{K}_v) = 0$ by equivariance of μ, so if $K_{v,z} = p_*(\tilde{K}_v)_z = 0$, then $d(p, \mu)(\tilde{K}_v) = 0$ along $p^{-1}(z)$, hence $\tilde{K}_v = 0$ along $p^{-1}(z)$. $\qquad\square$

Hence for any $\alpha \in \mathcal{D}^0$, the infinitesimal stabilizer $\mathfrak{stab}_{\mathfrak{t}_N}(\alpha) = \mathfrak{stab}_{\mathfrak{t}_N}(p(\alpha))$. Since $\ker \kappa \leq \ker q_{\mathcal{D}} \circ \kappa = \Theta$, for any $z \in N$, $\mathfrak{stab}_{\mathfrak{t}_N}(z) \leq \Theta_z$ and hence $\mathcal{E}_z \leq \mathfrak{stab}_{\mathfrak{t}_N}(z)^0$.

2.2. **Symplectic slices for torus actions.** Let $(N, \mathcal{D})$ be a contact manifold of rank m and codimension ℓ, and let $\mathbb{T}_N = \mathfrak{t}_N/2\pi\Lambda$ be a (real) torus with (abelian) Lie algebra $\mathfrak{t}_N = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$, where Λ is the lattice of circle subgroups of $\mathbb{T}_N$.

Definition 8. A *contact torus action* of $\mathbb{T}_N$ on N is a local contact action $\mathbf{K}\colon \mathfrak{t}_N \to \mathfrak{con}(N, \mathcal{D})$ which integrates to an effective (i.e., faithful) action of $\mathbb{T}_N$.

We recall the construction of a symplectic slice along an $\mathbb{T}_N$-orbit $\mathbb{T}_N \cdot \alpha$ for $\alpha \in U_{\mathcal{D}}$. The isotropy representation of $\mathrm{Stab}_{\mathbb{T}_N}(\alpha)$ on $T_\alpha U_{\mathcal{D}}$ induces an effective linear action, symplectic with respect to $\omega = \Omega_\alpha^{\mathcal{D}}$, of any subtorus $H \leq \mathrm{Stab}_{\mathbb{T}_N}(\alpha)$. Since $\mathbb{T}_N \cdot \alpha$ is isotropic with respect to $\Omega^{\mathcal{D}}$, the isotropy representation has an H-invariant filtration

$$(6) \qquad 0 \leq T_\alpha(\mathbb{T}_N \cdot \alpha) \leq T_\alpha(\mathbb{T}_N \cdot \alpha)^\perp \leq T_\alpha U_{\mathcal{D}},$$

where $\perp$ denotes the orthogonal space with respect to $\omega = \Omega_\alpha^{\mathcal{D}}$.

Remark 4. There is also [2, 10, 24] an H-invariant ω-compatible complex structure on $T_\alpha U_{\mathcal{D}}$ and a decomposition $(T_\alpha U_{\mathcal{D}}, \omega) = \bigoplus_{i=0}^k (W_i, \omega_i)$ into H-invariant ω-orthogonal symplectic subspaces, so the induced representation of H (or its Lie algebra $\mathfrak{h}$) on W_i has weight $\beta_i \in \mathfrak{h}^*$ for $i \in \{0, \ldots k\}$. We assume $\beta_0 = 0$ and $\beta_i \neq 0$ for $i > 0$, so $W_0 \leq T_\alpha U_{\mathcal{D}}$ is the tangent space to the fixed point set of H at α. Since the H-action is effective, the weights $\beta_1, \ldots \beta_k$ span the *weight lattice* of $\mathfrak{h}^*$, dual to the lattice $\Lambda \cap \mathfrak{h}$. Note that $T_\alpha(\mathbb{T}_N \cdot \alpha) \leq W_0$ and $T_\alpha(\mathbb{T}_N \cdot \alpha)^\perp + W_0 = T_\alpha U_{\mathcal{D}}$.

When H is the identity component H_α of $\mathrm{Stab}_{\mathbb{T}_N}(\alpha)$ and $\mathfrak{h}_\alpha := \mathfrak{stab}_{\mathfrak{t}_N}(\alpha)$,

$$T_\alpha(\mathbb{T}_N \cdot \alpha) \cong \mathfrak{t}_N/\mathfrak{h}_\alpha \quad \text{and} \quad T_\alpha U_{\mathcal{D}}/T_\alpha(\mathbb{T}_N \cdot \alpha)^\perp \cong (\mathfrak{t}_N/\mathfrak{h}_\alpha)^* \cong \mathfrak{h}_\alpha^0 \leq \mathfrak{t}_N^*.$$

Definition 9. The middle composition factor in (6),

$$W^\alpha := T_\alpha(\mathbb{T}_N \cdot \alpha)^\perp/T_\alpha(\mathbb{T}_N \cdot \alpha),$$

is called the *symplectic isotropy representation*. By Remark 4, $W^\alpha \cong W_0^\alpha \oplus \bigoplus_{i=1}^k W_i$, where $W_0^\alpha = (T_\alpha(\mathbb{T}_N \cdot \alpha)^\perp \cap W_0)/T_\alpha(\mathbb{T}_N \cdot \alpha)$ and W_i are the nonzero weight spaces of H_α (with weights $\beta_i \in \mathfrak{h}_\alpha^*$). The momentum map $\mu_W\colon W^\alpha \to \mathfrak{h}_\alpha^*$ of the H_α-action is

$$(7) \qquad \mu_W(w_0, w_1, \ldots w_k) = \sum_{i=1}^k |w_i|^2 \beta_i$$

for all $w_0 \in W_0^\alpha$ and $(w_1, \ldots w_k) \in \bigoplus_{i=1}^k W_i$. The image of μ_W is thus the convex cone $\mathcal{C}^\alpha$ in $\mathfrak{h}_\alpha^*$ generated by the nonzero weights $\beta_1, \ldots \beta_k$ of W^α.

To construct a model for a neighbourhood of $\mathbb{T}_N \cdot \alpha$, observe that $T^*\mathbb{T}_N \times W^\alpha \cong \mathbb{T}_N \times \mathfrak{t}_N^* \times W^\alpha$ is a symplectic manifold with commuting hamiltonian actions of $\mathbb{T}_N$ (the right action on $T^*\mathbb{T}_N$) and $\mathrm{Stab}_{\mathbb{T}_N}(\alpha)$ (the diagonal left action), where the latter has momentum map $(g, \xi, v) \mapsto \mu_W(v) - \xi|_{\mathfrak{h}_\alpha}$. By the orbit-stabilizer theorem $\mathbb{T}_N$ is a principal $\mathrm{Stab}_{\mathbb{T}_N}(\alpha)$-bundle over $\mathbb{T}_N \cdot \alpha \cong \mathbb{T}_N/\mathrm{Stab}_{\mathbb{T}_N}(\alpha)$, and if we choose a splitting $\chi \colon \mathfrak{t}_N \to \mathfrak{h}_\alpha$, we may identify the symplectic quotient of $T^*\mathbb{T}_N \times W^\alpha$ by $\mathrm{Stab}_{\mathbb{T}_N}(\alpha)$ at the zero momentum level $(\xi|_{\mathfrak{h}_\alpha} = \mu_W(v))$ with the associated vector bundle

$$(8) \qquad \mathcal{Y} := \mathbb{T}_N \times_{\mathrm{Stab}_{\mathbb{T}_N}(\alpha)} (W^\alpha \oplus \mathfrak{h}_\alpha^0)$$

over $\mathbb{T}_N/\mathrm{Stab}_{\mathbb{T}_N}(\alpha)$. The induced action of $\mathbb{T}_N$ is hamiltonian with momentum map

$$(9) \qquad \mu_\chi([g, v + \xi]) = \mu(\alpha) + \chi^\top(\mu_W(v)) + \xi,$$

where $g \in \mathbb{T}_N$, $v \in W^\alpha$ and $\xi \in \mathfrak{h}_\alpha^0$. The Symplectic Slice Theorem [3, 19, 21, 22, 24] now asserts the following.

Lemma 3. *For any splitting χ, there is a $\mathbb{T}_N$-invariant neighbourhood $\mathcal{U}$ of the orbit $\mathbb{T}_N \cdot \alpha \subseteq U_\mathcal{D}$ and a $\mathbb{T}_N$-equivariant symplectomorphism Ψ_χ from $(\mathcal{U}, \Omega)$ to a neighbourhood of the zero section of $(\mathcal{Y}, \Omega_\chi)$ such that $\mu|_\mathcal{U} = \mu_\chi \circ \Psi_\chi$.*
Thus $\mu - \mu(\alpha)$ maps $\mathcal{U}$ to a neighbourhood of $0 \in \mathfrak{h}_\alpha^0 \oplus \chi^\top(\mathcal{C}^\alpha)$.

2.3. **Orbit stratification and convexity.** The theory of effective proper abelian group actions [3, 14] implies that for $H \leq \mathbb{T}_N$,

$$N_{(H)} = \{z \in N \mid H = \mathrm{Stab}_{\mathbb{T}_N}(z)\} \leq N^H = \{z \in N \mid H \leq \mathrm{Stab}_{\mathbb{T}_N}(z)\}$$

is an open submanifold of a closed submanifold of N, and if $N_{(H)}$ is nonempty (for which H must be a closed subgroup) then it is dense in the fixed point set N^H of H.

Definition 10. The connected components of $N_{(H)}$, for $H \leq \mathbb{T}_N$, and their closures (which, if nonempty, are connected components of N^H) are called the *open* and *closed orbit strata* of $(N, \mathbf{K})$. (Thus the open orbit strata partition N.) The *combinatorics* Φ_N of $(N, \mathbf{K})$ is the poset of closed orbit strata, partially ordered by inclusion.

