STABILITY AND EXTREMAL METRICS ON TORIC AND PROJECTIVE BUNDLES

VESTISLAV APOSTOLOV, DAVID M. J. CALDERBANK, PAUL GAUDUCHON, AND CHRISTINA W. TØNNESEN-FRIEDMAN

Abstract. This survey on extremal Kähler metrics is a synthesis of several recent works—of G. Székelyhidi [39, 40], of Ross–Thomas [35, 36], and of ourselves [5, 6, 7]—but embedded in a new framework for studying extremal Kähler metrics on toric bundles following Donaldson [13] and Szekelyhidi [40]. These works build on ideas Donaldson, Tian, and many others concerning stability conditions for the existence of extremal Kähler metrics. In particular, we study notions of relative slope stability and relative uniform stability and present examples which suggest that these notions are closely related to the existence of extremal Kähler metrics.

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1. K-stability for extremal Kähler metrics

1.1. **Introduction to stability.** For a Hodge manifold (M, Ω), the existence of a CSC Kähler metric in Ω is conjectured to be equivalent to a notion of *stability* [9, 12, 13, 34, 41] for the polarized projective variety (M, L), where L is a line bundle on M with $c_1(L) = \Omega/2\pi$. This conjecture is drawn from a detailed formal picture which makes clear an analogy with the well-established relation between the polystability of vector bundles and the existence of Einstein–Hermitian connections.

At present the most promising candidate for the conjectured stability criterion is 'K-polystability', in the form given by Donaldson [13], following Tian [41]: a polarized projective variety (M, L) is K-polystable if any 'test configuration' for (M, L) has nonpositive Futaki invariant with equality iff the test configuration is a product. We shall explain this definition shortly. We also discuss an idea of Ross and Thomas [35, 36], who focus on test configurations arising as 'deformations to the normal cone' of subschemes of (M, L), leading to a notion of 'slope' K-polystability analogous to the slope polystability of vector bundles.

(Note that some authors use the term *K-stable* rather than K-polystable, but the latter term agrees better with pre-existing notions of stability.)

G. Székelyhidi [39] has extended the theory of K-polystability to cover extremal Kähler metrics, not just CSC Kähler metrics. We explain his ideas here.

1.2. **Finite dimensional motivation.** Let $(X, \mathcal{L}, \Omega)$ be a polarized Kähler manifold with a hermitian metric on $\mathcal{L}$ with curvature $-i\Omega$ (thus $c_1(\mathcal{L}) = \Omega/2\pi$). Suppose a compact connected group G acts holomorphically on X and there is a momentum map $\mu\colon X \to \mathfrak{g}^*$ for the action (i.e., $d\langle\mu, \xi\rangle = -\Omega(K_\xi, \cdot)$, where K_ξ is the vector field on X corresponding to $\xi \in \mathfrak{g}$, the Lie algebra of G). There is a lift of the action to $\mathcal{L}$ generated by $\tilde{K}_\xi + \langle\mu, \xi\rangle K$ for each $\xi \in \mathfrak{g}$, where $\langle\mu, \xi\rangle$ is pulled back to $\mathcal{L}$, $\tilde{K}_\xi$ is the horizontal lift, and K generates the standard $U(1)$ action on $\mathcal{L}$. The action of $\mathfrak{g}$ on X and $\mathcal{L}$ extends to an action of the complexification $\mathfrak{g}^c$ and we assume this integrates to an action of a complex Lie group G^c.

By a well-known result of Kempf–Ness and Kirwan, for any $x \in X$, there exists $g \in G^c$ such that $\mu(g\cdot x) = 0$ if and only if for some (hence any) nonzero lift $\tilde{x}$ of x to $\mathcal{L}^*$, the orbit $G^c \cdot \tilde{x} = 0$ is closed (and then any nonzero lift will have this property). Such points x are said to be *polystable*. If X^{ps} denotes the set of polystable points in X, we then have an equality between X^{ps}/G^c, the polystable quotient of X by G^c, and the symplectic quotient $X//G = \mu^{-1}(0)/G$.

$G^c \cdot \tilde{x}$ is closed if and only if $\alpha(\mathbb{C}^\times) \cdot \tilde{x}$ is closed for any one parameter subgroup $\alpha\colon \mathbb{C}^\times \hookrightarrow G^c$. This leads to the Hilbert–Mumford criterion for polystability: x is said to be *semistable* if for any one parameter subgroup $\alpha\colon \mathbb{C}^\times \hookrightarrow G^c$, the linear action of $\mathbb{C}^\times$ on $\mathcal{L}^*_{x_0}$ has nonpositive weight $w_{x_0}(\alpha) \leq 0$, where $x_0 = \lim_{\lambda\to 0} \alpha(\lambda)\cdot x$ is the limit point; x is then polystable if it is semistable and $w_{x_0}(\alpha) = 0$ only when $x_0 = x$; finally x is *stable* if it is polystable and has zero dimensional isotropy subgroup.

Suppose now that the Lie algebra $\mathfrak{g}$ is equipped with a G-invariant inner product $\langle,\rangle$. In this situation G. Székelyhidi [39] shows that stability conditions can detect not only the G^c orbit of $\mu^{-1}(0)$, which (assuming it is nonempty) is the set of absolute minima of $||\mu||^2$, but also the G^c orbit of the set of critical points of $||\mu||^2$. For this note that the weight w_x of the linear action of the isotropy algebra $\mathfrak{g}_x$ on $\mathcal{L}^*_x$ is given by $w_x = \langle\beta_x, \cdot\rangle\colon \mathfrak{g}_x \to \mathbb{C}^\times$ for some $\beta_x \in \mathfrak{g}_x$, which is the orthogonal projection of $\mu(x)$ onto $\mathfrak{g}_x$. We refer to β_x (or rather the induced vector field on

X) as the *extremal vector field*: for in the infinite dimensional setting it agrees with the extremal vector field of Futaki and Mabuchi up to a normalization convention.

Clearly x is a critical point of $||\mu||^2$ if and only if β_x is in $\mathfrak{g}_x$. Using this, Székelyhidi shows that x is in the G^c orbit of a critical point of $||\mu||^2$ if and only if it is polystable *relative to the extremal vector field*, i.e., for the action of the subgroup of G^c whose Lie algebra is the subspace $\beta_x^\perp$ of the centralizer of β_x. The Hilbert–Mumford criterion may then be modified as follows: the *modified weight* $w_{x_0}(\alpha) - \langle\alpha, \beta_x\rangle w_{x_0}(\beta_x)/\langle\beta_x, \beta_x\rangle$ of the limit point x_0 should be nonpositive for any one parameter subgroup α of the centralizer of β_x, with equality if and only if $x_0 = x$.

1.3. **The infinite dimensional analogue.** The finite dimensional picture described in the previous subsection will now be applied formally to an infinite dimensional setting in which X is the space of compatible complex structures on a compact symplectic manifold (M, ω) with $H^1(M) = 0$. The space X has a natural Kähler metric with respect to which the group G of symplectomorphisms of M acts holomorphically with a momentum map $\mu\colon X \to C_0^\infty(M, \mathbb{R})$ given by the scalar curvature of the corresponding Kähler metric on M, modified by a constant in order to lie in $\mathfrak{g}^* \cong \mathfrak{g} = C_0^\infty(M, \mathbb{R})$, the functions with total integral zero, which is the Lie algebra of the symplectomorphism group equipped with the L_2-inner product. A quick way to see this is to observe that the Mabuchi K-energy of M is a Kähler potential for the metric on X: the gradient on X of the Mabuchi K-energy is the scalar curvature [20].

There is no group whose Lie algebra is the complexification $\mathfrak{g}^c$, but one can still consider the foliation of X given by the vector fields induced by $\mathfrak{g}^c$. The complex structures in a given leaf are all biholomorphic by a diffeomorphism in the connected component of the identity, and pulling back the symplectic form ω by these biholomorphisms, we may identify the leaf with the set of all Kähler metrics in a fixed Kähler class, compatible with a fixed complex structure on M. Hence the problem of finding a critical point of $||\mu||^2$ in a fixed G^c-orbit reduces to the search for *extremal Kähler metrics*, since these are the critical points for the L_2-norm of the scalar curvature for metrics in a fixed Kähler class on a complex manifold (M, J). In particular if the momentum map vanishes on a given leaf, there should be a CSC Kähler metric in the corresponding Kähler class. By analogy with the finite dimensional setting, the existence of CSC or extremal Kähler metrics in a given Kähler class should be equivalent to a suitable stability condition.

In order to make precise this infinite dimensional analogue, we formalize what is meant by the orbit of a 1-parameter subgroup in terms of 'test configurations' and give a Hilbert–Mumford formulation of stability in terms of the weight of limit points. This is what we do next.

1.4. **Test configurations.** Let (M, Ω) be a Hodge manifold, viewed as a polarized projective variety with respect to a line bundle L with $c_1(L) = \Omega/2\pi$. Let G be a maximal torus in $H_0(M)$ and χ be the (Futaki–Mabuchi) extremal vector field of (M, L, G). In [39], Székelyhidi makes the following definition, following Donaldson [13].

Definition 1. A *test configuration* for (M, L, χ) is a polarized scheme $(X, \mathcal{E})$ over $\mathbb{C}$ together with

- a flat proper morphism $p\colon X \to \mathbb{C}$ such that the fibre $(X_t = p^{-1}(t), \mathcal{E}|_{X_t})$ is isomorphic to (M, L) for $t \neq 0$;

• an extension, also denoted χ, of the extremal vector field on X_t for $t \neq 0$, to all of X;

• a $\mathbb{C}^\times$ action α preserving χ and covering the $\mathbb{C}^\times$ action on $\mathbb{C}$ by scalar multiplication.

$(X_0, \mathcal{E}|_{X_0})$ is called the *central fibre*. Since $0 \in \mathbb{C}$ is fixed by the action, $(X_0, \mathcal{E}|_{X_0})$ inherits a $\mathbb{C}^\times$ action from α, also denoted by α. Similarly, χ also denotes its restriction to X_0.

A test configuration is said to be a *product configuration* if $X = M \times \mathbb{C}$ and α is given by a $\mathbb{C}^\times$ action on M (and scalar multiplication on $\mathbb{C}$).

Since relevant properties of test configurations are unchanged if we replace $\mathcal{E}$ by $\mathcal{E}^r$ for a positive integer r, we can let $\mathcal{E}$ be a $\mathbb{Q}$-line bundle in the definition above (i.e., $\mathcal{E}$ denotes a 'formal root' of a line bundle $\mathcal{E}^r$ for some positive integer r).

A particularly important class of test configurations are those associated to a subscheme of (M, L), as studied by Ross and Thomas [35, 36]. We shall state it here for complex submanifolds of (M, L), but the same definition actually makes sense for subschemes.

Definition 2 (*Deformation to the normal cone*)**.** For a polarized complex manifold (M, L), the *normal cone* of a complex submanifold Z is $\hat{M} \cup_E P$, where $\hat{M}$ is the blow-up of M along Z with exceptional divisor $E = P(\nu_Z)$, $P = P(\mathcal{O} \oplus \nu_Z)$ and ν_Z is the normal bundle to Z in M. This is a singular projective variety (for example the normal cone of a point $p \in \mathbb{C}P^1$ is $\mathbb{C}P^1 \cup_p \mathbb{C}P^1$, which is a line-pair in $\mathbb{C}P^2$).

The normal cone is the central fibre of the family $p \colon X \to \mathbb{C}$ obtained by blowing up $M \times \mathbb{C}$ along $Z \times \{0\}$ (where p is the projection of the blow-down to $\mathbb{C}$) called the *deformation to the normal cone* of Z in M. We equip this with the polarization $\mathcal{E}_c = \pi^* L \otimes \mathcal{O}(-cP)$, where $\mathcal{O}(P)$ is the line bundle associated to the exceptional divisor P, $\pi \colon X \to M$ is the projection of the blow-down to M, and c is a positive rational number such that $\mathcal{E}_c$ is an ample $\mathbb{Q}$-line bundle. This last condition gives an upper bound ε on c, called the *Seshadri constant* of Z with respect to L.

We let α be the $\mathbb{C}^\times$ action coming from the trivial action on M and multiplication on $\mathbb{C}$. This clearly defines an action on X with a lift α_c to $\mathcal{E}_c$. Let us suppose further that the (a given) extremal vector field χ vanishes on Z. Then it extends to X and so the deformation to the normal cone determines a family of test configurations, parameterized by $c \in (0, \varepsilon) \cap \mathbb{Q}$.

Remark 1. Nakagawa shows [32] that the (Futaki–Mabuchi) extremal vector field associated to a Hodge Kähler manifold (M, Ω) has closed orbits, and therefore defines an effective $\mathbb{C}^\times$ action which we will refer to as the *extremal* $\mathbb{C}^\times$ action of (M, Ω, G).

1.5. **The modified Futaki invariant.** The notion of K-stability will be defined using a Hilbert–Mumford weight of a test configuration, which will involve the Futaki invariant of the central fibre; however, since the latter is typically a singular projective variety, we need an algebraic geometric definition of the Futaki invariant. Such a definition has been given by Donaldson [13].

Let V be a scheme of dimension n over $\mathbb{C}$ (in our examples it will be be a singular projective variety) polarized by an ample line bundle L and suppose that α is a $\mathbb{C}^\times$ action on V with a lift to L. Then α acts on the vector spaces $H_k = H^0(V, L^k)$, $k \in \mathbb{Z}^+$. If $w_k(\alpha)$ denotes the weight of the highest exterior power of H_k (that is, the trace $\operatorname{Tr} A_k$ of the infinitesimal generator A_k of the action) and d_k denotes the dimension of H_k then $w_k(\alpha)$ and d_k are given by polynomials in k for sufficiently large k, of degrees at most $n+1$ and n respectively. For sufficiently large k the quotient $w_k(\alpha)/(kd_k)$ can be expanded into a power series with rational coefficients and no positive powers, and its residue at $k = 0$, i.e., the coefficient of the k^{-1} term in the resulting expansion, is a rational number which turns out to be independent of the choice of lift of α to L.

