SELFDUAL EINSTEIN METRICS AND CONFORMAL SUBMERSIONS

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ABSTRACT. Weyl derivatives, Weyl-Lie derivatives and conformal submersions are defined, then used to generalize the Jones-Tod correspondence between self-dual 4-manifolds with symmetry and Einstein-Weyl 3-manifolds with an abelian monopole. In this generalization, the conformal symmetry is replaced by a particular kind of conformal submersion with one dimensional fibres, and the gauge group of the monopole becomes a group of diffeomorphisms of a 1-manifold. Special cases are studied in which the conformal submersion is affine or projective. All scalar-flat Kähler metrics with a holomorphic conformal submersion, and all four dimensional hypercomplex structures with a compatible Einstein metric are obtained from affine monopoles, while projective monopoles encompass Hitchin's twistorial construction of selfdual Einstein metrics from an Einstein-Weyl conformal infinity. The results include a linear equation for hyperKähler metrics generalizing the Gibbons-Hawking Ansatz, and an explicit formula for Hitchin's metrics. New selfdual Einstein metrics depending explicitly on an arbitrary holomorphic function of one variable or an arbitrary axially symmetric harmonic function are obtained. The former generically have no continuous symmetries.

INTRODUCTION

The aims of this paper are threefold: firstly, to advertise the notion of a Weyl derivative both as a simple, but useful, tool in differential geometry, and also as an object of study in its own right; secondly, to apply this tool to the theory of conformal submersions, with particular attention to the case of selfdual conformal 4-manifolds; and thirdly to give explicit constructions of selfdual Einstein metrics. The key discovery is a class of conformal submersions with one dimensional fibres which admit a holomorphic interpretation on the twistor space. This class includes the conformal submersions generated by conformal vector fields and provides a natural setting for a generalized Jones-Tod correspondence [17] encompassing Hitchin's construction of selfdual Einstein metrics with Einstein-Weyl conformal infinity [15].

A Weyl derivative on a manifold M is nothing more than a covariant derivative on a real line bundle naturally associated to the differential geometry of M. Weyl derivatives can be used to define Lie derivatives along foliations with one dimensional leaves, generalizing the usual Lie derivative along a vector field. They also occur naturally in the geometry of conformal submersions. These two situations have in common the conformal submersions with one dimensional fibres, to which most of this paper is devoted. I focus on the case that the total space is a selfdual

4-manifold M and define the notion of a selfdual conformal submersion. In Theorem I (4.6), the base B of such a submersion is shown to be not just conformal, but Einstein-Weyl, generalizing the Jones-Tod correspondence, and yielding an explicit construction, in Theorem II (4.9), of selfdual spaces from solutions of the Einstein-Weyl Bogomolny equation, where the gauge group acts on a 1-manifold.

The central part of the paper deals with special cases of this construction. In Theorem III (4.11), shear-free geodesic congruences on B are shown to correspond to antiselfdual complex structures on M which are invariant with respect to the conformal submersion (i.e., the submersion is holomorphic), and the hypercomplex structures and scalar-flat Kähler metrics arising in this way are identified (generalizing [8, 12]). In Theorem IV (4.13), an antiselfdual complex structure is constructed from a generic selfdual conformal submersion on a selfdual Einstein-Weyl 4-manifold—for a selfdual Einstein metric with a Killing field, this reduces to the construction of Tod [24].

The notion of an affine conformal submersion is defined, characterized, and shown, in Theorem V (5.2), to provide a method for constructing selfdual spaces from coupled linear differential equations, which will be called *affine monopole equations.* In Theorem VI (6.1), I show that the generalized Jones-Tod constructions for scalar-flat Kähler and hyperKähler metrics arise in this way—this includes as special cases, the Gibbons-Hawking Ansatz [13], LeBrun's construction of scalar flat metrics with Killing vector fields [19], and the construction of such metrics with homothetic vector fields [8, 12]. Projective conformal submersions are analysed in a similar way and the *projective monopole equations* are identified in Theorem VII (7.3) with the $SL(2,\mathbb{R})$ Einstein-Weyl Bogomolny equation.

In Theorem VIII (8.2), I prove that Hitchin's version of LeBrun's $\mathcal{H}$-space construction, defined twistorially in [15], is a special case of the generalized Jones-Tod correspondence, and this special case is characterized. The Hitchin-LeBrun construction, although simple from a twistor point of view, has always been difficult to carry out explicitly due to a lack of a direct construction: the known examples are, to the best of my knowledge, those of [18, 15, 22, 9]. The main result of the final portion of the paper is an explicit formula for the selfdual Einstein metric associated to an *arbitrary* Einstein-Weyl structure.

To this end, I first study selfdual Einstein-Weyl 4-manifolds, which are known to be either Einstein or locally hypercomplex [4, 23]. Theorem IX (9.2) states that a pair of compatible selfdual Einstein-Weyl structures determine a selfdual conformal submersion, and this is used, in Theorem X (10.3), to prove that all selfdual Einstein metrics with a compatible hypercomplex structure admit an affine conformal submersion over a hyperCR Einstein-Weyl space. This special case of the Hitchin-LeBrun construction yields new examples of selfdual Einstein metrics, and motivates the final result, Theorem XI (11.1), in which the formula for the Hitchin-LeBrun construction is obtained.

The paper is organized as follows. In section 1, Weyl derivatives and Weyl-Lie derivatives are introduced. Although it is possible to present some of the later results without this formalism, the proofs are simpler and more natural when one makes systematic use of the affine space of Weyl derivatives and the product rule

for Weyl-Lie derivatives. Indeed, many of the results of this paper would have been impossible to find (for the author at least) without the geometric guidance provided by working in a gauge-independent way. In order to familiarize the reader with this language, I have presented a few simple applications of Weyl derivatives, and discussed the Weyl-Lie derivative on natural bundles. The key formula from this section is the Weyl-Lie derivative of a torsion-free covariant derivative on a natural bundle. In section 2, after recalling basic facts from conformal geometry, I present another arrow in the Weyl geometer's quiver: the linearized Koszul formula.

In the third section, the notion of a conformal submersion is defined, but most of the local properties are studied within the more general framework of conformal almost product structures. I prove a simple proposition which shows that there is a canonically defined Weyl derivative in this setting, which will be called the *minimal Weyl derivative*. The main interest, however, is in conformal submersions with one dimensional fibres, which may be analyzed locally using the congruences and Weyl-Lie derivatives of section 1. I focus on this case for the second half of section 3 and characterize basic objects using the minimal Weyl-Lie derivative.

The generalized Jones-Tod correspondence (Theorem I) is established in section 4. In an earlier version of this paper [5], my proof followed closely the proof of the "classical" Jones-Tod correspondence given in [7, 8]. However, Paul Gauduchon has recently obtained a cleaner proof by exploiting more thoroughly the isomorphism between the bundle of antiselfdual 2-forms on the conformal 4-manifold M and the pullback of the tangent bundle of the 3-dimensional quotient B. This approach also integrates nicely with the fact that invariant antiselfdual complex structures on M correspond to shear-free geodesic congruences on B [8], and so I adapt it here to the context of conformal submersions.

The defining equation for conformal submersions, and the monopole equation arising in the inverse construction, are nonlinear, but there is a special case in which the Bogomolny equation linearizes: affine conformal submersions. Here M is an affine bundle over B such that the nonlinear connection and relative length scale induced by the conformal structure are affine. Such submersions are studied in section 5, then applied in section 6 to scalar-flat Kähler and hyperKähler metrics. The new Ansatz is illustrated by some examples, taken from [9]. Section 7 treats projective conformal submersions in a similar way.

The motivation for several results in this paper originally came from twistor theory. I explain this in section 8, and also discuss the Hitchin-LeBrun construction. The twistorial point of view suggests that the conformal submersion arising here will be affine if the Einstein-Weyl conformal infinity is hyperCR, and projective in general. In the former case, the selfdual Einstein metric admits a compatible hypercomplex structure and this is studied first. A key tool is the general observation that two compatible Einstein-Weyl structures on a conformal manifold define a conformal submersion, and in four dimensions, this submersion is selfdual if the conformal structure is. This is proven in section 9 and then, in section 10, selfdual Einstein metrics admitting a compatible hypercomplex structure are characterized explicitly in terms of affine monopoles. Examples with no continuous symmetries are given, which depend on an arbitrary holomorphic function of one variable.

In the final section, a projective gauge transformation is applied to obtain a solution of the projective monopole equations which makes sense on any Einstein-Weyl space. In section 11, I show that the explicit metric given by the induced projective conformal submersion is Einstein. The resulting direct method for carrying out the Hitchin-LeBrun construction is illustrated by a family of selfdual Einstein metrics depending on an arbitrary axially symmetric harmonic function on $\mathbb{R}^3$.

1. Weyl derivatives and Weyl-Lie derivatives

If V is a real n-dimensional vector space and w any real number, then the oriented one dimensional linear space $L^w = L^w(V)$ carrying the representation $A \mapsto |\det A|^{w/n}$ of $\mathrm{GL}(V)$ is called the space of *densities of weight w* or *w-densities*. It can be constructed canonically as the space of maps $\rho \colon (\Lambda^n V) \smallsetminus 0 \to \mathbb{R}$ such that $\rho(\lambda\omega) = |\lambda|^{-w/n}\rho(\omega)$ for all $\lambda \in \mathbb{R}^\times$ and $\omega \in (\Lambda^n V) \smallsetminus 0$.

The same construction can be carried out pointwise on any vector bundle E to give, for each $w \in \mathbb{R}$, the oriented real line bundle L_E^w whose fibre at x is $L^w(E_x)$. Applying this to the tangent bundle gives the following definition.

1.1**. Definition.** Suppose M is any n-manifold. Then the *density line bundle* $L^w = L_{TM}^w$ of M is defined to be the bundle whose fibre at $x \in M$ is $L^w(T_x M)$. Equivalently it is the associated bundle $\mathrm{GL}(M) \times_{\mathrm{GL}(n)} L^w(n)$ where $\mathrm{GL}(M)$ is the frame bundle of M and $L^w(n)$ is the space of w-densities of $\mathbb{R}^n$.

The density bundles are oriented (hence trivializable) real line bundles, but there is no preferred trivialization. Sections of $L = L^1$ may be thought of as scalar fields with dimensions of length. This geometric dimensional analysis may also be applied

to tensors: the tensor bundle $L^w \otimes (TM)^j \otimes (T^*M)^k$ (and any subbundle, quotient bundle, element or section) is said to have *weight* $w + j - k$, or dimensions of $[length]^{w+j-k}$. Note that $L^{w_1} \otimes L^{w_2}$ is canonically isomorphic to $L^{w_1+w_2}$ and L^0 is the trivial bundle. When tensoring a vector bundle with some L^w (or any real line bundle), I shall often omit the tensor product sign. Note also that an orientation of M may be viewed as a unit section of $L^n \Lambda^n T^*M$, defining an isomorphism between L^{-n} and $\Lambda^n T^*M$. A nonvanishing (usually positive) section of L^1 (or L^w for $w \neq 0$) is called a *length scale* or *gauge* (of weight w).

1.2. Definition. A *Weyl derivative* is a covariant derivative D on L^1. It induces covariant derivatives on L^w for all w. The curvature of D is a real 2-form F^D called the *Faraday curvature*.

If $F^D = 0$ then D is said to be *closed*. There are then local length scales μ with $D\mu = 0$. If such a length scale exists globally then D is said to be *exact*. Conversely, a length scale μ induces an exact Weyl derivative D^μ such that $D^\mu \mu = 0$. Note that $D^{c\mu} = D^\mu$ for any constant $c \neq 0$. Weyl derivatives form an affine space modelled on the linear space of 1-forms, while closed and exact Weyl derivatives are affine subspaces modelled on the linear spaces of closed and exact 1-forms respectively.

A *gauge transformation* on M is a positive function e^f which rescales a gauge $\mu \in C^\infty(M, L^w)$ to give $e^{wf}\mu$. It acts on Weyl derivatives by $e^f \cdot D = e^f \circ D \circ e^{-f} = D - df$, so that $e^f \cdot D^\mu = D^{e^f\mu}$ for $\mu \in C^\infty(M, L^1)$. If D is any Weyl derivative, then $D = D^\mu + \omega^\mu$ for the 1-form $\omega^\mu = \mu^{-1}D\mu$, and consequently, $\omega^{e^f\mu} = \omega^\mu + df$.

On an oriented manifold, Weyl derivatives may be viewed as a generalization of volume forms, since the exact Weyl derivatives correspond to volume forms up to constant multiples. For instance, let $\Omega \in L^2\Lambda^2 T^*M$ be a weightless 2-form on M^{2m} such that Ω^m is an orientation. Then Ω is nondegenerate and one would like to find a length scale μ such that $\mu^{-2}\Omega$ is symplectic. If $2m > 2$ this may not be possible.

1.3. Proposition. (cf. [20]) *Let M be an n-manifold ($n = 2m > 2$) and let $\Omega \in L^2\Lambda^2 T^*M$ be nondegenerate. Then there is a unique Weyl derivative D such that $d^D\Omega$ is tracefree with respect to Ω, in the sense that $\sum d^D\Omega(e_i, e_i', \cdot) = 0$, where e_i, e_i' are frames for $L^{-1}TM$ with $\Omega(e_i, e_j') = \delta_{ij}$.*

This can be proven [8] by picking any Weyl derivative D^0 and setting $D = D^0 + \gamma$ for some 1-form γ, then $d^D\Omega = d^{D^0}\Omega + 2\gamma \wedge \Omega$ and so the traces differ by a multiple of γ. It follows that there is a unique γ such that $d^D\Omega$ is tracefree.

There are therefore two local obstructions to finding μ with $d(\mu^{-2}\Omega) = 0$, namely $d^D\Omega$ and F^D [20]. In four dimensions $d^D\Omega$ automatically vanishes. In general if $d^D\Omega = 0$ then $F^D \wedge \Omega = 0$. In six or more dimensions this implies $F^D = 0$, so D is closed, but it need not be exact. In four dimensions, however, D need not even be closed. This construction is of particular interest in Hermitian geometry [25].

There is also a version of this in contact geometry. A *contact structure* on M is codimension one subbundle $\mathcal{H}$ of TM which is maximally nonintegrable, in the sense that the Frobenius tensor $\Omega_{\mathcal{H}}: \Lambda^2\mathcal{H} \to TM/\mathcal{H}$ is nondegenerate. In this context, one defines a (slightly generalized) Weyl derivative to be a covariant derivative on the real line bundle $TM/\mathcal{H}$. The following result generalizes the fact that a

contact form (which corresponds to a section of $TM/\mathcal{H}$) defines a Reeb vector field complementary to the contact distribution.

1.4. Proposition. *Let $M, \mathcal{H}$ be a contact manifold. Then there is a bijection between complementary subspaces to $\mathcal{H}$ in TM and Weyl derivatives on $TM/\mathcal{H}$ such that the horizontal part of F^D is tracefree with respect to $\Omega_{\mathcal{H}}$.*

Proof. Let $\eta_{\mathcal{H}} \colon TM \to TM/\mathcal{H}$ be the twisted contact 1-form whose kernel defines $\mathcal{H}$. Then given any Weyl derivative D, one can define $d^D \eta_{\mathcal{H}}$ and if $D = D^0 + \gamma$ then $d^D \eta_{\mathcal{H}} = d^{D^0} \eta_{\mathcal{H}} + \gamma \wedge \eta_{\mathcal{H}}$. Note that $d^D \eta_{\mathcal{H}}|_{\mathcal{H}}$ is well defined, being equal to $\Omega_{\mathcal{H}}$. Since $\Omega_{\mathcal{H}}$ is nondegenerate, given D there is a unique complementary subspace (spanned by ξ, say) such that $d^D \eta_{\mathcal{H}}(\xi, \cdot) = 0$, and such a complementary subspace fixes D up to 1-forms γ with $\eta_{\mathcal{H}} \wedge \gamma = 0$.

Now note that if $D = D^0 + \mu^{-1} \eta_{\mathcal{H}}$ for a section μ of $TM/\mathcal{H}$, then $F^D|_{\mathcal{H}} = F^{D^0}|_{\mathcal{H}} + \mu^{-1}\Omega_{\mathcal{H}}$ and consequently, D may be found uniquely with tracefree horizontal Faraday curvature. $\qquad\square$

The first part of the proof also gives an *affine* bijection between complementary subspaces and "horizontal" Weyl derivatives (i.e., covariant differentiation is only defined along directions in $\mathcal{H}$).