Proposition 4. *Let $\mathbf{K}$ be a contact torus action of $\mathbb{T}_N$ on a contact manifold $(N, \mathcal{D})$ of codimension ℓ with locally Reeb type, let $i \colon N' \to N$ be the inclusion of a closed orbit stratum $N' \in \Phi_N$, let $\mathcal{D}' = TN' \cap \mathcal{D}$, and let H be the kernel of the induced $\mathbb{T}_N$-action on N'. Then $(N', \mathcal{D}')$ is a contact manifold of codimension ℓ, with $U_{\mathcal{D}'} \cong i^*U_\mathcal{D}$, and $\mathbf{K}$ induces a contact torus action of $\mathbb{T}_N/H$ on N' which is locally Reeb type.*

Proof. The local Reeb type condition means that $\mathcal{D} \cap \mathcal{K}$ has codimension ℓ in $\mathcal{K}$ (the tangent distribution to the $\mathbb{T}_N$ orbits). Since $i^*\mathcal{K} \leq TN'$, $\mathcal{D}'$ has codimension ℓ in TN', and $i_*^\top \colon i^*T^*N \to T^*N'$ restricts to an isomorphism $i^*\mathcal{D}^0 \to (\mathcal{D}')^0$, which identifies $U_{\mathcal{D}'}$ with $i^*U_\mathcal{D}$ (since the induced map $(\mathcal{D}')^0 \to \mathcal{D}^0$ pulls the tautological 1-form $\tau^\mathcal{D}$ back to the corresponding $\tau^{\mathcal{D}'}$). Hence $(N', \mathcal{D}')$ is contact and the rest is immediate. $\qquad \square$

Note that $H = \mathrm{Stab}_{\mathbb{T}_N}(z)$ for any z in the open orbit stratum corresponding to N'.

If a contact torus action $\mathbf{K}$ of $\mathbb{T}_N$ on N has locally Reeb type, Proposition 3 implies that the hamiltonian $\mathbb{T}_N$-action on $U_\mathcal{D}$ is transverse to the fibres of $p \colon \mathcal{D}^0 \to N$. Thus for any subtorus $H \leq \mathbb{T}_N$, the fixed point set of H in $U_\mathcal{D}$ is $p^{-1}(N^H)$.

Recall that the *critical submanifolds* of a smooth function $f\colon N \to \mathbb{R}$ are the connected components of the zero-set of df; then f is called a *Morse–Bott function* if along any critical submanifold, its transverse hessian is nondegenerate.

Lemma 4. *Suppose $(N, \mathcal{D}, \mathbf{K})$ has Reeb type. Then for any nondegenerate Levi pair $(\mathfrak{g}, \lambda)$ and any $v \in \mathfrak{t}_N$, $f := \eta^\lambda(K_v)$ is a Morse–Bott function on N whose critical submanifolds all have even index.*

Proof cf. [2]. Since $d(\eta(K_v)) = -(d\eta)(K_v, \cdot)$ and $d\eta^\lambda$ has kernel $\mathcal{K}^{\mathfrak{g}}$, a point $z \in N$ is critical for f iff $K_{v,z} \in \mathcal{K}^{\mathfrak{g}}$, in which case $d(\eta(K_v))_z = 0$. This holds iff $\exists\, w \in \mathfrak{g}$ such that K_{v-w} vanishes at z, and since $\eta^\lambda(K_{v-w}) = f - \langle w, \lambda \rangle$ has the same critical submanifolds as f, we may assume $w = 0$. Let Z be the critical submanifold of f containing z; since $\eta(K_v)_z = 0$ and $d(\eta(K_v))|_Z = 0$, it follows that K_v vanishes along Z and hence Z is a component of the fixed point set N^H of the subtorus H of $\mathbb{T}_N$ generated by $\exp(K_v)$. By Remark 4, with $\alpha = \eta_z^\lambda$, the tangent space $T_\alpha U_{\mathcal{D}}$ decomposes into symplectic weight spaces and the zero weight space is the tangent space to $p^{-1}(Z)$. Hence the normal bundle to Z in N is isomorphic to the sum of the nonzero weight spaces. The transverse hessian of f at z is the hermitian form $(w_1, \ldots w_k) \mapsto \sum_i \beta_i(v)|w_i|^2$, which is nondegenerate of even index because K_v generates H. $\qquad\square$

Proposition 5. *Let $(N, \mathcal{D})$ be a compact connected contact manifold of rank m and codimension ℓ with a contact action of a torus $\mathbb{T}_N$ with Lie algebra $\mathfrak{t}_N$. Assume that the action is Reeb type with Levi pair $(\mathfrak{g}, \lambda)$.*

For any $v_1, \ldots v_k \in \mathfrak{t}_N$, the map $f\colon N \to \mathbb{R}^k$ with components $f_i = \eta^\lambda(K_{v_i})$ satisfies

  (A) *all fibres $f^{-1}(x)$ are empty or connected;*
  (B) *the image $f(N)$ is convex.*

Further, if $Z_j : j \in \mathcal{I}$ are the connected components of the set of common critical points of f_i, then $f(Z_j) = \{p_j\}$ is a single point and $f(N)$ is the convex hull of $\{p_j \mid j \in \mathcal{I}\}$.

The proof is essentially identical to the proof of the symplectic convexity theorem by M. Atiyah [2] (see also [15]): the key ingredient is Lemma 4, which makes essential use of Condition 1 (the transversality property of the subalgebra $\mathfrak{g}$). Without this assumption, convexity may fail even in codimension one [20], although in that case, the conditions needed for convexity are well understood [9, 21, 22, 23]. A similar convexity result for transverse symplectic foliations has been obtained recently by Ishida [17].

3. Toric contact manifolds

3.1. **The quotient manifold with corners.** If $(N, \mathcal{D})$ is contact of rank m, then $\operatorname{rank} \mathcal{K} \cap \mathcal{D} \le m$, and hence $\dim \mathfrak{t}_N \le m + \ell$. On any open set where $\operatorname{rank} \mathcal{K} = m + \ell$, $(N, \mathcal{D}, \mathbf{K})$ is locally Reeb type.

Definition 11. A compact contact manifold $(N, \mathcal{D})$ of rank m and codimension ℓ with a contact torus action $\mathbf{K}$ of $\mathbb{T}_N$ is *toric* if it is locally Reeb type and $\dim \mathbb{T}_N = m + \ell$.

When $(N, \mathcal{D}, \mathbf{K})$ is toric, $U_{\mathcal{D}}$ is a (noncompact) toric symplectic manifold under the lifted hamiltonian $\mathbb{T}_N$-action $\tilde{\mathbf{K}}$ of Observation 1.

Lemma 5. *Let $(N, \mathcal{D}, \mathbf{K})$ be a toric contact manifold of codimension ℓ. Then its closed orbit strata are toric contact of codimension ℓ, and for any $z \in N$, its stabilizer $H := \operatorname{Stab}_{\mathbb{T}_N}(z)$ is connected (i.e., a subtorus). For any $\alpha \in p^{-1}(z)$, $H = \operatorname{Stab}_{\mathbb{T}_N}(\alpha)$ and the normal bundle in N to the fixed point set N^H is isomorphic to the symplectic*

isotropy representation of H, which is a direct sum of 2-dimensional symplectic subrepresentations whose weights form a basis for the weight lattice of $\mathfrak{stab}_{\mathfrak{t}_N}(z)^*$.

Proof cf. [10, 24]. Let m be the rank of N, $z \in N$, $\alpha \in p^{-1}(z)$ and $\dim \mathfrak{stab}_{\mathfrak{t}_N}(z) = k$. Thus $\mathbb{T}_N/H$ has dimension $m + \ell - k$, $U_{\mathcal{D}}$ has dimension $2(m + \ell)$ and the symplectic isotropy representation W^α has dimension $2(m + \ell) - 2(m + \ell - k) = 2k$. The nonzero weights span $\mathfrak{stab}_{\mathfrak{t}_N}(\alpha)^*$, so they are linearly independent with 2-dimensional weight spaces, and $W_0^\alpha = 0$. Hence the open orbit stratum through z has dimension $2(m-k)+\ell$ and its closure is toric by Proposition 4. Since maximal tori in $\mathrm{Sp}(2k)$ are maximal closed abelian subgroups, H is connected, and is therefore equal to $\mathrm{Stab}_{\mathbb{T}_N}(\alpha)$. $\qquad\square$

Thus inverse images under p of orbit strata in N are orbit strata in $U_{\mathcal{D}}$, and Lemma 3 specializes as follows.

Proposition 6. *For any $\alpha \in U_{\mathcal{D}}$, there is a $\mathbb{T}_N$-invariant neighbourhood $\mathcal{U}$ of the orbit $\mathbb{T}_N \cdot \alpha \subseteq U_{\mathcal{D}}$ on which image of the momentum map $\mu|_{\mathcal{U}} - \mu(\alpha)$ is a neighbourhood of 0 in the product of a quadrant with $\mathfrak{stab}_{\mathfrak{t}_N}(\alpha)^0$ in $\mathfrak{t}_N^*$, it is a submersion over the interior of this cone, and the fibres (in $\mathcal{U}$) are $\mathbb{T}_N$-orbits.*

Karshon and Lerman [19] observe that these charts make $U_{\mathcal{D}}/\mathbb{T}_N$ into a manifold with corners. Recall (see e.g. [18]) that an m-dimensional *manifold with corners* is a Hausdorff topological space Q equipped with a subsheaf of the continuous functions which is locally isomorphic (as a space with a sheaf of rings) to a quadrant in a (finite dimensional, real) vector space, hence also to $[0, \infty)^m \subseteq \mathbb{R}^m$. Here, a function f on a subset B of vector space is smooth iff every $v \in B$ has an open neighbourhood U such that $f|_{B \cap U}$ is the restriction of a smooth function on U. Any $x \in Q$ has a neighbourhood isomorphic to a neighbourhood of the origin in the product of a quadrant and a vector space, i.e., to $[0, \infty)^k \times \mathbb{R}^{m-k} \subseteq \mathbb{R}^m$ for a well-defined $k \in \mathbb{N}$ called the *depth* of x. The subspace Q_k of all points with depth k is a smooth manifold of dimension $m - k$. The closures in Q of the connected components of Q_k are called the $(m - k)$-dimensional faces of Q. The *combinatorics* Φ_Q of Q is the poset of all faces, ordered by inclusion. Faces of depth 1 are called *facets*.