The *Futaki invariant* $\mathfrak{F}(\alpha)$ is is defined to -4 times this residue. (When V is a manifold, this definition coincides with Futaki's original definition with our normalization convention.) We note that Donaldson [13] uses the opposite sign convention.

We next define a modified Futaki invariant of a polarized scheme (V, L) (of dimension n over $\mathbb{C}$) relative to a $\mathbb{C}^\times$ action β. We first need to define an inner product between such actions.

Assume then that V has two $\mathbb{C}^\times$ actions α and β with lifts to L and infinitesimal generators A_k and B_k of the actions on H_k. Then for k sufficiently large, $\mathrm{Tr}\,(A_k B_k)$ is a polynomial of degree at most $n + 2$. The inner product $\langle \alpha, \beta \rangle$ is defined to be the coefficient of k^{n+2} of the expansion of $\mathrm{Tr}\,(A_k B_k) - w_k(\alpha) w_k(\beta)/d_k$ for large k, which is independent of the lifts of α and β to L: indeed it depends only on the trace-free parts of A_k and B_k. (When V is a manifold, this inner product coincides with Futaki–Mabuchi bilinear form [19] up to a normalization convention.)

Finally, the (*modified*) *Futaki invariant* [39] $\mathfrak{F}_\beta(\alpha)$ of α relative to β is defined to be $\mathfrak{F}_\beta(\alpha) := \mathfrak{F}(\alpha) - \langle \alpha, \beta \rangle$.

1.6. **K-stability.** The ingredients of the previous two subsections can now be put together to yield Székelyhidi's extension [39] of K-stability to the context of extremal Kähler metrics. The analogue of the Hilbert–Mumford weight will be a negative multiple of a modified Futaki invariant.

Definition 3. The Futaki invariant of a test configuration is defined to be the Futaki invariant $\mathfrak{F}_\chi(\alpha)$ of the central fibre relative to the extremal vector field χ, where α denotes the induced $\mathbb{C}^\times$ action.

A Hodge manifold (M, L, χ) is said to be K-*polystable* if the Futaki invariant of any test configuration is nonnegative, and equal to zero if and only if the test configuration is a product configuration.

Here we prefer the term K-polystable to K-stable, since it is analogous to the corresponding notion in the finite dimensional case, intermediate between stability and semistability.

As we shall see later, an example in [7] strongly suggests that K-polystability is not a strong enough notion to detect the existence of extremal Kähler metrics. Székelyhidi [40] has a stronger notion of uniform K-stability, such as the following, might address this issue.

Definition 4. A Hodge manifold (M, L, G) is said to be L_2-*uniformly* K-*polystable* if there is a constant $\lambda > 0$ such that the Futaki invariant $F_\chi(\alpha)$ of any test configuration satisfies $F_\chi(\alpha) \geq \lambda ||\mathrm{pr}_G^\perp(\alpha)||$, where $\mathrm{pr}_G^\perp$ is the L_2-projection, with respect to the Futaki–Mabuchi inner product, of α away from the subspace induced by the generators of the action of G^c (where G is the maximal torus), and $|| \cdot ||$ is the corresponding L_2-norm.

Such a uniform bound on the Futaki invariant appears already in the work of Donaldson [13] on toric surfaces, with the L_2-norm replaced by a boundary integral over the momentum polygon. Using Donaldson's work, Székelyhidi [40] shows that K-polystability is equivalent to uniform K-polystability for toric surfaces. With this case in mind, he suggests that that the L_2-norm needs to be replaced by an

$L_{m/(m-1)}$ for Kähler $2m$-manifolds, but we do not find his argument compelling. We believe that the boundary integral of Donaldson is a technical device which works in the case of toric surfaces, but does not necessarily have a wider significance.

In the context of slope K-stability, introduced by Ross and Thomas [35, 36], there is an alternative way to strengthen the notion of stability. Slope K-stability is essentially the notion of stability obtained by considering only test configurations $(X, \mathcal{E}_c)$ arising from a deformation to a normal cone of a submanifold Z. In this case, Ross and Thomas show that the Futaki invariants $\mathfrak{F}(\alpha_c)$ are rational in $c \in (0, \varepsilon) \cap \mathbb{Q}$, where ε is the Seshadri constant, and so can be extended to $c \in (0, \varepsilon)$. When the extremal vector field vanishes on Z, the same is true for the relative Futaki invariant $\mathfrak{F}_\chi(\alpha_c)$. We thus have the following analytic notion of *slope* K-polystability, extending the notion of Ross and Thomas [35, 36] to the extremal context.

Definition 5. A Hodge manifold (M, L, χ) is said to be *slope K-polystable* if for the deformation to the normal cone of any submanifold on which the extremal vector field vanishes, the Futaki invariant $\mathfrak{F}_\chi(\alpha_c)$ of the corresponding family $(X, \mathcal{E}_c)$ of test configurations is positive for $c \in (0, \varepsilon)$.

Actually, the definition in [35, 36] is more subtle, since it requires that $\mathfrak{F}_\chi(\alpha_\varepsilon) > 0$ unless ε is rational and the semi-ample configuration $(X, \mathcal{E}_\varepsilon)$ is the pullback by a contraction of a product configuration. We shall not worry about this refinement.

At any rate, the motivation of §1.2 suggests the following conjecture [39].

Conjecture 1. *Let (M, Ω, L) be a polarized Hodge manifold and G a maximal torus in $H_0(M)$. Then there is a G-invariant extremal Kähler metric in $\Omega = 2\pi c_1(L)$ if and only if (M, L) is L_2-uniformly K-polystable relative to the extremal $\mathbb{C}^\times$ action of (M, Ω, G).*

Following Ross and Thomas [36], one might hope that slope K-polystability implies uniform K-polystability. This would imply the following corollary of the above conjecture.

Conjecture 2. *If (M, L) is slope K-polystable relative to the extremal vector field, then there is an extremal Kähler metric in $2\pi c_1(L)$.*

2. Rigid toric bundles with semisimple base

2.1. **The rigid Ansatz.** Let $\pi \colon M \to S$ be a bundle of toric kählerian manifolds of the form $M = P \times_T V$, for an ℓ-torus T, a principal T-bundle P over a kählerian base S of dimension $2d$, and a toric 2ℓ-manifold V with Delzant polytope $\Delta \subset \mathfrak{t}^*$. We let Δ_i $(i = 1, \dots n)$ denote the codimension one faces of Δ and u_i the primitive inward normals (with respect to Λ).

Let $2m = 2(d + \ell)$ be the dimension of M. By choosing a connection 1-form on P with curvature $\phi \in \Omega^2(S, \mathfrak{t})$, and letting $\theta \in \Omega^1(M, \mathfrak{t})$ be the induced connection on M, it follows that M admits Kähler metrics (g, ω) of the form

$$
\begin{aligned}
g &= g_h(z) + \langle dz, (H^g)^{-1}, dz \rangle + \langle \theta, H^g, \theta \rangle, \\
\omega &= \omega_h(z) + \langle dz \wedge \theta \rangle, \qquad \omega_h = \langle \phi, z \rangle + \psi, \qquad d\theta = \phi
\end{aligned}
$$

where:

- $z \in C^\infty(M, \mathfrak{t}^*)$ is the momentum map of the T action with image Δ;

- $H^g \in C^\infty(\Delta, S^2\mathfrak{t}^*)$ is a matrix valued function which, firstly, satisfies the boundary conditions that on any codimension one face Δ_i, there is a function h_i with

$$\sum_t H^g_{st}(y)(u_i)_t = 0, \qquad \sum_t \frac{\partial H^g_{st}}{\partial z_r}(y)(u_i)_t = (u_i)_r h_i(y)_s$$

and $\langle h_i(y), u_i\rangle := \sum_s h_i(y)_s(u_i)_s = 2$ for all $y \in \Delta_i$; secondly the inverse $(H^g)^{-1} \in C^\infty(\Delta, S^2\mathfrak{t})$ of H^g is the hessian of a function U_g on Δ; thirdly, H^g induces a positive definite metric on the interior of each face F of Δ (as an element of $S^2(\mathfrak{t}/\mathfrak{t}_F)^*$, where $\mathfrak{t}_F$ is the isotropy algebra of F);
- ψ is a 2-form on S such that $\omega_h = \langle \phi, z\rangle + \psi$ is positive for $z \in \Delta$, and g_h is the associated family of Kähler metrics.

Throughout, angle brackets denote natural contractions of $\mathfrak{t}$ with $\mathfrak{t}^*$, and we omit pullbacks by z and π. In particular z itself will denote the standard $\mathfrak{t}^*$-valued coordinate on Δ, as well as its pullback to M.

U_g is called a *symplectic potential* for H^g. The remaining (boundary and positivity) conditions can be formulated in terms of U_g [2, 15] by requiring that it is smooth and strictly convex on the interior of each face F of Δ, and that in a neighbourhood in Δ of this interior face, U_g is equal to $\frac{1}{2}\sum_i(\langle u_i, z\rangle - v_i)\log(\langle u_i, z\rangle - v_i)$ plus a smooth function, where the sum is over the codimension one faces containing F and $\langle u_i, z\rangle = v_i$ on F. We let $\mathcal{S}$ denote the space of these symplectic potentials on Δ. (Note that in [1, 13, 43], the strict convexity condition on the interior of the proper faces is omitted: this condition is essential. In [2], it is realised equivalently as a condition on the determinant of $\operatorname{Hess} U_g$.)

We shall assume that the metrics $(g_h(z), \omega_h(z))$ are fixed and have constant scalar curvature for each $z \in \Delta$. (More generally, we could assume that these metrics are extremal, but this would complicate our formulae.) Note that ϕ and u_j determine the bundle M and its complex structure, while ψ and v_j determine a Kähler class Ω on M.

Remark 2. Because the generators of the T-action have constant inner products for each fixed value of z (which is equivalent to the fact that the connection on $\pi\colon M \to S$ is induced by a principal T-connection), the T-action is said to be *rigid.*

We note that for metrics of this form, we have

$$\begin{aligned} Scal_g &= Scal_h(z) - \frac{1}{p(z)}\sum_{r,s}\frac{\partial^2}{\partial z_r \partial z_s}(p(z)H^g_{rs}) \\ \mathrm{Vol}_\omega &= p(z)\mathrm{Vol}_S \wedge \langle dz \wedge \theta\rangle^{\wedge \ell} \end{aligned}$$

where Vol_S is a fixed volume form on S, $\mathrm{Vol}_{\omega_h(z)} = p(z)\mathrm{Vol}_S$ and $Scal_h(z)$ is the (constant) scalar curvature of $\omega_h(z)$ for each fixed z. It follows that integrals over M of functions of z (pullbacks from Δ) are given by integrals on Δ with respect to the volume form $p(z)\,dv$, where dv is a constant volume form on $\mathfrak{t}^*$. The normalization of dv will be largely irrelevant in the following, and it is often more convenient to tak dv to be the volume form of the lattice of circle subgroups of T.

Remark 3. If H^g is not the inverse hessian of a function U_g, then the above metric is an almost Kähler metric on M, and one can show (at least in the semisimple case below) that the above formula for $Scal_g$ actually computes the hermitian scalar curvature of g (the trace of the curvature of the Chern connection on the canonical bundle).

2.2. **The semisimple case.** Suppose that the first Chern class of P is diagonalizable with respect to a Kähler class $[\omega_S]$ on S. By this we mean that (S, ω_S) is covered by a product of Kähler manifolds (S_j, ω_j) such that $2\pi c_1(P)$ pulls back to $\sum_j [\omega_j] \otimes b_j$ for some constants $b_j \in \mathfrak{t}$.

In this case we can take the curvature $\Phi = \sum_j b_j \omega_j$, and the Ansatz becomes:

$$g = \sum_j (\langle b_j, z\rangle + c_j) g_j + \langle dz, (H^g)^{-1}, dz\rangle + \langle \theta, H^g, \theta\rangle,$$
$$\omega = \sum_j (\langle b_j, z\rangle + c_j)\omega_j + \langle dz \wedge \theta\rangle, \qquad d\theta = \sum_j b_j \omega_j,$$

Because the Kähler quotient metrics on S are simultaneously diagonalizable with constant eigenvalues, we say S is *semisimple.* In order that $\omega_h(z) = \sum_j (\langle b_j, z\rangle + c_j)\omega_j$ has constant scalar curvature for each z, we require that the metrics (g_j, ω_j) have constant scalar curvature. The constants b_j (together with the u_i) then determine the bundle M as a complex manifold, while the constants c_j (together with the v_i) determine the Kähler class Ω.

We let $\mathrm{Vol}_S = \big(\bigwedge_a \omega_j^{\wedge d_j}\big)$ be the volume of ω_S, where $2d_j$ is the dimension of S_j (so $\sum_j d_j = d = m - \ell$). Then:

$$Scal_h(z) = \sum_j \frac{Scal_j}{\langle b_j, z\rangle + c_j}, \qquad p(z) = \prod_j (\langle b_j, z\rangle + c_j)^{d_j}$$

2.3. **The isometry Lie algebra in the semisimple case.** For a compact Kähler manifold (M, g), we denote by $\mathfrak{i}_0(M, g)$ the Lie algebra of all Killing vector fields with zeros. Since M is compact this is equivalently the Lie algebra of all hamiltonian Killing vector fields.

Proposition 1. *Let g be a compatible metric on $M \xrightarrow{p} S$ and equip S and with the metric (g_S, ω_S) induced by $\prod_j \omega_j$. Let $\mathfrak{z}(K, g)$ be the centralizer in $\mathfrak{i}_0(M, g)$ of the ℓ-torus generated by $K = J\,\mathrm{grad}_g z \in C^\infty(M, TM) \otimes \mathfrak{t}^*$.*

Then the vector space $\mathfrak{z}(K, g)$ is the direct sum of a lift of $\mathfrak{i}_0(S, g_S)$ and the span of K in such a way that $p_* \colon \mathfrak{i}_0(M, g) \to \mathfrak{i}_0(S, g_S)$ is the natural projection.