Much of the rest of the paper is concerned with the dual situation of one dimensional subbundles of TM (the complementary subspaces arising above being an example). The integral manifolds of such a distribution define a foliation of M with one dimensional leaves. This will be viewed as an unparameterized version of a vector field by thinking of such a subbundle as an inclusion $\xi \colon \mathcal{V} \to TM$ of a real line bundle $\mathcal{V}$, and hence as a "twisted" vector field. Sections of $\mathcal{V}$ correspond to vector fields tangent to the foliation. It is therefore natural to consider covariant derivatives on $\mathcal{V}$, which will again be referred to as Weyl derivatives. I will also use the following terminology from relativity.

1.5. Definition. A *congruence* on a manifold M is a nonvanishing section ξ of $\mathcal{V}^{-1}TM$ for some oriented real line bundle $\mathcal{V}$. It defines an oriented one dimensional subbundle of TM and hence a foliation with oriented one dimensional leaves.

No use will be made of the orientation of $\mathcal{V}$ in this section.

1.6. Proposition. *Let E be a vector bundle associated to the frame bundle of M with induced representation ρ of $\mathfrak{gl}(TM)$. Then for a congruence ξ, a Weyl derivative D (on $\mathcal{V}$), and a section s of E, the formula*

$$\mu^{-1} \mathcal{L}_{\mu\xi} s + \rho(\mu^{-1} D\mu \otimes \xi) s$$

is independent of the choice of a nonvanishing section μ of $\mathcal{V}$, and will be called the Weyl-Lie derivative $\mathcal{L}_{\xi}^{D} s$ of s along ξ.

Proof. For any vector field X and function f, $\mathcal{L}_{fX} s = f\mathcal{L}_X s - \rho(df \otimes X)s$. $\qquad\square$

Note that $\mathcal{L}_{\xi}^{D} s$ is a section of $\mathcal{V}^{-1}E$. If D is exact, then trivializing $\mathcal{V}$ by a parallel section gives back the usual Lie derivative. The dependence of $\mathcal{L}_{\xi}^{D}$ on D is

clearly given by $\mathcal{L}_\xi^{D+\gamma} s = \mathcal{L}_\xi^D s + \rho(\gamma \otimes \xi)s$. In particular, for functions (ρ trivial), $\mathcal{L}_\xi^D f = df(\xi)$, which is a section of $\mathcal{V}^{-1}$ independent of D.

In order to compute the Weyl-Lie derivative, it is convenient to find a formula in terms of a torsion-free connection inducing covariant derivatives ∇ on any associated bundle E. In the case of the usual Lie derivative, $\mathcal{L}_K X = \nabla_K X - \nabla_X K$ for vector fields X, and so $\mathcal{L}_K s = \nabla_K s - \rho(\nabla K)s$ on sections of E. This readily yields the following generalization to Weyl-Lie derivatives:

$$(1.1) \qquad \mathcal{L}_\xi^D s = \nabla_\xi s - \rho\big((D \otimes \nabla)\xi\big)s,$$

where $(D \otimes \nabla)\xi$ denotes the *twisted* or *tensor sum* covariant derivative of ξ as a section of $\mathcal{V}^{-1}TM = \mathcal{V}^{-1} \otimes TM$. More generally $D \otimes \nabla$ will denote the twisted covariant derivative on $\mathcal{V}^{-1}E$. Formula (1.1) immediately gives the following.

1.7. Proposition. *Let ξ be a congruence and D be a Weyl derivative (on $\mathcal{V}$). Then the Weyl-Lie derivative may be computed in terms of an arbitrary torsion-free covariant derivative ∇ as follows:*

- *For a w-density μ, $\mathcal{L}_\xi^D \mu = \nabla_\xi \mu - \frac{w}{n}(\mathrm{div}^{D \otimes \nabla} \xi)\mu$.*
- *For a vector field X, $\mathcal{L}_\xi^D X = \nabla_\xi X - (D \otimes \nabla)_X \xi$.*
- *For a 1-form α, $\mathcal{L}_\xi^D \alpha = \nabla_\xi \alpha + \alpha\big((D \otimes \nabla)\xi\big) = d\alpha(\xi, \cdot) + D(\alpha(\xi))$.*
- *For a k-form α, $\mathcal{L}_\xi^D \alpha = \iota_\xi d\alpha + d^D(\iota_\xi \alpha)$.*

Here $\mathrm{div}^{D \otimes \nabla} \xi$ denotes the trace of $(D \otimes \nabla)\xi$, i.e., the divergence has been twisted by D on $\mathcal{V}^{-1}$ and ∇ on L^n.

The above treatment only deals with the Weyl-Lie derivative for zero and first order geometric objects (functions and sections of bundles associated to the first order frame bundle). It may be extended to differential operators (higher order geometric objects) using the product rule, but because the Weyl-Lie derivative of a section of E is not a section of E, the differential operators have to be twisted by D. Consequently, some natural differential operators may have nonzero Weyl-Lie derivative. In particular $(\mathcal{L}_\xi^D d)\alpha = \mathcal{L}_\xi^D(d\alpha) - d^D(\mathcal{L}_\xi^D \alpha) = F^D \wedge \iota_\xi \alpha$, i.e., the Weyl-Lie derivative only commutes with exterior differentiation on functions in general. Similarly, for the Lie bracket, $\mathcal{L}_\xi^D[,](X, Y) = -F^D(X, Y)\xi$. Of course, this is compatible with the definition of d in terms of $[,]$.

If ∇ is any torsion-free covariant derivative then its Weyl-Lie derivative $\mathcal{L}_\xi^D \nabla = \mathcal{L}_\xi^D \circ \nabla - (D \otimes \nabla) \circ \mathcal{L}_\xi^D$ evaluates to

$$(1.2) \qquad (\mathcal{L}_\xi^D \nabla)_X = R_{\xi,X}^\nabla + (D \otimes \nabla)_X\big((D \otimes \nabla)\xi\big) \quad \in \mathcal{V}^{-1}\mathfrak{gl}(TM).$$

This acts on an associated bundle E via the corresponding representation ρ of $\mathfrak{gl}(TM)$. The vanishing of $\mathcal{L}_\xi^D \nabla$ defines a notion of invariance, along ξ, of ∇ on E. Also note that $\nabla_\xi - \mathcal{L}_\xi^D = \rho\big((D \otimes \nabla)\xi\big)$ is a kind of Higgs field. If the Higgs field vanishes, I will say ∇ is *horizontal*, for reasons that will become clear later. In particular if ∇ is invariant and horizontal on E, then $\rho(R_{\xi,X}^\nabla) = 0$.

2. Conformal geometry

In the previous section, the term "Weyl derivative" was sometimes applied in a generalized sense, when the oriented real line bundle was not necessarily L^1. Such a distinction disappears when one introduces a conformal structure.

2.1. Definition. A *conformal structure* c on M is a metric on $L^{-1}TM$, whose determinant at each $x \in M$ (defined up to sign) is a unit element of $L_x^n \Lambda^n T_x^* M$.

A conformal structure identifies any oriented one dimensional subbundle or quotient bundle of TM with L^1. In particular if M is conformal and ξ is a congruence, there is a unique oriented isomorphism between $\mathcal{V}$ and L^1 such that $\xi \in \mathrm{C}^\infty(M, L^{-1}TM)$ is a weightless unit vector field. A congruence will now be viewed as an injective linear map $\xi \colon L^1 \to TM$ and $\mathcal{V}$ will denote its image.

A conformal structure is equivalently a fibrewise inner product $\langle \cdot, \cdot \rangle$ on TM with values in L^2: compatible Riemannian metrics therefore correspond to length scales. As in the previous section, it is natural to replace length scales with Weyl derivatives, and the Koszul formula for the Levi-Civita connection generalizes to give an induced torsion-free conformal connection, called a *Weyl connection*, on M:

$$(2.1) \qquad \begin{aligned} 2\langle D_X Y, Z \rangle = \;& D_X \langle Y, Z \rangle + D_Y \langle X, Z \rangle - D_Z \langle X, Y \rangle \\ & + \langle [X,Y], Z \rangle - \langle [X,Z], Y \rangle - \langle [Y,Z], X \rangle. \end{aligned}$$

This gives an affine bijection between Weyl derivatives and Weyl connections on TM. The corresponding linear map sends a 1-form γ to the $\mathfrak{co}(TM)$-valued 1-form Γ defined by $\Gamma_X = [\![\gamma, X]\!] = \gamma(X) id + \gamma \vartriangle X$, where $(\gamma \vartriangle X)(Y) = \gamma(Y) X - \langle X, Y \rangle \gamma$.

2.2. Remark. Here, and elsewhere, I freely identify a 1-form γ with a vector field of weight -1 using the isomorphism $\sharp \colon T^*M \to L^{-2}TM$ given by the conformal structure. Similarly, vector fields of any weight are identified with 1-forms of the same weight. Also, a skew linear map J on TM (of any weight) corresponds to a 2-form Ω_J (of the same weight) via $J(X) = \sharp(\iota_X \Omega_J)$; this identifies $\gamma \vartriangle X$ with $\gamma \wedge \langle X, \cdot \rangle$. It will sometimes, but not always, be helpful to maintain a distinction between a skew endomorphism and the corresponding 2-form. The bracket $[\![\cdot, \cdot]\!]$ is part of an algebraic Lie bracket on $TM \oplus \mathfrak{co}(TM) \oplus T^*M$, and the same notation will be used for the commutator bracket on $\mathfrak{co}(TM)$.

The curvature R^D of D, as a $\mathfrak{co}(TM)$-valued 2-form, decomposes as follows:

$$(2.2) \qquad R^D_{X,Y} = W_{X,Y} - [\![r^D(X), Y]\!] + [\![r^D(Y), X]\!].$$

Here W is the *Weyl curvature* of the conformal structure, an $\mathfrak{so}(TM)$-valued 2-form, and r^D is a covector valued 1-form, the *normalized Ricci tensor* of D.

If ξ is a congruence, then the Weyl-Lie derivative of D on L^1 reduces to

$$(2.3) \qquad (\mathcal{L}_\xi D)^{L^1}_X \mu = F^D(\xi, X) \mu + \frac{1}{n} D^\xi_X (\mathrm{div}^{D^\xi \otimes D} \xi) \mu,$$

where D^ξ is the Weyl derivative used to define $\mathcal{L}_\xi$. Linearizing (2.1) gives a formula for the Weyl-Lie derivative on other bundles:

$$(\mathcal{L}_\xi D)_X = D_X(\mathcal{L}_\xi \mathsf{c}) + alt\big(D(\mathcal{L}_\xi \mathsf{c})(X)\big) + [\![\mathcal{L}_\xi D^{L^1}, X]\!] + \tfrac{1}{2}\xi \vartriangle F^\xi(X) + \tfrac{1}{2}\langle \xi, X \rangle F^\xi.$$

In this *linearized Koszul formula*, F^ξ is the Faraday curvature of D^ξ: these F^ξ terms come from the Weyl-Lie derivative of the Lie bracket. All the terms apart from the first belong the conformal Lie algebra $\mathfrak{co}(TM)$. This reflects the fact that if $\mathcal{L}_\xi \mathsf{c} = 0$, the Weyl-Lie derivative preserves conformal subbundles of natural vector bundles and so extends to any bundle associated to the conformal frame bundle.

3. Conformal submersions

3.1. Definition. Let $\pi \colon M \to B$ be a smooth surjective map between conformal manifolds and let the *horizontal bundle* $\mathcal{H}$ be the orthogonal complement to the vertical bundle $\mathcal{V}$ of π in TM. Then π will be called a *conformal submersion* iff for all $x \in M$, $d\pi_x|_{\mathcal{H}_x}$ is a nonzero conformal linear map.

Since the main aim is to study the local geometry on M, it is not really necessary for the base to be a manifold: it could be an orbifold, or be replaced altogether by the horizontal geometry of a foliation.

A bundle $\mathcal{H}$ complementary to $\mathcal{V}$ is often called a *(nonlinear) connection* on π: it is determined by a projection $\eta \colon TM \to \mathcal{V}$, the *connection 1-form*.

3.2. Proposition. *If $\pi \colon M \to B$ is a submersion onto a conformal manifold B, then conformal structures on M making π into a conformal submersion correspond bijectively to triples $(\mathcal{H}, \mathsf{c}^{\mathcal{V}}, w)$, where $\mathcal{H}$ is a connection on π, $\mathsf{c}^{\mathcal{V}}$ is a conformal structure on the fibres, and $w \colon \pi^* L^1_{TB} \cong L^1_{\mathcal{H}} \to L^1_{\mathcal{V}}$ is a (positive) isomorphism.*

Proof. $d\pi \colon \mathcal{H} \to \pi^* TB$ is certainly an isomorphism, so $L^2_{\mathcal{H}} \cong \pi^* L^2_{TB}$ and the conformal structure on $\mathcal{H}$ is obtained by pullback. Combining this with $\mathsf{c}^{\mathcal{V}}$ gives an $L^2_{\mathcal{V}}$ valued inner product $\mathsf{c} = \mathsf{c}^{\mathcal{V}} \oplus w^2 \mathsf{c}^{\mathcal{H}}$, which in turn determines an isomorphism between $L_{\mathcal{V}}$ and L_{TM} such that c becomes a conformal structure on M. $\qquad\square$

The final ingredient w in this construction will be called a *relative length scale*, since it allows vertical and horizontal lengths to be compared. The freedom to vary w generalizes the so-called "canonical variation" of a Riemannian submersion, in which the fibre metric is rescaled, while the base metric remains constant.

A natural generalization of conformal submersions, following Gray [14], is a *conformal almost product structure*. On a conformal manifold M, this is a nontrivial orthogonal direct sum decomposition $TM = \mathcal{V} \oplus_\perp \mathcal{H}$, i.e., $\mathcal{V}$ and $\mathcal{H}$ are nontrivial subbundles of TM and are orthogonal complements with respect to the conformal structure. Although the roles of $\mathcal{V}$ and $\mathcal{H}$ are interchangeable, I will call the corresponding tangent directions vertical and horizontal.

Given any Weyl derivative D, observe that that the vertical component $(D_X Y)^{\mathcal{V}}$ for $X, Y \in \mathcal{H}$ is tensorial in Y and so defines a tensor in $\mathcal{H}^* \otimes \mathcal{H}^* \otimes \mathcal{V}$. I will write $(D_X Y)^{\mathcal{V}} = \mathrm{I\!I}^D_{\mathcal{H}}(X, Y) + \frac{1}{2}\Omega_{\mathcal{H}}(X, Y)$, where $\mathrm{I\!I}^D_{\mathcal{H}}$ is symmetric and $\Omega_{\mathcal{H}}$ is skew, and extend these fundamental forms by zero to $T^*M \otimes T^*M \otimes TM$. Similarly the horizontal component $(D_U V)^{\mathcal{H}}$ for $U, V \in \mathcal{V}$ defines tensors $\mathrm{I\!I}^D_{\mathcal{V}}$ and $\Omega_{\mathcal{V}}$ in $\mathcal{V}^* \otimes \mathcal{V}^* \otimes \mathcal{H} \leqslant T^*M \otimes T^*M \otimes TM$. Since D is torsion-free, $\Omega_{\mathcal{H}}$ and $\Omega_{\mathcal{V}}$ are the Frobenius tensors of the distributions $\mathcal{H}$ and $\mathcal{V}$, which vanish iff the distributions are tangent to foliations. On the other hand, the fundamental forms $\mathrm{I\!I}^D_{\mathcal{H}}$ and $\mathrm{I\!I}^D_{\mathcal{V}}$ do depend on D and can be used to find a distinguished Weyl derivative D^0.

3.3. Proposition. *Suppose M is conformal with a conformal almost product structure $TM = \mathcal{V} \oplus_\perp \mathcal{H}$. Then there is a unique Weyl derivative D^0 such that $\mathcal{V}$ and $\mathcal{H}$ are minimal, in the sense that the fundamental forms, denoted $\mathbb{II}_\mathcal{H}^0$ and $\mathbb{II}_\mathcal{V}^0$, are tracefree. It may be computed from an arbitrary Weyl derivative D via the formula:*

$$D^0 = D + \frac{tr\,\mathbb{II}_\mathcal{H}^D}{\dim \mathcal{H}} + \frac{tr\,\mathbb{II}_\mathcal{V}^D}{\dim \mathcal{V}}.$$

Proof. Observe that if $\tilde{D} = D + \gamma$ then $\langle tr\,\mathbb{II}_\mathcal{H}^{\tilde{D}}, U \rangle = \langle tr\,\mathbb{II}_\mathcal{H}^D, U \rangle - (\dim \mathcal{H})\gamma(U)$ for all $U \in \mathcal{V}$, and similarly for $\mathbb{II}_\mathcal{V}$. This shows that the formula for D^0 is independent of D. Substituting $D = D^0$ shows that $\mathbb{II}_\mathcal{H}^0$ and $\mathbb{II}_\mathcal{V}^0$ are tracefree. $\qquad\square$

I shall refer to D^0 as the *minimal Weyl derivative*; this usage is consonant both with minimal submanifolds and minimal coupling. The minimal Weyl derivative need not be closed: I shall write F^0 for its Faraday curvature.