In our situation, the momentum map μ also maps the fibre $p^{-1}(y)$, for $y \in N$, to the linear subspace $\mathcal{E}_y \leq \mathfrak{stab}_{\mathfrak{t}_N}(y)^0 \leq \mathfrak{t}_N^*$. Since $\mathrm{Stab}_{\mathfrak{t}_N}(y)$ fixes this fibre pointwise, Proposition 6 implies that $\mu(\mathcal{U})$ is foliated by its intersection with the linear subspaces $\mathcal{E}_{p(\beta)} : \beta \in \mathcal{U}$. On the other hand $\mu(\mathcal{U})$ is an open neighbourhood of $z = \mu(\alpha)$, and shrinking $\mathcal{U}$ if necessary, we may assume that the foliation of $\mu(\mathcal{U})$ is regular, so that the leaf space is isomorphic to an open neighbourhood of zero in the product of a quadrant with $\mathfrak{stab}_{\mathfrak{t}_N}(\alpha)^0/\mathcal{E}_z$.

Corollary 1. *The orbit space $N/\mathbb{T}_N$ has the structure of a manifold with corners such that the quotient map induces a poset isomorphism from Φ_N to $\Phi_{N/\mathbb{T}_N}$.*

Remark 5. We shall also need to know that the symplectic slices in Lemma 3 may be chosen compatible with the foliation of $U_{\mathcal{D}}$ over M. For this recall that the Symplectic Slice Theorem proceeds by constructing a differentiable slice to $\mathbb{T}_N \cdot \alpha$ in $U_{\mathcal{D}}$ and then using the Equivariant Relative Darboux Theorem to standardize the symplectic form on $\mathcal{Y} = \mathbb{T}_N \times_{\mathrm{Stab}_{\mathbb{T}_N}(\alpha)} (W^\alpha \oplus \mathfrak{h}_\alpha^0)$. Now although the $\mathbb{T}_N$ orbit of the fibre over $z \in N$ is $\mu_\chi^{-1}(\mathcal{E}_z)$, with $\mathcal{E}_z \leq \mathfrak{h}_\alpha^0$, there is no reason to suppose a priori that the foliation of this orbit over M is compatible with the fibration over $\mathbb{T}_N \cdot \alpha \cong \mathbb{T}_N/\mathrm{Stab}_{\mathbb{T}_N}(\alpha)$.

To establish this, we use the transitive action of equivariant symplectomorphisms on angular coordinates. Since we have not found a local proof of this in the literature,

we sketch it here, using arguments from [26]. First, we recall from Lemma 5 that W^α is a direct sum of 2-dimensional weight spaces $W_1, \ldots W_k$ where $0 \leq k = \dim \mathfrak{h}_\alpha \leq m$. Compatible complex structures on each W_j yield $\mathbb{T}_N$-invariant angular coordinates $\theta_1, \ldots \theta_k$ on $\mathcal{Y}^0 = \{[g, w_1 + \cdots w_k + \xi] \in \mathcal{Y} \mid w_i \neq 0\}$. The level surfaces of $\theta_1, \ldots \theta_k$ form a $\mathbb{T}_N$-invariant coisotropic foliation of $\mathcal{Y}^0$. Extending by angular coordinates on $\mathbb{T}_N \cdot \alpha$ (pulled back to $\mathcal{Y}^0$, we obtain an $\mathbb{T}_N$-invariant lagrangian foliation $\mathcal{F}_0$ of $\mathcal{Y}^0$, singular along the special orbits $w_i = 0$. The fibres of $U_\mathcal{D} \to M$ need not lie in these leaves, but they do lie in the coisotropic foliation. Thus the coisotropic foliation contains an $\mathbb{T}_N$-invariant lagrangian foliation $\mathcal{F}_1$ containing the fibres of $U_\mathcal{D}$ and tangent to $\mathcal{F}_0$ along the singular orbits. On a neighbourhood of the zero section of $\mathcal{Y}$, there is therefore an $\mathbb{T}_N$-equivariant diffeomorphism, equal to the identity along the zero section, pulling back $\mathcal{F}_1$ to $\mathcal{F}_0$. The pullback of Ω_χ is thus a $\mathbb{T}_N$-invariant symplectic form Ω_1 agreeing with Ω_χ along the zero section, and $\mathcal{F}_0$ is lagrangian for both Ω_1 and Ω_χ.

We now apply the Equivariant Moser Lemma, using the observation that for equivariant differential forms on the total space of an equivariant vector bundle, the homotopy operator can be defined by

$$I(\beta) = \int_0^1 \sigma_t^*(\iota_{X_t}\beta)dt,$$

where σ_t denotes fibrewise scalar-multiplication by t and X_t is its derivative (the radial vector field). Now, following [26], we set $\beta = \Omega_1 - \Omega_\chi$ and $\Omega_t = \Omega_\chi + t\beta$, noting that Ω_t is symplectic for $t \in [0, 1]$ and that $\beta = d\alpha$ with $\alpha = I(\beta)$, so we may define Y_t by $\alpha = -\iota_{Y_t}\Omega_t$; the local flow ϕ_t then satisfies that $\phi_t^*\Omega_t$ is independent of t, so that $\phi_1^*\Omega_1 = \phi_0^*\Omega_0 = \Omega_\chi$. However for any ξ tangent to $\mathcal{F}_0$,

$$\Omega_t(Y_t, \xi) = -\alpha(\xi) = -\int_0^1 \sigma_t^*(\Omega_1 - \Omega_\chi)(X_t, \xi) = 0$$

since X_t is radial, hence also tangent to $\mathcal{F}_0$, and $\mathcal{F}_0$ is isotropic with respect to both Ω_1 and Ω_χ. Indeed it is *maximal* isotropic, and so it follows that Y_t is tangent to $\mathcal{F}_0$ for all t. Hence ϕ_t preserves $\mathcal{F}_0$. We now have a symplectic slice for $U_\mathcal{D}$ along $\mathbb{T}_N \cdot \alpha$ in which the fibre over z is contained in $\mathcal{F}_0 \cap \mu_\chi^{-1}(\mathcal{E}_z)$, hence in a fibre of $\mathcal{Y}$ over $\mathbb{T}_N \cdot \alpha$.

3.2. **The toric symplectic cone and Levi pairs.** We combine the local theory of the previous section with Proposition 5 to establish global properties of toric contact manifolds of Reeb type. We use the setting of Strategem 1 in which pairs $(\mathfrak{g}, \lambda)$ are given by linear maps $\mathbf{L} \colon \mathfrak{t}_N \to \mathfrak{h}$ with $d \circ \mathbf{L} = \mathbf{u} \colon \mathfrak{t}_N \to \mathfrak{k}$. Recall that a convex polytope Δ in the m-dimensional affine space $\mathcal{A} = (\varepsilon^\top)^{-1}(1) \subseteq \mathfrak{h}^*$ is a subset of the form

$$\Delta := \{\xi \in \mathcal{A} \mid \forall s \in \mathcal{S}, \ L_s(\xi) \geq 0\}$$

where $\mathcal{S}$ is a finite set, and $L_s \in \mathfrak{h}$ (an affine function on $\mathcal{A}$) for each $s \in \mathcal{S}$.

Definition 12. Given $\Delta \subseteq \mathcal{A}$ as above, and $\xi \in \mathcal{A}$, let $S_\xi = \{s \in \mathcal{S} \mid L_s(\xi) = 0\}$ and $\Phi_\Delta = \{S_\xi \subseteq \mathcal{S} \mid \xi \in \Delta\}$ with the induced partial ordering. We assume $\varnothing \in \Phi_\Delta$ (so $\Delta \subseteq \mathcal{A}$ has nonempty interior) and all singletons $\{s\} : s \in \mathcal{S}$ belong to Φ_Δ (otherwise we may discard s without changing Δ). The poset Φ_Δ is then called the *combinatorics* of Δ; it is isomorphic to the poset of closed faces of F of Δ via the map sending $S \in \Phi_\Delta$ to $F_S = \{\xi \in \Delta \mid S \subseteq S_\xi\} = \{\xi \in \Delta \mid \forall s \in S, \ L_s(\xi) = 0\}$—in particular $S_\varnothing = \Delta$, and $s \in \mathcal{S}$ may be identified with the facet (codimension one face) $F_s := F_{\{s\}}$. Any closed face is the intersection of the facets containing it: $F_S = \bigcap_{s \in S} F_s$. We say Δ is *simple* if every vertex is m-valent (or equivalently Φ_Δ is a simplicial set: if $S' \subseteq S \in \Phi_\Delta$ then $S' \in \Phi_\Delta$). In this case Δ is a manifold with corners.

