PARABOLIC SUBALGEBRAS, PARABOLIC BUILDINGS AND PARABOLIC PROJECTION

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ABSTRACT. Reductive (or semisimple) algebraic groups, Lie groups and Lie algebras have a rich geometry determined by their parabolic subgroups and subalgebras, which carry the structure of a building in the sense of J. Tits. We present herein an elementary approach to the geometry of parabolic subalgebras, over an arbitrary field of characteristic zero, which does not rely upon the structure theory of semisimple Lie algebras. Indeed we derive such structure theory, from root systems to the Bruhat decomposition, from the properties of parabolic subalgebras. As well as constructing the Tits building of a reductive Lie algebra, we establish a "parabolic projection" process which sends parabolic subalgebras of a reductive Lie algebra to parabolic subalgebras of a Levi subquotient. We indicate how these ideas may be used to study geometric configurations and their moduli.

Parabolic subgroups and their Lie algebras are fundamental in Lie theory and the theory of algebraic groups [2, 16, 17, 19, 21, 25, 29, 35, 38]. They play a key role in combinatorial, differential and integrable geometry through Tits buildings and parabolic invariant theory [1, 3, 5, 8, 31, 33, 36]. Traditional approaches define parabolic subalgebras of semisimple or reductive Lie algebras as those containing a Borel (maximal solvable) subalgebra in some field extension, and typically develop the theory of parabolic subalgebras using the root system associated to a Cartan subalgebra of such a Borel subalgebra. Such approaches are far from elementary, and provide limited insight when the field is not algebraically closed.

The present paper is motivated by a programme to study geometric configurations in projective spaces and other generalized flag manifolds (*i.e.*, adjoint orbits of parabolic subalgebras) using a process called *parabolic projection.* This process relies upon two observations: first, if $\mathfrak{p}, \mathfrak{q}$ are parabolic subalgebras of a (reductive or semisimple) Lie algebra $\mathfrak{g}$, then so is $\mathfrak{p} \cap \mathfrak{q} + \mathfrak{nil}(\mathfrak{q})$ (where $\mathfrak{nil}(\mathfrak{q})$ is the nilpotent radical of $\mathfrak{q}$); secondly, if $\mathfrak{r}$ is a Lie subalgebra of $\mathfrak{q}$, and $\mathfrak{q}$ is parabolic subalgebra of $\mathfrak{g}$, then $\mathfrak{r}$ is a parabolic subalgebra of $\mathfrak{g}$ if and only if it is the inverse image of a parabolic subalgebra of the reductive Levi quotient of $\mathfrak{q}_0 := \mathfrak{q}/\,\mathfrak{nil}(\mathfrak{q})$—thus two senses in which $\mathfrak{r}$ might be called a parabolic subalgebra of $\mathfrak{q}$ coincide. These observations point towards a theory of parabolic subalgebras of arbitrary Lie algebras, which may be developed using straightforward methods of linear algebra over an arbitrary field of characteristic zero, without relying on the structure theory of semisimple Lie algebras. Indeed, much of the latter can be derived as a consequence.

This development combines two approaches to parabolic subalgebras. In the first approach, inspired by work of V. Morozov [22, 23, 27] and other work emphasising the role of nilpotent elements in Lie algebra theory (see also [2, 21, 28, 33]), the Borel subalgebras in the traditional definition of parabolic subalgebras are replaced by the normalizers of maximal nil subalgebras (*i.e.*, subalgebras contained in the nilpotent cone). In the second approach, initiated by A. Grothendieck [15] and developed by F. Burstall [4] and others (see [5, 6, 7, 10]), a subalgebra of a semisimple Lie algebra is parabolic if its Killing perp is a nilpotent subalgebra. The latter definition appears to rely upon the Killing form of a semisimple Lie algebra, but is easily adapted to the reductive case.