3.4. Remark. The curvature of D^0 on TM can be related to the curvatures of horizontal and vertical connections on $\mathcal{H}$ and $\mathcal{V}$. One defines a horizontal connection on $\mathcal{H}$ by $D_X^\mathcal{H} Y = (D_X^0 Y)^\mathcal{H}$ for $X, Y \in \mathcal{H}$; similarly $D_U^\mathcal{V} V = (D_U^0 V)^\mathcal{V}$ for $U, V \in \mathcal{V}$. The (modified) curvature of $D^\mathcal{H}$ is defined by

$$R_{X,Y}^\mathcal{H} Z = D_X^\mathcal{H} D_Y^\mathcal{H} Z - D_Y^\mathcal{H} D_X^\mathcal{H} Z - D_{[X,Y]^\mathcal{H}}^\mathcal{H} Z - \big[[X,Y]^\mathcal{V}, Z\big]^\mathcal{H},$$

where $X, Y, Z \in \mathcal{H}$. The definition of $R^\mathcal{V}$ is analogous, and O'Neill-type formulae [21] relating R^{D^0} to $R^\mathcal{H}$ and $R^\mathcal{V}$ follow directly as in [14].

Some of the properties of the minimal Weyl derivative may be elucidated by comparing it with partial Lie derivatives. If X is a horizontal vector field and U is vertical, then $[U,X]^\mathcal{H}$ is tensorial in U, which defines a partial covariant derivative on $\mathcal{H}$ in vertical directions. This extends naturally to horizontal forms in $\Lambda^k \mathcal{H}^*$ and to densities in $L_\mathcal{H}^1$. One says a horizontal vector field, form, or density is invariant if its partial covariant derivative along $\mathcal{V}$ vanishes; if $\mathcal{V}$ is a tangent to the fibres of a submersion, this means the horizontal vector field, form, or density is basic.

Similarly, there is a partial covariant derivative on $\mathcal{V}$, $\Lambda^k \mathcal{V}^*$ and $L_\mathcal{V}^1$ in horizontal directions. Since the conformal structure identifies $L_\mathcal{H}^1$ with $L_\mathcal{V}^1$, putting these together gives a Weyl derivative. In order to verify that this is the minimal Weyl derivative defined above, introduce an arbitrary Weyl connection D so that $[U,X]^\mathcal{H} = (D_U X)^\mathcal{H} - (D_X U)^\mathcal{H}$ and the second term is tensorial: since $\langle D_X U, Y \rangle = -\langle U, D_X Y \rangle$ this tensor is essentially $\mathbb{II}_\mathcal{H}^D + \frac{1}{2}\Omega_\mathcal{H}$. Therefore the vertical partial connections on $L_\mathcal{H}^1$ induced by $[U,X]^\mathcal{H}$ and $(D_U X)^\mathcal{H}$ agree if and only if $tr\,\mathbb{II}_\mathcal{H}^D = 0$. Similarly the horizontal partial connections on $L_\mathcal{V}^1$ induced by $[X,U]^\mathcal{V}$ and $(D_X U)^\mathcal{V}$ agree if and only if $tr\,\mathbb{II}_\mathcal{V}^D = 0$.

According to this discussion, the following definition for densities is compatible with the identification of L^1 with $L_\mathcal{H}^1$ and $L_\mathcal{V}^1$.

3.5. Definition. A density $\mu \in C^\infty(M, L^1)$ on a conformal manifold M with a conformal almost product structure $(\mathcal{V}, \mathcal{H})$ is *invariant along $\mathcal{V}$* iff $D_U^0 \mu = 0$ for all vertical U, and *invariant along $\mathcal{H}$* iff $D_X^0 \mu = 0$ for all horizontal X.

I specialize now to the case that $\mathcal{V}$ is one dimensional and oriented. Then the positively oriented weightless unit vector field ξ spanning $L^{-1}\mathcal{V} \leqslant L^{-1}TM$ is a congruence and the minimal Weyl derivative D^0 is characterized by $D^0_\xi \xi = 0$ and $tr\, D^0\xi = 0$ (note that $\langle D^0\xi, \xi \rangle = 0$ since ξ has unit length). The formula for computing D^0 reduces to $D^0 = D - \frac{1}{n-1}(div^D\, \xi)\xi + (d^D\xi)(\xi, \cdot)$. Also note that $\Omega_\mathcal{V}$ and $\mathrm{I\!I}^0_\mathcal{V}$ both vanish, so Ω and $\mathrm{I\!I}^0$ will denote the fundamental forms of $\mathcal{H}$. I denote the minimal Weyl-Lie derivative along ξ by $\mathcal{L}^0_\xi$.

3.6. Proposition. *Let ξ be a congruence with minimal Weyl derivative D^0 and let D be an arbitrary Weyl derivative.*

(i) *$\frac{1}{2}\mathcal{L}^0_\xi \mathsf{c} = sym_0(D^0 \otimes D)\xi = sym\, D^0\xi = -\langle \xi, \mathrm{I\!I}^0 \rangle$.*
(ii) *D^0 is exact iff $\xi = K/|K|$ for some (nonvanishing) vector field K which is divergence-free and geodesic with respect to the metric $g = |K|^{-2}\mathsf{c}$.*
(iii) *If D^0 is exact and $\mathcal{L}^0_\xi \mathsf{c} = 0$ then K is a conformal vector field, and hence is a Killing field of the metric $g = |K|^{-2}\mathsf{c}$.*

Conversely if K is a nonvanishing conformal vector field then $\xi = K/|K|$ is a congruence with $\mathcal{L}^0_\xi \mathsf{c} = 0$ and $D^0|K| = 0$.

Proof. For the first part, note that for any vector fields X, Y,

$$(\mathcal{L}^0_\xi \mathsf{c})(X,Y) = \mathcal{L}^0_\xi \langle X,Y \rangle - \langle \mathcal{L}^0_\xi X, Y \rangle - \langle X, \mathcal{L}^0_\xi Y \rangle$$
$$= D_\xi \langle X,Y \rangle - \tfrac{2}{n}(div^{D^0 \otimes D}\, \xi)\langle X,Y \rangle$$
$$- \langle D_\xi X - (D^0 \otimes D)_X \xi, Y \rangle - \langle X, D_\xi Y - (D^0 \otimes D)_Y \xi \rangle$$
$$= \langle (D^0 \otimes D)_X \xi, Y \rangle + \langle X, (D^0 \otimes D)_Y \xi \rangle - \tfrac{2}{n}(div^{D^0 \otimes D}\, \xi)\langle X,Y \rangle.$$

This is $2(sym_0(D^0 \otimes D)\xi)(X,Y) = 2(sym\, D^0\xi)(X,Y)$, since D was arbitrary. If either X or Y is parallel to ξ, $(sym\, D^0\xi)(X,Y)$ vanishes automatically, because $D^0_\xi \xi = 0 = \langle D^0\xi, \xi \rangle$. On the other hand, if X and Y are orthogonal to ξ, it is equal to $-\frac{1}{2}\langle \xi, D^0_X Y + D^0_Y X \rangle = -\langle \xi, \mathrm{I\!I}^0(X,Y) \rangle$.

For the second and third parts, observe that an exact D^0 preserves a length scale μ. Then $K = \mu\xi$ and D^0 is the Levi-Civita connection of $g = \mu^{-2}\mathsf{c}$. $\qquad\square$

I shall now assume that the congruence ξ is tangent to the one dimensional fibres of a submersion π over a manifold B; this is always true locally. A horizontal vector field, form or density is then basic if it is invariant in the sense above. The Weyl-Lie derivative $\mathcal{L}^0_\xi$ provides an efficient way to characterize such basic objects.

3.7. Proposition. *Let ξ be a congruence with minimal Weyl derivative D^0 generating a submersion π of M over B. Then a horizontal vector field X is basic iff $\mathcal{L}^0_\xi X = 0$. Similarly a horizontal form α is basic iff $\mathcal{L}^0_\xi \alpha = 0$. Finally a density μ is basic iff $\mathcal{L}^0_\xi \mu = 0$.*

Proof. It suffices to check that $\mathcal{L}^0_\xi X = 0$ is equivalent to $[U, X]$ being vertical for all vertical U. If $U = \lambda\xi$ then away from the zero set of λ, $D^\lambda = D^0 + \gamma$ for some 1-form γ and so $[U, X] = \mathcal{L}_{\lambda\xi} X = \lambda\mathcal{L}^{D^\lambda}_\xi X = \lambda\mathcal{L}^0_\xi X + \gamma(X)\lambda\xi$. Hence $\mathcal{L}^0_\xi X$ is

vertical if and only if $[U, X]$ is vertical for all vertical U. For the vertical component observe that $\langle \xi, \mathcal{L}^0_\xi X \rangle = -\mathcal{L}^0_\xi \mathsf{c}(\xi, X)$ which vanishes by the previous Proposition. The result for forms follows from the product rule.

For densities, $\mathcal{L}^0_\xi \mu = D^0_\xi \mu$, and this means μ is basic as a section of $L^1_\mathcal{H}$. $\square$

The product rule means that other basic objects are characterized by vanishing Weyl-Lie derivative. For instance the submersion is conformal (i.e., the horizontal part of the conformal structure is basic) iff $\mathcal{L}^0_\xi \mathsf{c} = 0$.

A connection ∇ on E (associated to the frame bundle, or the conformal frame bundle if $\mathcal{L}^0_\xi \mathsf{c} = 0$) is horizontal if $\nabla_\xi = \mathcal{L}^0_\xi$, i.e., invariant sections are covariant constant along the fibres. The horizontal part of ∇ is $\nabla - \langle \xi, \cdot \rangle \otimes \rho\big((D^0 \otimes \nabla)\xi\big)$, and this is basic (i.e., a pullback connection) iff ∇ is invariant, i.e., $\mathcal{L}^0_\xi \nabla = 0$.

The horizontal part of a Weyl derivative D on M is basic iff $0 = \mathcal{L}^0_\xi D = F^D(\xi, \cdot) + \frac{1}{n} D^0(div^{D^0 \otimes D}\, \xi)$. If D is horizontal this reduces to $F^D(\xi, \cdot)$. In particular D^0 itself is basic iff $F^0(\xi, \cdot) = 0$. If $\mathcal{L}^0_\xi \mathsf{c} = 0$, the only nonzero fundamental form is $\Omega = \Omega_\mathcal{H}$:

$$\langle \Omega(X, Y), \xi \rangle = 2\langle D^0_X Y, \xi \rangle = -2\langle D^0_X \xi, Y \rangle = -(d^0\xi)(X, Y).$$

Note that $\mathcal{L}^0_\xi(d^0\xi) = \iota_\xi(d^0)^2\xi = F^0 - \xi \wedge \big(F^0(\xi, \cdot)\big)$ and so Ω is basic iff $\xi \wedge F^0 = 0$.

Note also that the linearized Koszul formula with respect to the minimal Weyl derivative along the fibres of the conformal submersion reduces to:

$$(3.1) \qquad (\mathcal{L}^0_\xi D)_X = [\![\mathcal{L}^0_\xi D^{L^1}, X]\!] + \tfrac{1}{2}\xi \triangle F^0(X) + \tfrac{1}{2}\langle \xi, X \rangle F^0.$$

Hence invariance on L^1 does not imply invariance on other natural bundles.

4. The Jones-Tod correspondence

Suppose that ξ is a congruence on an oriented conformal 4-manifold M defining a conformal submersion π onto a manifold B. Let D^0 be the minimal Weyl derivative of ξ, define $\omega = -(*d^D\xi)(\xi, \cdot)$ (which can be computed using any Weyl derivative D) and let $D^{sd} = D^0 + \frac{1}{2}\omega$ and $D^B = D^0 + \omega$.

4.1. *Remarks.* The definition of ω uses the natural extension of the star operator to 2-forms of any weight. The star operator on 1-forms (of any weight) is a 3-form of the same weight defined by $\iota_X *\alpha = *(\langle X, \cdot \rangle \wedge \alpha)$ for any vector field X. In general the star operator on any manifold will be defined in terms of the orientation $*1$ so that a similar relation holds between the star operator, wedge product and interior multiplication, with no signs. As remarked in [8], this is more convenient in computations than the usual choice. Note that $*^2 = +1$ in four dimensions, whereas $*^2_B = -1$ in three dimensions.

Since the star operator is an involution on 2-forms in four dimensions, $\Lambda^2 T^*M = \Lambda^2_+ T^*M \oplus \Lambda^2_- T^*M$. The selfdual and antiselfdual parts of a 2-form are denoted $F = F_+ + F_-$. A skew endomorphism J of TM may be identified with a weightless 2-form $\Omega_J \in L^2\Lambda^2 T^*M \cong L^{-2}\Lambda^2 TM$ via $\Omega_J(X, Y) = \langle JX, Y \rangle$ and J is said to be *selfdual* or *antiselfdual* if Ω_J is. If J is antiselfdual, then $\Omega_J = \eta \wedge J\eta - *\eta \wedge J\eta$ for any weightless unit 1-form η. It follows that, for a 1-form α (of any weight),

$*\alpha = J\alpha \wedge \Omega_J$ for any antiselfdual endomorphism J with $J^2 = -id$. Note that if F is a selfdual 2-form, then $X \wedge F(Y) - Y \wedge F(X)$ is also selfdual.

4.2. Proposition. $\langle (D^0 \otimes D^{sd})\xi, \cdot \rangle$ *is a selfdual 2-form of weight* -1.

Proof. In terms of an arbitrary Weyl derivative D,

$$D^0 = D - \tfrac{1}{4}(div^{D^0 \otimes D}\xi)\xi + \tfrac{1}{2}(d^{D^0 \otimes D}\xi)(\xi, \cdot)$$

$$\omega = -(*d^{D^0 \otimes D}\xi)(\xi, \cdot)$$

and so $\qquad D^{sd} = D - \tfrac{1}{4}(div^{D^0 \otimes D}\xi)\xi + \tfrac{1}{2}(d^{D^0 \otimes D}\xi)(\xi, \cdot) - \tfrac{1}{2}(*d^{D^0 \otimes D}\xi)(\xi, \cdot).$

Substituting $D = D^{sd}$ into this formula gives the result. $\qquad\qquad\square$

4.3. *Remark.* This property clearly characterizes D^{sd}. One can characterize D^B in a similar way by the vanishing of the trace of $D^B\xi$ and the selfduality of $alt\, D^B\xi$.

4.4. Proposition. *The Weyl derivative* D^B *is basic iff* D^0 *has selfdual Faraday curvature.*

Proof. Since $\omega(\xi) = 0$, D^B is basic iff $F^B(\xi, \cdot) = 0$, where F^B is the Faraday curvature of D^B. Now $D^B = D^0 + \omega$ and so

$$F^B(\xi, \cdot) = F^0(\xi, \cdot) + d\omega(\xi, \cdot) = F^0(\xi, \cdot) + \mathcal{L}^0_\xi \omega.$$

Writing $\omega = -(*d^0\xi)(\xi, \cdot) = -*(\xi \wedge d^0\xi)$ yields

$$\mathcal{L}^0_\xi \omega = -*(\xi \wedge \mathcal{L}^0_\xi d^0\xi) = -*(\xi \wedge F^0) = -(*F^0)(\xi, \cdot).$$

So $F^B(\xi, \cdot) = (F^0 - *F^0)(\xi, \cdot)$. Since $F^0 - *F^0$ is antiselfdual, this contraction with ξ vanishes iff $F^0 = *F^0$. $\qquad\qquad\square$

When D^0 has selfdual Faraday curvature, the conformal submersion is said to be *selfdual*. In this case D^B is a basic Weyl derivative on $L^1 \cong \pi^* L^1_B$ and the induced Weyl structure on B is sometimes called the *Jones-Tod Weyl structure*. It follows from the Koszul formula that the induced Weyl connection on TB pulls back to the conformal connection on $\mathcal{H} \cong \pi^* TB$ given by the horizontal part of the Weyl connection induced by D^B on TM. The same observation holds for $L^{-1}\mathcal{H} \cong \pi^* L^{-1}_B TB$.