Recall also from Definition 10 that the combinatorics of N is the poset Φ_N of closed orbit strata of N. The analogue of a facet is a closed orbit stratum stabilized by a circle, so we take $\mathcal{S}$ to be (in bijection with) this subset of Φ_N, i.e., for each $s \in \mathcal{S}$, $N_s \in \Phi_N$ is a connected component of N^{H_s} where $H_s = \{\exp(te_s) : t \in \mathbb{R}\} \le \mathbb{T}_N$ is a circle with primitive generators $\pm e_s \in \Lambda \subseteq \mathfrak{t}_N$.

Definition 13. Let $(N, \mathcal{D}, \mathbf{K})$ be a connected toric contact manifold. For any nondegenerate Levi pair $(\mathfrak{g}, \lambda)$, we denote by $U^{(\mathfrak{g}, \lambda)}$ the connected component of $U_\mathcal{D}$ containing $\eta^\lambda(N)$. We say $U \subseteq U_\mathcal{D}$ is a *Reeb component* of $U_\mathcal{D}$ if $U = U^{(\mathfrak{g}, \lambda)}$ for some nondegenerate Levi pair $(\mathfrak{g}, \lambda)$.

Clearly a Reeb component exists if and only if the contact action has Reeb type.

Theorem 1. *Suppose $(N, \mathcal{D}, \mathbf{K})$ is a (compact, connected) toric contact manifold, and let $N_s : s \in \mathcal{S}$ index the closed orbit strata stabilized by a circle.*

(i) *For any Reeb component U of $U_\mathcal{D}$, the signs of the primitive generators e_s may be chosen uniquely so that for all $s \in \mathcal{S}$, $\langle \mu, e_s \rangle \ge 0$ on U. Furthermore, for any $\alpha \in U$, the e_s with $\langle \mu(\alpha), e_s \rangle = 0$ are linearly independent elements of $\mathcal{E}^0_{p(\alpha)} \le \mathfrak{t}_N$.*

(ii) *If $U = U^{(\mathfrak{g}, \lambda)}$ for a nondegenerate Levi pair $(\mathfrak{g}, \lambda)$, then the image of the horizontal momentum map $\mu^\lambda \colon N \to \mathcal{A} \subseteq \mathfrak{h}^*$ is the compact simple convex polytope $\Delta = \Delta_{\mathfrak{g}, \lambda}$ in $\mathcal{A}$ defined by the affine functions $L_s := \mathbf{L}(e_s)$. Furthermore the fibres of μ^λ are $\mathbb{T}_N$-orbits, and if $\mu^\lambda(z)$ is in the interior of a face F then the annihilator of the tangent space to F at z is $\mathbf{u}(\mathfrak{stab}_{\mathfrak{t}_N}(z)) \le \mathfrak{k} \cong T^*_{\mu^\lambda(z)}\mathcal{A}$.*

In particular, μ^λ induces a poset isomorphism of Φ_N with Φ_Δ, sending N_s to F_s for all $s \in \mathcal{S}$ (so both Φ_N and Φ_Δ are simplicial sets), and the restriction of μ^λ to an orbit stratum is a submersion over the interior of the corresponding face.

Proof. First note that by Lemma 5, the orbit strata have connected stabilizers, so $N_s : s \in \mathcal{S}$ are the maximal proper strata in Φ_N; furthermore $\langle \mu(N_s), e_s \rangle = 0$.

Proposition 6 shows that the connected components of the fibres of $\mu \colon U_\mathcal{D} \to \mathfrak{t}^*_N$ are $\mathbb{T}_N$ orbits, and if $\alpha \in N_s$ with stabilizer $<\exp(te_s)>$, then μ maps a neighbourhood of $\mathbb{T}_N \cdot \alpha$ to a half space bounded by $\langle \mu, e_s \rangle = 0$.

Given a nondegenerate Levi pair $(\mathfrak{g}, \lambda)$, Proposition 5 implies, by choosing a basis for a complement to the image of $\ker \lambda$ in $\mathfrak{t}_N$, that the image Δ of μ^λ is the convex hull of the points $\mu^\lambda(z)$ where $\mathcal{K}^{\mathfrak{g}}_z = \mathcal{K}_z$, and that the fibres of μ^λ are connected. Since $\mathbf{L}^\top \circ \mu^\lambda = \mu \circ \eta^\lambda$ is the restriction of μ to $\eta^\lambda(N) = \mu_{\mathfrak{g}}^{-1}(\lambda)$, the fibres of μ^λ are $\mathbb{T}_N$-orbits and we may choose the signs of e_s so that $L_s := \mathbf{L}(e_s) \ge 0$ on Δ.

Since μ maps the fibres of $U_\mathcal{D} \subseteq \mathcal{D}^0 \to N$ linearly onto $\mathfrak{t}^*_N$ (sending $\mathcal{D}^0_z$ to $\mathcal{E}_z \le \mathfrak{t}^*_N$) these signs ensure $\langle \mu, e_s \rangle \ge 0$ on the Reeb component containing $\eta^\lambda(N)$, proving (i).

The explicit local description of Proposition 6 now shows that Δ is the compact convex polytope on which $L_s \ge 0$ and that this polytope is simple with the stated face and submersive properties, proving (ii). The last part now follows. $\square$

We may assume, if necessary, that the primitive generators $e_s : s \in \mathcal{S}$ span $\mathfrak{t}_N$: otherwise, there is a nontrivial subtorus H of $\mathbb{T}_N$ acting freely on N and transversely to $\mathcal{D}$. Thus N is a principal H-bundle over a toric contact manifold $(N/H, \mathcal{D}/H)$. Conversely, any principal torus bundle with connection over a toric contact manifold is toric contact as well. There is no reason to suppose that $e_s : s \in \mathcal{S}$ are linearly dependent, or even distinct, but the case that they form a basis is of particular interest.

Example 4. Theorem 1 associates to every Reeb component of a toric contact manifold, a simple convex polytope

$$\Delta := \{\xi \in \mathcal{A} = (\varepsilon^\top)^{-1}(1) \subseteq \mathfrak{h}^* \mid \forall s \in \mathcal{S}, \ L_s(\xi) \geq 0\}$$

which is *labelled* in the sense that the affine functions $L_s \in \mathfrak{h}$, parametrized by the set $\mathcal{S}$ of facets of Δ, are given, whereas in general they are only determined by Δ up to rescaling by a positive real number. Let $\mathbb{Z}_\mathcal{S}$ be the free abelian group generated by $\mathcal{S}$, let $\mathfrak{t}_\mathcal{S} = \mathbb{Z}_\mathcal{S} \otimes_\mathbb{Z} \mathbb{R}$ and $\mathbb{C}_\mathcal{S} = \mathbb{Z}_\mathcal{S} \otimes_\mathbb{Z} \mathbb{C}$ be the corresponding free vector spaces over $\mathbb{R}$ and $\mathbb{C}$, and let $\mathbb{T}_\mathcal{S} = \mathfrak{t}_\mathcal{S}/2\pi\mathbb{Z}_\mathcal{S}$. Denote the generators of $\mathbb{Z}_\mathcal{S} \subseteq \mathfrak{t}_\mathcal{S} \subseteq \mathbb{C}_\mathcal{S}$ by $e_s : s \in \mathcal{S}$. The affine functions L_s are encoded by the induced linear map $\mathbf{L} \colon \mathfrak{t}_\mathcal{S} \to \mathfrak{h}$ with $\mathbf{L}(e_s) = L_s$.

Conversely, given a simple labelled polytope $(\Delta, \mathbf{L})$, we now exhibit many toric contact manifolds with a Reeb component inducing these data. For this, we first observe that $\mathbb{T}_\mathcal{S}$ and $\mathbb{T}_\mathcal{S}^\mathbb{C} = \mathbb{C}_\mathcal{S}/2\pi\mathbb{Z}_\mathcal{S} \cong \mathbb{C}_\mathcal{S}^\times$ act diagonally on $\mathbb{C}_\mathcal{S}$, via $[\sum_s t_s e_s] \cdot (\sum_s z_s e_s) = \sum_s \exp(it_s) z_s e_s$. The action of $\mathbb{T}_\mathcal{S}$ on $\mathbb{C}_\mathcal{S}$ is hamiltonian with respect to the standard symplectic form $\omega_\mathcal{S}$ on $\mathcal{S}$ and has a momentum map $\boldsymbol{\sigma} \colon \mathbb{C}_\mathcal{S} \to \mathfrak{t}_\mathcal{S}^*$ defined by

$$(10) \qquad \langle \boldsymbol{\sigma}(z), e_s \rangle = \sigma_s(z) = \tfrac{1}{2}|z_s|^2,$$

where $z_s \colon \mathbb{C}_\mathcal{S} \to \mathbb{C}$ denote the standard (linear) complex coordinates on $\mathbb{C}_\mathcal{S}$. We define $\mathfrak{g} = \ker(d \circ \mathbf{L} \colon \mathfrak{t}_\mathcal{S} \to \mathfrak{k})$ and let $\iota_\mathfrak{g} \colon \mathfrak{g} \hookrightarrow \mathfrak{t}_\mathcal{S}$ denote the inclusion. Since $\mathbf{L} \circ \iota_\mathfrak{g}$ takes values in $\ker d$ there is a unique $\lambda \in \mathfrak{g}^*$ with $\mathbf{L} \circ \iota_\mathfrak{g} = \varepsilon \circ \lambda$. We now define

$$N := (F \circ \boldsymbol{\sigma})^{-1}(0) \quad \text{with} \quad F = \iota_\mathfrak{g}^\top - \lambda \colon \mathfrak{t}_\mathcal{S}^* \to \mathfrak{g}^*.$$