Proof. Let X be a holomorphic vector field on S which is hamiltonian with respect to ω_S; then the projection X_j of X onto the distribution $\mathcal{H}_a$ (induced by TS_j on the universal cover $\prod_j S_j$ of S) is a Killing vector field with zeros, so $\iota_{X_j}\omega_S = -df_j$ for some function f_j (with integral zero). Thus $\sum_j f_j b_j$ is a family of hamiltonians for X with respect to the family of symplectic forms covered by $\sum_j b_j \omega_j$: since this is the curvature $d\theta$ of the connection on M^0, X lifts to a holomorphic vector field $\tilde{X} = X_H + \sum_j f_j \langle b_j, K\rangle$ on M^0, which is hamiltonian with potential $\sum_j (\langle b_j, z\rangle + c_j) f_j$ and commutes with the components of K. (Here X_H is the horizontal lift to M^0 with respect to θ.) $\tilde{X}$ and its potential extend to M since $M \setminus M^0$ has codimension ≥ 2 and $\tilde{X}$ has zeros.

Converse the image of any element V of $\mathfrak{z}(K, g)$ in the normal bundle to the fibres of $p \colon M \to S$ is holomorphic hence constant on the (compact) fibres by Liouville's Theorem (the normal bundle p^*TS is trivial on each fibre), so V is projectable; since V is the pullback of a Killing vector field which commutes with K, it maps to zero iff it comes from a constant multiple of K. This gives a projection to $\mathfrak{i}_0(S, g_S)$ splitting the inclusion just defined. $\square$

In particular a maximal torus G can be taken to be the product of a maximal torus in the group of hamiltonian isometries $\mathrm{Isom}_0(S, g_S)$ and the ℓ-torus generated by K.

2.4. **The extremal vector field.** If the extremal vector field lies in the ℓ-torus tangent to the fibres of M over S (as it does in the semisimple case, by Proposition 1, since it is central, and S has constant scalar curvature), then the projection of $Scal_g$ orthogonal to the Killing potentials of g takes the form:

$$\mathrm{pr}_g^{\perp} Scal_g = \langle A, z\rangle + B + Scal_g,$$

where

$$\begin{cases} \sum_s \alpha_s A_s + \alpha B + 2\beta &= 0,\\ \sum_s \alpha_{rs} A_s + \alpha_r B + 2\beta_r &= 0,\end{cases}$$

and

$$\alpha = \int_\Delta p(z)dv, \qquad \alpha_r = \int_\Delta z_r p(z)dv, \qquad \alpha_{rs} = \int_\Delta z_r z_s p(z)dv,$$

$$\beta = \frac{1}{2}\int_\Delta Scal_g p(z)dv = \int_{\partial\Delta} p(z)d\sigma + \frac{1}{2}\int_\Delta Scal_h(z)p(z)dv,$$

$$\beta_r = \frac{1}{2}\int_\Delta Scal_g z_r p(z)dv = \int_{\partial\Delta} z_r p(z)d\sigma + \frac{1}{2}\int_\Delta Scal_h(z) z_r p(z)dv.$$

Here $d\sigma$ is the $(n-1)$-form on $\partial\Delta$ with $u_i \wedge d\sigma = dv$ on the face Δ_i with normal u_i. These formulae are immediate once one applies the divergence theorem and the boundary conditions for H^g, noting that the normals are inward normals, which introduces a sign compared to the usual formulation of the divergence theorem.

It follows that the extremal vector field is $-\langle A, K\rangle$, where $K \in C^\infty(M, TM)\otimes \mathfrak{t}^*$ is the generator of the T action. Notice that this computation does not use the fact that $(H^g)^{-1}$ is the hessian of a function. It follows that the extremal vector field can be computed from any almost Kähler metric compatible with ω.

A Killing potential of integral zero for the extremal vector field is given by $-\langle A, z\rangle + \sum_s \alpha_s A_s/\alpha = -(\langle A, z\rangle + B + 2\beta/\alpha)$. The Futaki invariant of the vector field with affine Killing potential $f(z) = \langle C, z\rangle + D$ is therefore

$$\sum_{rs} C_r(\alpha_r\alpha_s - \alpha_{rs}\alpha)A_s/\alpha = \sum_r 2C_r(\beta_r\alpha - \beta\alpha_r)/\alpha.$$

Of course the modified Futaki invariant of any such vector field is zero.

2.5. **K-energy and the Futaki invariant.** If we parameterize compatible Kähler metrics g by their (relative) symplectic potentials U (satisfying suitable boundary conditions), then (under the same assumption as the previous subsection) the modified (Mabuchi–Guan–Simanca) K-energy $\mathcal{E}^\Omega$ on this space satisfies the functional equation

$$\begin{aligned}(d\mathcal{E}^\Omega)_g(\dot U) &= \int_\Delta (\mathrm{pr}_g^\perp Scal_g)\dot U(z)p(z)dv\\ &= \int_\Delta \Big(\big(\langle A, z\rangle + B + Scal_h(z)\big)p(z) - \frac{\partial^2}{\partial z_r \partial z_s}(p(z)H^g_{rs})\Big)\dot U(z)dv\\ &= 2\int_{\partial\Delta}\dot U(z)p(z)d\sigma + \int_\Delta \big(\langle A, z\rangle + B + Scal_h(z)\big)\dot U(z)p(z)dv\\ &\quad - \int_\Delta \langle H^g, \mathrm{Hess}\,\dot U(z)\rangle p(z)dv.\end{aligned}$$

This last equality reduces to Donaldson's formula [13] in the toric case (when S is a point). The relationship between the (modified) Futaki invariant and the derivative of the (modified) K-energy shows that if $f(z)$ is an affine linear function then

$$\mathcal{F}^\Omega(f) := \int_{\partial\Delta} f(z)p(z)d\sigma + \frac{1}{2}\int_\Delta \big(\langle A, z\rangle + B + Scal_h(z)\big)\, f(z)p(z)dv = 0$$

as one can easily check directly by writing $f(z) = \langle C, z\rangle + D$ as above.

Using the fact that the derivative of $\log\det V$ is $\operatorname{tr} V^{-1}dV$ we obtain the following generalization of Donaldson's formula for $\mathcal{E}^\Omega$:

$$\mathcal{E}^\Omega(U) = 2\mathcal{F}^\Omega(U) - \int_\Delta (\log\det \operatorname{Hess} U(z))p(z)dv.$$

(In case of doubt about the convergence of the integrals, one can introduce a reference potential U_c and a relative version $\mathcal{E}^\Omega_{g_c}$ of $\mathcal{E}^\Omega$, but in fact, as Donaldson shows, the convexity of U ensures that the positive part of $\log\det\operatorname{Hess} U(z)$ is integrable, hence $-\log\det\operatorname{Hess} U(z)$ has a well defined integral in $(-\infty, \infty]$.)

2.6. **Stability under small perturbations.** On a given manifold M of the type we consider in the semisimple case, finding a compatible extremal metric g is equivalent to solving the equation (for a unknown symplectic potential $U^g \in \mathcal{S}(\Delta)$ [1])

$$(1)\quad P_{c_j,b_j,Scal_j}(U^g) = \langle A, z\rangle + B + \sum_j \frac{Scal_j}{c_j + \langle b_j, z\rangle} - \frac{1}{p(z)}\sum_{r,s}\frac{\partial^2}{\partial z_r \partial z_s}\big(p(z)H^g_{rs}\big) = 0,$$

where $H^g = (\text{Hess } U^g)^{-1}$, $c_j, b_j, Scal_j$ are fixed constants, $p(z) = p_{c_j,b_j,Scal_j}(z) = \prod_j (c_j + \langle b_j, z\rangle)^{d_j}$ is trictly positive on Δ, and $A = A_{c_j,b_j,Scal_j}$, $B = B_{c_j,b_j,Scal_j}$ are determined in § 2.4.

If we leave the parameters $b_j, Scal_j$ unchanged and move c_j a little bit, solutions to the equation (1) correspond to compatible extremal Kähler metrics in nearby Kähler classes on M. Following LeBrun–Simanca [26], we will use a Banach space implicit function theorem argument to show that the existence of *compatible* extremal Kähler metrics is an open condition.

To do so, let us suppose U^{g_0} is one such solution, corresponding to a compatible extremal Kähler metric g_0 on M, with parameters $t_0 = (c^0_j, b^0_j, Scal^0_j) \in \mathbb{R}^{(\ell+2)N}$. We want to establish the existence of solutions of (1) for arbitrary t close to t_0. Note that, by the general theory of extremal metrics, for $t = t_0$ the linear system in § 2.4 has a unique solution A_{t_0}, B_{t_0}. The same is true, therefore, for any t close to t_0; moreover, the corresponding solution (A_t, B_t) depends smoothly on t.Thus, fot t close to t_0, (1) defines a smooth family of forth order quasi-linear differential operators acting on $\mathcal{S}(\Delta)$.

Proposition 2. *Let (g_0, ω_0) be a compatible extremal Kähler on M, with symplectic potential U^{g_0} and parameters $t_0 = (c^0_j, b^0_j, Scal^0_j)$. Then there exists $\varepsilon > 0$ such that for any $t \in \mathbb{R}^{(\ell+2)N}$ with $|t - t_0| < \varepsilon$ there exists a symplectic potential $U^g \in \mathcal{S}(\Delta)$ such that $P_t(U^g) = 0$.*

Proof. To avoid dealing with boundary conditions on Δ, we will reformulate our equation (1) on the closed toric 2ℓ-manifold V.

Recall that [2, 15] the space of symplectic potentials $\mathcal{S}(\Delta)$ parametrizes T-invariant Kähler metrics on V, which are compatible with a fixed symplectic form ω^V_0 (equal to the restriction of ω_0 to a fibre). In particular, the potential U^{g_0} defines a compatible Kähler metric (g^V_0, J_0, ω^V_0) on V (equal to the restriction of g_0 to a fibre).

[1] $\mathcal{S}(\Delta)$ stands for the space of symplectic potentials of compatible toric Kähler metrics on V, cf. [2, 15]

Recall also that, by a well-known result of Schwarz [38], the space of $\mathcal{C}^\infty_T(V)$ of T-invariant smooth functions on V is identified with pull-backs (via the momentum map z) of smooth functions on Δ.

In terms of these identifications, we have the following

Lemma 1. *Let (g^V, J, ω_0^V) be a T-invariant Kähler metric on V with symplectic potential U^g. Then for $t = (c_j, b_j, Scal_j)$*

$$\begin{aligned} P_t(U^g) =& \langle A_t, z\rangle + B_t + \sum_j \frac{Scal_j}{c_j + \langle b_j, z\rangle} \\ &+ Scal(g^V) - \frac{1}{p_t(z)} \sum_{r,s} \Big(\Big(\frac{\partial^2 p_t}{\partial z_r \partial z_s}\Big)(z) g^V(K_r, K_s)\Big) \\ &+ \frac{2}{p_t(z)} \sum_r \Big(\Big(\frac{\partial p_t}{\partial z_r}\Big)(z) \Delta^{g^V} z_r\Big), \end{aligned}$$

where Δ^{g^V} is the Laplacian of g^V, and $dz_r = -\iota_{K_r}\omega_0^V$.

Proof. On an open dense subset of V, the Kähler structure (g^V, ω_0^V) has the form [22, 2]

$$g^V = \sum_{r,s} \Big(G_{rs}(z) dz_r dz_s + H_{rs}(z) dt_r dt_s\Big),$$

$$\omega_0^V = \sum_r dz_r \wedge dt_r,$$

where $t_1, \cdots, t_m$ are coordinates on T, $K_1 = \frac{\partial}{\partial t_1}, \cdots, K_m = \frac{\partial}{\partial t_m}$ are some (fixed) generators of the hamiltonian ℓ-torus T, $z_1, \cdots, z_m$ are the corresponding momentum coordinates, and $G_{rs}(z) = \frac{\partial^2 U^g}{\partial z_r \partial z_s}$, $(H_{rs}(z)) = (G_{rs}(z))^{-1}$. Note that $J_0 dt_r = -\sum_k G_{rk} dz_k$, $J_0 dz_r = \sum_k H_{kr} dt_k$, and therefore

$$\Delta^{g^V}(z_r) = -\ell\Big(dd^c z_r \wedge (\omega_0^V)^{\ell-1}/(\omega_0^V)^\ell\Big) = -\sum_k \frac{\partial H_{rk}}{\partial z_k}.$$

According to [2],

$$Scal(g^V) = -\sum_{r,s} \Big(\frac{\partial^2 H_{rs}}{\partial z_r \partial z_s}\Big),$$

from where the lemma follows. ☐

The above lemma allows us to reformulate our problem as an existence result on the space $\mathcal{M}^T_\Omega \cong \{f \in \mathcal{C}^\infty_T(V) : \omega_0 + dd^c f > 0\}$ (an open set in $\mathcal{C}^\infty_T(V)$ with respect to $||\cdot||_{\mathcal{C}^2}$) of T-invariant Kähler metrics on V, which are compatible with the fixed complex structure J_0 and whose Kähler form is in the cohomology class $\Omega := [\omega_0^V]$. Indeed, if (g_f^V, ω_f^V) is such a metric with $\omega_f^V = \omega_0^V + dd^c f$, $f \in \mathcal{M}^T_\Omega$, we then want to solve

$$\begin{aligned} Q_t(f) :=& \frac{p_t(z^f)}{p_{t_0}(z^f)} \Big[\langle A_t, z^f\rangle + B_t + \sum_j \frac{Scal_j}{c_j + \langle b_j, z^f\rangle} \\ &+ Scal(g_f^V) - \frac{1}{p_t(z^f)} \sum_{r,s} \Big(\Big(\frac{\partial^2 p_t}{\partial z_r \partial z_s}\Big)(z^f) g_f^V(K_r, K_s)\Big) \\ &+ \frac{2}{p_t(z^f)} \sum_r \Big(\Big(\frac{\partial p_t}{\partial z_r}\Big)(z^f) \Delta^{g_f^V}(z_r^f)\Big)\Big] = 0, \end{aligned} \tag{2}$$

where $z^f := z - (J_0K)\cdot f$ is the momentum map of T with respect to (g_f^V, ω_f^V). The principal part of Q_t is concentrated in $Scal(g_f^V)$; it follows from [26] that it is a quasi-linear 4-th order elliptic operator acting on $\mathcal{M}_\Omega^T$. Note also that, by the equivariant Moser lemma, g_f^V is T- equivariantly isometric to a Kähler metric compatible with ω_0^V; moreover, by the Delzant theorem, $z^f(M) = \Delta$. The positive factor $\frac{p_t(z^f)}{p_{t_0}(z^f)}$ is introduced so that $Q_t(f)$ is L_2-orthogonal with respect to the measure $p_{t_0}(z^f)v_{g_f^V}$ to all affine functions of z^f (see Lemma 1 and the preceding section).