The set of parabolic subalgebras of a reductive Lie algebra $\mathfrak{g}$ has a rich structure: it may be decomposed into flag varieties under the action of the adjoint group G of $\mathfrak{g}$, but is also equipped with an incidence geometry relating these varieties. Let us illustrate this

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in the case that $G = PGL(V)$ is the projective general linear group of a vector space V of dimension $n+1$. Among the parabolic subalgebras of $\mathfrak{g}$, the maximal (proper) parabolic subalgebras play a distinguished role. The corresponding flag varieties are the grassmannians $Gr_k(V)$ $(1 \leqslant k \leqslant n)$ of k-dimensional subspaces of V. Two such subspaces are incident if one contains the other, and more general parabolic subalgebras (and their flag varieties) can be described using sets of mutually incident subspaces (called "flags"). In particular, the minimal parabolic subalgebras of $\mathfrak{g}$ are the infinitesimal stabilizers of complete flags, which are nested chains of subspaces of V, one of each dimension.

The general picture is similar. Among adjoint orbits of parabolic subalgebras (generalized flag varieties), the maximal ones are distinguished. For algebraically closed fields, these may be identified with the nodes of the Dynkin diagram, and with the nodes of the Coxeter diagram of the (restricted) root system in general. Other parabolic subalgebras may be described by the maximal (proper) parabolic subalgebras containing them (which are mutually incident), so that their adjoint orbits correspond to subsets of nodes of the Coxeter or Dynkin diagram. This situation is abstracted by J. Tits' theory of buildings [36], which deserves (in our opinion) a more central place in Lie theory and representation theory than it currently enjoys.

Our purpose in this paper is to present a self-contained treatment of parabolic subalgebras and their associated Lie theory, sufficient to give a novel proof that the set of minimal parabolic subalgebras of a Lie algebra are the chambers of a strongly transitive Tits building. In the algebraically closed case, this establishes the conjugacy theorems for Cartan subalgebras and Borel subalgebras, together with the Bruhat decomposition, using no algebraic geometry and very little structure theory. It also provides a natural framework in which to develop the basic properties of parabolic projection. In subsequent work, we shall apply parabolic projection to the construction of geometric configurations and discrete integrable geometries.

Although the results herein are essentially algebraic in nature, we wish to emphasise the geometry behind them. We thus begin in Section 1 by introducing incidence systems, and two examples which we shall use throughout the paper to illustrate the theory. Example 1A is the incidence geometry of vector (or projective) subspaces of a vector (or projective) space over an arbitrary field, while Example 1B is the incidence geometry of isotropic subspaces of a real inner product space of indefinite signature.

Section 2 provides a self-contained treatment of the Lie algebra theory we need, modulo some basic facts about Jordan decompositions and invariant bilinear forms which we summarize in Appendix A. In order to work over an arbitrary field of characteristic zero, we develop the nilpotent–reductive dichotomy for Lie algebras rather than the solvable–semisimple dichotomy. We thus ignore Lie's theorem, Weyl's theorem, the Levi–Malcev decomposition, and even most of the representation theory of $\mathfrak{sl}_2$. Instead, we emphasise the role played by filtrations and Engel's theorem, nilpotency ideals and the nilpotent cone, trace forms and Cartan's criterion (which, in its most primitive form, concerns nilpotency rather than solvability). In Theorem 2.20 we thus establish, in a novel way, the basic result that (in characteristic zero) a Lie algebra is reductive if and only if it admits a nondegenerate trace form. Using this, Proposition 2.22 extends one of Cartan's criteria from $\mathfrak{gl}(V)$ to reductive Lie algebras, a result which we have not been able to find in the literature.

We define parabolic subalgebras in Section 3 as those subalgebras containing the normalizer of a maximal nil subalgebra, but immediately obtain, in Theorem 3.4 several equivalent definitions using "admissible" trace forms and nilpotency ideals, some of which are new, cf. [2, 4]. We then consider pairs of parabolic subalgebras, their "oppositeness", and their incidence properties. Intersections of opposite minimal parabolic subalgebras define minimal Levi subalgebras, also known as anisotropic kernels, which govern the structure theory of root systems for semisimple Lie algebras in characteristic zero—see Theorem 3.28. The theory of such "restricted" root systems is well known in the context of algebraic groups [35] or when the underlying field is the real numbers [16, 19, 25, 38]; here, though, we are forced to discard concepts such as Cartan decompositions which are particular to the real case.