Since Δ is simple, λ is a regular value of $\iota_\mathfrak{g}^\top \circ \boldsymbol{\sigma}$ which is a momentum map for the action of $\mathfrak{g}$. Thus we get that $\mathcal{K}^\mathfrak{g}$ has full rank everywhere on N. Denote by $\tau_\mathcal{S}$, a primitive of $\omega_\mathcal{S}$ such that $\tau_\mathcal{S}(K_w) = \langle \boldsymbol{\sigma}, w \rangle$ for every $w \in \mathfrak{t}_\mathcal{S}$ (for e.g. take $\tau_\mathcal{S} = \sum_{s \in \mathcal{S}} d^c \boldsymbol{\sigma}_s$). By definition, $N = \{z \in \mathbb{C}_\mathcal{S} \mid \langle \boldsymbol{\sigma}_z, w \rangle = \langle \lambda, w \rangle \forall w \in \mathfrak{g}\}$. Hence, on N, for any $w \in \mathfrak{g}$ we have $\tau_\mathcal{S}(K_w) = \langle \lambda, w \rangle$ and $\mathcal{K}^{\ker \lambda} \subseteq \ker \tau_\mathcal{S}$. Observe that the convexity of Δ implies that $\mathfrak{g} \cap (\mathbb{R}_{>0})_\mathcal{S}$ is not empty. In particular, there exists $w \in \mathfrak{g}$ such that $\tau_\mathcal{S}(K_w) = \sum_{s \in \mathcal{S}} w_s |z_s|^2 \neq 0$ which ensures that the rank of $(\ker \tau_\mathcal{S}) \cap TN$ is $2m + \ell - 1$. Any complementary bundle $\mathcal{D}$ of $\mathcal{K}^{\ker \lambda}$ in $(\ker \tau_\mathcal{S}) \cap TN$ is contact and admits $(\mathfrak{g}, \lambda)$ as a Levi pair. Indeed, for any local sections $X, Y \in \Gamma(\mathcal{D})$, where $\mathcal{D} \subseteq \ker \tau_\mathcal{S}$ we have

$$\omega(X, Y) = d\tau_\mathcal{S}(X, Y) = -\tau_\mathcal{S}([X, Y]),$$

whereas ω is nondegenerate on $TN/\mathcal{K}^\mathfrak{g}$ (by symplectic reduction) and thus on $\mathcal{D}$.

If $\mathcal{D}$ is $\mathbb{T}_\mathcal{S}$-invariant we get a contact toric manifold $(N, \mathcal{D}, \mathbf{K})$ together with a Levi pair $(\mathfrak{g}, \lambda)$ for which the pull back of $\tau_\mathcal{S}$ is the contact form η^λ. Recall that the momentum map μ of $\mathbb{T}_\mathcal{S}$ at $\alpha \in \mathcal{D}^0$ is defined by $\langle \mu_\alpha, v \rangle = \alpha(K_v)$ for all $v \in \mathfrak{t}_\mathcal{S}^*$. Thus $\langle \mathbf{L}^\top \circ \mu^\lambda, v \rangle = \langle \mu \circ \tau_\mathcal{S}, v \rangle = \sum_{s \in \mathcal{S}} v_s |z_s|^2$ and $\mathbf{L}^\top(\operatorname{im} \mu^\lambda) = (\iota_\mathfrak{g}^\top)^{-1}(\lambda) \cap \operatorname{im} \boldsymbol{\sigma}$ coincides with $\mathbf{L}^\top(\Delta)$. Such contact distributions exist—for instance $\mathcal{D} = (\ker \tau_\mathcal{S}) \cap \left(\bigcap_{w \in \ker \lambda} \ker(d^c \langle \boldsymbol{\sigma}, w \rangle) \right)$ works.

In this example, there is a close connection with the Delzant–Lerman–Tolman construction of toric symplectic orbifolds. More generally, we have the following.

Remark 6. If $(\mathfrak{g}, \lambda)$ be a nondegenerate Levi pair for $(N, \mathcal{D}, \mathbf{K})$ then there is a subtorus $G \leq \mathbb{T}_N$ with Lie algebra $\mathfrak{g}$ if and only if the lattice of circle subgroups in $\mathfrak{t}_N$ is mapped, via $\mathbf{u}$, to a lattice Λ in $\mathfrak{k}$ containing $\operatorname{span}_\mathbb{Z}\{u_s = \mathbf{u}(e_s) \mid s \in \mathcal{S}\}$. In this case, the quotient $(N/G, d\eta^\lambda|_\mathcal{D}, \mathbb{T}_N/G)$ is a toric symplectic orbifold, and μ^λ descends to a momentum map on N/G. If $[z] \in N/G$ such that $\mu^\lambda(z) \in F_S$ for some $S \subseteq \mathcal{S}$ in Φ_Δ, then the stabilizer of z in $\mathbb{T}_N = \mathfrak{t}_N/2\pi\Lambda$ is connected and $\mathfrak{stab}_{\mathfrak{t}_N}(z) = \operatorname{span}_\mathbb{R}\{e_s \mid s \in S\}$. An element $[v] = [\sum_{s \in S} t_s e_s]$ of the stabilizer of z in $\mathbb{T}_N$ is in G iff $\sum_{s \in S} t_s u_s \in \Lambda$, and is zero iff

$t_s \in \mathbb{Z}$ for all $s \in S$. Hence the orbifold structure group of $[z]$ (the stabilizer of z in G) is isomorphic to $(\Lambda \cap \mathrm{span}_{\mathbb{R}}\{u_s \mid s \in S\})/\mathrm{span}_{\mathbb{Z}}\{u_s \mid s \in S\}$.

The Delzant–Lerman–Tolman correspondence then associates the symplectic toric orbifold N/G with $(\Delta, \mathbf{L})$. However, if $\mathfrak{g}$ is not the Lie algebra of a closed subtorus of $\mathbb{T}_N$, the image of Λ, the lattice of circle subgroups in $\mathfrak{t}_N$, is not sent to a lattice in $\mathfrak{k}$ via $\mathbf{u}$ and there is no preferred lattice in $\mathfrak{k}$. In particular, there is no reason for the polytope Δ to be rational with respect to any lattice.

3.3. **The grassmannian image.** Let $(N, \mathcal{D}, \mathbf{K})$ be a (compact, connected) codimension ℓ toric contact manifold of Reeb type, and let $N/\mathbb{T}_N$. Corollary 1 and Theorem 1 imply that $N/\mathbb{T}_N$ is a compact connected manifold with corners of dimension m, diffeomorphic to a simple polytope Δ. However, Example 4 shows that even the *labelled* polytope $(\Delta, \mathbf{L})$ is insufficient data to recover $(N, \mathcal{D}, \mathbf{K})$.

To remedy this we consider instead the grassmannian momentum map

$$\mathcal{E} \colon N \to \mathrm{Gr}_\ell(\mathfrak{t}_N^*)$$
$$z \mapsto \mathcal{E}_z = \mu(\mathcal{D}_z^0)$$

introduced in Definition 7: recall that if $(N, \mathcal{D}, \mathbf{K})$ is locally Reeb type, then μ maps the fibres of $\mathcal{D}^0$ to ℓ-dimensional linear subspaces of $\mathfrak{t}_N^*$. If $(\mathfrak{g}, \lambda)$ is a nondegenerate Levi pair, then by Remark 3, $\mathcal{E}$ takes values in $\mathrm{Gr}_\ell(\mathfrak{t}_N^*)$ and $\langle \psi_{\mathfrak{g}} \circ \mathcal{E}, \lambda \rangle = \mathbf{L}^\top \circ \mu^\lambda$. Since the latter is an orbit map and $\mathcal{E}$ is $\mathbb{T}_N$-invariant, $\mathcal{E}$ induces an embedding of $N/\mathbb{T}_N$ into $\mathrm{Gr}_\ell(\mathfrak{t}_N^*)$. This is the *grassmannian image* that we will use to characterize $(N, \mathcal{D}, \mathbf{K})$.

Definition 14. Let $\mathfrak{t}$ be the Lie algebra of a compact $(m + \ell)$-torus $\mathbb{T} = \mathfrak{t}/2\pi\Lambda$. We say that a manifold with corners $\Xi \subseteq \mathrm{Gr}_\ell(\mathfrak{t}^*)$, with facets $\Xi_s : s \in \mathcal{S}$, is

- *polyhedral*, if for all $s \in \mathcal{S}$, there is a 1-dimensional subspace $E_s \leq \mathfrak{t}$ such that the facet Ξ_s lies in the subgrassmannian $\mathrm{Gr}_\ell(E_s^0) \subseteq \mathrm{Gr}_\ell(\mathfrak{t}^*)$;
- *labelled* by $e = (e_s)_{s \in \mathcal{S}}$ if for all $s \in \mathcal{S}$, $e_s \in \mathfrak{t} \setminus 0$ and $\Xi_s \subseteq \mathrm{Gr}_\ell(e_s^0)$.

A labelling e of Ξ determines a (possibly empty) cone

$$(11) \qquad \mathcal{C}_e := \{x \in \mathfrak{t}^* \mid \forall\, s \in \mathcal{S},\ \langle x, e_s \rangle \geq 0 \text{ and } S_x \in \Phi_\Xi\},$$

where $S_x = \{s \in \mathcal{S} \mid \langle x, e_s \rangle = 0\}$. A labelled polyhedral manifold with corners (Ξ, e) is

- *rational* if $e_s \in \Lambda$ for all $s \in \mathcal{S}$;
- *Delzant* if for each $S \in \Phi_\Xi$, $(e_s)_{s \in S}$ is a $\mathbb{Z}$-basis for $\Lambda \cap \mathrm{span}_{\mathbb{R}}\{e_s \mid s \in S\}$.
- *of Reeb type* if there is an ℓ-dimensional subspace $\iota_{\mathfrak{g}} \colon \mathfrak{g} \hookrightarrow \mathfrak{t}$ and $\lambda \in \mathfrak{g}^*$ such that $\Xi \subseteq \mathrm{Gr}_\ell^{\mathfrak{g}}(\mathfrak{t}^*)$ and $\langle \psi_{\mathfrak{g}}, \lambda \rangle \colon \mathrm{Gr}_\ell^{\mathfrak{g}}(\mathfrak{t}^*) \to \mathfrak{t}^*$ maps Ξ bijectively to the convex polytope

$$(12) \qquad \Delta_{\mathfrak{g}, \lambda} := (\iota_{\mathfrak{g}}^\top)^{-1}(\lambda) \cap \mathcal{C}_e.$$

Note that the Delzant and Reeb type conditions for $(\mathfrak{g}, \lambda)$ determine the labelling e.