In the spirit of [26], we have

Lemma 2. *Let Π_0^T be the orthogonal L_2-projection of $\mathcal{C}_T^\infty(V)$ to the finite dimensional sub-space of (pull backs to V of) the affine functions of z, with respect to the global inner product relative to the measure $\mu_0 = p_{t_0}(z)\frac{(\omega_0^V)^\ell)}{\ell!}$. There exists a $\delta > 0$ such that if a function $f \in \mathcal{M}_\Omega^T$ with $||f||_{\mathcal{C}^1} < \delta$ satisfies $\Big(Q_t - \Pi_0^T \circ Q_t\Big)(f) = 0$, then $Q_t(f) = 0$.*

Proof. Denote by Π_f^T the orthogonal projection of $\mathcal{C}_T^\infty(V)$ to the finite dimensional sub-space $\mathcal{A}_f^T(V)$ of affine functions of z^f, with respect to the global inner product on V relative to the measure $\mu_f = p_{t_0}(z^f)\frac{(\omega_f^V)^\ell)}{\ell!}$. The orthogonal projection $\Pi_f^T : \mathcal{A}_0^T \to \mathcal{A}_f^T$ depends continuousely on f; since for $f = 0$ it is the identity, it is invertible for any $||f||_{\mathcal{C}^1} < \delta$. For such an f, therefore, $\Pi_f^T \circ \Pi_0^T Q_t(f) = 0$ if and anly of $\Pi_0^T(Q_t(f)) = 0$, from where our claim follows. □

Following [26], we introduce the space $\tilde{\mathcal{C}}_T^\infty(V)$ of T-invariant smooth functions on V, which are L_2-orthogonal to (the pull backs to V) of affine functions of z with respect to the measure $\mu_0 := p_0(z)v_{g_0^V}$. Let $\tilde{W}_T^k(V)$ be the closure of $\tilde{\mathcal{C}}_T^\infty(V)$ with respect to the Sobolev norm $||\cdot||_2^k$. For k big enough we have an embedding $\tilde{W}_T^{k+4}(V) \subset C_T^3(V)$ which allows us to extend the quasi-linear elliptic operator Q_t to a $\mathcal{C}^1$-map from a neighbourhood of $(t_0, 0) \in \mathbb{R}^{(2+\ell)N} \times \tilde{W}_T^{k+4}(V)$ into $W_T^k(V)$ with $Q_{t_0}(0) = 0$; thus $\tilde{Q}(t, f) := (t, (\mathrm{Id} - \Pi_0^T)(Q_t(f)))$ is a $\mathcal{C}^1$-map from a neighbourhood of $(t_0, 0) \in \mathbb{R}^{(2+\ell)N} \times \tilde{W}_T^{k+4}(V)$ into $\mathbb{R}^{(\ell+2)N} \times \tilde{W}_T^{k,p}(V)$ with $\tilde{Q}(t_0, 0) = (t_0, 0)$. The inverse function theorem, Lemma 2 and the standard elliptic theory imply that for Q_t to have a *smooth* solution $f_t \in \tilde{\mathcal{C}}_T^\infty(V)$ for any $|t - t_0| < \varepsilon$ is sufficient to establish the following

Lemma 3. *Let $L_0 : \tilde{\mathcal{C}}_T^\infty(V) \to \tilde{\mathcal{C}}_T^\infty(V)$ be the linearization at $0 \in \mathcal{M}_\Omega^T(V)$ of Q_{t_0}. Then $L_0 : \tilde{\mathcal{C}}_T^\infty(V) \to \tilde{\mathcal{C}}_T^\infty(V)$ is an isomorphism of Fréchet spaces.*

Proof. Let (M, g_0, J_0, ω_0) be the extremal Kähler manifold corresponding to the initial value $t = t_0$.

The momentum map $z : M \to \Delta$ allows us to lift any smooth function on Δ (or, equivalently [38], any T-invariant smooth function on V) to a T-invariant smooth function on M, constant on the level sets of z. Conversely, any smooth T-invariant function on M which is constant on the level sets of z defines a T-invariant smooth function on V. Furthermore, if G denotes the maximal torus of $\mathrm{Isom}_0(M, g_0, J_0)$ defined in § 2.3, then the lifts of T-invariant smooth functions on V will be automatically invariant under G; since our identification is a homothety with respect to the global inner product on V, relative to $\mu_0 = p_0(z)v_{g_0{}^V}$, and the global inner product on M induced by g_0, it follows that $f \in \tilde{\mathcal{C}}_T^\infty(V)$ if and only if the corresponding lift to M is in $\tilde{\mathcal{C}}_G^\infty(M)$, where $\tilde{\mathcal{C}}_G^\infty(M)$ is the space of G-invariant smooth functions on M which are L_2-orthogonal (with respect to g_0) to all Killing potentials of vectors in Lie(G) (these potentials are explicitly identified in the proof of Proposition 1 from where the claim follows easily).

In view of the above correspondence, for any $f \in \mathcal{M}_\Omega^T(V)$ we can define a Kähler metric g_f on M, with Kähler form $\omega_f = \omega + dd^c f$. According to [6], T acts in a rigid and semisimple way with respect to (g_f, ω_f) and, therefore, the expression $Q_{t_0}(f)$ is equal to the normalized scalar curvature (with respect to of the maximal torus G) of g_f (see [5] and § 2.4 above). It follows from [26, 20] that the linearisation L_0 of Q_{t_0} (at g_0), viewed as an operator acting on the sub-space of lifted functions $\hat{\mathcal{C}}_T^\infty(V)$), is equal to -2 times the Lichnerowicz operator

$$\mathbb{L}(f) := \frac{1}{2}\Delta_{g_0}^2 f + g_0(dd^c f, \rho_{g_0}) + \frac{1}{2} g_0(df, dScal_{g_0}).$$

Note that $\mathbb{L}$ is a (formally) self-adjoint 4-th order elliptic operator acting on $\mathcal{C}^\infty(M)$; standard elliptic theory gives an L_2-orthogonal splitting $\mathcal{C}^\infty(M) = \ker(\mathbb{L}) \oplus \mathrm{im}(\mathbb{L})$, where $\ker(\mathbb{L})$ is the space of all Killing potentials with respect to g_0 [26]. Since $\mathbb{L}$ is G-invariant, the latter splitting refines to $\mathcal{C}_G^\infty(M) = \ker(\mathbb{L}^G) \oplus \mathrm{im}(\mathbb{L}^G)$, where $\mathbb{L}^G$ stands for the restriction of $\mathbb{L}$ to the subspace $\mathcal{C}_G^\infty(M)$ of G-invariant smooth functions on M. Observe that , since G is a maximal torus, $\ker(\mathbb{L}^G)$ is the space of all Killing potentials of vector fields in Lie(G). LeBrun and Simanca show [26] that $\mathbb{L}^G$ is an isomorphism of $\tilde{\mathcal{C}}_G^\infty(M) = \ker(\mathbb{L}^G)^\perp$ $(= \mathrm{im}(\mathbb{L}^G))$.

Denote by $\mathbb{L}^V = -\frac{1}{2}L_0$ the restriction of $\mathbb{L}$ to the yet smaller subspace $\hat{\mathcal{C}}_T^\infty(V)$. It follows from our discussion above that $\ker(\mathbb{L}^G)$ is the space of lifted affine functions of z, while its L_2-orthogonal complement, $\ker(\mathbb{L}^G)^\perp$, is nothing but the lifted space $\hat{\tilde{\mathcal{C}}}_T^\infty(V)$ of $\tilde{\mathcal{C}}_T^\infty(V)$. Our lemma claims that $\mathbb{L}^V$ is an isomorphism of $\hat{\tilde{\mathcal{C}}}_T^\infty(V)$. The only missing piece to establish this from the facts mentioned above is the surjectivity of $\mathbb{L}^V$ on the space $\hat{\tilde{\mathcal{C}}}_T^\infty(V)$. In what follows we shall establish this.

Suppose for a contradiction that $\mathbb{L}^V : \hat{\mathcal{C}}_T^\infty(V) \to \hat{\tilde{\mathcal{C}}}_T^\infty(V)$ is not surjective. Considering the extension of $\mathbb{L}$ to a linear operator between the Sobolev spaces $W^4(M) \to L_2(M)$ (by the elliptic theory $\mathbb{L}$ is a closed operator), our assumption is equivalent to the existence of a non-zero function $u \in L_2(M)$, which is in the closure of $\hat{\tilde{\mathcal{C}}}_T^\infty(V)$, and such that

$$\int_M \mathbb{L}^V(\phi) u v_{g_0} = 0, \tag{3}$$

for any T-invariant smooth function ϕ on V. We claim that (3) implies

$$\int_M \mathbb{L}^G(\phi) u v_{g_0} = 0 \tag{4}$$

for any G-invariant smooth function on M. This would be impossible because $\mathbb{L}^G$ is surjective by [26].

Since any sequence of functions converging in $L_2(M)$ has a point-wise converging sub-sequence, we can assume that $u = u(z)$ is a L_2-function on Δ. It is enough to establish (4) on $M^0 = z^{-1}(\Delta^0)$ (which is the complement of the union of sub-manifolds of real co-dimension at least 2). Let ϕ be any G-invariant (and hence T-invariant) smooth function on M. It can be written as a function depending on z and S. Using [5, Prop. 7] and the specific form of g_0, one can easily see that on M^0

$$\mathbb{L}(\phi) = \mathbb{L}^V(\phi) + \mathbb{L}_z^S(\phi) + 2\Delta_z^S(\Delta^V(\phi)) + \sum_j R_j(z)\Delta^{S_j}(\phi) + P_j(z)\Delta^{S_j}(\Delta^V(\phi)),$$

where $\mathbb{L}^V$ and Δ^V are the Lichnerowicz and Laplacian operators of (V, g_0^V), for any fixed value of z $\mathbb{L}_z^S$ stands for the Lichnerowicz operator of S with respect to the Kähler metric $h(z) = \sum_j (c_j + \langle b_j^0, z\rangle) g_j$, Δ^{S_j} is the Laplacian of g_j, and $R_j(z), P_j(z)$ are functions of z only, depending on g_0. If we integrate this expression against $u(z)$, with respect to the volume form $v_{g_0} = p_{t_0}(z)\Big(\bigwedge_a \omega_j^{\wedge d_j}\Big) \wedge \langle dz \wedge \theta\rangle^{\wedge \ell}\Big)$, we get

$$
\begin{aligned}
&\mathrm{const} \int_{M^0} \mathbb{L}(\phi) u(z) v_{g_0} \\
&= \prod_j \int_{S_j} \Big(\int_{\Delta^0} \mathbb{L}^V(\phi) u(z) p_{t_0}(z) dz\Big) \omega_j^{\wedge d_j} + \int_{\Delta^0} u(z) p_{t_0}(z) \Big(\int_S \mathbb{L}_z^S(\phi) v_{h(z)}\Big) dz \\
&\qquad + \sum_j \Big[\int_{\prod_{k\neq j} S_k} \Big(\int_{\Delta^0} u(z) p_{t_0}(z) \big(R_j(z) \int_{S_j} \Delta^{S_j}(\phi) \omega_j^{\wedge d_j} \\
&\qquad\qquad + P_j(z) \int_{S_j} \Delta^{S_j}(\Delta^V(\phi)) \omega_j^{\wedge d_j}\big) dz\Big) \bigwedge_{k\neq j} \omega_k^{\wedge d_k}\Big].
\end{aligned}
$$

□

All the terms vanish, by using (3), and the fact that $\mathbb{L}_z^S(\phi)$ and Δ^{S_j} are orthogonal (with respect to the global inner products) to constants. This concludes the proof of the proposition. □

Corollary 1. *The existence of a compatible extremal Kähler metric is an open condition on admissible Kähler classes (parametrized by the constants c_j).*

To give another application, we observe that the operator P_t has the symmetry $P_{\lambda t} = P_t$ for any real number $\lambda \neq 0$. For instance, start with $M = \mathbb{C}P^\ell \times \Sigma = P(\mathcal{O} \otimes \mathbb{C}^{\ell+1}) \to \Sigma$, where Σ is a compact complex curve of genus $\mathbf{g} > 0$. We then have a product CSC solution g_0 corresponding to the data $t_0 = (c_0, 0 \cdots 0, 2(1-\mathbf{g})$. The perturbation result implies that for any integers $p_0, p_1, \cdots, p_\ell$ there is a solution with respect to the data $t_n = (c_0, \frac{p_0}{n}, \cdots \frac{p_\ell}{n}, 2(1-\mathbf{g}))$ if $n >> 1$. For any such n, this is a solution with respect to the data $(nc_0, p_0, \cdots p_\ell, 2n(1-\mathbf{g}))$ too, because $P_{nt_n} = P_{t_n}$. Now such a solution defines a compatible extremal Kähler metric on the manifold $M_n = P(L_0 \oplus \cdots \oplus L_\ell) \to \Sigma_n$, where Σ_n is any curve of genus $\mathbf{g}_n = 1 - n(1-\mathbf{g})$ and L_i are holomorphic line bundles over Σ_n with $\deg(L_i) = p_i$.