Now observe that the map $\Omega_J \mapsto J\xi$ is an isomorphism from $L^2 \Lambda^2_- T^*M$ to $L^{-1}\mathcal{H}$ with inverse $\chi \mapsto \xi \wedge \chi - *(\xi \wedge \chi) \in L^{-2}\Lambda^2_- TM \cong L^2 \Lambda^2_- T^*M$. If $J^2 = -id$ then $|J\xi| = 1$ and so this isomorphism is an isometry up to a constant multiple (conventionally, $|\Omega_J|^2 = 2$ when $J^2 = -id$). Also note that $[\![J_1, J_2]\!]\xi = -2*(\xi \wedge J_1\xi \wedge J_2\xi)$.

The action of $\mathfrak{co}(TM)$ on antiselfdual endomorphisms is by commutator, and so only the antiselfdual part contributes, since selfdual and antiselfdual endomorphisms commute. Therefore Proposition 4.2 shows that $\mathcal{L}^0_\xi J = D^{sd}_\xi J$, i.e., D^{sd} is horizontal on $L^2 \Lambda^2_- T^*M$. The linearized Koszul formula may be used to show that if π is selfdual, then D^{sd} is invariant on $L^2 \Lambda^2_- T^*M$. More precisely,

$$\mathcal{L}^0_\xi D^{sd}_X = \tfrac{1}{2}F^0(\xi, X)id + \tfrac{1}{2}\xi \triangle F^0(X) - \tfrac{1}{2}X \triangle F^0(\xi) + \tfrac{1}{2}\langle \xi, X \rangle F^0,$$

which has selfdual skew part. Hence D^{sd} is basic on $L^2\Lambda^2_- T^*M \cong \pi^* L_B^{-1} TB$. The following Proposition identifies it with D^B (which gives another proof of invariance).

4.5. Proposition. *If J is an antiselfdual endomorphism, then for any horizontal vector field X, $\big(D^B_X(J\xi)\big)^{\mathcal{H}} = (D^{sd}_X J)\xi$.*

Proof. Write $\chi = J\xi$ so that $\Omega_J = \xi \wedge \chi - *(\xi \wedge \chi)$. Then

$$D^{sd}_X \Omega_J = D^{sd}_X \xi \wedge \chi + \xi \wedge D^{sd}_X \chi - *(D^{sd}_X \xi \wedge \chi + \xi \wedge D^{sd}_X \chi).$$

Now $D^{sd} = D^0 + \frac{1}{2}\omega = D^B - \frac{1}{2}\omega$ and so

$$D^{sd}_X \xi = \tfrac{1}{2} *(X \wedge \xi \wedge \omega),$$
$$D^{sd}_X \chi = D^B_X \chi + \tfrac{1}{2}\omega(\chi)X - \tfrac{1}{2}\langle \chi, X\rangle \omega.$$

Therefore $\qquad D^{sd}_X \xi \wedge \chi = -\tfrac{1}{2}\langle \chi, X\rangle *(\xi \wedge \omega) + \tfrac{1}{2}\omega(\chi) *(\xi \wedge X)$

and $\qquad \xi \wedge D^{sd}_X \chi = \xi \wedge D^B_X \chi + \tfrac{1}{2}\omega(\chi)\xi \wedge X - \tfrac{1}{2}\langle \chi, X\rangle \xi \wedge \omega$

which gives $\xi \wedge D^{sd}_X \chi - *(D^{sd}_X \xi \wedge \chi) = \xi \wedge D^B_X \chi$.

Taking the antiselfdual part and contracting with ξ completes the proof. $\qquad\square$

A generalized Jones-Tod correspondence follows readily from these observations, following an approach due to Gauduchon. Recall [15] that a Weyl connection is said to be *Einstein-Weyl* iff the symmetric traceless part of its Ricci tensor vanishes.

4.6. Theorem I. *Suppose (M, c) is an oriented conformal 4-manifold and ξ generates a selfdual conformal submersion π over a manifold B. Then $D^B = D^0 + \omega$ is Einstein-Weyl on B if and only if c is selfdual.*

Proof. Since π is selfdual, D^B descends to a Weyl connection on B. If $\pi^* D^B$ denotes the pullback of D^B to $\pi^* L_B^{-1} TB \cong L^{-1}\mathcal{H}$ then Proposition 4.5 implies that $D^{sd} J = (\pi^* D^B)(J\xi)$ for any antiselfdual endomorphism J, and so $[\![R^{sd}_{X,Y}, J]\!]\xi = (\pi^* R^B)_{X,Y}(J\xi)$, where R^B is the curvature of D^B on B and X, Y are arbitrary vector fields. Here I have used the fact that D^{sd} is horizontal, and so $D^{sd}_\xi J = \mathcal{L}^0_\xi J = (\pi^* D^B)_\xi(J\xi)$ by definition of pullback; horizontality and the definition of pullback likewise imply that $[\![R^{sd}_{X,Y}, J]\!]$ and $(\pi^* R^B)_{X,Y}(J\xi)$ vanish if X or Y is vertical. (Recall that $\mathcal{L}^0_\xi D^{sd} = R^{sd}_{\xi,X} + (D^0 \otimes D^{sd})_X(D^0 \otimes D^{sd})\xi$.)

Let $R^{sd,-}\colon \Lambda^2 TM \to L^2\Lambda^2_- T^*M$ denote the antiselfdual part of R^{sd} and let $R^{B,0}\colon \Lambda^2 TB \to L_B^2\Lambda^2 T^*B$ denote the skew part of R^B. Then, omitting pullbacks, $R^{sd,-}(J)\xi = -\tfrac{1}{2}*_B R^{B,0}(*_B J\xi)$, since $R^{sd,-}(\xi \wedge X) = 0$. The symmetric traceless part of $J \mapsto R^{sd,-}(J)$ is the antiselfdual Weyl tensor W^- and the symmetric traceless part of $\chi \mapsto *_B R^{B,0}(*_B \chi)$ is the symmetric traceless Ricci tensor of D^B, which proves the theorem. $\qquad\square$

An explicit formula for the relationship between r^{sd} and r^B will be useful later.

4.7. Proposition. *Let ξ generate a selfdual conformal submersion π from a selfdual space M to an Einstein-Weyl space B. Then $F_-^{sd} = \frac{1}{4}(F^B - *F^B)$ and*

$$\operatorname{sym} r^{sd} = \tfrac{1}{12}\operatorname{scal}^B(\mathrm{id} - 2\xi \otimes \xi) + \tfrac{1}{4}\big(*_B F^B \otimes \xi + \xi \otimes *_B F^B\big).$$

Proof. Let X, Y be basic vector fields. Since $\langle R^{sd,-}(\xi \wedge X), \xi \wedge Y \rangle = 0$, it follows that

$$r^{sd}(X,Y) + r^{sd}(\xi,\xi)\langle X,Y\rangle + *\big(\xi \wedge r^{sd}(\xi,\cdot) \wedge X \wedge Y\big) = 0.$$

On the other hand, since $\langle R^{sd}(*\xi \wedge X), \xi \wedge Y - *(\xi \wedge Y)\rangle = \langle R^{B,0}(*_B JX), *_B JY\rangle$, it follows that

$$r^{sd}(X,Y) - (\operatorname{tr}_{\mathcal{H}} r^{sd})\langle X,Y\rangle - *\big(\xi \wedge r^{sd}(\cdot,\xi) \wedge X \wedge Y\big)$$
$$= r^B(X,Y) - (\operatorname{tr} r^B)\langle X,Y\rangle = -\tfrac{1}{2}F^B(X,Y) - \tfrac{1}{6}\operatorname{scal}^B\langle X,Y\rangle.$$

The stated formulae follow easily from these. □

Note that $F^{sd} = \frac{1}{2}(F^0 + F^B)$ and so the formula for F_-^{sd} follows immediately from the fact that $F_-^0 = 0$. On the other hand F^0 and F_+^{sd} are not basic in general.

The form of the Jones-Tod construction stated in Theorem I gives a procedure for constructing Einstein-Weyl spaces from selfdual spaces. For the inverse construction, the following reformulation is useful.

4.8. Proposition. *Suppose that (M, c) is an oriented conformal 4-manifold, that ξ generates a conformal submersion π over an Einstein-Weyl manifold B, and that $D^0 = \pi^*D^B - \omega$, where D^B is the Weyl derivative on L_B^1 and $\omega = -(*d^D\xi)(\xi,\cdot)$ (computed using any Weyl derivative D). Then (M, c) is selfdual and ξ is selfdual.*

This follows immediately from Theorem I and Proposition 4.4: $\pi^*D^B - \omega$ has selfdual Faraday curvature since π^*D^B is basic. Therefore, an inverse Jones-Tod construction will be obtained if the equation $D^0 = \pi^*D^B - \omega$ can be interpreted as an equation on B. There are two ways to do this. Recall from Proposition 3.2, that a conformal structure on a fibre bundle $\pi \colon M \to B$ over a conformal manifold B is determined by a connection 1-form $\eta \colon TM \to \mathcal{V}$ and a relative length scale $w \colon \pi^*L_B^1 \to \mathcal{V}$, where I have assumed the fibres are oriented and one dimensional so that $L_{\mathcal{V}}^1 = \mathcal{V}$. Choosing a fibre coordinate t identifies M with an open subset of $B \times \mathbb{T}$, for a connected 1-manifold $\mathbb{T}$, and provides trivialization of $\mathcal{V}$ and a flat connection 1-form dt, so that $\eta = dt + A$.

The inverse construction can now be formulated as a nonlinear evolution equation on B, or as a monopole equation with infinite dimensional gauge group. This idea is closely related to Ward's constructions of hyperKähler metrics [27].

4.9. Theorem II. *Let (M, c) be an oriented conformal 4-manifold with a conformal submersion over an Einstein-Weyl space (B, c_B, D^B). Then $\pi \colon M \to B$ is locally conformal to $\pi_1 \colon B \times \mathbb{T} \to B$ with conformal structure*

$$\mathsf{c} = \pi^*\mathsf{c}_B + w^{-2}(dt + A)^2$$

for $A \in \mathrm{C}^\infty(B \times \mathbb{T}, \pi^*T^*B)$ and $w \in \mathrm{C}^\infty(B \times \mathbb{T}, \pi^*L_B^{-1})$.

Furthermore M and π are selfdual if and only if (A, w) satisfy the evolution equation

$$*_B(D^B w + \dot{A}w - A\dot{w}) = dA + \dot{A} \wedge A,$$

where $w\colon \mathbb{T} \to \mathrm{C}^\infty(B, L_B^{-1})$ *and* $A\colon \mathbb{T} \to \mathrm{C}^\infty(B, T^*B)$ *are viewed as a time-dependent density and 1-form on B, so that a dot denotes differentiation with respect to t, while $D^B w$ and dA are the derivatives on B.*

Equivalently, writing instead $w = w(t)d/dt\colon L_B^1 \to \mathrm{Vect}(\mathbb{T})$ *and* $A = A(t) = A(t)d/dt\colon TB \to \mathrm{Vect}(\mathbb{T})$, *the equation is the Einstein-Weyl Bogomolny equation*

$$*_B(D^B w + [A, w]) = F^A := dA + \tfrac{1}{2}[A \wedge A]$$

where $[\cdot, \cdot]$ denotes the Lie bracket of vector fields in $\mathrm{Vect}(\mathbb{T})$.

Proof. A conformal submersion certainly has the form given and the two equations on B are clearly equivalent. I will show that the evolution equation is equivalent to the fact that $\pi_1^* D^B = D^0 + \omega$. To do this, I will work in the (arbitrarily chosen) gauge $g = w^2 \mathsf{c}$ and rewrite the equation $\pi_1^* D^B = D^0 + \omega = D^g - \frac{1}{3}(div^g \xi)\xi + (d^g \xi)(\xi, \cdot) - (*d^g \xi)(\xi, \cdot)$ using the fact that $D^g w = 0$.

In the chosen gauge, $w\xi = dt + A$ and so

$$wd^g\xi = d(w\xi) = dA + dt \wedge \dot{A} = w\xi \wedge \dot{A} + dA + \dot{A} \wedge A.$$

It follows that $(d^g\xi)(\xi, \cdot) = \dot{A}$ and

$$(*d^g\xi)(\xi, \cdot)w = *(\xi \wedge d^g\xi)w = -*_B(dA + \dot{A} \wedge A).$$

Writing div^g in terms of $*d^g*$ readily yields $\frac{1}{3}\, div^g\, \xi = \dot{w}$. Therefore:

$$\begin{aligned}
0 = D^g w &= (\pi_1^* D^B)w - \dot{w}w\xi + \dot{A}w + *_B(dA + \dot{A} \wedge A) \\
&= D^B w + \dot{w}dt - \dot{w}(dt + A) + \dot{A}w + *_B(dA + \dot{A} \wedge A) \\
&= D^B w - A\dot{w} + \dot{A}w + *_B(dA + \dot{A} \wedge A).
\end{aligned}$$

This completes the proof and also shows that $D^0 = D^g - \dot{w}\xi + \dot{A}$.   □

4.10. *Remarks.* A gauge theoretic interpretation is obtained by viewing M as an open subset of an associated bundle $\tilde{M} = P \times_{Diff(\mathbb{T})} \mathbb{T}$, where $Diff(\mathbb{T})$ is the group of orientation preserving diffeomorphisms of $\mathbb{T}(= S^1$ or $\mathbb{R})$ and P is the principal $Diff(\mathbb{T})$-bundle whose fibre at $x \in B$ consists of the orientation preserving diffeomorphisms $\mathbb{T} \to \tilde{M}_x$. Choosing a gauge, i.e., a (local) section of P, identifies $\tilde{M}$ (locally) with $B \times \mathbb{T}$. Thus the gauge freedom is essentially the choice of t coordinate made above. If $\tilde{t} = f(b, t)$ for a function f on $B \times \mathbb{T}$ with $\dot{f} \neq 0$ then $w(t) = \tilde{w}(\tilde{t})/\dot{f}$ and $A(t) = (\tilde{A}(\tilde{t}) + d_B f)/\dot{f}$.

The classical Jones-Tod construction arises by reduction to a one dimensional translational subgroup S^1 or $\mathbb{R}$. In this case, there is a preferred gauge in which to work: the constant length gauge of the conformal vector field K ($D^0 = D^{|K|}$ is exact). The section $t = 0$ of M over B makes it into a line bundle and M may be

recovered from a linear differential equation on B:

$$\mathsf{c} = \pi^*\mathsf{c}_B + w^{-2}(dt + A)^2$$

where
$$*_B D^B w = dA \quad \text{for} \quad w \in C^\infty(B, L_B^{-1}).$$

This equation for (A, w) is often called the (abelian) monopole equation. Other finite dimensional subgroups of $Diff(\mathbb{T})$ will be studied later.

Now suppose that J is an antiselfdual almost complex structure on M which is invariant with respect to ξ, i.e., $\mathcal{L}_\xi^0 J = 0$. Let $\Omega_J(X, Y) = \langle JX, Y \rangle$ be the conformal Kähler form of J and D the unique Weyl derivative such that $d^D \Omega_J = 0$. Then it is well known that J is integrable if and only if $DJ = 0$; D is then called the *Kähler-Weyl* connection of J. If $DJ = 0$ then $(\mathcal{L}_\xi^0 J)X = J(D^0 \otimes D)_X \xi - (D^0 \otimes D)_{JX}\xi$, and so J is invariant iff ξ is *holomorphic* in the sense that $(D^0 \otimes D)\xi$ is complex linear. The following generalizes a theorem of [8] to conformal submersions.