Corollary 2. *The grassmannian image Ξ of a compact toric contact manifold of Reeb type $(N, \mathcal{D}, \mathbf{K})$ is polyhedral in $\mathrm{Gr}_\ell(\mathfrak{t}_N^*)$, and any Reeb component $U \subseteq U_{\mathcal{D}}$ induces a labelling e such that (Ξ, e) is Delzant of Reeb type.*

We would like to show that the grassmannian image classifies toric contact manifolds of Reeb type, following a well-known line of argument [19, 24] in the symplectic and codimension one case, which builds on [16]. However, in higher codimension, local invariants obstruct a straightforward description of $\mathfrak{con}(N, \mathcal{D})$, and so the method yields

less explicit conclusions. Let $\mathrm{Con}_0^{\mathbb{T}}(\mathcal{D})$ be the sheaf over $N/\mathbb{T}_N$ of $\mathbb{T}_N$-equivariant contactomorphisms of N preserving the orbits (i.e., inducing the identity on $N/\mathbb{T}_N$) and let $\mathfrak{con}^{\mathbb{T}}(\mathcal{D})$ be the sheaf over $N/\mathbb{T}_N$ of $\mathbb{T}_N$-invariant contact vector fields on $(N, \mathcal{D})$.

Note that if a contact vector field X is $\mathbb{T}_N$-invariant, then so is the hamiltonian f_X for its lift $\tilde{X}$ to $U_{\mathcal{D}}$, i.e., $\Omega^{\mathcal{D}}(\tilde{X}, \tilde{K}_v) = -df_X(\tilde{K}_v) = 0$ for any $v \in \mathfrak{t}_N$. Since the latter span a lagrangian subspace on a dense open set, $\tilde{X}_\alpha = \tilde{K}_{\tilde{v}(\alpha),\alpha}$ for some $\mathfrak{t}_N$-valued function of the form $\tilde{v} = p^*v$ (since $\tilde{X}$ is $\mathfrak{t}_N$-invariant). Hence for any $z \in N$, $X_z = K_{v(z),z}$ is tangent to the $\mathbb{T}_N$-orbit through z.

Lemma 6. *There is an exact sequence of sheaves over $N/\mathbb{T}_N$:*

(13) $$0 \to \underline{2\pi\Lambda} \to \mathfrak{con}^{\mathbb{T}}(\mathcal{D}) \to \mathrm{Con}_0^{\mathbb{T}}(\mathcal{D}) \to 0,$$

where the first nontrivial map is the inclusion of the locally constant sheaf associated to $2\pi\Lambda \subseteq \mathfrak{t}_N$ into $\mathfrak{con}^{\mathbb{T}}(\mathcal{D})$ and the second is the time 1 flow of a contact vector field.

Proof. Exactness at the first two steps is straightforward, following [19, 24]; surjectivity of the second map is less so. However, given a contactomorphism ϕ in $\mathrm{Con}_0^{\mathbb{T}}(U, \mathcal{D})$ over a $\mathbb{T}_N$-invariant open subset U of N, the lift $\tilde{\phi}$ to $p^{-1}(U)$ preserves $\tau^{\mathcal{D}}$, hence the momentum map μ, hence the $\mathbb{T}_N$-orbits. Thus [16, Theorem 3.1] shows that for U sufficiently small, $\tilde{\phi}(\alpha) = \exp(\tilde{v}(\alpha)) \cdot \alpha$ for a smooth $\mathbb{T}_N$-invariant function $\tilde{v} \colon p^{-1}(U) \to \mathfrak{t}_N$. Since $p(\tilde{\phi}(\alpha)) = \phi(p(\alpha))$, $\exp(\tilde{v}(\alpha)) \cdot p(\alpha) = \phi(p(\alpha))$ and so we may assume $\tilde{v} = p^*(v)$ for a smooth $\mathbb{T}_N$-invariant function $v \colon U \to \mathfrak{t}_N$. Thus $\tilde{\phi} = \psi_1$, where ψ_t is the flow of the projectable vector field $\tilde{X}_\alpha = \tilde{K}_{v(p(\alpha)),\alpha}$ (this flow exists because $\tilde{X}$ is tangent to the compact $\mathbb{T}_N$ orbits).

We now show that $\tilde{X}$ is the lift to $p^{-1}(U)$ of a infinitesimal contactomorphism. First note that since the $\mathbb{T}_N$-orbits are isotropic, $\Omega^{\mathcal{D}}(\tilde{X}, K_w) = 0$ for all $w \in \mathfrak{t}_N$ and hence $\mathcal{L}_{\tilde{X}}\tau^{\mathcal{D}} = \iota_{\tilde{X}}\Omega^{\mathcal{D}} + d\iota_{\tilde{X}}\tau^{\mathcal{D}}$ is a basic 1-form with respect to the $\mathbb{T}_N$ action. Since $\psi_1 = \tilde{\psi}$ and $\psi_0 = Id$ preserve $\tau^{\mathcal{D}}$,

$$0 = \psi_1^*\tau^{\mathcal{D}} - \psi_0^*\tau^{\mathcal{D}} = \int_0^1 \frac{d}{dt}\psi_t^*\tau^{\mathcal{D}}\, dt = \int_0^1 \psi_t^*(\mathcal{L}_{\tilde{X}}\tau^{\mathcal{D}})\, dt = \int_0^1 \mathcal{L}_{\tilde{X}}\tau^{\mathcal{D}}\, dt = \mathcal{L}_{\tilde{X}}\tau^{\mathcal{D}},$$

where in the penultimate step, we have used that ψ_t induces the identity on the $p^{-1}(U)/\mathbb{T}_N$, and so preserves basic 1-forms. $\square$

Identifying $N/\mathbb{T}_N$ with Ξ via the grassmannian momentum map, we may view (13) as an exact sequence of sheaves on Ξ.

Proposition 7. *Let $(N, \mathcal{D}, \mathbf{K})$ be a toric contact manifold of Reeb type with grassmannian image Ξ. Then toric contact manifolds of Reeb type with the same grassmannian image are parametrized up to isomorphism by $H^1(\Xi, \mathfrak{con}^{\mathbb{T}}(\mathcal{D}))$.*

Proof. Suppose N' has the same grassmannian image. This induces a diffeomorphism between $N/\mathbb{T}$ and $N'/\mathbb{T}$. It follows from Remark 5 that N and N' are locally isomorphic by a fibre preserving contactomorphism. Thus a standard argument [16, 19, 24] shows that N' determines and is determined up to isomorphism by an element of $H^1(\Xi, \mathrm{Con}_0^{\mathbb{T}}(\mathcal{D}))$. However, since N has Reeb type, Ξ is diffeomorphic to a simple convex polytope, and hence is contractible, so $H^i(\Xi, \underline{2\pi\Lambda}) = 0$ for $i \geq 1$. Thus by the long exact sequence associated to (13), $H^1(\Xi, \mathrm{Con}_0^{\mathbb{T}}(\mathcal{D})) \cong H^1(\Xi, \mathfrak{con}^{\mathbb{T}}(\mathcal{D}))$. $\square$

In order to understand the sheaf $\mathfrak{con}^{\mathbb{T}}(\mathcal{D})$, it is convenient to introduce a transversal subalgebra $\mathfrak{g} \leq \mathfrak{t}_N$ which is the Lie algebra of a subtorus $G \leq \mathbb{T}_N$ and has $\mathcal{C}_{\mathfrak{g}} \subseteq \mathfrak{g}^*$

nonempty. Thus N is a principal G-bundle over a compact orbifold $M = N/G$, and $\mathcal{D}$ defines a $\mathbb{T}_M := \mathbb{T}_N/G$-invariant principal G-connection η on $N \to M$, cf. Example 2.

Definition 15. A *toric ℓ-symplectic manifold* (or *orbifold*) is a $2m$-manifold (or orbifold) M with an effective action of an m-torus $\mathbb{T}_M$ together with a $\mathbb{T}_M$-invariant principal G-bundle $\pi\colon N \to M$ with connection η, for G an ℓ-torus, such that for some $\lambda \in \mathfrak{g}^*$, $\langle \omega, \lambda \rangle$ is nondegenerate, where $\omega \in \Omega^2(M, \mathfrak{g})$ is the curvature of η.

In this setting, $(N, \mathcal{D} = \ker \eta)$ is a toric contact manifold (or orbifold) of dimension ℓ under an extension $\mathbb{T}_N$ of $\mathbb{T}_M$ by G, the induced action of $\mathfrak{g}$ is transversal, and $\langle \Omega, \lambda \rangle$ is nondegenerate (hence a symplectic form on M) iff $\lambda \in \mathcal{C}_{\mathfrak{g}}$, i.e., $(\mathfrak{g}, \lambda)$ is a Levi pair.