2.7. **Existence?** It is natural to speculate (as Donaldson does at the end of his paper [13]) that there are $S^2\mathfrak{t}^*$-valued functions H on Δ satisfying the same boundary conditions as H^g and such that the double divergence of pH is equal to

$$\big(\langle A, z\rangle + B + Scal_h(z)\big)\, p(z)$$

on Δ^0. For any such H, we have

$$\mathcal{F}^\Omega(f) = \int_\Delta \langle H, \operatorname{Hess} f\rangle(z) p(z) dv \tag{5}$$

If such an H exists, then so do many because the double divergence is underdetermined. Indeed there are locally exact adjoint complexes of linear differential operators on $\Delta \subset \mathfrak{t}^*$:

$$
\begin{aligned}
&C^\infty(\mathbb{R}) \to C^\infty(S^2\mathfrak{t}) \to C^\infty(\Lambda^2\mathfrak{t} \odot \mathfrak{t}) \to \cdots \\
&C^\infty(\mathbb{R}) \leftarrow C^\infty(S^2\mathfrak{t}^*) \leftarrow C^\infty(\Lambda^2\mathfrak{t}^* \odot \mathfrak{t}^*) \leftarrow \cdots,
\end{aligned} \tag{6}
$$

where $\Lambda^2\mathfrak{t} \odot \mathfrak{t}$ denotes the alternating-free tensors in $\Lambda^2\mathfrak{t} \otimes \mathfrak{t}$ (the kernel of the projection, alternation, to $\Lambda^3\mathfrak{t}$). The first two arrows in the top line are the hessian and the exterior derivative (of a $\mathfrak{t}$ valued 1-form). The adjoint operators in the bottom line are the double divergence and the symmetrized divergence.

Notice that if H is positive definite then $\mathcal{F}^{\Omega}(f)$ is nonnegative for convex f. It is natural to conjecture, following Donaldson [13] that the converse holds, and that if the Futaki invariant $\mathcal{F}^{\Omega}(f)$ of any analytic toric test configuration (given by piecewise linear convex f—see below) is nonnegative, with equality for product configurations (given by affine linear f), then there are positive definite H satisfying the above conditions. Now, since $\log\det$ is strictly convex on positive definite matrices, the function $\int_\Delta (\log\det H(z))p(z)dv$ is strictly convex on these potentials and therefore has at most one minimum $H = H^{\Omega}$. Such a minimum would automatically have its inverse equal to the hessian of a function U_Ω, by the adjointness of the above complexes and the fact that the derivative of $\log\det H$ is $\operatorname{tr} H^{-1}dH$. Setting $H^g = H^{\Omega}$ would then give an extremal Kähler metric in the given Kähler class.

It is natural to wonder if a (not necessarily positive definite) H^{Ω} might exist without any positivity assumptions on this invariant, generalizing the extremal polynomial when $\ell = 1$: $F^{\Omega}(z) = p(z)H^{\Omega}(z)$ would be the precise generalization.

2.8. **The homogeneous formalism and projective invariance.** The momentum map of a hamiltonian torus action is only defined up to translation, and so it is natural to consider that it takes values in an affine space. For this, suppose that $\mathfrak{t}^*$ is a codimension one subspace of a vector space W. Then the momentum map can be considered to take values in $P(W) \setminus P(\mathfrak{t}^*)$ which can be identified with any affine space parallel to $\mathfrak{t}^*$ in W. Many aspects of the geometry we have already discussed simplify in this formalism, especially with a particular choice of affine slice, say $\psi = 1$, where $\psi \in W^*$ has kernel $\mathfrak{t}^*$. Let w denote the tautological W-valued coordinate on Δ, viewed as a subset of the affine slice $\psi = 1$, and also the pullback of this function to M.

The sequences (6) of linear differential operators are affine invariant. In fact, after tensoring the first sequence by $\mathcal{O}(1)$ and the second by $\mathcal{O}(-\ell-2)$, and replacing $\mathfrak{t}$ with $T^*\Delta$ and $\mathfrak{t}^*$ with $T\Delta$, they are even projectively invariant. Powers of ψ, viewed as a section of $\mathcal{O}(1)$, can be used to apply these operators, e.g., to $p(w)H^g$.

The constants $c_j \in \mathbb{R}$ and $b_j \in \mathfrak{t}$ can be replaced by constants $a_j \in W^*$ so that the Kähler quotient metric is $\sum_j \langle a_j, w\rangle g_j$. It follows that $p(w) = \prod_a \langle a_j, w\rangle^{d_j}$ is naturally a section of $\mathcal{O}(d)$. Note also that $\mathfrak{t}$ is a quotient of W^*, and the constants b_j are the quotient classes of the a_j.

The scalar curvature and a potential for the extremal vector field are given by

$$Scal_g = \sum_j \frac{Scal_j}{\langle a_j, w\rangle} - \frac{1}{p(w)} \mathrm{Hess}^*(p(w)H^g)$$

and
$$\mathrm{pr}_g Scal_g = \langle C, w\rangle,$$
where
$$\sum_s \tilde{\alpha}_{rs} C_s + 2\tilde{\beta}_r = 0,$$
and
$$\tilde{\alpha}_{rs} = \int_\Delta w_r w_s p(w) dv,$$
$$\tilde{\beta}_r = \frac{1}{2}\int_\Delta Scal_g w_r p(w) dv = \int_{\partial\Delta} w_r p(w) d\sigma + \frac{1}{2}\int_\Delta \Big(\sum_j \frac{Scal_j}{\langle a_j, w\rangle}\Big) w_r p(w) dv.$$

Powers of ψ can be used to make these formulae projectively invariant if desired. The Futaki invariant

$$\mathcal{F}^{\Omega}(f) := \int_{\partial\Delta} f(w)p(w)d\sigma + \frac{1}{2}\int_{\Delta}\Big(\langle C, w\rangle + \sum_j \frac{Scal_j}{\langle a_j, w\rangle}\Big) f(w)p(w)dv$$

now vanishes on linear functions.

When the Delzant polytope is a simplex, it is particularly natural to take the slice $\psi(w) = w_0 + \cdots + w_\ell$ in $\mathbb{R}^{\ell+1}$ so that the simplex is the subset of $w_0 + \cdots + w_\ell = 1$ where w_i is nonnegative for all i. In this case elements of $S^2\mathfrak{t}^*$ can be identified with $(\ell+1)\times(\ell+1)$ symmetric matrices whose rows (and columns) sum to zero.

2.9. **Examples.** (i) Let M be $\mathbb{C}P^m$, fibred over a point, with the Kähler class and momentum map normalized so that Δ is the standard simplex. Then $H_{rs} = z_r\delta_{rs} - z_r z_s$.

(ii) More generally if M is a weighted projective space with a Bochner-flat metric, then, according to Abreu (the orbifolds paper) H_{rs} is an inhomogeneous cubic in z, the homogeneous cubic term having $\langle A, z\rangle$ as a factor. (This is easy to check in the orthotoric case using Vandermonde identities—see below.)

(iii) The examples of [7] on (possibly) blown-down projective line bundles fit into our framework as follows. Recall (with a minor change of notation) that the metric has the form

$$\begin{aligned} g &= \sum_a (1 + x_a z)(g_a/x_a) + (1+z)g_0 + (1-z)g_\infty + \frac{dz^2}{\Theta(z)} + \Theta(z)\theta^2,\\ \omega &= \sum_a (1 + x_a z)(\omega_a/x_a) + (1+z)\omega_0 + (1-z)\omega_\infty + dz\wedge\theta, \end{aligned}$$

where $d\theta = \sum_a \omega_a + \omega_0 - \omega_\infty$, (g_0, ω_0) and $(g_\infty, \omega_\infty)$ are Fubini–Study metrics of scalar curvature $2d_0(d_0+1)$ and $2d_\infty(d_\infty+1)$, $\Theta(z) = (1-z^2)(1+q(z)(1-z^2)/p(z))$, $p(z) = \prod_a (1 + x_a z)^{d_a}$ and $q(z)$ is a polynomial of degree at most $d-1$ satisfying an ODE. By choosing toric structures for ω_0 and ω_∞, this metric is defined a toric $\mathbb{C}P^{d_0+d_\infty+1}$-bundle. In particular, when $q = 0$ and $g_a = 0$ for all a, this is the Fubini–Study metric on $\mathbb{C}P^{d_0+d_\infty+1}$. In general, the metric on the torus fibres gets perturbed (compared to the Fubini–Study metric) by a rank one metric $q(z)((1-z^2)\theta)^2/p(z)$. Note that the range of the index a does not include $\{0,\infty\}$, so $\sum_a d_a = d$ is the complex dimension of the base of the toric projective bundle, and the polynomial p_c of [7] is $p(z)(1+z)^{d_0}(1-z)^{d_\infty}$.

If $\omega_0 = \langle dy_0 \wedge dt_0\rangle$ and $\omega_\infty = \langle dy_\infty \wedge dt_\infty\rangle$, then we can take $\theta = \theta' + \langle y_0, dt_0\rangle - \langle y_\infty, dt_\infty\rangle$ with $d\theta' = \sum_a \omega_a$. Hence

$$\omega = \sum_a (1 + x_a z)(\omega_a/x_a) + dz\wedge\theta' + \langle d((1+z)y_0)\wedge dt_0\rangle + \langle d((1-z)y_\infty)\wedge dt_\infty\rangle$$

A momentum map for the torus action is therefore given by (z, z_0, z_∞), where $z_0 = (1+z)y_0$, $z_\infty = (1-z)y_\infty$. Thus, on pulling back to the torus fibres, we can write

$$\phi := (1-z^2)\theta = (1-z^2)dt + (1-z)\langle z_0, dt_0\rangle + (1+z)\langle z_\infty, dt_\infty\rangle.$$

Hence $q(z)((1-z^2)\theta)^2$ has coefficients of degree at most $(d-1)+4 = d+3$ in the momenta.

Note that the image of (z, z_0, z_∞) is a rather nonstandard simplex. This description arises from a standard simplex in homogeneous coordinates by splitting the (nonnegative) variables into two groups $w_0, \ldots w_{d_0}$ and $x_0, \ldots x_{d_\infty}$, where

$\sum_j w_j + \sum_k x_k = 1$, and then setting $1 + z = 2\sum_j w_k$ and $1 - z = 2\sum_k x_k$. Then we can take $z_0 = (w_1, \dots w_{d_0})$ and $z_\infty = (x_1, \dots x_{d_\infty})$. After making the dual transformation of the angle variables, and introducing homogeneous coordinates, ϕ has the form:

$$\phi = \textstyle\sum_{j,k} 2x_k w_j (dt_j - du_k).$$

Obviously this vanishes on the $U(1)$ generator $\sum_j \partial_{t_j} + \sum_k \partial_{u_k}$ and hence defines a 1-form on the quotient torus. Note also that along any codimension one face, ϕ vanishes on the corresponding normal. This ensures that $H + \phi^2$ satisfies the boundary conditions whenever H does.

The Fubini–Study metric on the simplex is the pullback of $\sum_j dw_j \otimes dw_j/(2w_j) + \sum_k dx_k \otimes dx_k/(2x_k)$, where $\{dw_j, dx_k\}$ is dual to $\{dt_j, du_k\}$, so the contraction of this metric with ϕ is $X := \sum_{j,k}(x_k dw_j - w_j dx_k)$. A further contraction with ϕ yields $\phi(X) = 2(\sum_j w_j)(\sum_k x_k)(\sum_j w_j + \sum_k x_k) = 2\sum_{j,k} w_j x_k$ (on the simplex, where this reduces to $(1 - z^2)/2$ in the old coordinates).

Now to find the perturbation of the Fubini–Study metric on the simplex, and hence check the existence of a symplectic potential, we have to invert $I - f\phi \otimes X$ for a function f (such as $-q(z)/p(z)$). We readily check that the inverse is $I + f\phi \otimes X/(1 - f\phi(X))$ (found by a geometric series argument), and so the perturbation of the Fubini–Study metric on the simplex is (the pullback of) $fX \otimes X/(1 - f\phi(X))$. By construction this ought to reduce to something like $dz^2/\Theta(z) - dz^2/(1 - z^2)$ in our setting, but one wonders more generally when such an expression is the hessian of a function.

(iv) Extremal Kähler metrics with a hamiltonian 2-form of higher order would provide further examples on projective bundles if we knew that they existed. There are at least solutions to the extremal equations on $\mathbb{C}P^2$-bundles over a Riemann surface, albeit for inadmissible data. In this case, following the notation of [5]–[7], the momentum map z is denoted σ and its components are the elementary symmetric functions of coordinates $(\xi_1, \dots \xi_\ell)$. The polynomial $p(\sigma)$ is $\prod_a p_{nc}(\eta_a)^{d_a} = \pm \prod_j p_c(\xi_j)$. The metric on the torus takes the form

$$H^g_{rs} = \sum_{j=1}^{\ell} \frac{F(\xi_j)\sigma_{r-1}(\hat{\xi}_j)\sigma_{s-1}(\hat{\xi}_j)}{p_c(\xi_j)\Delta_j}$$

where $F(t)$ has degree $\leq m+2$. To compute the degree of $p(\sigma)H^g_{rs}$ in σ, we suppose that $d_a = 1$ for all a (the general case will follow by a limiting argument) so a ranges from 1 to d and introduce formal variables ξ_0, ξ_∞. Then

$$\sum_{r,s=1}^{\ell} (-1)^{r+s} p(\sigma) H^g_{rs} \xi_0^{\ell-r} \xi_\infty^{\ell-s}$$
$$= p_{nc}(\xi_0) p_{nc}(\xi_\infty) \big(\textstyle\prod_a p_{nc}(\eta_a)\big) \displaystyle\sum_{j=1}^{\ell} \frac{F(\xi_j)}{\Delta_j p_c(\xi_j)(\xi_j - \xi_0)(\xi_j - \xi_\infty)}.$$

Since F has degree at most $m + 2$, it now follows by Vandermonde identities using the $m+2$ variables $\xi_0, \xi_1, \dots \xi_\ell, \xi_\infty, \eta_1, \dots \eta_d$ that this expression is a polynomial of degree $\leq d + 3$ in σ. This will still be the case if the η_a are not distinct, which is the limiting argument mentioned above.