4.11. Theorem III. *Let M be an oriented conformal 4-manifold, let ξ generate a selfdual conformal submersion over a manifold B and let J be an invariant antiselfdual almost complex structure on M. Define $D = D^{sd} - \kappa\xi - \tau\chi$ for basic sections τ and κ of L^{-1}, where $\chi = J\xi$. Then $DJ = 0$ iff $D^B\chi = \tau(id - \chi \otimes \chi) + \kappa *_B \chi$ on B. Hence J is integrable if and only if χ is a shear-free geodesic congruence, in the sense that $D^B\chi$ has the above form.*

Proof. Since J is invariant, χ is invariant, hence basic, since it is horizontal. Note that $D_\xi J = D_\xi^{sd} J - \tau[\![\xi \triangle \chi, J]\!] = D_\xi^{sd} J = \mathcal{L}_\xi^0 J = 0$, so it remains to compute $D_X J$ for horizontal vector fields X. Since $\Omega_J = \xi \wedge \chi - *(\xi \wedge \chi)$ it follows that

$$D_X\Omega_J = D_X\xi \wedge \chi + \xi \wedge D_X\chi - *(D_X\xi \wedge \chi + \xi \wedge D_X\chi),$$

where
$$D_X\xi \wedge \chi = D_X^{sd}\xi \wedge \chi - \kappa X \wedge \chi$$

and
$$\xi \wedge D_X\chi = \xi \wedge D_X^{sd}\chi - \tau\xi \wedge \left(X - \langle\chi, X\rangle\chi\right).$$

Therefore (see the proof of 4.5)

$$\xi \wedge D_X\chi - *(D_X\xi \wedge \chi) = \xi \wedge D_X^B\chi - \tau\xi \wedge \left(X - \langle\chi, X\rangle\chi\right) + \kappa *(X \wedge \chi).$$

Since the right hand side is a vertical 2-form, it follows that $D_X J = 0$ iff

$$D_X^B\chi - \langle D_X^B\chi, \xi\rangle = \tau(X - \langle\chi, X\rangle\chi) + \kappa \iota_X *_B \chi.$$

To prove the final statement, suppose that J is invariant and integrable with Kähler-Weyl connection D. Then $(D^0 \otimes D)\xi = -\kappa\, id + \frac{1}{2}\tau J + \frac{1}{2}(d^{D^0 \otimes D}\xi)^+$, where $(d^{D^0 \otimes D}\xi)^+$ is a selfdual 2-form and κ, τ are sections of L^{-1}. It follows that

$$(d^{D^0 \otimes D}\xi)(\xi, \cdot) = \tau\chi + (d^{D^0 \otimes D}\xi)^+(\xi, \cdot)$$
$$(*d^{D^0 \otimes D}\xi)(\xi, \cdot) = -\tau\chi + (d^{D^0 \otimes D}\xi)^+(\xi, \cdot).$$

Therefore $D^{sd} = D + \kappa\xi + \tau\chi$. It remains to check that κ and τ are basic. Since $DJ = 0$ and $\mathcal{L}_\xi^0 J = 0$ it follows that $[\![\mathcal{L}_\xi^0 D_X, J]\!] = 0$. By the linearized Koszul

formula, this implies $\mathcal{L}_\xi^0 D = \frac{1}{2}F^0(\xi)$ on L^1 and therefore

$$R_{\xi,X}^D + (D^0 \otimes D)_X\left(-\kappa\,id + \tfrac{1}{2}\tau J + \tfrac{1}{2}(d^{D^0 \otimes D}\xi)^+\right)$$
$$= \mathcal{L}_\xi^0 D_X = \tfrac{1}{2}F^0(\xi,X)id + \tfrac{1}{2}F^0(\xi) \vartriangle X - \tfrac{1}{2}F^0(X) \vartriangle \xi + \tfrac{1}{2}\langle \xi, X\rangle F^0.$$

The identity and J components of this formula give $D^0\kappa + \frac{1}{2}F^0(\xi) = F^D(\xi)$ and $D^0\tau = \rho^D(\xi)$ where $\rho^D(X,Y)$ is the Ricci form of D, defined to be the contraction of $R_{X,Y}^D$ with J. In particular $D_\xi^0\kappa = 0 = D_\xi^0\tau$. $\square$

4.12. *Remark.* τ and κ are called the *divergence* and *twist* of the congruence χ.

Assume now that W is selfdual. Then so are F^D and ρ^D (see e.g. [8]), and hence they are uniquely determined by their contractions with ξ. It follows that (M, J) is locally hypercomplex iff $D^0\tau = 0$ iff $\tau = 0$ or D^0 is exact and τ is constant in this gauge. This implies that a hyperCR structure on the Einstein-Weyl quotient B (the case $\tau = 0$) induces a hypercomplex structure on M, generalizing a result of Gauduchon and Tod [12] to conformal submersions. On the other hand, (M, J) is locally scalar-flat Kähler iff $D^0\kappa + \frac{1}{2}F^0(\xi) = 0$, so the presence of F^0 obstructs a naive generalization of LeBrun's work [19] to this context. This will be remedied in the next two sections.

I next generalize a result of Mason and Tod, which was used by Tod [24] to give a general description of selfdual Einstein metrics with a Killing vector field.

4.13. **Theorem IV.** *Let (M, c, D^{ew}) be a selfdual Einstein-Weyl 4-manifold and let ξ generate a selfdual conformal submersion with minimal Weyl derivative D^0. Then M admits a canonical compatible Kähler-Weyl structure on the open set where the antiselfdual part of $(D^0 \otimes D^{ew})\xi$ is nonzero.*

More precisely, if this antiselfdual part is τJ where $J^2 = -id$ then J is integrable, with Kähler-Weyl connection $D = D^{ew} - \tau^{-1}D^0\tau = D^{ew} + D^0 - D^\tau$, where D^τ is defined by $D^\tau\tau = 0$.

Proof. It suffices to prove that $DJ = 0$. Observe first that $(D_X^0\tau)J + \tau D_X^{ew}J$ is the antiselfdual part of $(D^0 \otimes D^{ew})_X(D^0 \otimes D^{ew})\xi = \mathcal{L}_\xi^0 D_X^{ew} - R_{\xi,X}^{ew}$, where R^{ew} is the curvature of D^{ew}. Expanding the curvature and using the linearized Koszul formula to compute the Weyl-Lie derivative gives

$$(D_X^0\tau)J + \tau D_X^{ew}J$$
$$= \left[\tfrac{1}{2}F^{ew}(\xi) \vartriangle X + \tfrac{1}{2}F^{ew}(X) \vartriangle \xi + \tfrac{1}{12}scal^{ew}\,\xi \vartriangle X - D^0\kappa \vartriangle X + \tfrac{1}{2}\xi \vartriangle F^0(X)\right]^-$$
$$= \left[(F^{ew}(\xi) + \tfrac{1}{12}scal^{ew}\,\xi - D^0\kappa - \tfrac{1}{2}F^0(\xi)) \vartriangle X\right]^-,$$

where $[...]^-$ denotes the antiselfdual part, κ is minus the identity component of $(D^0 \otimes D^{ew})\xi$, $scal^{ew}$ is the scalar curvature of D^{ew}, F^{ew} is the Faraday curvature of D^{ew}, and I have used the fact that F^0 and F^{ew} are selfdual. The precise form of this expression is now not important: it suffices to observe that it is of the form $[J\alpha \vartriangle X]^-$ for some 1-form α. Since $*(J\alpha \vartriangle X) = -\alpha \vartriangle JX + \alpha(X)J$, it follows that $J\alpha \vartriangle X - *(J\alpha \vartriangle X) = -[\![\alpha \vartriangle X, J]\!] - \alpha(X)J$ and the commutator term is

orthogonal to J. Since $D^{ew}J$ is also orthogonal to J, $D^0_X\tau = -\alpha(X)$ and $\tau D^{ew}_X J = -[\![\alpha \triangle X, J]\!] = [\![D^0\tau \triangle X, J]\!]$, i.e., J is parallel with respect to $D^{ew} - \tau^{-1}D^0\tau$.   □

The theorems of this section reduce to known results when the conformal submersion is generated by a symmetry. However, the defining equation for conformal submersions and the generalized monopole equation, unlike the conformal Killing equation and the abelian monopole equation, are nonlinear. For the results of this section to be interesting, it is therefore essential to find situations in which these equations can be solved. This will be done next.

5. AFFINE CONFORMAL SUBMERSIONS

An affine structure on a submersion $\pi\colon M \to B$ is a flat torsion-free connection on each fibre. This identifies M, at least locally, with an affine bundle modelled on the vector bundle whose fibre at each point of B is the space of parallel vector fields on the corresponding fibre of M. For conformal submersions with oriented one dimensional fibres, the vertical bundle $\mathcal{V}$ of M is isomorphic to L^1 and so the vertical part D_ξ of any Weyl derivative D defines an affine structure on M.

There are many choices of affine structure on M, but such a choice is only helpful if the conformal structure on M is affine; that is, in terms of Proposition 3.2, the nonlinear connection on M is an affine connection, and the relative length scale is affine along the fibres. If such a "good" affine structure can be found, I will say that π is an *affine conformal submersion*.

More precisely, a nonlinear connection $\mathcal{H}$ induces a linearized connection on the infinite dimensional vector space of vertical vector fields defined by $\mathcal{D}_X U = [\tilde{X}, U]$, where U is a vertical vector field, X is a vector field on B and $\tilde{X}$ is its horizontal lift, so that $[\tilde{X}, U]$ is vertical. $\mathcal{H}$ is affine iff the parallel vertical vector fields on each fibre are preserved by $\mathcal{D}$; this then induces the linearized connection on the model vector bundle. Affine connections form an affine space modelled on 1-forms on B with values in the affine vector fields on $\mathcal{V}$. Similarly, the relative length scale $w\colon \pi^*L^1_B \to \mathcal{V}$ is affine iff it maps basic densities to affine vector fields, in which case it may be viewed as a (-1)-density on B with values in the affine vector fields.

The above approach and the next proposition arose from discussions with Paul Gauduchon in a joint effort to understand affine conformal submersions.

5.1. Proposition. *Let M be a conformal manifold and let ξ generate be a conformal submersion π over B with minimal Weyl derivative D^0. Define an affine structure D_ξ on π by the Weyl derivative $D = D^0 + \lambda\xi$ where λ is a section of L^{-1}.*

  (i) *The connection $\mathcal{H}$ on $M \to B$ is affine with respect to D_ξ iff $F^D(\xi) = 0$.*
  (ii) *The relative length scale is affine with respect to D_ξ iff λ is basic.*

Proof. The D_ξ-parallel vertical vector fields are defined by identifying $\mathcal{V}$ with L^1 using ξ. Hence the linearized connection may be defined on $\mu \in C^\infty(M, L^1)$ by

$$(\mathcal{D}_X\mu)\xi = [\tilde{X}, \mu\xi] = D^0_{\tilde{X}}(\mu\xi) - \mu D^0_\xi\tilde{X} = (D^0_{\tilde{X}}\mu)\xi - \mu(D^0_\xi\tilde{X} - D^0_\xi\tilde{X})$$
$$= (D^0_{\tilde{X}}\mu)\xi + \mu\mathcal{L}^0_\xi\tilde{X} = (D_{\tilde{X}}\mu)\xi$$

since $\tilde{X}$ is invariant and $D - D^0$ is vertical. Hence $\mathcal{H}$ is affine iff $D_\xi(D_{\tilde{X}}\mu) = 0$ for all μ with $D_\xi\mu = 0$. Since $[U, \tilde{X}]$ is vertical for U vertical, $D_\xi(D_{\tilde{X}}\mu) = F^D(\xi, \tilde{X})\mu$ and so $\mathcal{H}$ is affine iff $F^D(\xi) = 0$.

The relative length scale is section w of $\pi^* L_B^{-1} \otimes \mathcal{V}$. This is affine iff its vertical derivative with respect to the affine structure, as a section of $\pi^* L_B^{-1} \otimes \mathcal{V}^* \otimes \mathcal{V} \cong \pi^* L_B^{-1}$, is basic. Identifying $\pi^* L_B^1$ and $\mathcal{V}$ with L^1 identifies w with the identity map in $L^{-1} \otimes L^1$ but its vertical derivative must be computed with respect to the covariant derivative $D^0 \otimes D$ and so w is affine iff $0 = D_\xi^0(D^0 \otimes D)_\xi id = D_\xi^0 \lambda$. $\square$

Since $F^D(\xi) = F^0(\xi) + d(\lambda\xi)(\xi) = F^0(\xi) + (D_\xi^0 \lambda)\xi - D^0\lambda$, it follows immediately that the conformal submersion is affine if $D^0\lambda = F^0(\xi)$. Hence there is an obstruction to the existence of a good affine structure for a conformal submersion: since $-F^0\lambda = d^0(F^0(\xi)) = \mathcal{L}_\xi^0 F^0$, the Weyl-Lie derivative of F^0 must be a multiple of F^0. If a good affine structure exists, it is essentially unique: any two differ by a section μ of L^{-1} with $D^0\mu = 0$ which implies that the affine structures are equal or D^0 is exact and μ is constant.

I now return to four dimensions and the Jones-Tod construction.

5.2. Theorem V. *Let (B, c_B, D^B) be Einstein-Weyl and let*

$$\mathsf{c} = \pi^*\mathsf{c}_B + (tw_1 + w_0)^{-2}(dt + tA_1 + A_0)^2$$

where

$$*_B D^B w_1 = dA_1$$

and

$$*_B(D^B w_0 + A_1 w_0 - A_0 w_1) = dA_0 + A_1 \wedge A_0$$

for some $w_0, w_1 \in C^\infty(B, L_B^{-1})$ and $A_0, A_1 \in C^\infty(B, T^*B)$. Then c is selfdual with a selfdual affine conformal submersion ξ over B, and $D^0 w_1 = F^0(\xi)$.

A selfdual conformal structure with a selfdual conformal submersion ξ arises locally in this way if and only if it admits such a section w_1 of L^{-1}.

Proof. An affine conformal submersion certainly has the form given. It remains to apply this Ansatz to the equations of Theorem II, by writing $w = tw_1 + w_0$ and $A = tA_1 + A_0$. Now

$$D^B w + \dot{A}w - A\dot{w} = tD^B w_1 + D^B w_0 + w_0 A_1 - w_1 A_0$$

$$dA + \dot{A} \wedge A = tdA_1 + dA_0 + A_1 \wedge A_0$$

and the linear and constant terms (in t) of these equations prove the result. Note that $D^0 = D^g - w_1\xi + A_1$ and so $F^0(\xi) = -d(w_1\xi)(\xi) = D^0 w_1 - (D_\xi^0 w_1)\xi = D^0 w_1$ since w_1 is basic. The characterization follows from the previous proposition. $\square$

5.3. *Remarks.* The freedom in the choice of affine coordinate t corresponds to the gauge freedom for the affine monopole equations. If $\tilde{t} = at + b$ for basic functions a, b, write $w_1 = \tilde{w}_1$, $w_0 = a^{-1}(\tilde{w}_0 + b\tilde{w}_1)$, $A_1 = \tilde{A}_1 + a^{-1}da$ and $A_0 = a^{-1}(\tilde{A}_0 + b\tilde{A}_1 + db)$. One immediately verifies, by substituting into the affine monopole equations, that $(\tilde{A}, \tilde{w})$ is a solution if (A, w) is. Note that $t\mu_g = tw^{-1}$ is a well defined section of L^1 up to translation by a basic section of L^1: it may be fixed by choosing a

section of $M \to B$. The induced exact Weyl derivative is $D^{t\mu_g} = D^g - t^{-1}dt = D^0 - t^{-1}w_0\xi + t^{-1}A_0$.

The equations show that the affine Jones-Tod correspondence reduces to the classical case in two ways. Firstly the *linear part* of the affine monopole equation is an abelian monopole equation for (A_1, w_1), and if (A_0, w_0) is zero, the affine bundle M is isomorphic to the model line bundle. Secondly if (A_1, w_1) is zero, the *translational part* of the of the affine monopole equation is an abelian monopole equation for (A_0, w_0): the model line bundle is trivial, and so M is a principal $\mathbb{R}$-bundle. On the other hand if the solution (A_1, w_1) is nontrivial, then it gives a linearization of M, i.e., a selfdual space with an affine conformal submersion is affinely modelled on a selfdual space with a conformal vector field.

6. Scalar-flat Kähler and hyperKähler metrics

The affine monopole equations give give a new method for constructing selfdual spaces from linear differential equations, since the translational part is linear once a solution of the linear part is chosen. In particular, this method gives all scalar-flat Kähler metrics with a holomorphic selfdual conformal submersion.

6.1. Theorem VI. *Suppose (c_B, D^B) is an Einstein-Weyl space and let χ be a shear-free geodesic congruence with twist κ and divergence τ. Then for any solution $\rho \in \mathrm{C}^\infty(B, L_B^{-2})$, $\Phi \in \mathrm{C}^\infty(B, L_B^{-1}T^*B)$ of the linear differential equation*

$$*_B(D^B\rho + 2\tau\chi\rho + 2\kappa\Phi) = d^B\Phi + 2\tau\chi \wedge \Phi,$$

the metric

$$g = (\rho - 2\mu_t^{-1}\kappa)\pi^*\mathsf{c}_B + \frac{\left(D^B(\mu_t^{-1}) + 2\tau\chi\mu_t^{-1} + \Phi\right)^2}{\rho - 2\mu_t^{-1}\kappa}$$

on an affine bundle π over B, where μ_t^{-1} is a section of L^{-1} increasing along the fibres, is scalar-flat Kähler, and is hyperKähler if and only if $D^0\tau = 0$ (i.e., iff $\tau = 0$ or D^0 is exact and τ is constant in this gauge).

Any 4-dimensional scalar-flat Kähler metric admitting a holomorphic selfdual conformal submersion arises locally in this way.