Proposition 8. *Let M be a toric ℓ-symplectic manifold with associated toric contact manifold $(N, \mathcal{D}, \mathbb{T}_N)$. For $Z \in \mathfrak{con}^{\mathbb{T}}(\mathcal{D})$, viewed as a sheaf on N, we may write $\eta(Z) = \pi^* f_Z$ and this defines an isomorphism between $\mathfrak{con}^{\mathbb{T}}(\mathcal{D})$ and the sheaf of $\mathbb{T}_M$-invariant $\mathfrak{g}$-valued functions f on M such that $df = -\iota_X \omega$ for some vector field X.*

Proof. For $Z \in \mathfrak{con}^{\mathbb{T}}(\mathcal{D})$, $\eta(Z)$ is G-invariant, hence of the form $\pi^* f_Z$ for some $\mathfrak{g}$-valued function f_Z on M, which is $\mathbb{T}_M$-invariant because $\eta(Z)$ is $\mathbb{T}_N$-invariant. However, since Z is contact $0 = \mathcal{L}_Z \eta = \iota_Z d\eta + \pi^* df_Z$, so if $f_Z = 0$ on an open subset of M, $\iota_Z d\eta = 0$ on its inverse image in N, and hence $Z = 0$ (since $d\eta$ is nondegenerate on the kernel of η). Since $\iota_Z d\eta$ is G-invariant, and vanishes on generators of the G-action, it is basic. However $d\eta = \pi^* \omega$, so Z is projectable, to a vector field X with $\iota_X \omega = -df_Z$. Conversely, for any $\mathbb{T}_M$-invariant $\mathfrak{g}$-valued functions f with $df = -\iota_X \omega$ for some vector field X, we may take $Z = X^H + \langle \pi^* f, \mathbf{K}^{\mathfrak{g}} \rangle$, where X^H is the horizontal lift of X to $\mathcal{D}$ and $\mathbf{K}^{\mathfrak{g}} \colon \mathfrak{g} \to \mathfrak{con}^{\mathbb{T}}(\mathcal{D})$ generate the infinitesimal action of $\mathfrak{g}$ on N. Then Z is $\mathbb{T}_N$-invariant with $\eta(Z) = \pi^* f$, and $\mathcal{L}_Z \eta = \iota_Z d\eta + \pi^* df_Z = \pi^*(\iota_X \omega + df) = 0$, so Z is contact. $\square$

Note that the vector field X here is ℓ-hamiltonian, in the sense that it has a hamiltonian with respect to every (nondegenerate) component of ω. When $\ell = 1$, or more generally, when the Levi form has rank one image (the ℓ-contact manifolds of [6, 11]—cf. Examples 1 (iii)), this places no condition on f, so $\mathfrak{con}^{\mathbb{T}}(\mathcal{D})$ is isomorphic to the sheaf of functions on $M/\mathbb{T}_M = \Xi$, which is a fine sheaf, so $H^1(\Xi, \mathfrak{con}^{\mathbb{T}}(\mathcal{D})) = 0$.

We also have the following generalization.

Proposition 9. *If N is a product of codimension one contact manifolds of Reeb type, then $H^1(\Xi, \mathfrak{con}^{\mathbb{T}}(\mathcal{D})) = 0$.*

Proof. We take G to be the product of transverse actions on each factor, so that $M = N/G$ is a product of symplectic manifolds $(M_i, \omega_i) : i \in \{1, \ldots \ell\}$ and $\omega = \omega_1 \oplus \cdots \oplus \omega_\ell \in \Omega^2(M, \mathbb{R}^\ell)$. Now for any vector field X on M, $-\iota_X \omega = (\alpha_1, \ldots \alpha_\ell)$ where α_i is a section of the pullback of $T^* M_i$ to M, viewed as a subbundle of $T^* M$. For this to be of the form $(df_1, \ldots df_\ell)$ for functions $f_1, \ldots f_\ell$ on M, we must have $df_i(X_j) = 0$ whenever X_j is tangent to M_j for $j \neq i$. Hence $f_1, \ldots f_\ell$ are locally pullbacks from $M_1, \ldots M_\ell$, and are otherwise constrained only by $\mathbb{T}_N$-invariance. Thus $\mathfrak{con}^{\mathbb{T}}(\mathcal{D})$ is a product of pullbacks of fine sheaves, so has $H^1(\Xi, \mathfrak{con}^{\mathbb{T}}(\mathcal{D})) = 0$. $\square$

This result applies in particular to products of spheres.

3.4. **Existence.** It remains to construct a toric contact manifold of Reeb type from a Delzant, Reeb type, labelled polyhedral manifold with corners (Ξ, e) in $\mathrm{Gr}_\ell(\mathfrak{t}^*)$.

Theorem 2. *Let $\mathfrak{t}$ be the Lie algebra of a $(\ell + m)$-dimensional torus $\mathbb{T} = \mathfrak{t}/2\pi\Lambda$ and (Ξ, e) be a Delzant, Reeb type, labelled polyhedral manifold with corners in $\mathrm{Gr}_\ell(\mathfrak{t}^*)$. There is a codimension ℓ compact toric contact manifold with a Reeb component whose associated labelled manifold with corners is (Ξ, e).*

Proof. The Reeb type condition implies there exist $\iota_\mathfrak{g} \colon \mathfrak{g} \hookrightarrow \mathfrak{t}$ and $\lambda \in \mathfrak{g}^*$ such that $\langle \psi_\mathfrak{g}, \lambda \rangle \colon \Xi \to \Delta_{\mathfrak{g}, \lambda}$ is a diffeomorphism, with $\Delta_{\mathfrak{g}, \lambda}$ defined by (12). This condition is open in λ (by compactness of Ξ, cf. Remark 2), so for λ' in a contractible open neighbourhood $\mathcal{N}$ of $\lambda \in \mathfrak{g}^*$, $\langle \psi_\mathfrak{g}, \lambda' \rangle$ is a diffeomorphism from Ξ to $\Delta_{\mathfrak{g}, \lambda}$. Thus

$$V := (\iota_\mathfrak{g}^\top)^{-1}(\mathcal{N}) \cap \mathcal{C}_e$$

(with $\mathcal{C}_e$ defined by (11)) is foliated by its intersections with $\xi \in \Xi$, i.e., it has a foliation $\mathcal{F}$ with leaf space Ξ and leaves $\mathcal{F}_\xi := V \cap \xi$ for $\xi \in \Xi$. Note that $\mathcal{F}$ is compatible with the face decomposition of V: for any $S \in \Phi_\Xi$ and any $\xi \in \Xi_S$, $\mathcal{F}_\xi \subseteq V_S := \{x \in V \mid \forall s \in S, \langle x, e_s \rangle = 0\}$. Let $\widetilde{V}$ be open in $\mathfrak{t}^*$ with $V = \widetilde{V} \cap \mathcal{C}_e$.

We now use a variant of the Delzant construction, cf. [19], to build a toric symplectic $2(m + \ell)$-manifold (M, ω) under $\mathbb{T}$. First, since $\widetilde{V} \times \mathbb{T} \hookrightarrow \mathfrak{t}^* \times \mathbb{T} \simeq T^*\mathbb{T}$, it is a toric symplectic manifold with momentum map

$$x \colon \widetilde{V} \times \mathbb{T} \to \mathfrak{t}^*$$

induced by the embedding of the first factor into $\mathfrak{t}^*$. It has $\mathbb{T}$-invariant symplectic form $\tilde{\omega} = \langle dx \wedge d\theta \rangle$, where $\theta \colon \mathbb{T} \to \mathfrak{t}/2\pi\Lambda$ are the tautological angular coordinates on $\mathbb{T}$.

Next, as in Example 4, let $\mathbb{Z}_\mathcal{S}$ be the free abelian group generated by $\mathcal{S}$, let $\mathfrak{t}_\mathcal{S} = \mathbb{Z}_\mathcal{S} \otimes_\mathbb{Z} \mathbb{R}$ and $\mathbb{C}_\mathcal{S} = \mathbb{Z}_\mathcal{S} \otimes_\mathbb{Z} \mathbb{C}$ be the corresponding free vector spaces over $\mathbb{R}$ and $\mathbb{C}$, and let $\mathbb{T}_\mathcal{S} = \mathfrak{t}_\mathcal{S}/2\pi\mathbb{Z}_\mathcal{S}$. Since (Ξ, e) is rational, the map $s \mapsto e_s \in \Lambda$ induces a group homomorphism $\mathbb{Z}_\mathcal{S} \to \Lambda$ and hence also homomorphisms $\pi \colon \mathbb{T}_\mathcal{S} \to \mathbb{T}$ and $\pi_* \colon \mathbb{R}_\mathcal{S} \to \mathfrak{t}$.