In view of this evidence, it is natural to conjecture that $F^\Omega(z) := p(z)H^\Omega(z)$ is polynomial of degree $\leq d + 3$. Unfortunately, Calabi's extremal metrics on $P(\mathcal{O} \oplus \mathcal{O}(k)) \to \mathbb{C}P^1$, regarded as toric metrics by choosing a circle action on $\mathbb{C}P^1$, already show that $F^\Omega(z)$ (if it exists) is not a polynomial in general. However, it remains possible that a polynomial F^Ω exists on $\mathbb{C}P^\ell$-bundles, as it does when $\ell = 1$. Let us then consider the next simplest case, $\ell = 2$, and take the base S to be a Riemann surface ($d = 1$).

We work in the homogeneous formalism with $\psi(w) = w_0 + w_1 + w_2$. We therefore seek a 3×3 matrix polynomials of degree 4 satisfying the boundary, extremality and integrability conditions (we do not concern ourselves with positivity at this stage, nor with integrality conditions on the constants y).

We first impose the boundary conditions by writing F^Ω as a perturbation of the Fubini–Study matrix. The r, s component is then

$$2\psi(w)^2 p(w) w_r \delta_{rs} - 2\psi(w)p(w)w_r w_s + w_r w_s p_{rs}$$

where $p(w) = a_0 w_0 + a_1 w_1 + a_2 w_2$. In order that the rows and columns sum to zero, the quadratic perturbation $p_{rs}(= p_{sr})$ is given by

$$\begin{aligned}
p_{00} &= x_2 w_1^2 + x_1 w_2^2 + 2y_0 w_1 w_2,\\
p_{11} &= x_0 w_2^2 + x_2 w_0^2 + 2y_1 w_2 w_0,\\
p_{22} &= x_1 w_0^2 + x_0 w_1^2 + 2y_2 w_0 w_1,\\
p_{01} &= y_2 w_2^2 - x_2 w_0 w_1 - y_0 w_2 w_0 - y_1 w_1 w_2,\\
p_{02} &= y_1 w_1^2 - x_1 w_2 w_0 - y_2 w_1 w_2 - y_0 w_0 w_1,\\
p_{12} &= y_0 w_0^2 - x_0 w_1 w_2 - y_1 w_0 w_1 - y_2 w_2 w_0,
\end{aligned}$$

for six unknown constants $(x_0, x_1, x_2, y_0, y_1, y_2)$. The extremality condition is then:

$$\begin{aligned}
&\psi(w)^2(4a_0 + 4a_1 + 4a_2 + s)\\
&\qquad + 2(y_0 + x_1 + x_2)(w_0^2 - 2w_0w_1 - 2w_2w_0 + 2w_1w_2)\\
&\qquad + 2(y_1 + x_2 + x_0)(w_1^2 - 2w_1w_2 - 2w_0w_1 + 2w_2w_0)\\
&\qquad + 2(y_2 + x_0 + x_1)(w_2^2 - 2w_2w_0 - 2w_1w_2 + 2w_0w_1)\\
&\qquad\qquad = p(w)(24\psi(w) + C_0w_0 + C_1w_1 + C_2w_2),
\end{aligned}$$

where s is the scalar curvature of the base S. This is a linear system of rank 3 in $(x_0, x_1, x_2, y_0, y_1, y_2)$. The condition that the inhomogeneous term is in the image of the linear map determines $C = (C_0, C_1, C_2)$, and is of course the condition that $C_0w_0 + C_1w_1 + C_2w_2$ is a potential for the extremal vector field.

The integrability condition that H^Ω is the inverse of a hessian is harder to compute, but eventually reduces to six equations:

$$\begin{gathered}
x_0 = y_1 + y_2, \qquad x_1 = y_2 + y_0, \qquad x_2 = y_0 + y_1,\\
y_0y_1 + 2y_0(a_1 - a_2) + 2y_1(a_0 - a_2) = 0,\\
y_1y_2 + 2y_1(a_2 - a_0) + 2y_2(a_1 - a_0) = 0,\\
y_2y_0 + 2y_2(a_0 - a_1) + 2y_0(a_2 - a_1) = 0.
\end{gathered}$$

These last three are not independent, and imply in particular that

$$y_0y_1 + y_1y_2 + y_2y_0 = 0. \tag{7}$$

We can use the equations for the x_j to reduce the extremality condition to a linear system for the y_j, which is readily solved. We obtain

$$
\begin{aligned}
y_0 &= K(a_0-a_2)(a_0-a_1)(a_0+2a_1+2a_2)(4a_0+4a_1+4a_2+s)\\
y_1 &= K(a_1-a_0)(a_1-a_2)(2a_0+a_1+2a_2)(4a_0+4a_1+4a_2+s)\\
y_2 &= K(a_2-a_1)(a_2-a_0)(2a_0+2a_1+a_2)(4a_0+4a_1+4a_2+s),
\end{aligned}
\tag{8}
$$

for some common denominator K. Substituting (8) into (7), we find that

$$((a_0-a_1)(a_1-a_2)(a_2-a_0)(4a_0+4a_1+4a_2+s))^2 = 0.$$

It is easy to check that the vanishing of any of these factors ensures all the integrability conditions are satisfied. The solutions with $a_j = a_k$ for some $k \neq j$ have a hamiltonian 2-form of order one, whereas the solution with $4a_0+4a_1+4a_2+s=0$ is trivial (the Fubini–Study matrix being unperturbed).

By the openness result for the existence of solutions, we see that even in this simple case, there are solutions which are not polynomial of degree ≤ 4.

3. K-stability for rigid toric bundles

3.1. **Toric test configurations.** Let us suppose that $[\sum_j c_j\omega_j/2\pi] = c_1(L_S)$ for a positive line bundle L_S, so S is projective, and that Δ has integral vertices, so M is a bundle of toric varieties over S. Then M is also projective, with polarization $L = \pi^* L_S \otimes (P \times_T L_V)$, where L_V is the polarization of V defined by Δ.

Thus $\pi_* L^k = L_S^k \otimes \big(\bigoplus_{\zeta \in \Delta \cap \frac{1}{k}\mathbb{Z}^\ell} \Lambda_{k\zeta}\big)$, where Λ_χ is the line bundle $P \times_T \mathbb{C}_\chi$, $\mathbb{C}_\chi$ being the 1-dimensional representation of T with weight $\chi \in \mathbb{Z}^\ell \subset \mathfrak{t}^*$.

Define $P(\zeta) = c_1(L_S \otimes \Lambda_\zeta)^d/d!$ and $Q(\zeta) = c_1(L_S\otimes\Lambda_\zeta)^{d-1} \cup c_1(S)/(2\,(d-1)!)$. For sufficiently large k we have, by Riemann–Roch, that

$$
\begin{aligned}
\dim H^0(S, L_S^k \otimes \Lambda_{k\zeta}) &= \chi(S, L_S^k\otimes\Lambda_{k\zeta})\\
&= \tfrac{1}{d!}\big(c_1(L_S^k\otimes\Lambda_{k\zeta}) + \tfrac12 c_1(\mathcal{K}_S^{-1})\big)^d[S] + O(k^{d-2})\\
&= P(\zeta)k^d + Q(\zeta)k^{d-1} + O(k^{d-2}).
\end{aligned}
$$

Hence, using the standard formula for summation over a lattice in a polytope [13, 43, 40], we have

$$
\begin{aligned}
\dim H^0(M, L^k) &= \dim H^0(S, \pi_* L^k)\\
&= k^m \int_\Delta P\, dv + k^{m-1}\Big(\frac12 \int_{\partial\Delta} P\, d\sigma + \int_\Delta Q\, dv\Big) + O(k^{m-2})
\end{aligned}
$$

for sufficiently large k.

Suppose f is a positive rational piecewise-linear (PL) concave function on Δ, i.e., f is the minimum of a finite collection of positive affine linear functions on Δ with rational coefficients. Then, $\Delta' = \{(z,t) : z \in \Delta, t \in (0, f(z))\} \subset \mathfrak{t}^* \oplus \mathbb{R}$ is a convex rational polytope corresponding (up to homothety) to polarized toric variety V' for (the complexification of) the $(\ell+1)$-torus $T' = T \times S^1$. Following work of Donaldson, Zhou and Zhu, and Székelyhidi [13, 43, 40], it turns out that, with $P' = P \times S^1 \to S$, $X = \pi' \colon P' \times_{T'} V' = P \times_T V' \to S$ is a test configuration for M, polarized by $\mathcal{E} = (\pi')^* L_S \otimes (P' \times_{T'} L_{V'})$. Let $L_0 = \mathcal{E}|_{X_0}$ where $\pi_0 \colon X_0 \to S$ is the central fibre. Then $(\pi_0)_* L_0^k$ can be identified with $\pi_* L^k$ (since $P' = P \times S^1$), and the $\mathbb{C}^\times$ action on X_0 has weight $kf(\zeta)$ on $\Lambda_{k\zeta}$ for k sufficiently large and divisible [43].

Thus the total weight of the action on $H^0(X_0, L_0^k)$ is

$$\begin{aligned} w_k &= \sum_{\zeta\in\Delta\cap\frac{1}{k}\mathbb{Z}^\ell} kf(\zeta)\big(P(\zeta)k^d + Q(\zeta)k^{d-1} + O(k^{d-2})\big) \\ &= k^{m+1}\int_\Delta fP\,dv + k^m\Big(\frac{1}{2}\int_{\partial\Delta} fP\,d\sigma + \int_\Delta fQ\,dv\Big) + O(k^{m-1}). \end{aligned}$$

Combined with the formula above for $d_k = \dim H^0(M, L^k)$, it follows that the residue of $w_k/(kd_k)$ at $k = 0$ is

$$\frac{1}{2}\int_{\partial\Delta} fP\,d\sigma + \int_\Delta fQ\,dv - \frac{\frac{1}{2}\int_{\partial\Delta} P\,d\sigma + \int_\Delta Q\,dv}{\int_\Delta P\,dv}\int_\Delta fP\,dv,$$

as shown by Székelyhidi [40]. In our setting, we can compute P and Q using the base metrics on S to provide representatives for $c_1(L_S\otimes\Lambda_\zeta)$ and $c_1(S)$. We thus obtain, up to a common multiple,

$$P(z) = p(z), \qquad Q(z) = \frac{1}{4}p(z)Scal_h(z)$$

It then follows that the Futaki invariant of $\mathbb{C}^\times$ action on X_0 is a negative multiple of

$$\int_{\partial\Delta} f(z)p(z)d\sigma + \frac{1}{2}\int_\Delta Scal_h(z)f(z)p(z)dv - \frac{\beta}{\alpha}\int_\Delta f(z)p(z)dv.$$

The modified Futaki invariant is then obtained by subtracting from the Futaki invariant the (correctly normalized) L^2 inner product of f and the Killing potential $-(\langle A, z\rangle + B + 2\beta/\alpha)$, with integral zero, for the extremal vector field [39, 40]. The result is easily computed to be a negative multiple of $\mathcal{F}^\Omega(f)$. Replacing the concave function f by the convex function $-f$ (plus a constant if desired), it follows that the $\mathcal{F}^\Omega(f)$, which vanishes for affine linear functions, computes the Futaki invariant of toric test configurations for convex rational PL functions.

3.2. **K-energy and K-stability.** Let $\mathcal{C}$ be the set of continuous convex functions on Δ (continuity follows from convexity on the interior of Δ), $\mathcal{C}_\infty$ the subset of those functions which are smooth on the interior of each face of Δ (including Δ itself of course), and $\mathcal{S}\subset\mathcal{C}_\infty$ the set of symplectic potentials. Note that if $u\in\mathcal{S}$ and $f\in\mathcal{C}_\infty$, with f smooth on all of Δ, then $u+f\in\mathcal{S}$.

The affine linear functions act on $\mathcal{C}$ and $\mathcal{C}_\infty$ by translation. Let $\mathcal{C}_\infty^*$ be a slice for the action on $\mathcal{C}_\infty$ which is linear (i.e., closed under positive linear combinations) so any f in $\mathcal{C}_\infty$ can be written uniquely as $f = \pi(f) + g$, where g is affine linear and $\pi(f)\in\mathcal{C}_\infty^*$ for a linear projection π. Functions in $\mathcal{C}_\infty^*$ are sometimes said to be *normalized.*

Let $||\cdot||$ be any semi-norm on $\mathcal{C}_\infty$ inducing a norm (in the obvious sense) on $\mathcal{C}_\infty^*$ which bounds the L_1 norm $\int_\Delta |f|p\,dv$ and such that the functions in $\mathcal{C}_\infty^*$ which are smooth on Δ are dense. The first condition implies in particular that $\mathcal{F}^\Omega$ is continuous on $\mathcal{C}_\infty$. Donaldson [13] shows that $\mathcal{E}^\Omega$ is also well defined as a function on $\mathcal{C}_\infty$ (in fact on a slightly larger space) taking values in $(-\infty, +\infty]$.

Donaldson also shows that the L_1 boundary integral on Δ satisfies the required assumptions, where $\mathcal{C}_\infty^*$ consists of those functions in $\mathcal{C}_\infty$ which vanish to first order at a chosen basepoint in the interior of Δ. However, we find it more flexible to abstract the setting as above.