Proof. Suppose g is a scalar-flat Kähler metric and ξ generates a holomorphic selfdual conformal submersion. Then by Theorem III, $D^g = D^{sd} - \kappa\xi - \tau\chi$ where $D^B\chi = \tau(\mathrm{id} - \chi \otimes \chi) + \kappa *_B \chi$, and furthermore, $D^0\kappa + \frac{1}{2}F^0(\xi) = 0$, since $F^{D^g} = 0$. Hence $D^0_\xi - 2\kappa$ defines an affine structure on M making the submersion affine and $w_1 = -2\kappa$.

The analysis of shear-free geodesic congruences in [8] shows that $*_B D^B\kappa = \frac{1}{2}F^B - d(\tau\chi)$, and hence, choosing any gauge μ_B, one can take $A_1 = -\omega_B + 2\tau\chi$. The translational part of the monopole equation is therefore:

$$*_B(D^B w_0 - \omega_B w_0 + 2\tau\chi w_0 + 2\kappa A_0) = dA_0 + (-\omega_B + 2\tau\chi) \wedge A_0.$$

Now the Levi-Civita connection of the affine gauge is $D^0 + w_1\xi - A_1 = D^0 - 2\kappa\xi - 2\tau\chi + \omega_B$, whereas the Levi-Civita connection of the μ_B-gauge is $D^B - \omega_B$. The

barycentre of these is $D^{sd} - \kappa\xi - \tau\chi = D^g$, which iidentifies the scalar-flat Kähler metric within the conformal class as

$$g = (w_0 - 2t\kappa)\pi^* g_B + (w_0 - 2t\kappa)^{-1}(dt - t\omega_B + 2\tau\chi t + A_0)^2$$

and the gauge-invariant form of the theorem is obtained by setting $\rho = \mu_B^{-1} w_0$, $\Phi = \mu_B^{-1} A_0$ and $\mu_t^{-1} = t\mu_B^{-1}$.   □

When $\kappa = 0$ this is LeBrun's construction of scalar-flat Kähler metrics with Killing fields [19], including the Gibbons-Hawking Ansatz for hyperKähler metrics as the special case $\tau = 0$ [13]. On the other hand, when $(A_0, w_0) = 0$, this theorem reduces to the construction of scalar-flat Kähler metrics with homothetic vector fields [8], including, as the special case $\tau = 0$, the hyperKähler metrics of [12].

There is a plentiful supply of shear-free geodesic congruences on any Einstein-Weyl space [8], and many hyperCR Einstein-Weyl spaces (admitting a shear-free divergence-free geodesic congruence) are known explicitly. The only remaining problem, therefore, is to solve a linear differential equation, and I now discuss some nontrivial examples. In [9], the following Einstein-Weyl structures were found from solutions of the SU(∞) Toda field equation.

$$g_B = (z + h)(z + \overline{h})g_{S^2} + dz^2, \qquad \omega_B = -\frac{2z + h + \overline{h}}{(z + h)(z + \overline{h})}dz,$$

where h is a holomorphic function on an open subset of S^2 and $D^B = D^g + \omega$. Note that the weightless unit vector field dual to dz is a shear-free geodesic congruence with vanishing twist ($\tau \neq 0, \kappa = 0$). These spaces also admit shear-free geodesic congruences with vanishing divergence ($\tau = 0, \kappa \neq 0$), i.e., they are hyperCR, and they are called the *hyperCR-Toda spaces*.

Applying the classical Jones-Tod construction to these spaces gives conformal structures of the form $\mathsf{c} = \pi^* \mathsf{c}_B + w^{-2}(\beta + v\,dz)^2$, where $\beta = dt + \theta$ for a 1-form θ on B orthogonal to dz, and $*_B D^B w = dA$ with $A = \theta + v\,dz$.

These conformal structures admit a compatible scalar-flat Kähler metric and also a compatible hypercomplex structure. The conformal vector field $\partial/\partial t$ is a Killing field of the scalar-flat Kähler metric and triholomorphic with respect to the hypercomplex structure.

For certain solutions of the abelian monopole equation, $\partial/\partial z$ defines a conformal submersion. To see this, write $\mathsf{c} = \varepsilon_0^2 + \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2$ where ε_0 and ε_3 are the weightless unit 1-forms corresponding to $w\,dz$ and $\beta + v\,dz$. The weightless unit 1-form dual to $\partial/\partial z$ is $\xi = (w\varepsilon_0 + v\varepsilon_3)/\sqrt{w^2 + v^2}$ and so $\varepsilon_0^2 + \varepsilon_3^2 - \xi^2 = (v\varepsilon_0 - w\varepsilon_3)^2/(w^2 + v^2) = w^2\beta^2/(w^2 + v^2)$. Hence if β and $(w^2 + v^2)|z + h|^2$ are independent of z, then $\partial/\partial z$ will define a conformal submersion with quotient $(w^2 + v^2)|z + h|^2 g_{S^2} + \beta^2$.

Now Ian Strachan has pointed out [9] that for any holomorphic function f,

$$w = \frac{1}{2}\left(\frac{f}{z + h} + \frac{\overline{f}}{z + \overline{h}}\right), \quad v = \frac{1}{2i}\left(\frac{f}{z + h} - \frac{\overline{f}}{z + \overline{h}}\right), \quad d\beta = \tfrac{1}{2}(f + \overline{f})vol_{S^2}$$

defines a solution of the monopole equation. Clearly β and $(w^2 + v^2)|z + h|^2$ are independent of z and so $\partial/\partial z$ defines a conformal submersion. Explicitly, c has a

compatible metric

$$g = \left(\frac{z+h}{2f} + \frac{z+\overline{h}}{2\overline{f}}\right)^2 (|f|^2 g_{S^2} + \beta^2) + \left[dz + i\left(\frac{z+h}{2f} - \frac{z+\overline{h}}{2\overline{f}}\right)\beta\right]^2$$

and so this submersion is obviously affine, with affine coordinate z. The quotient conformal 3-manifold $\tilde{B}$ admits an Einstein-Weyl structure:

$$g_{\tilde{B}} = |f|^2 g_{S^2} + \beta^2, \qquad \omega_{\tilde{B}} = \frac{i}{2}\left(\frac{1}{f} - \frac{1}{\overline{f}}\right)\beta.$$

These are the Einstein-Weyl spaces with *geodesic symmetry* described in [8]. One easily checks that g is given by a solution of the affine monopole equations with $w_1 = -2\kappa_s$ where κ_s is the twist of the geodesic symmetry on $\tilde{B}$, i.e., $d\beta = 2\kappa_s *_B \beta$. Hence these scalar-flat Kähler metrics with compatible hypercomplex structures could have been constructed directly as selfdual affine conformal submersions over the Einstein-Weyl spaces with geodesic symmetry. When $f = ah + b$ for $a, b \in \mathbb{R}$, these metrics are conformally Einstein [9] and will feature again in section 10.

7. Projective conformal submersions

A natural generalization of an affine conformal submersion is a projective conformal submersion. A projective structure on a 1-manifold is a second order linear differential operator from $L^{1/2}$ to $L^{-3/2}$ which has no first order term with respect to any Weyl derivative, and the same definition may be applied fibrewise to a congruence ξ. Hence any Weyl derivative D induces a projective structure $\mu \mapsto D_\xi(D_\xi\mu)$. Note that $(D+\gamma)_\xi\big((D+\gamma)_\xi\mu\big) = (D_\xi - \frac{1}{2}\gamma(\xi))(D_\xi\mu + \frac{1}{2}\gamma(\xi)\mu) = D_\xi(D_\xi\mu) + \frac{1}{2}D_\xi(\gamma(\xi))\,\mu - \frac{1}{4}(\gamma(\xi))^2\,\mu$, verifying that the condition of vanishing first order term is independent of the Weyl derivative.

A conformal submersion π will be called *projective* iff there is a projective structure on π such that the connection $\mathcal{H}$ is projective and the relative length scale w takes values in the projective vector fields: recall that these are characterized as being quadratic in any projective coordinate.

7.1. *Remark.* A curve in a conformal manifold, with weightless unit tangent ξ has a canonical projective structure given by $(D_\xi)^2 + \frac{1}{2}r^D(\xi, \xi) + \frac{1}{4}|D_\xi\xi|^2$, and is called a conformal geodesic if $D_\xi(D_\xi\xi) + |D_\xi\xi|^2\xi - r^D(\xi) + r^D(\xi, \xi)\xi = 0$; these expressions are independent of the Weyl derivative D. However, if ξ generates a projective conformal submersion, there is no reason for the projective structure to equal the canonical one, nor will the fibres be conformal geodesics in general.

7.2. **Proposition.** *Let M be a conformal manifold and let ξ generate a conformal submersion over B with minimal Weyl derivative D^0. Then the conformal submersion is projective with respect to the projective structure $(D^0_\xi)^2 + \frac{1}{2}\rho$, for a section ρ of L^{-2}, iff $D^0\rho = \mathcal{L}^0_\xi F^0(\xi)$.*

Proof. Write the projective structure as $(D_\xi)^2$ where $D_\xi = D^0_\xi + \lambda$ is a compatible affine structure, so that $\rho = D^0_\xi\lambda - \frac{1}{2}\lambda^2$. Then the connection $\mathcal{H}$ is projective iff it maps D_ξ-parallel vertical vector fields to D_ξ-affine vertical vector fields, i.e., iff

$D_\xi \mu = 0 \implies \partial_\xi(D_\xi(D_X \mu)) = 0$ for basic vector fields X. This condition reduces easily to $(\mathcal{L}_\xi^0 - \lambda)(F^0(\xi, X) - D_X^0 \lambda) = 0$.

The relative length scale is projective iff $0 = (D^0 \otimes D)_\xi(D_\xi^0 \lambda) = (\mathcal{L}_\xi^0 - \lambda)(D_\xi^0 \lambda)$. Hence the conformal submersion is projective iff

$$
\begin{aligned}
0 &= (\mathcal{L}_\xi^0 - \lambda)(F^0(\xi) - D^0 \lambda) \\
&= (\mathcal{L}_\xi^0 - \lambda)F^0(\xi) - (\mathcal{L}_\xi^0 D^0)\lambda - D^0(\mathcal{L}_\xi^0 \lambda) + \lambda D^0 \lambda \\
&= \mathcal{L}_\xi^0 F^0(\xi) - D^0(D_\xi^0 \lambda - \tfrac{1}{2}\lambda^2)
\end{aligned}
$$

since $\mathcal{L}_\xi^0 D^0 = -F^0(\xi)$ on L^{-1}. $\qquad\qquad\qquad\qquad\qquad\square$

There is an obstruction to solving this for ρ, since it implies $-2F^0(\xi)\rho = \mathcal{L}_\xi^0 \mathcal{L}_\xi^0 F^0(\xi)$.

7.3. Theorem VII. *Let (B, c_B, D^B) be Einstein-Weyl and let*

$$
\mathsf{c} = \pi^* \mathsf{c}_B + (t^2 \mathsf{w}_2 + t \mathsf{w}_1 + \mathsf{w}_0)^{-2}(dt + t^2 A_2 + t A_1 + A_0)^2,
$$

where $\qquad\qquad *_B(D^B + \mathrm{ad} A)\mathsf{w} = F^A := dA + A \wedge A$

for $\qquad\qquad A = \begin{pmatrix} \tfrac{1}{2}A_1 & A_0 \\ -A_2 & -\tfrac{1}{2}A_1 \end{pmatrix}, \qquad \mathsf{w} = \begin{pmatrix} \tfrac{1}{2}\mathsf{w}_1 & \mathsf{w}_0 \\ -\mathsf{w}_2 & -\tfrac{1}{2}\mathsf{w}_1 \end{pmatrix}$

with $\mathsf{w}_0, \mathsf{w}_1, \mathsf{w}_2 \in \mathrm{C}^\infty(B, L_B^{-1})$ *and* $A_0, A_1, A_2 \in \mathrm{C}^\infty(B, T^*B)$.

Then c is selfdual with a selfdual projective conformal submersion ξ over B, and the projective structure is $(D_\xi^0)^2 + \tfrac{1}{2}\rho$ where $\rho = 2\mathsf{w}_0 \mathsf{w}_2 - \tfrac{1}{2}\mathsf{w}_1^2$.

A selfdual conformal structure with a selfdual projective conformal submersion ξ arises locally in this way. Note that the equation is the $\mathrm{SL}(2, \mathbb{R})$ Einstein-Weyl Bogomolny equation for an $\mathfrak{sl}(2, \mathbb{R})$-connection A and $\mathfrak{sl}(2, \mathbb{R})$-valued density w.

Proof. As in the proof of Theorem VI, this amounts to computing the equations of Theorem II, now with $\mathsf{w} = t^2 \mathsf{w}_2 + t \mathsf{w}_1 + \mathsf{w}_0$ and $A = t^2 A_2 + t A_1 + A_0$. This leads to the quadratic expressions

$$
\begin{aligned}
D^B \mathsf{w} + \dot{A}\mathsf{w} - A\dot{\mathsf{w}} &= t^2\big(D^B \mathsf{w}_2 + A_2 \mathsf{w}_1 - A_1 \mathsf{w}_2\big) \\
&\quad + 2t\big(\tfrac{1}{2}D^B \mathsf{w}_1 + A_2 \mathsf{w}_0 - A_0 \mathsf{w}_2\big) + D^B \mathsf{w}_0 + A_1 \mathsf{w}_0 - A_0 \mathsf{w}_1
\end{aligned}
$$

$$
dA + \dot{A} \wedge A = t^2(dA_2 + A_2 \wedge A_1) + 2t(\tfrac{1}{2}dA_1 + A_2 \wedge A_0) + dA_0 + A_1 \wedge A_0.
$$

Equating coefficients (in t) of the resulting equations and expanding the Einstein-Weyl Bogomolny equation completes the proof. $\qquad\qquad\qquad\square$

Note that $D_\xi^g = D_\xi^0 + 2t\mathsf{w}_2 + \mathsf{w}_1$ and $D^{t\mu_g} = D^0 + (t\mathsf{w}_2 - t^{-1}\mathsf{w}_0)\xi - tA_2 + t^{-1}A_0$.

8. Twistor theory of conformal submersions

The constructions discussed so far have a natural interpretation on the *twistor space Z* of M. This is a complex manifold fibering over M whose fibres are the antiselfdual complex structures on each tangent space of M (see [2, 3, 10]). The antipodal map on each fibre defines a real structure (antiholomorphic involution) σ on Z, so the fibres of Z are real, i.e., σ-invariant. Each fibre Z_x has normal bundle

$N^{(x)} \cong \mathcal{O}(1) \oplus \mathcal{O}(1)$ and so the fibres are precisely the real lines amongst the "twistor lines", which are the holomorphic deformations of a typical fibre. Hence real holomorphic sections of $N^{(x)}$ over Z_x are constant as maps from Z_x to $T_x M$.

Each Weyl derivative D induces a connection on $\pi_Z \colon Z \to M$ and hence a projection $v^D \colon TZ \to VZ$ onto the vertical bundle of Z. Under a change of Weyl derivative, $v^{D+\gamma}(U) = v^D(U) + [\![\gamma \vartriangle d\pi_Z(U), J]\!]$ for $U \in T_J Z$. If K is any vector field on M and $J \in Z_x$, then the commutator $[\![DK - \frac{1}{2}\mathcal{L}_K \mathsf{c}, J]\!]$ is a skew endomorphism of $T_x M$ anticommuting with J (since $\mathcal{L}_K \mathsf{c} = 2\,\mathrm{sym}_0 DK$) and hence an element of $V_J Z = T_J(Z_x)$. The lift $K^{\mathbb{C}}$ of K to Z defined by $v^D(K^{\mathbb{C}}) = [\![DK - \frac{1}{2}\mathcal{L}_K \mathsf{c}, J]\!]$ is easily seen to be independent of the choice of D, and is a holomorphic vector field iff K is a conformal vector field, in which case $v^D(K^{\mathbb{C}}) = [\![DK, J]\!]$ (see [11]). This generalizes to congruences.

8.1. Proposition. *Congruences ξ on M (up to a sign) are in bijective correspondence with complex line subbundles L^ξ of TZ which are σ-invariant and transverse to the real twistor lines.*

L^ξ is a holomorphic subbundle of TZ iff ξ is a selfdual conformal submersion. The holomorphic structure on this line bundle corresponds, under the Ward correspondence, to the minimal Weyl derivative D^0.