The core results of the paper appear in Section 4, where we show that the parabolic subalgebras of a reductive Lie algebra $\mathfrak{g}$ form the simplices of a Tits' building. This theory has a formidable reputation, but has become more approachable in recent years as the key concepts have become better understood. More recent approaches emphasise chamber systems [13, 30, 37, 39] rather than simplicial complexes [14, 36]. In particular, these approaches make explicit the labelling corresponding to the nodes of the Coxeter–Dynkin diagram in the parabolic case. Unfortunately, the modern definition of buildings incorporates an abstraction of the Bruhat decomposition, which is a nontrivial result in representation theory. In order to address these issues, we adopt a concise hybrid approach to buildings, which combines the original viewpoint (using "apartments") with more recent approaches using chamber systems. Following [1, 30, 39], we develop sufficient theory to derive an abstract Bruhat decomposition (Theorem 4.21) for strongly transitive buildings. We use this to obtain (in Theorem 4.22) the Bruhat decomposition from conjugacy results for minimal parabolic subalgebras and their Levi subalgebras, which we also prove. This result is well-known in the context of algebraic groups, using Tits' (B, N)-*pairs*, but our methods are almost entirely different.

In Section 5, we turn finally to *parabolic projection*, which, for a given fixed parabolic subalgebra $\mathfrak{q} \leqslant \mathfrak{g}$, projects arbitrary parabolic subalgebras of $\mathfrak{g}$ onto parabolic subalgebras of the reductive Levi quotient $\mathfrak{q}_0$ of $\mathfrak{q}$. This idea was originally developed by A. Macpherson and the first author [20] using root systems and standard parabolic subgroups. The analysis here is based instead on Proposition 3.16 which gives an explicit formula for the projection. The main result is Theorem 5.11, in which parabolic projection is shown to be a morphism of chamber systems when restricted to weakly opposite subalgebras. This is the basis for constructions of geometric configurations (see [24]) that we shall pursue elsewhere.

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1. Elements of incidence geometry

Incidence geometry is conveniently described using graph theory, where graphs (herein) are undirected with no loops or multiple edges. We use $|\mathit{\Gamma}|$ and $E_{\mathit{\Gamma}}$ to denote the vertex and edge sets of a graph $\mathit{\Gamma}$; an edge is determined by its two endpoints, so $E_{\mathit{\Gamma}}$ may be viewed as a collection of two element subsets of $|\mathit{\Gamma}|$, or equivalently, a symmetric irreflexive relation on $|\mathit{\Gamma}|$. For $v, w \in |\mathit{\Gamma}|$, we write v—w for the reflexive closure of this relation (which is the structure preserved by graph morphisms). By a "subgraph", we always mean a subset of vertices with the induced relation.

Definition 1.1 ([31, 32, 39])**.** An *incidence system* over a (usually finite) set $\mathcal{I}$ is an $\mathcal{I}$-multipartite graph $\mathit{\Gamma}$, *i.e.*, a graph equipped with a type function $t = t_{\mathit{\Gamma}} \colon |\mathit{\Gamma}| \to \mathcal{I}$ such that $\forall\, v, w \in |\mathit{\Gamma}|$, v—w and $t(v) = t(w)$ imply $v = w$. An *incidence morphism* $\mathit{\Gamma}_1 \to \mathit{\Gamma}_2$ of incidence systems over $\mathcal{I}$ is a type-preserving graph morphism.