In this general situation, two elementary arguments of Donaldson [13], together with an enhancement by Zhou–Zhu [43], can be used to prove a surprisingly strong result.

Lemma 4. *For any $\lambda > 0$ the following are equivalent*:

(i) $\mathcal{F}^\Omega(f) \geq \lambda ||\pi(f)||$ *for all* $f \in \mathcal{C}_\infty$;

(ii) *for all* $0 \leq \delta < \lambda$ *there exists* C_δ *such that* $\mathcal{E}^\Omega(u) \geq \delta||\pi(u)|| + C_\delta$ *for all* $u \in \mathcal{S}$.

Proof. As $\mathcal{F}^\Omega(f)$ and and $\mathcal{E}^\Omega(u)$ are unchanged by the addition of an affine linear function, it suffices to prove the equivalence for normalized f and u.

(i)$\Rightarrow$(ii) For any bounded function a on Δ, one can define a generalized Futaki invariant $\mathcal{F}_a$ by replacing the second integral (over Δ), in the formula for $\mathcal{F}^\Omega$, by $\int_\Delta a(z)f(z)p(z)dv$. Similarly one can define a generalized K-energy $\mathcal{E}_a$ using $\mathcal{F}_a$ instead of $\mathcal{F}^\Omega$.

For any bounded functions a, b, there is a constant $C = C_{a,b} > 0$ with $|\mathcal{F}_a(f) - \mathcal{F}_b(f)| \leq C||f||$ for all $f \in \mathcal{C}^*_\infty$, because $||\cdot||$ bounds the L_1 norm on $\mathcal{C}^*_\infty$. Let us write $C = (1+k)C - kC$ for an arbitrary $k \geq 0$ and take b to be the bounded function such that $\mathcal{F}_b = \mathcal{F}^\Omega$.

Then, by assumption, $|\mathcal{F}_a(f) - \mathcal{F}^\Omega(f)| \leq C\lambda^{-1}(1+k)\mathcal{F}^\Omega(f) - kC||f||$ for all $f \in \mathcal{C}^*_\infty$ and so $\mathcal{F}_a(f) \leq (1 + C\lambda^{-1}(1+k))\mathcal{F}^\Omega(f) - kC||f||$. Turning this around,

$$\mathcal{F}^\Omega(f) \geq \varepsilon \mathcal{F}_a(f) + \delta||f||,$$

where $0 < \varepsilon := (1 + C\lambda^{-1}(1+k))^{-1} < 1$ and $\delta := kC\lambda(\lambda + C(1+k))^{-1}$. Notice that δ is an injective function of $k \in [0,\infty)$ with range $[0,\lambda)$. Now we can estimate

$$\begin{aligned}\mathcal{E}^\Omega(u) &= -\frac{1}{2}\int_\Delta \log\det(\operatorname{Hess} u)p\,dv + \mathcal{F}^\Omega(u)\\ &\geq -\frac{1}{2}\int_\Delta \log\det(\operatorname{Hess} u)p\,dv + \varepsilon\mathcal{F}_a(u) + \delta||u||\\ &= \mathcal{E}_a(\varepsilon u) + \delta||u|| + m\log\varepsilon.\end{aligned}$$

As in [13], we can choose a so that $\mathcal{E}_a$ is bounded below on $\mathcal{C}_\infty$ and we are done.

(ii)$\Rightarrow$(i) Suppose $\mathcal{E}^\Omega(u) \geq \delta||u|| + C_\delta$ for all normalized $u \in \mathcal{S}$. By density and continuity, it suffices to prove (i) only for $f \in \mathcal{C}^*_\infty$ which are smooth on Δ. Then for fixed $u \in \mathcal{S}$ and all $k > 0$, $u + kf \in \mathcal{S}$ and so $\mathcal{E}^\Omega(u+kf) \geq \delta||u+kf|| + C_\delta$. By comparing with $\mathcal{E}^\Omega(u)$, we find

$$\begin{aligned}k\mathcal{F}^\Omega(f) &\geq \delta||u+kf|| + C_\delta + \frac{1}{2}\int_\Delta \log\frac{\det\operatorname{Hess}(u+kf)}{\det\operatorname{Hess} u}p\,dv - \mathcal{E}^\Omega(u)\\ &\geq \delta||u+kf|| + \tilde{C}_\delta\end{aligned}$$

with $\tilde{C}_\delta = C_\delta - \mathcal{E}^\Omega(u)$, since the ratio of the determinants is at least one. Dividing by k and letting $k \to \infty$ (for fixed $u \in \mathcal{S}$) we obtain $\mathcal{F}^\Omega(f) \geq \delta||f||$. Since this is true for all $0 \leq \delta < \lambda$, we have $\mathcal{F}^\Omega(f) \geq \lambda||f||$ for all $f \in \mathcal{C}^*_\infty$. □

In the case that the norm is the L_1 boundary integral, (i)$\Rightarrow$(ii) is due to Donaldson [14] when $\delta = 0$, and to Zhou–Zhu [43] for some $\delta > 0$—we have just abstracted their arguments and found the range of δ. (ii)$\Rightarrow$(i) also extends an elementary argument of Donaldson from the case of $\delta = 0$ and the L_1-boundary integral.

Donaldson's normalization condition has the disadvantage that there is not a unique way to normalize a convex function which is not smooth (for instance a PL convex function). It seems more natural to use instead the L_2 norm on Δ with $\mathcal{C}^*_\infty$ equal to the functions in $\mathcal{C}_\infty$ which are L_2 orthogonal to the affine linear functions. Then π is the L_2 projection onto $\mathcal{C}^*_\infty$. Let us say that the K-energy is L_2-proper on compatible metrics if, modulo a constant, it is bounded below by a multiple of the L_2 norm on symplectic potentials which are L_2 orthogonal to affine linear functions. Then the above lemma essentially establishes an equivalence between L_2-uniform K-polystability and L_2-properness of the K-energy.

Theorem 1. *Let M be a rigid toric bundle over a semisimple base with a compatible Kähler class Ω. Then (M,Ω) is L_2-uniformly K-polystable with respect to toric degenerations if and only if the K-energy is L_2-proper on compatible metrics in Ω.*

Proof. If M is L_2-uniformly K-polystable with respect to toric degenerations, then there is a $\lambda > 0$ such that $\mathcal{F}^{\Omega}(f) \geq \lambda ||\pi(f)||_2$ for all rational PL $f \in \mathcal{C}$. Such an estimate then clearly also holds without the assumption of rationality. Now functions in $\mathcal{C}$ are uniformly continuous, because Δ is compact. It follows that an argument of Donaldson, which he uses to establish a L_1 density result for PL convex functions [13, (5.2.8)], actually shows that the PL convex functions are dense in $\mathcal{C}$ in the L_∞ norm, hence in the L_q-norm for any $1 \leq q \leq \infty$. (Given $f \in \mathcal{C}$, Donaldson's construction provides a sequence f_n of PL convex functions which are bounded by values of f in cubes of side lengths $\delta_n \to 0$ as $n \to \infty$. Since f is uniformly continuous, it follows that the convergence of f_n to f is uniform.) Thus $\mathcal{F}^{\Omega}(f) \geq \lambda ||\pi(f)||_2$ for all $f \in \mathcal{C}$, hence for all f in $\mathcal{C}_\infty$. Hence the K-energy of (M,Ω) is L_2-proper on compatible metrics by the lemma.

Conversely, if the K-energy is L_2-proper, then by the lemma, $\mathcal{F}^{\Omega}(f) \geq \lambda ||\pi(f)||_2$ for all $f \in \mathcal{C}_\infty$. However, the smooth convex functions are dense in $\mathcal{C}$ with respect to the L_q norm for any $1 \leq q < \infty$ (by the usual convolution argument, together with a dilation argument about an interior point of Δ to maintain convexity near the boundary), so that in particular $\mathcal{F}^{\Omega}(f) \geq \lambda ||\pi(f)||_2$ for all rational PL $f \in \mathcal{C}$. Hence (M,Ω) is L_2-uniformly K-polystable. $\square$

Of course the lemma actually shows more: the modulus of L_2-properness of the K-energy and the modulus of L_2-uniform K-polystability agree in an obvious sense.

It is traditional in the literature (following Tian) to define properness of the K-energy using one of Aubin's functionals, which have an L_1 nature in the Kähler potential. Since the L_2 norm bounds the L_1 norm, an argument of Zhou and Zhu [43] shows that L_2-properness implies properness in this traditional sense. For a converse, one would need to change the norm and notion of uniform K-polystability.

A compatible extremal metric provides a global minimum for $\mathcal{E}^{\Omega}$ on $\mathcal{C}_\infty$. This gives immediately a K-semistability result.

Corollary 2. *If there exists a compatible extremal metric then $\mathcal{F}^{\Omega}(f) \geq 0$ for all PL convex functions f.*

In fact, the K-energy is minimized by any extremal Kähler metric, so we don't actually need to assume compatibility in this corollary.

3.3. **A formula for slope stability.** Consider the deformation to the normal cone $(X, \mathcal{E}_c, \alpha)$ of a submanifold Z in a Kähler $2n$-manifold (M,ω) polarized by a line bundle L with $\omega \in \Omega = 2\pi c_1(L)$.

Denote the Seshadri constant of this polarization by ε, so that $c \in (0,\varepsilon) \cap \mathbb{Q}$. Suppose that the $\mathbb{C}^\times$ action β preserves Z, so that X is compatible. We use the letters α, β to denote also the corresponding actions on the (polarized) central fibre (X_0, L_0) and on the vector space $H^0(X_0, L_0^k)$, where $L_0 = \mathcal{E}_c|_{X_0}$.

Let us calculate the Futaki invariant of this configuration. For this we first note that if $\mathcal{I}_Z \subset \mathcal{O}_M$ is the ideal sheaf of holomorphic functions vanishing on Z, then for any $p \geq 0$, $\mathcal{I}_Z^p/\mathcal{I}_Z^{p+1}$ is supported on Z, and its restriction is $S^p\nu_Z^*$, where ν_Z is the normal bundle to Z in M.

Therefore, for k sufficiently large, we have, as in [35, 39]

$$H^0(X_0, L_0^k) = \bigoplus_{i=0}^{(\varepsilon-c)k} H^0(Z, L|_Z^k \otimes S^{\varepsilon k-i}\nu_Z^*) \oplus \bigoplus_{j=1}^{ck} H^0(Z, L|_Z^k \otimes S^{ck-j}\nu_Z^*),$$

where α acts on the first direct sum with weight 0 and on the components of the second direct sum with weight $-j$. We can choose the lift of β to L so that the weight of the induced action on $H^0(Z, L|_Z^k \otimes S^{uk+v}\nu_Z^*)$ is $(u-\delta)k+v$ for an arbitrary fixed integer δ: $\delta = 0$ is an obvious choice, but it will be useful later to take $\delta = 1$.

Now $S^p\nu_Z^*$ is the direct image $q_*\mathcal{O}(p)_{\nu_Z}$, where $\mathcal{O}(-1)_{\nu_Z}$ is the (fibrewise) tautological bundle of $q\colon E = P(\nu_Z) \to Z$ (this is the exceptional divisor of the blow-up of Z in M). Let i be the composite map $E \to Z \to M$ and $\mathcal{L} = \mathcal{O}(1)_{\nu_Z}$. We shall also choose $\omega_E \in 2\pi c_1(\mathcal{L})$. We now have

$$\begin{aligned} H^0(X_0, L_0^k) &= \bigoplus_{i=0}^{(\varepsilon-c)k} H^0(E, i^*L^k \otimes \mathcal{L}^{\varepsilon k-i}) \oplus \bigoplus_{j=1}^{ck} H^0(E, i^*L^k \otimes \mathcal{L}^{ck-j}) \\ &= \bigoplus_{i=0}^{\varepsilon k} H^0(E, i^*L^k \otimes \mathcal{L}^{\varepsilon k-i}). \end{aligned}$$

To compute d_k, $\operatorname{Tr} A_k$, $\operatorname{Tr} B_k$, $\operatorname{Tr} A_kB_k$, $\operatorname{Tr} B_k^2$, and thereby $\mathfrak{F}_\beta(\alpha)$, we need only the dimensions of these vector spaces. We note that we only need to compute d_k, $\operatorname{Tr} A_k$ and $\operatorname{Tr} B_k$ to subleading order in k, whereas for $\operatorname{Tr} A_kB_k$ and $\operatorname{Tr} B_k^2$ the leading order term suffices. Consequently we will be dropping lower order terms without further comment.

By the ampleness of i^*L (in fact it is only semiample unless $Z = E$, but we can apply a limiting argument in this case, as in [35]), for sufficiently large k we have, by Riemann–Roch, that

$$\begin{aligned} h^0(E, i^*L^k \otimes \mathcal{L}^{xk}) &= \chi(E, i^*L^k \otimes \mathcal{L}^{xk}) \\ &= \tfrac{1}{(n-1)!}\big(c_1(i^*L^k \otimes \mathcal{L}^{xk}) + \tfrac{1}{2}c_1(\mathcal{K}_E^{-1})\big)^{n-1}[E] + O(k^{n-3}) \\ &= P(x)k^{n-1} + Q(x)k^{n-2} + O(k^{n-3}) \end{aligned}$$

where $P(x)$ and $Q(x)$ are polynomials in x independent of k, which, for integer x, are given by

$$\begin{aligned} P(x) &= \frac{1}{(2\pi)^{n-1}(n-1)!}\int_E (i^*\omega + x\omega_E)^{n-1} \\ Q(x) &= \frac{1}{2(2\pi)^{n-1}(n-2)!}\int_E \rho \wedge (i^*\omega + x\omega_E)^{n-2}. \end{aligned}$$

We shall use this expansion with $x = u + v/k$ for various u, v. In order to carry out the summations over i and j we use the trapezium rule, as in [35, Lemma 4.7].