Proof. Given ξ, let L^ξ be the complex span of those vectors U in $T_J Z$ with $d\pi_Z(U) = \mu\xi$ and $v^D(U) = \mu[\![(D^0 \otimes D)\xi - \frac{1}{2}\mathcal{L}^0_\xi \mathsf{c}, J]\!]$ for some element μ of $L^1_{\pi_Z(J)}$ (recall that $\frac{1}{2}\mathcal{L}^0_\xi \mathsf{c} = \mathrm{sym}_0(D^0 \otimes D)\xi$). This line subbundle L^ξ of TZ naturally isomorphic to $\pi_Z^* L^1 \otimes \mathbb{C}$ and is clearly transverse to the real twistor lines. Conversely, a σ-invariant complex line subbundle L^ξ transverse to a real twistor line Z_x must be degree 0 and the real sections define a congruence ξ up to sign, and identify L^ξ with $\pi_Z^* L^1 \otimes \mathbb{C}$.

By the Ward correspondence, a $\bar{\partial}$-operator on L^ξ corresponds to a Weyl derivative D^0 on M, and L^ξ holomorphic if D^0 is selfdual [11]. If so, the inclusion of L^ξ into TZ may be viewed as a section $\xi^{\mathbb{C}}$ of the holomorphic bundle $(L^\xi)^{-1} \otimes TZ$ and the invariance of the construction implies that $\xi^{\mathbb{C}}$ is holomorphic iff $\mathcal{L}^0_\xi \mathsf{c} = 0$. $\square$

This gives a twistorial explanation for the theorems of section 4.

I. The distribution L^ξ on Z given by a selfdual conformal submersion ξ integrates a holomorphic foliation with one dimensional leaves. Since L^ξ is trivial on each twistor line, the twistor lines map to rational curves (called "minitwistor lines") with normal bundle $\mathcal{O}(2)$ in the (local) quotient space $\mathcal{S}$, which is the "minitwistor space" that gives rise to the Einstein-Weyl structure on M/ξ [15].

II. Conversely, Z is obtained as a fibre bundle over $\mathcal{S}$ which is trivial on the minitwistor lines. In the affine or projective case $Z \to \mathcal{S}$ is an affine or projective line bundle. Such bundles arise as affine subspaces or projectivizations of rank two vector bundles trivial on minitwistor lines. The Einstein-Weyl Bogomolny equation now arises from a generalized Hitchin-Ward correspondence [16], and the same idea applies formally to the general, infinite dimensional, case.

III. The condition that ξ is holomorphic with respect to an antiselfdual complex structure J is simply the condition that the image of J (as a section of Z) is a union

of leaves of the foliation determined by ξ. This divisor $\mathcal{D}$ in Z therefore descends to a divisor $\mathcal{C}$ in the minitwistor space, which in turn determines a shear-free geodesic congruence [8]. The correspondence between hypercomplex and hyperCR spaces follows from the fact that $[\mathcal{D} - \overline{\mathcal{D}}]$ is trivial if $[\mathcal{C} - \overline{\mathcal{C}}]$ is trivial: if so this gives a map from $\mathcal{S}$ to $\mathbb{C}P^1$ and hence from Z to $\mathbb{C}P^1$. On the other hand, the correspondence between scalar-flat Kähler metrics and Toda Einstein-Weyl spaces has a more subtle generalization because the canonical bundle K_Z is no longer the pullback of $K_{\mathcal{S}}$ and so $[\mathcal{D} + \overline{\mathcal{D}}]K_Z^{1/2}$ is not the pullback of $[\mathcal{C} + \overline{\mathcal{C}}]K_{\mathcal{S}}^{1/2}$.

IV. Compatible Einstein-Weyl structures D on selfdual conformal manifolds are Einstein or locally hypercomplex [4, 23] and correspond to holomorphic rank two distributions $H^D = \ker v^D$ on Z. The twisted 1-form $\theta \in H^0\big(Z, (L^D)^{-1}K_Z^{-1/2}T^*Z\big)$ defining H^D can be contracted with $\xi^{\mathbb{C}} \in H^0\big(Z, (L^\xi)^{-1}TZ\big)$ to give a holomorphic section $\theta(\xi^{\mathbb{C}})$ of $(L^D)^{-1}(L^\xi)^{-1}K_Z^{-1/2}$, which has degree two on each twistor line. If this section is not identically zero, then the corresponding divisor gives rise to a complex structure. If the section *is* identically zero, then L^ξ is a subbundle of H^D and so $[\![(D^0 \otimes D)\xi, J]\!] = 0$ for all antiselfdual almost complex structures J.

When the Einstein-Weyl structure is Einstein with nonzero scalar curvature, H^D is a contact distribution, whereas when it is locally hypercomplex, H^D is integrable. Therefore, if the skew symmetric part of $(D^0 \otimes D)\xi$ is selfdual (i.e., if L^ξ is a subbundle of H^D), I will say that ξ is *Legendrian* or *triholomorphic* in the case that the scalar curvature is nonzero or zero respectively.

In the Legendrian case, the leaves of the foliation of Z given by ξ are Legendrian curves in a contact manifold. Let $\mathcal{S}$ be the (local) quotient and let $Z^{(y)}$ be the leaf corresponding to a point $y \in \mathcal{S}$. Then at each point J of $Z^{(y)}$, the contact distribution projects onto a one dimensional subbundle of $T_y\mathcal{S}$, giving a holomorphic map of complex curves from $Z^{(y)}$ to $P(T_y\mathcal{S})$. This map cannot be constant, as the contact distribution is non-integrable, so it is locally an isomorphism. By its very definition, this isomorphism identifies the contact distribution on Z with the canonical contact distribution on $P(T\mathcal{S}) \cong P(T^*\mathcal{S})$, where a line in $T_y\mathcal{S}$ is identified with its annihilator in $T_y^*\mathcal{S}$. Therefore, Z can be locally identified (in a neighbourhood of any twistor line) with the projectivized cotangent bundle of $\mathcal{S}$. This is Hitchin's construction of the selfdual Einstein metric (with nonzero scalar curvature) "filling in" a 3-dimensional Einstein-Weyl space [15]. LeBrun [18] has given such a construction for any real analytic conformal 3-manifold B, and Hitchin observes that the choice of a compatible Einstein-Weyl structure on B (if one exists) equips the selfdual Einstein metric with a conformal submersion onto B. The discussion here *characterizes* the conformal submersions arising in this way.

8.2. Theorem VIII. *Let M be a selfdual Einstein manifold with nonzero scalar curvature. Then M arises from the Hitchin-LeBrun construction iff it admits a Legendrian selfdual conformal submersion.*

The scalar curvature is usually taken to be negative: it is in such a real slice that the original conformal 3-manifold appears as a conformal infinity.

When the Einstein-Weyl space is hyperCR, its minitwistor space fibres over $\mathbb{C}P^1$ and the vertical bundle of this fibration is transverse to the minitwistor lines. This line subbundle of $T\mathcal{S}$ defines a section X of $P(T\mathcal{S})$ which does not intersect the lifted minitwistor lines, and hence does not intersect nearby twistor lines. On removing this section $P(T\mathcal{S}) \smallsetminus X(\mathcal{S})$ is an affine bundle over $\mathcal{S}$ (and is still a twistor space). Hence one can expect to carry out the Hitchin-LeBrun construction explicitly in this case, using the affine monopole equations.

In general, $P(T\mathcal{S}) = P(K_{\mathcal{S}}^{1/2} T\mathcal{S})$ is at least a projective bundle, and so, corresponding to $K_{\mathcal{S}}^{1/2} T\mathcal{S}$, which is trivial on minitwistor lines and has trivial determinant, there should be a canonical solution of the $\mathrm{SL}(2, \mathbb{R})$ Einstein-Weyl Bogomolny equation on any Einstein-Weyl space, yielding a general formula for the Hitchin-LeBrun construction on any Einstein-Weyl space. In the final section I shall find this canonical solution directly.

9. Einstein-Weyl structures and conformal submersions

If g is a Riemannian metric and $D = D^g + \omega$ is Einstein-Weyl, then it is well known that $D^g - \omega$ is Einstein-Weyl if and only if ω is dual, with respect to g, to a conformal vector field. If ω is also divergence-free with respect to g, i.e., g is a *Gauduchon gauge* for D, then ω is dual to a Killing field of g. (See e.g. [7, 8, 11, 23] for more information on Einstein-Weyl geometry.)

9.1. **Proposition.** *Suppose that (M, c) is a conformal manifold and (D^+, D^-) are compatible Einstein-Weyl structures on M. Define a 1-form $\theta := \frac{1}{2}(D^+ - D^-)$ and a Weyl derivative $D := \frac{1}{2}(D^+ + D^-)$ (the barycentre) so that $D^\pm = D \pm \theta$. Then on the open set where θ is nonvanishing, $\xi = \theta/|\theta|$ locally generates a conformal submersion with minimal Weyl derivative $D^0 = 2D - D^{|\theta|} = D - |\theta|^{-1} D|\theta|$.*

(Here $D^{|\theta|}$ is the exact Weyl derivative defined by $D^{|\theta|}|\theta| = 0$. Since $|\theta|$ is a section of L^{-1}, this means that $D^{|\theta|} = D + |\theta|^{-1} D|\theta|$.)

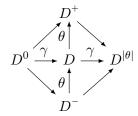*Proof.* The standard formula for the dependence of the (normalized) Ricci tensor on the Weyl derivative gives

$$r^{D^\pm} = r^D \mp D\theta + \theta \otimes \theta - \tfrac{1}{2}|\theta|^2 id$$

and hence $D\theta = \frac{1}{2}(r^{D^-} - r^{D^+})$. Since $D^\pm$ are both Einstein-Weyl, $sym_0 D\theta = 0$ and so one can write $D\theta = \sigma\, id + F$ where σ is a section of L^{-2} and F is a skew endomorphism of weight -2. Direct calculations give

$$D^0 = D - |\theta|^{-1}(\sigma\xi - F(\xi))$$
$$D^{|\theta|} = D + |\theta|^{-1}(\sigma\xi - F(\xi)).$$

One now readily checks that ξ is conformal. $\qquad\square$

The following diagram in the affine space of Weyl derivatives summarizes this Proposition, where $\gamma = |\theta|^{-1}D|\theta| = |\theta|^{-1}(\sigma\xi - F(\xi))$.

$$
\begin{array}{ccccc}
 & & D^+ & & \\
 & \nearrow & \theta\uparrow & \searrow & \\
D^0 & \xrightarrow{\gamma} & D & \xrightarrow{\gamma} & D^{|\theta|} \\
 & \searrow & \theta\uparrow & \nearrow & \\
 & & D^- & &
\end{array}
$$

In particular D^0 is exact iff D is exact. This holds, for instance if $D^\pm$ are both Levi-Civita derivatives of Einstein metrics.
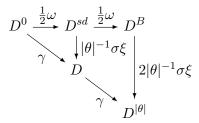
I now specialize to four dimensions and Einstein-Weyl structures with selfdual Faraday curvature. On a selfdual conformal 4-manifold, Einstein-Weyl structures necessarily have selfdual Faraday curvature [4] and are either Einstein or locally hypercomplex [23].

9.2. Theorem IX. *Let $D^\pm$ be Einstein-Weyl structures whose Faraday curvatures $F^{D^\pm}$ are selfdual. Then, with the notation of the previous proposition, ξ locally generates a selfdual conformal submersion, and also $\mathrm{alt}(D^0 \otimes D^\pm)\xi$ is selfdual.*

Proof. Since F^{D^+} and F^{D^-} are both selfdual, so is F^D, and hence so is F^0, since $D^{|\theta|}$ is exact. This means that ξ locally generates a selfdual conformal submersion. Furthermore F is selfdual, since it is equal to $\frac{1}{4}(F^{D^+} - F^{D^-})$. Consequently $\omega = -2|\theta|^{-1}F(\xi)$ and so $D = D^{sd} + |\theta|^{-1}\sigma\xi$. Therefore $D^\pm$ both differ from D^{sd} by a vertical 1-form, and so the antiselfdual part of $(D^0 \otimes D^\pm)\xi$ vanishes. $\square$

In the language of the previous section, the final condition on ξ means that ξ is Legendrian if $D^\pm$ is the Levi-Civita derivative of an Einstein metric, and triholomorphic if $D^\pm$ is the Obata derivative of a hypercomplex structure.

Another picture in the affine space of Weyl derivatives may be helpful.

$$
\begin{array}{ccccc}
D^0 & \xrightarrow{\frac{1}{2}\omega} & D^{sd} & \xrightarrow{\frac{1}{2}\omega} & D^B \\
 & \searrow^{\gamma} & \downarrow{|\theta|^{-1}\sigma\xi} & & \Big\downarrow{2|\theta|^{-1}\sigma\xi} \\
 & & D & & \\
 & & & \searrow^{\gamma} & \\
 & & & & D^{|\theta|}
\end{array}
$$

When both Einstein-Weyl structures are (locally) hypercomplex $\sigma = 0$ and so $D^B = D^{|\theta|}$ and the Einstein-Weyl structure on B is Einstein. Indeed, since the Einstein-Weyl quotient B admits two hyperCR structures, it must be the round 3-sphere metric [12]. Such bi-hypercomplex structures have been studied by Apostolov and Gauduchon [1] and they have an elegant construction of the conformal submersion that arises in this case, which I briefly describe.

If the hypercomplex structures corresponding to D^+ and D^- are (I_1^+, I_2^+, I_3^+) and (I_1^-, I_2^-, I_3^-), then since both give oriented orthonormal frames for $L^2\Lambda_-^2 T^*M$, they are related by an SO(3)-valued function: $I_i^+ = A_{ij}I_j^-$. Applying D to this

equation gives $dA_{ij}(X)I_j^- = -2[\![\theta \bigtriangleup X, A_{ij}I_j^-]\!]$, which implies that $dA_{ij}(X) = -A_{ik}\varepsilon_{jk\ell}\theta(I_\ell^- X)$. In particular, $dA_{ij}(\xi) = 0$ so this map to SO(3) factors through the conformal submersion ξ. To see that the map to SO(3) actually *is* the conformal submersion generated by ξ, one computes $dA_{ij}(X)dA_{ij}(X) = 2(|\theta|^2|X|^2 - \theta(X)^2)$.

Conversely, by Theorem III, any selfdual conformal submersion over the round 3-sphere metric is bi-triholomorphic with respect to a bi-hypercomplex structure. The generalized monopole equation in the gauge $w = |\theta| = \mu_{S^3}^{-1}$ reduces to $*_B\dot{A} = dA + \dot{A} \wedge A$, which has been also obtained by Belgun and Moroianu.

10. Selfdual Einstein metrics and hypercomplex structures

In this section I study selfdual conformal 4-manifolds M admitting a compatible Einstein metric with nonzero scalar curvature and a compatible hypercomplex structure. All such structures are obtained by applying the Hitchin-LeBrun construction to a hyperCR Einstein-Weyl space B and the Einstein metric will be found explicitly in terms of the Einstein-Weyl structure on B and a special solution of the affine monopole equations.

The work of the previous section shows that if $D^g = D - \theta$ is the Levi-Civita derivative of the Einstein metric and $D^{Ob} = D + \theta$ is the Obata derivative of the hypercomplex structure, then $\xi = \theta/|\theta|$ locally generates a Legendrian triholomorphic selfdual conformal submersion. The first goal in this section is to add one more adjective to this list and prove that the submersion is affine.

To do this, a section w_1 of L^{-1} must be found with $D^0 w_1 = F^0(\xi)$: w_1 is then linear part of the affine monopole w and the affine structure is given by $D_\xi^0 + w_1$.

10.1. Proposition. *Write $D^{Ob} = D^{sd} - \kappa\xi$. Then $D^0(2\kappa) = F^0(\xi)$.*

Proof. First recall that κ is basic, i.e., $D_\xi^0\kappa = 0$. Now D^0 and D^{Ob} are gauge equivalent, since D^g and $D^{|\theta|}$ are both exact. This implies that

$$0 = d(\omega - 2\kappa\xi)(\xi) = d\omega(\xi) + D^0(2\kappa) = F^B(\xi) - F^0(\xi) + D^0(2\kappa)$$

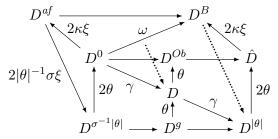which proves the proposition since $F^B(\xi) = 0$. □

This shows that the linear part of the affine monopole is twice the κ monopole of the hyperCR space $B = M/\xi$, and so it satisfies $*_B D^B(2\kappa) = F^B$ [12, 8]. Fix a gauge on B so that the Einstein-Weyl structure is given by a metric g_B and a 1-form ω_B. Then $w_1 = 2\kappa$ and one can take $A_1 = \omega_B$. In order to fix the affine gauge on M note that σ is a nonzero constant multiple of the scalar curvature of g and so $D^g\sigma = 0$ since g is Einstein. In particular σ is nonvanishing by assumption.