A *flag* or *clique* in $\mathit{\Gamma}$ of type $J \in P(\mathcal{I})$ (*i.e.*, $J \subseteq \mathcal{I}$) is a set of mutually incident elements, one of each type $j \in J$ (a J-*flag*). We denote the set of J-flags by $\mathcal{F}\mathit{\Gamma}(J)$; together with the obvious "face maps" $\mathcal{F}\mathit{\Gamma}(J_2) \to \mathcal{F}\mathit{\Gamma}(J_1)$ for $J_1 \subseteq J_2$, these form an (abstract, $\mathcal{I}$-labelled) simplicial complex $\mathcal{F}\mathit{\Gamma}$ (a presheaf or functor $P(\mathcal{I})^{op} \to \mathbf{Set}$) called the *flag complex*—$\mathcal{F}\mathit{\Gamma}$ is also an incidence system over $P(\mathcal{I})$: two flags are incident iff their union is a flag. A *full flag* is an $\mathcal{I}$-flag $\sigma \in \mathcal{F}\mathit{\Gamma}(\mathcal{I})$, *i.e.*, σ contains one element of each type $j \in \mathcal{I}$.

Example 1A. The proper nonempty subsets B of an $n + 1$ element set $\mathcal{S}$ form an incidence system $\mathit{\Gamma}^{\mathcal{S}}$ over $\mathcal{I}_n := \{1, 2, \dots n\}$, where $t(B)$ is the number of elements of B, and B_1—B_2 iff $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. We "linearize" this example as follows.

Let V be a vector space of dimension $n+1$ over a field $\mathbb{F}$. The proper nontrivial subspaces $W \leqslant V$ are the elements of an incidence system Γ^V over $\mathcal{I}_n$, where $t(W) = \dim W$ and W_1—W_2 iff $W_1 \leqslant W_2$ or $W_2 \leqslant W_1$. For $J \subseteq \mathcal{I}_n$, a J-flag is a family of subspaces $W_j : j \in J$ of V with $\dim W_j = j$ and $W_j \leqslant W_k$ for $j \leqslant k$. Thus a full flag is a nested sequence $0 \leqslant W_1 \leqslant W_2 \leqslant \cdots \leqslant W_n \leqslant V$ with $\dim W_j = j$.

Example 1B. Let U be a vector space of dimension $2n+k$ over $\mathbb{R}$ equipped with a quadratic form Q_U of signature $(n+k, n)$, where $k \geqslant 1$. The nontrivial isotropic subspaces of U (on which Q_U is identically zero) have $1 \leqslant \dim U \leqslant n$, and are the elements (typed by dimension) of an incidence system Γ^{U,Q_U} over $\mathcal{I}_n$, where the incidence relation is again given by containment. We shall provide a discrete model for Γ^{U,Q_U} in Example 4B.

2. Elementary Lie theory

2.1. Lie algebra notions and notations. Recall that a *Lie algebra* $\mathfrak{g}$ over a field $\mathbb{F}$ is an $\mathbb{F}$-vector space equipped with a skew-symmetric bilinear operation $[\cdot,\cdot]\colon \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ satisfying the Jacobi identity $[x,[y,z]] = [[x,y],z]+[y,[x,z]]$. The commutator bracket $(\alpha,\beta) \mapsto [\alpha,\beta] = \alpha\circ\beta-\beta\circ\alpha$ makes $End_{\mathbb{F}}(V)$ into a Lie algebra, denoted $\mathfrak{gl}(V)$. A *representation* of Lie algebra $\mathfrak{g}$ on a vector space V is a Lie algebra homomorphism $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ (*i.e.*, a linear map with $\rho([x,y]) = [\rho(x),\rho(y)]$). We write $\rho(x,v)$ or $x\cdot v$ as a shorthand for the action $\rho(x)(v)$ of $x \in \mathfrak{g}$ on $v \in V$. For subspaces $\mathfrak{p} \subseteq \mathfrak{g}$ and $U \subseteq V$, we define $\rho(\mathfrak{p},U) = \mathfrak{p}\cdot U$ to be the span of $\{\rho(x)(u) \mid x \in \mathfrak{p},\ u \in U\}$, and introduce shorthands $\rho(x,U) = x\cdot U$ and $\rho(\mathfrak{p},u) = \mathfrak{p}\cdot u$ when $\mathfrak{p} = span\{x\}$ or $U = span\{u\}$.