Lemma 5. *Let $f(x)$ be a polynomial and b a rational number. Then for $\eta \in \{0, 1\}$ and for $k \in \mathbb{Z}^+$ such that bk is a positive integer, we have*

$$\sum_{i=\eta}^{bk} f(i/k) = k\int_0^b f(x)\,dx + \frac{1}{2}(f(b) + (-1)^\eta f(0)) + O(k^{-1}).$$

The proof is easy (see e.g. [35]): by linearity we can assume $f(x) = x^m$ and then use $\sum_{i=1}^N i^m = N^{m+1}/(m+1) + N^m/2 + O(N^{m-1})$ (which in turn is an easy induction on N).

Using $\eta = 0$, $b = u = \varepsilon$, $v = -i$, we then obtain (up to an overall multiple), that for any $r \geq 0$,

$$\begin{aligned} k^{-d-r}\mathrm{Tr}\, B_k^r &= k\int_0^\varepsilon (\varepsilon-\delta-x)^r P(\varepsilon - x)\,dx + \tfrac12((\varepsilon-\delta)^r P(\varepsilon) + (-\delta)^r P(0)) \\ &\qquad + \int_0^\varepsilon (\varepsilon-\delta-x)^r Q(\varepsilon-x)\,dx + O(1/k) \\ &= k\alpha_r + \tfrac12\beta_r + O(1/k) \end{aligned}$$

where we define α_r and β_r by

$$\begin{aligned} \alpha_r &= \int_0^\varepsilon (x-\delta)^r P(x)\,dx \\ \beta_r &= (\varepsilon-\delta)^r P(\varepsilon) + (-\delta)^r P(0) + 2\int_0^\varepsilon (x-\delta)^r Q(x)\,dx. \end{aligned}$$

Note that if we extend δ to all real numbers then $\partial\alpha_r/\partial\delta = -r\alpha_{r-1}$ and $\partial\beta_r/\partial\delta = -r\beta_{r-1}$.

Similarly, using $\eta = 1$, $b = u = c$, $v = -j$ we obtain obtain

$$\begin{aligned} k^{-d-1}\mathrm{Tr}\, A_k &= k\int_0^c -xP(c-x)\,dx - \tfrac12 cP(0) + \int_0^c -xQ(c-x)\,dx + O(1/k) \\ &= k\int_0^c (x-c)P(x)\,dx - \tfrac12 cP(0) + \int_0^c (x-c)Q(x)\,dx + O(1/k) \\ k^{-d-r-1}\mathrm{Tr}\, A_k^r &= \int_0^c (x-c)^r P(x)\,dx + O(1/k) \\ k^{-d-3}\mathrm{Tr}\, A_k B_k &= \int_0^c -x(c-\delta-x)P(c-x)\,dx + O(1/k) \\ &= \int_0^c (x-c)(x-\delta)P(x)\,dx + O(1/k). \end{aligned}$$

Now we are ready to calculate $\langle\beta,\beta\rangle$, $\langle\alpha,\beta\rangle$, $\mathfrak{F}(\beta)$, and $\mathfrak{F}(\alpha)$. (We omit the dependence on c for convenience.)

$$\begin{aligned} \langle\beta,\beta\rangle &= \frac{\alpha_2\alpha_0 - \alpha_1^2}{\alpha_0} \\ \langle\alpha,\beta\rangle &= \int_0^c P(x)(x-\delta)(x-c)\,dx - \frac{\alpha_1}{\alpha_0}\int_0^c P(x)(x-c)\,dx \\ \mathfrak{F}(\alpha) &= \mathrm{Res}_{k=0}\frac{(\mathrm{Tr}\, A_k)_1 + (\mathrm{Tr}\, A_k)_0/k}{\alpha_0(1+\beta_0/(2k\alpha_0))} = \frac{\alpha_0(\mathrm{Tr}\, A_k)_0 - \frac12\beta_0(\mathrm{Tr}\, A_k)_1}{\alpha_0^2} \\ &= \Big(\alpha_0\int_0^c Q(x)(x-c)\,dx - \tfrac12\alpha_0 cP(0) - \tfrac12\beta_0\int_0^c P(x)(x-c)\,dx\Big)/\alpha_0^2 \\ \mathfrak{F}(\beta) &= \mathrm{Res}_{k=0}\frac{\alpha_1 + \beta_1/2k}{\alpha_0(1+\beta_0/(2k\alpha_0))} = \frac{\beta_1\alpha_0 - \beta_0\alpha_1}{2\alpha_0^2}. \end{aligned}$$

Finally, we can calculate the Futaki invariant for our test configuration.

$$\begin{aligned} \alpha_0^2\mathfrak{F}_\beta(\alpha) &= \alpha_0^2\big(\mathfrak{F}(\alpha) - \langle\alpha,\beta\rangle\mathfrak{F}(\beta)/\langle\beta,\beta\rangle\big) \\ &= \alpha_0\int_0^c Q(x)(x-c)\,dx - \tfrac12\alpha_0 cP(0) - \tfrac12\beta_0\int_0^c P(x)(x-c)\,dx \\ &\quad - \frac{\alpha_0(\beta_1\alpha_0 - \beta_0\alpha_1)}{2(\alpha_2\alpha_0 - \alpha_1^2)}\Big(\int_0^c P(x)(x-\delta)(x-c)\,dx - \frac{\alpha_1}{\alpha_0}\int_0^c P(x)(x-c)\,dx\Big) \end{aligned}$$

$$
\begin{aligned}
&= \alpha_0\Big(\int_0^c Q(x)(x-c)\,dx - \tfrac12 cP(0) - \frac{\beta_1\alpha_0-\beta_0\alpha_1}{2(\alpha_2\alpha_0-\alpha_1^2)}\int_0^c P(x)(x-\delta)(x-c)\,dx\Big)\\
&\quad + \frac{\alpha_1(\beta_1\alpha_0-\beta_0\alpha_1)-\beta_0(\alpha_2\alpha_0-\alpha_1^2)}{2(\alpha_2\alpha_0-\alpha_1^2)}\int_0^c P(x)(x-c)\,dx\\
\alpha_0\mathfrak{F}_\beta(\alpha) &= \int_0^c Q(x)(x-c)\,dx - \tfrac12 cP(0)\\
&\quad - \frac{\beta_1\alpha_0-\beta_0\alpha_1}{2(\alpha_2\alpha_0-\alpha_1^2)}\int_0^c P(x)(x-\delta)(x-c)\,dx + \frac{\alpha_1\beta_1-\beta_0\alpha_2}{2(\alpha_2\alpha_0-\alpha_1^2)}\int_0^c P(x)(x-c)\,dx
\end{aligned}
$$

It follows easily that if we view the (modified) Futaki invariant $\mathfrak{F}_\beta(\alpha)$ as a function of δ and extend δ to all real numbers, then $\partial\mathfrak{F}_\beta(\alpha)/\partial\delta = 0$. Hence, as expected from general principles, $\mathfrak{F}_\beta(\alpha)$ does not depend on δ and therefore we may choose δ as we like.

These formulae make sense even if ω is not necessarily rational. We may therefore still define the Futaki invariant with respect to a analytic submanifold of an arbitrary Kähler manifold, and hence (extending the Seshadri constant in suitable way [35, §4.4]), we may extend our notion of stability to this setting.

Definition 6. A polarized Hodge manifold (M, L) with nontrivial $\mathbb{C}^\times$ action β is said to be *slope K-polystable relative to* β if for any nontrivial analytic subscheme Z preserved by β, the Futaki invariant $\mathfrak{F}_\beta(\alpha_c)$ of Z is negative for $c \in (0, \varepsilon)$.

As with the definition of (absolute) slope K-polystability, strictly speaking, we should also require $\mathfrak{F}_\beta(\alpha_\varepsilon) < 0$ unless ε is rational and $(X, \mathcal{E}_\varepsilon)$ is the pullback by a contraction of a product configuration.

3.4. **Slope stability for admissible bundles.** Let us now specialize to the admissible case. We take Z to be the infinity section, and note that the $\mathbb{C}^\times$ action β induced by the vector field K preserves Z.

In this case, $\varepsilon = 2$ and we may take $\delta = 1$ and put $t = x - \delta = x - 1$, so that the range of t is $[-1, 1]$.

Also E is covered by $\prod_a S_a$ and equipped with the local Kähler product metric $\sum_a \omega_a/x_a$. We write $\rho = \sum_a \rho_a$ where ρ_a is the Ricci form of $\pm\omega_a$ and therefore has the form $s_a\omega_a$ plus a primitive part with respect to ω_a, where s_a is a constant with $2d_a s_a = Scal_{\omega_a}$ and $2d_a = \dim S_a$.

We take $\omega = dz\wedge\theta + \sum_a (1 + x_a z)\omega_a/x_a$, which represents the admissible Kähler class $\Xi + \sum_a[\omega_a]/x_a$. (Here $0 < |x_a| \le 1$ with equality iff $a \in \{0, \infty\}$) [7].

Hence $i^*\omega = \sum_a (1 - x_a)\omega_a/x_a$ and we can take $\omega_E = \sum_a \omega_a$ so that

$$
\begin{aligned}
P(x) &= \frac{1}{(2\pi)^{n-1}(n-1)!}\int_E \Big(\sum_a (1 + x_a t)\omega_a/x_a\Big)^{n-1}\\
&= \frac{\mathrm{Vol}(E, \prod_a \omega_a/x_a)}{(2\pi)^{n-1}}\prod_a (1 + x_a t)^{d_a}\\
Q(x) &= \frac{1}{2(2\pi)^{n-1}(n-2)!}\int_E \Big(\sum_a s_a\omega_a\Big)\wedge\Big(\sum_a (1 + x_a t)\omega_a/x_a\Big)^{n-2}\\
&= \frac{\mathrm{Vol}(E, \prod_a \omega_a/x_a)}{(2\pi)^{n-1}}\Big(\sum_a \frac{d_a s_a x_a}{1 + x_a t}\Big)\prod_a (1 + x_a t)^{d_a}
\end{aligned}
$$

We note that since the Futaki invariant is defined in terms of ratios, we can ignore any overall multiples. So after a rescaling, we have

$$\begin{aligned}P(x) &= p_c(t)\\ Q(x) &= \Big(\sum_a \frac{d_a s_a x_a}{1 - x_a t}\Big)p_c(t)\end{aligned}$$

with $t = x - 1$. Hence:

$$\begin{aligned}\alpha_r &= \int_0^2 (x-1)^r P(x)dx = \int_{-1}^1 t^r p_c(t)\,dt\\ \beta_r &= P(2) + (-1)^r P(0) + 2\int_0^2 (x-1)^r Q(x)\,dx\\ &= p_c(1) + (-1)^r p_c(-1) + \int_{-1}^1 \Big(\sum_a \frac{d_a s_a x_a}{1 - x_a t}\Big) t^r p_c(t)\,dt.\end{aligned}$$

Note that α_r and β_r are just rescales (by the same factor) of the values of same name in [7]. Substituting also $z = c - 1$, we then obtain

$$\begin{aligned}\alpha_0 \mathfrak{F}_\beta(\alpha) &= \int_0^c Q(x)(x-c)\,dx - \tfrac{1}{2}cP(0) + \frac{1}{4}\int_0^c P(x)(A(x-1)+B)(x-c)\,dx\\ &= -\tfrac{1}{2}(z+1)p_c(-1) - \frac{1}{4}\int_{-1}^z (At + B + \sum_a \frac{2d_a s_a x_a}{1 + x_a z}) p_c(t)(z-t)\,dt\end{aligned}$$

where A and B are exactly as in [7, Propn. 6]. Hence this is exactly $-1/4$ times the extremal polynomial $F_\Omega(z)$ given by e.g. [7, §2.4 Eq. (12)].

3.5. **Computation via Ross–Thomas slope.** From the proof of [35, Propn. 3.16] it follows that

$$P(x) = -a_0'(x) \tag{9}$$

and

$$Q(x) = -a_1'(x) - \frac{a_0''(x)}{2}, \tag{10}$$

where

$$a_0(x) = \frac{1}{n!}(L - xE)^n$$

and

$$a_1(x) = \frac{1}{2(n-1)!}c_1(\hat{M}) \cup (L - xE)^{n-1}.$$

In the general Kähler case, if $Z \in M$ is a smooth submanifiold preserved by the $\mathbb{C}^\times$ action β we define

$$a_0(x) := \frac{1}{n!}\int_{\hat{M}} (\pi^*\omega - xe)^n$$

and

$$a_1(x) := \frac{1}{2(n-1)!}\int_{\hat{M}} \hat{\rho} \wedge (\pi^*\omega - xe)^{n-1},$$

where $\hat{\rho}$ is a representative of $2\pi c_1(\hat{M})$ and e is a representative of the Poincaré dual of the exceptional divisor of the blow-up along Z, $\pi : \hat{M} \to M$. This generalizes (aside from a re-scaling factor of $(2\pi)^n$, which we will ignore) the previous definition of $a_0(x)$ and $a_1(x)$. We may also generalize the Seshadri constant to mean the least upper bound of the set of $x \in \mathbb{R}$ such that $\pi^*\omega - xe$ has non-negative volume on any analytic subvariety of $\hat{M}$ [35]. Thus we have a generalized definition of $P(x), Q(x), \alpha_r, \beta_r$, and finally the Futaki invariant $\mathfrak{F}_\beta(\alpha)$.

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Vestislav Apostolov, Département de Mathématiques, UQAM, C.P. 8888, Succ. Centre-ville, Montréal (Québec), H3C 3P8, Canada

E-mail address: apostolo@math.uqam.ca

David M. J. Calderbank, Department of Mathematics, University of York, Heslington, York YO10 5DD, England

E-mail address: dc511@york.ac.uk

Paul Gauduchon, Centre de Mathématiques, École Polytechnique, UMR 7640 du CNRS, 91128 Palaiseau, France

E-mail address: pg@math.polytechnique.fr

Christina W. Tønnesen-Friedman, Department of Mathematics, Union College, Schenectady, New York 12308, USA

E-mail address: tonnesec@union.edu