10.2. Proposition. *Define $D^{af} = D^0 + 2\kappa\xi$. Then $D_\xi^{af}(\sigma^{-1}|\theta|) = -2$ so $\sigma^{-1}|\theta|$ is an affine section of L^1 with respect to the affine structure.*

Proof. Observe that $(D^0 - 2\theta)(\sigma^{-1}|\theta|) = 0$, i.e., $D^{\sigma^{-1}|\theta|} = D^0 - 2\theta$ and so $D^{af} = D^{\sigma^{-1}|\theta|} + 2\theta + 2\kappa\xi = D^{\sigma^{-1}|\theta|} - 2|\theta|^{-1}\sigma\xi$. Hence $D^{af}(\sigma^{-1}|\theta|) = -2\xi$. □

One more diagram in the affine space of Weyl derivatives may again clarify the situation, where the Weyl derivative $\hat{D}$ with $\hat{D}(|\theta|^{-1}) = 2\xi$ has been introduced for completeness.

$$
\begin{array}{ccc}
D^{af} & \xrightarrow{\hspace{3cm}} & D^B \\
\end{array}
$$

It follows from Proposition 10.2 that the affine coordinate t can be taken to be a multiple of $\sigma^{-1}|\theta|$ and so $|\theta|^{-1}\sigma\xi$ is a multiple of $t^{-1}(dt + tA_1 + A_0)$. Now note that $D^B = D^{|\theta|} - 2|\theta|^{-1}\sigma\xi$ and so $-2d(|\theta|^{-1}\sigma\xi) = F^B$, $A_0 = 0$ and $|\theta|^{-1}\sigma\xi = -\frac{1}{2}(t^{-1}dt + \omega_B)$. The remaining gauge freedom is fixed by taking $w_0 = \mu_B^{-1}$.

Certainly $(1 + 2t\mu_B\kappa, t\omega_B)$ is a solution of the affine monopole equations on a hyperCR Einstein-Weyl space, yielding a selfdual space with a hypercomplex structure by Theorem III. The Einstein gauge is the barycentre of the $|\theta|^{-1}$ gauge and the $\sigma^{-1}|\theta|$ gauge (both of which have been identified in terms of the affine structure) and this encodes the Einstein equation, since D^{Ob} is Einstein-Weyl and $sym_0 D\theta = 0$. Hence, writing $\mu_t = t\mu_B$, the following theorem is obtained.

10.3. Theorem X. *Suppose* (c_B, D^B) *is a 3-dimensional hyperCR Einstein-Weyl space with twist* κ. *Then*

$$
g = \frac{1}{\mu_t^2}\big((1 + 2\mu_t\kappa)\pi^*\mathsf{c}_B + (1 + 2\mu_t\kappa)^{-1}(D^B\mu_t)^2\big)
$$

is a selfdual Einstein metric with nonzero scalar curvature a compatible hypercomplex structure. (Here μ_t *is a section of* L^1 *increasing along the fibres of a conformal submersion* $\pi\colon M \to B$ *and* c_B *is a conformal infinity at* $\mu_t = 0$.) *Any selfdual Einstein metric with a compatible hypercomplex structure arises locally in this way.*

Explicit selfdual Einstein metrics can be found by applying this construction to explicit hyperCR Einstein-Weyl spaces. To the best of my knowledge, the known examples are the Einstein-Weyl spaces with geodesic symmetry, and the hyperCR-Toda spaces. In the former case note that the twist of the hyperCR structure is *minus* the twist of the geodesic symmetry and so the examples of section 5.2 (with $h = f$) are reobtained [9]—these are the Pedersen metrics [22] when h is constant.

On the other hand, the hyperCR-Toda spaces yield new selfdual Einstein metrics with compatible hypercomplex structures and no continuous symmetries:

$$
g = \frac{1}{t^2}\big(H(|z + h|^2 g_{S^2} + dz^2) + H^{-1}(dt + t\omega_B)^2\big)
$$

where
$$
H = 1 + \frac{i(h - \overline{h})t}{|z + h|^2}, \qquad \omega_B = -\frac{2z + h + \overline{h}}{|z + h|^2}dz
$$

and h is holomorphic on an open subset of S^2.

HyperCR Einstein-Weyl spaces are not well understood: the results of this section perhaps provide motivation for further investigations. Alternatively, one may study hypercomplex selfdual Einstein 4-manifolds directly. Along these lines, Apostolov and Gauduchon [1] have characterized the Pedersen metrics as the hypercomplex selfdual Einstein 4-manifolds admitting a compatible (selfdual) Hermitian structure.

11. The Hitchin-LeBrun construction

There is an interesting gauge transformation one can apply to the metric of Theorem X: on replacing μ_t by the new projective coordinate $\mu_t/(1 - \mu_t\kappa)$, the metric becomes:

$$g = \frac{1}{\mu_t^2}\Big(\big(1 - \mu_t^2\kappa^2\big)\pi^*\mathsf{c}_B + \big(1 - \mu_t^2\kappa^2\big)^{-1}\big(D^B\mu_t + \mu_t^2 D^B\kappa\big)^2\Big).$$

Now on a hyperCR Einstein-Weyl space, $D^B\kappa = -\frac{1}{2}*_B F^B$ and $\kappa^2 = \frac{1}{6}\,scal^B$ (see [12]), so this form of the metric makes sense for any Einstein-Weyl space. In this final section, I prove the following theorem.

11.1. Theorem XI. *Let* (c_B, D^B) *be an arbitrary 3-dimensional Einstein-Weyl structure with Faraday curvature* F^B *and scalar curvature* $scal^B$*. Then*

$$g = \big(1 - \tfrac{1}{6}\mu_t^2\,scal^B\big)\mu_t^{-2}\pi^*\mathsf{c}_B + \big(1 - \tfrac{1}{6}\mu_t^2\,scal^B\big)^{-1}\big(\mu_t^{-1}D^B\mu_t - \tfrac{1}{2}\mu_t*_B F^B\big)^2$$

is a selfdual Einstein metric of nonzero scalar curvature, with a Legendrian selfdual conformal submersion π *over* B*. (Here* μ_t *is a section of* L^1 *increasing along the fibres of* $\pi\colon M \to B$ *and the conformal structure* c_B *is the conformal infinity at* $\mu_t = 0$*.) Any such selfdual Einstein metric arises locally in this way.*

Strictly speaking, the above metric is only positive definite for $\frac{1}{6}\mu_t^2\,scal^B < 1$ and in this region the scalar curvature is negative. For $\frac{1}{6}\mu_t^2\,scal^B > 1$ the negation of the above metric is positive definite and has positive scalar curvature.

Note that the Einstein metric can be written in a gauge μ_B by writing $\mu_t = t\mu_B$, $g_B = \mu_B^{-2}\mathsf{c}_B$ and $D^B = D^{\mu_B} + \omega_B$. Then

$$g = \frac{1}{t^2}\Big(\big(1 - \tfrac{1}{6}t^2\mu_B^2\,scal^B\big)g_B + \big(1 - \tfrac{1}{6}t^2\mu_B^2\,scal^B\big)^{-1}\big(dt + t\omega_B - \tfrac{1}{2}t^2\mu_B*_B F^B\big)^2\Big).$$

The theorem is proven using the $SL(2,\mathbb{R})$ Einstein-Weyl Bogomolny equation. The conformal structure is determined by a canonical monopole on $L_B^{1/2} \oplus L_B^{-1/2}$. In a gauge μ_B, the Higgs field and connection 1-form are given by:

$$w = \begin{pmatrix} 0 & \mu_B^{-1} \\ \frac{1}{6}\mu_B\,scal^B & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} \frac{1}{2}\omega_B & 0 \\ \frac{1}{2}\mu_B*_B F^B & -\frac{1}{2}\omega_B \end{pmatrix}.$$

The connection is therefore $D^B + \frac{1}{2}*_B F^B$, with $*_B F^B$ acting from $L_B^{1/2}$ to $L_B^{-1/2}$, while the Higgs field is $1 + \frac{1}{6}\,scal^B$ in $L_B^{-1} \otimes \big(\text{Hom}(L_B^{-1/2}, L_B^{1/2}) \oplus \text{Hom}(L_B^{1/2}, L_B^{-1/2})\big)$.

The projective monopole equations are:

$$-\tfrac{1}{6}*_B(D^B(\mu_B\,scal^B)-\omega_B\mu_B\,scal^B) = -\tfrac{1}{2}d(\mu_B*_B F^B)+\tfrac{1}{2}\omega_B\wedge\mu_B*_B F^B,$$

$$\tfrac{1}{2}F^B = \tfrac{1}{2}d\omega_B,$$

$$D^B(\mu_B^{-1})+\omega_B\mu_B^{-1}=0.$$

Since $D^B\mu_B=\omega_B\mu_B$ and $F^B=d\omega_B$, the only nontrivial equation is the first one, which reduces to $\tfrac{1}{6}D^B\,scal^B=-\tfrac{1}{2}*_B d^B*_B F^B=\tfrac{1}{2}\delta^B F^B$, where δ^B is the twisted exterior divergence. This equation is satisfied automatically: it is the differential Bianchi identity for the Weyl connection in Einstein-Weyl geometry [11, 7].

It follows from Theorem VII that the metric g is selfdual with a selfdual projective conformal submersion. It remains, therefore, to prove that g is Einstein and the submersion is Legendrian. To do this, observe that the Levi-Civita derivative of g is the barycentre of the Levi-Civita derivatives of μ_t and tw^{-1}. Simple calculations show that these are given by

$$D^B-\mu_t^{-1}D^B\mu_t = D^B-(1-\tfrac{1}{6}\mu_t^2\,scal^B)\mu_t^{-1}\xi-\tfrac{1}{2}\mu_t*_B F^B$$

and

$$D^w-t^{-1}dt = D^0-(1+\tfrac{1}{6}\mu_t^2\,scal^B)\mu_t^{-1}\xi+\tfrac{1}{2}\mu_t*_B F^B.$$

Hence $D^g=D^{sd}-\mu_t^{-1}\xi$ (which will give the Legendrian property) and consequently,

$$r^g=r^{sd}+D^B(\mu_t^{-1})\otimes\xi+\mu_t^{-1}(D^0\otimes D^{sd})\xi+\mu_t^{-2}(\xi\otimes\xi-\tfrac{1}{2}id),$$

where I have written $D^{sd}=D^B\otimes D^0\otimes D^{sd}$ on $T^*M=L^{-1}\otimes L^{-1}\otimes TM$. This simplifies to give

$$r^g=r^{sd}-\tfrac{1}{2}*_B F^B\otimes\xi+\tfrac{1}{6}scal^B\,\xi\otimes\xi-\tfrac{1}{2}\mu_t^{-2}id+\mu_t^{-1}(D^0\otimes D^{sd})\xi.$$

Finally substituting from Proposition 4.7 yields $sym\,r^g=\tfrac{1}{2}(\tfrac{1}{6}scal^B-\mu_t^{-2})id$, and so g is Einstein, with scalar curvature $-12\mu_t^{-2}(1-\tfrac{1}{6}scal^B\,\mu_t^2)$. This completes the proof of Theorem XI.

Examples arising from hyperCR Einstein-Weyl spaces have already been discussed. One source of further examples are the Ward-Toda spaces [26, 6]

$$g=(V_\rho^2+V_\eta^2)(d\rho^2+d\eta^2)+d\psi^2$$

$$\omega=\frac{2V_\rho V_\eta\,d\eta+(V_\rho^2-V_\eta^2)d\rho}{\rho(V_\rho^2+V_\eta^2)}$$

where V is an axially symmetric harmonic function: $(\rho V_\rho)_\rho+\rho V_{\eta\eta}=0$. These spaces admit a symmetry generated by $\partial/\partial\psi$ and hence so will their Hitchin-LeBrun metrics. Already in these examples, the Faraday and scalar curvatures are quite formidable, so these Einstein metrics are not at all simple. Nevertheless, they can be made completely explicit, and will undoubtedly repay further study.

References

[1] V. Apostolov and P. Gauduchon, *Selfdual Einstein Hermitian four manifolds*, Preprint, Ecole Polytechnique (2000).

[2] M. F. Atiyah, N. J. Hitchin and I. M. Singer, *Self-duality in four dimensional Riemannian geometry*, Proc. Roy. Soc. London **A 362** (1978) 425–461.

[3] A. L. Besse, *Einstein Manifolds*, Ergeb. Math. Grenzgeb. **10**, Springer, Berlin (1987).

[4] D. M. J. Calderbank, *The Faraday 2-form in Einstein-Weyl geometry*, Preprint 23, University of Warwick (1997), to appear in Math. Scand.

[5] D. M. J. Calderbank, *Weyl derivatives, conformal submersions and selfdual Einstein metrics*, Preprint MS-98-013, University of Edinburgh (1998).

[6] D. M. J. Calderbank, *The geometry of the Toda equation*, Preprint MS-99-003, University of Edinburgh (1999), to appear in J. Geom. Phys.

[7] D. M. J. Calderbank and H. Pedersen, *Einstein-Weyl geometry*, in *Essays on Einstein Manifolds* (C. R. LeBrun and M. Wang, eds.), Surveys in Differential Geometry **V**, International Press, Cambridge (1999).

[8] D. M. J. Calderbank and H. Pedersen, *Selfdual spaces with complex structures, Einstein-Weyl geometry and geodesics*, Ann. Inst. Fourier **50** (2000) 909–951.

[9] D. M. J. Calderbank and K. P. Tod, *Einstein metrics, hypercomplex structures and the Toda field equation*, Preprint MS-98-011, University of Edinburgh (1998), to appear in Diff. Geom. Appl.

[10] P. Gauduchon, *Structures de Weyl et théorèmes d'annulation sur une variété conforme auto-duale*, Ann. Sc. Norm. Sup. Pisa **18** (1991) 563–629.

[11] P. Gauduchon, *Structures de Weyl-Einstein, espaces de twisteurs et variétés de type $S^1 \times S^3$*, J. reine angew. Math. **469** (1995) 1–50.

[12] P. Gauduchon and K. P. Tod, *Hyperhermitian metrics with symmetry*, J. Geom. Phys. **25** (1998) 291–304.

[13] G. W. Gibbons and S. W. Hawking, *Gravitational multi-instantons*, Phys. Lett. **B 78** (1978) 430–432.

[14] A. Gray, *Pseudo-Riemannian almost product manifolds and submersions*, J. Math. Mech. **16** (1967) 715–737.

[15] N. J. Hitchin, *Complex manifolds and Einstein equations*, in *Twistor Geometry and Nonlinear Systems* (Primorsko, 1980, H. D. Doebner and T. D. Palev, eds.), Lecture Notes in Math. **970**, Springer, Berlin (1982), pp. 79–99.

[16] N. J. Hitchin, *Monopoles and geodesics*, Comm. Math. Phys. **83** (1982) 579–602.

[17] P. E. Jones and K. P. Tod, *Minitwistor spaces and Einstein-Weyl spaces*, Class. Quantum Grav. **2** (1985) 565–577.

[18] C. R. LeBrun, *$\mathcal{H}$-space with a cosmological constant*, Proc. Roy. Soc. London **A 380** (1982) 171–185.

[19] C. R. LeBrun, *Explicit self-dual metrics on $\mathbb{C}P^2 \# \cdots \# \mathbb{C}P^2$*, J. Diff. Geom. **34** (1991) 223–253.

[20] H.-C. Lee, *A kind of even-dimensional differential geometry and its application to exterior calculus*, Amer. J. Math. **65** (1943) 433–438.

[21] B. O'Neill, *The fundamental equations of a submersion*, Michigan Math. J. **13** (1966) 459–469.

[22] H. Pedersen, *Einstein metrics, spinning top motions and monopoles*, Math. Ann. **274** (1986) 35–39.

[23] H. Pedersen and A. Swann, *Riemannian submersions, four-manifolds and Einstein-Weyl geometry*, Proc. London Math. Soc. **66** (1993) 381–399.

[24] K. P. Tod, *The* SU(∞) *Toda field equation and special four-dimensional metrics*, in *Geometry and Physics* (Aarhus, 1995, J. E. Andersen, J. Dupont, H. Pedersen and A. Swann, eds.), Lecture Notes in Pure and Appl. Math. **184**, Marcel Dekker, New York (1997), pp. 307–312.

[25] I. Vaisman, *On locally conformal almost Kähler manifolds*, Israel J. Math. **24** (1976) 338–351.

[26] R. S. Ward, *Einstein-Weyl spaces and* SU(∞) *Toda fields*, Class. Quantum Grav. **7** (1990) L95–L98.

[27] R. S. Ward, *The* SU(∞) *chiral model and selfdual vacuum spaces*, Class. Quant. Grav. **7** (1990) L217–L222.

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