We now consider the action of $\gamma \in \mathbb{T}_\mathcal{S}$ on $\widetilde{V} \times \mathbb{T} \times \mathbb{C}_\mathcal{S}$ by

$$(14) \qquad \gamma \cdot (x, [\theta], z) = (x, \pi(\gamma) \cdot [\theta], \gamma \cdot z)$$

where $\mathbb{T}$ acts on $\mathbb{T}$ and $\mathbb{T}_\mathcal{S}$ on $\mathbb{C}_\mathcal{S}$ in the standard ways. This action is Hamiltonian for the symplectic form $\tilde{\omega} \oplus (-\omega_{\mathrm{std}})$ on $(\widetilde{V} \times \mathbb{T}) \times \mathbb{C}_\mathcal{S}$ with momentum map

$$(15) \qquad \begin{aligned} \phi \colon \widetilde{V} \times \mathbb{T} \times \mathbb{C}_\mathcal{S} &\to \mathbb{R}_\mathcal{S}^* \cong \mathbb{R}^\mathcal{S} \\ (x, [\theta], z) &\mapsto \left(l_s(x) - \tfrac{1}{2}|z_s|^2 \right)_{s \in \mathcal{S}}. \end{aligned}$$

where $l_s(x) := \langle x, e_s \rangle$ are the linear maps defining $\mathcal{C}_e$. We now show that $\mathbb{T}_\mathcal{S}$ acts freely on $\phi^{-1}(0)$. Indeed, if $x \in \widetilde{V} \backslash V$ then $\phi(x, [\theta], z) < 0$ and so

$$(16) \qquad \phi^{-1}(0) = \bigcup_{x \in V} \left(\{x\} \times \mathbb{T} \times \mathbb{T}_\mathcal{S}(x) \right)$$

where $\mathbb{T}_\mathcal{S}(x) := \prod_{s \in \mathcal{S}} \mathbb{S}^1_{l_s(x)}$ and $\mathbb{S}^1_{l_s(x)} = \left\{ z \in \mathbb{C} \mid \tfrac{1}{2}|z|^2 = l_s(x) \right\}$ is a circle of radius $\sqrt{2l_s(x)} \geq 0$. Hence if $\gamma \in \mathrm{Stab}_{\mathbb{T}_\mathcal{S}}(x, [\theta], z)$ then for all $s \in \mathcal{S}$, either $\gamma_s = 0$ or $l_s(x) = 0$. Now for $x \in V$, $S_x := \{s \in \mathcal{S} \mid l_s(x) = 0\}$ is in Φ_Ξ and so $e_s : s \in S_x$ are linearly independent. Indeed by the Delzant condition, they belong to a basis for Λ, and so if $\pi(\gamma) = 0$ then $\gamma = 0$.

Consequently, the symplectic reduction of $\widetilde{V} \times \mathbb{T} \times \mathbb{C}_\mathcal{S}$ at 0 with respect to the action (14) of $\mathbb{T}_\mathcal{S}$ is a smooth symplectic manifold (M, ω). Observe that the natural action

of $\mathbb{T}$ on the second factor of $\widetilde{V} \times \mathbb{T} \times \mathbb{C}_{\mathcal{S}}$ commutes with the action (14) of $\mathbb{T}_{\mathcal{S}}$ and descends to an effective action on M, whose momentum map $\mu \colon M \to \mathfrak{t}^*$ is determined by $\mu \circ q = x|_{\phi^{-1}(0)}$ where $q \colon \phi^{-1}(0) \to M$ is the quotient by $\mathbb{T}_{\mathcal{S}}$.

The foliation $\mathcal{F}$ of V induces a $\mathbb{T}_{\mathcal{S}}$-invariant isotropic foliation of $\phi^{-1}(0) \subseteq V \times \mathbb{T} \times \mathbb{C}_{\mathcal{S}}$, which descends to a $\mathbb{T}$-invariant isotropic foliation $\mathcal{G}$ of M. Note that $N_{\mathfrak{g},\lambda} := \mu^{-1}(\Delta_{\mathfrak{g},\lambda})$ is a global transversal submanifold, i.e., meets each leaf of $\mathcal{G}$ in a single point. Moreover, $N_{\mathfrak{g},\lambda}$ is smooth, compact (recall that μ is proper) and $\mathbb{T}$-invariant. Thus the space of leaves $N := M/\mathcal{G}$ is $\mathbb{T}$-equivariantly diffeomorphic to $N_{\mathfrak{g},\lambda}$. We denote by $f \colon M \to N$ the quotient map with $\mathcal{G}_z = f^{-1}(z)$. The momentum map $\mu \colon M \to V$ is an orbit map for $\mathbb{T}$, and it descends to a $\mathbb{T}$-invariant map $\mathcal{E} \colon N \to \Xi$, hence a rank ℓ subbundle $\mathcal{E} \le N \times \mathfrak{t}^*$ over N with fibre $\mathcal{E}_z = \mathcal{E}(z)$ such that, for $z \in N$, we have $\mu(\mathcal{G}_z) = \mathcal{F}_{\mathcal{E}_z}$.

On the other hand, the symplectic quotient construction provides action angle coordinates on $\mathring{M} := \mu^{-1}(\mathring{V})$, that is a splitting of the sequence

$$0 \to \mathring{M} \times \mathfrak{t} \xrightarrow{\kappa} T\mathring{M} \xrightarrow{d\mu} \mathring{M} \times \mathfrak{t}^* \to 0$$

which identifies

$$(17) \qquad\qquad T\mathring{M} \simeq \mathring{M} \times (\mathfrak{t} \oplus \mathfrak{t}^*)$$

such that the symplectic structure ω restricted on $\mathring{M}$ coincides with the fibrewise pairing on the left side. Now observe that via the identification (17), the pullback of $\mathcal{E}^0 \oplus \mathfrak{t}^* \le N \times (\mathfrak{t} \oplus \mathfrak{t}^*)$ on M, that is $f^*(\mathcal{E}^0 \oplus \mathfrak{t}^*) \to M$, coincides over $\mathring{M}$ with $T\mathcal{G}^{\perp_\omega}$. Hence $T\mathcal{G}^{\perp_\omega}$ is projectable on $\mathring{N} := f(\mathring{M})$. However a distribution containing $T\mathcal{G}$ is (locally) projectable if and only if it is preserved by Lie derivatives along vector fields in $T\mathcal{G}$. Hence $T\mathcal{G}^{\perp_\omega}$ is projectable on M by continuity.

We now show that $\mathcal{D} := f_*(T\mathcal{G}^{\perp_\omega}) \le TN$, a rank $2m$ distribution on N, is contact. If we denote $v = x\frac{\partial}{\partial x} \in TV$ the vector field induce from the $\mathbb{R}^*$-action of dilation, v is obviously tangent to each leaf of $\mathcal{F}$ and thus defines a vector field Y on M such that $\mathcal{L}_Y \omega = d\iota_Y \omega = \omega$. In particular, $\iota_Y \omega$ is a primitive 1-form whose kernel contains $T\mathcal{G}^{\perp_\omega}$, and by using the same argument as in Example 4, we conclude that $\mathcal{D}$ is contact. $\qquad\square$

Remark 7. If we begin the construction with (Ξ, e) rational instead of Delzant, we obtain instead a toric contact orbifold.

APPENDIX A. AFFINE GEOMETRY OF NATURAL MOMENTUM MAPS

Let (M, ω) be a connected symplectic manifold; then the *symplectic gradient* $\mathrm{grad}_\omega f$ of a smooth function f is the unique vector field X with $df = -i_X \omega$. The image of grad_ω is the Lie subalgebra $\mathfrak{ham}(M, \omega) \le C^\infty(M, TM)$ of *hamiltonian vector fields*, and there is an exact sequence

$$0 \to \mathbb{R} \to C^\infty(M, \mathbb{R}) \xrightarrow{\mathrm{grad}_\omega} \mathfrak{ham}(M, \omega) \to 0$$

of Lie algebras, where $C^\infty(M, \mathbb{R})$ (the space of smooth functions) is a Lie algebra under Poisson bracket, and $\mathbb{R}$ is included as the (central) ideal of constant functions. For any (local) action of a Lie algebra $\mathfrak{k}$ on M by hamiltonian vector fields, i.e., any Lie algebra morphism $\mathfrak{k} \to \mathfrak{ham}(M, \omega)$, this exact sequence pulls back to a central extension

$$0 \to \mathbb{R} \xrightarrow{\varepsilon} \mathfrak{h} \to \mathfrak{k} \to 0$$

of Lie algebras—if $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{ham}(M, \omega)$, then $\mathfrak{h} \le C^\infty(M, \mathbb{R})$ is the space of hamiltonian generators f of the action (i.e., with $\mathrm{grad}_\omega f \in \mathfrak{k}$). Such an extension

has an interpretation in affine geometry: dual to the inclusion $\varepsilon\colon \mathbb{R} \to \mathfrak{h}$, we have a surjection $\varepsilon^\top\colon \mathfrak{h}^* \to \mathbb{R}$ and hence an exact sequence

$$0 \to \mathfrak{k}^* \to \mathfrak{h}^* \xrightarrow{\varepsilon^\top} \mathbb{R} \to 0.$$

The inverse image of $1 \in \mathbb{R}$ under $\varepsilon^\top$ is an affine subspace $\mathcal{A} := (\varepsilon^\top)^{-1}(1)$ of $\mathfrak{h}^*$, modelled on $\mathfrak{k}^*$. The space of affine functions $f\colon \mathcal{A} \to \mathbb{R}$ is canonically isomorphic to $\mathfrak{h}$: the projection of f to $\mathfrak{k}$, viewed as a linear form on $\mathfrak{k}^*$, is its derivative df (at every point of $\mathcal{A}$), and the constant functions are $\varepsilon(c), c \in \mathbb{R}$.

Definition 16. Let $\mathfrak{k} \hookrightarrow \mathfrak{ham}(M,\omega)$ be a (local, effective) action by hamiltonian vector fields, and identify the extension $\mathfrak{h}$ of $\mathfrak{k}$ with its image in $C^\infty(M,\mathbb{R})$. The *natural momentum map* $\mu\colon M \to \mathcal{A} \subseteq \mathfrak{h}^*$ is defined by $\langle \mu(x), f \rangle = f(x)$ for $x \in M$ and $f \in \mathfrak{h}$.

A (local, effective) action by hamiltonian vector fields is called a *hamiltonian* action if the extension $\mathfrak{h} \to \mathfrak{k}$ has a (Lie algebra) splitting. However, if the splitting is not unique, it can be more convenient to work with the natural momentum map rather than its projection onto $\mathfrak{k}^*$ using a splitting. If $\mathfrak{k}$ is abelian, then the action is hamiltonian iff $\mathfrak{h}$ is also abelian, i.e., the hamiltonian generators Poisson commute.

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