The action of subspaces $\mathfrak{p} \subseteq \mathfrak{g}$ on subspaces $U \subseteq V$ has an upper adjoint in each variable:

$$\begin{aligned}\mathfrak{p}\cdot U \subseteq W \quad &\text{iff} \quad \mathfrak{p} \subseteq \mathfrak{c}_{\mathfrak{g}}(U,W) := \{x \in \mathfrak{g} \mid x\cdot U \subseteq W\}\\ &\text{iff} \quad U \subseteq \mathfrak{c}_V(\mathfrak{p},W) := \{v \in V \mid \mathfrak{p}\cdot v \subseteq W\}.\end{aligned}$$

In particular $\mathfrak{c}_{\mathfrak{g}}(U,U) = \mathfrak{stab}_{\mathfrak{g}}(U)$ is the *stabilizer* of U and $\mathfrak{c}_V(\mathfrak{p},0) = \ker\rho(\mathfrak{p})$ is the (*joint*) *kernel* of the $\mathfrak{p}$ action; U is $\mathfrak{p}$-*invariant* in V iff $\mathfrak{p}\cdot U \subseteq U$ iff $\mathfrak{p} \subseteq \mathfrak{stab}_{\mathfrak{g}}(U)$.

The *adjoint representation* $ad\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is defined by $ad(x)(y) = [x,y]$ for $x, y \in \mathfrak{g}$ (which is a representation by the Jacobi identity). Thus $ad(\mathfrak{p},\mathfrak{q})$ is the bracket $[\mathfrak{p},\mathfrak{q}]$ of subspaces $\mathfrak{p},\mathfrak{q} \subseteq \mathfrak{g}$ and we set $[x,\mathfrak{q}] := [span\{x\},\mathfrak{q}]$. The upper adjoint specializes to give:

$$[\mathfrak{p},\mathfrak{q}] \subseteq \mathfrak{r} \quad \text{iff} \quad \mathfrak{p} \subseteq \mathfrak{c}_{\mathfrak{g}}(\mathfrak{q},\mathfrak{r}) := \{x \in \mathfrak{g} \mid [x,\mathfrak{q}] \subseteq \mathfrak{r}\}.$$

In particular $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{q}) := \mathfrak{c}_{\mathfrak{g}}(\mathfrak{q},\mathfrak{q})$ is the *normalizer* of $\mathfrak{q}$, $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{q}) := \mathfrak{c}_{\mathfrak{g}}(\mathfrak{q},0)$ is the *centralizer* of $\mathfrak{q}$, and $\mathfrak{z}(\mathfrak{g}) := \mathfrak{c}_{\mathfrak{g}}(\mathfrak{g})$ is the *centre* of $\mathfrak{g}$. Thus $\mathfrak{p} \subseteq \mathfrak{g}$ is a *subalgebra* ($\mathfrak{p} \leqslant \mathfrak{g}$) iff $[\mathfrak{p},\mathfrak{p}] \subseteq \mathfrak{p}$ iff $\mathfrak{p} \subseteq \mathfrak{n}_{\mathfrak{g}}(\mathfrak{p})$, and an *ideal* ($\mathfrak{p} \trianglelefteq \mathfrak{g}$) iff $[\mathfrak{g},\mathfrak{p}] \subseteq \mathfrak{p}$ iff $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{p}) = \mathfrak{g}$. We note a useful lemma.

Lemma 2.1. *If* $\mathfrak{b} \subseteq \mathfrak{a} \subseteq \mathfrak{h}$ *and* $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, *where* $\mathfrak{h} \leqslant \mathfrak{g}$ *and* $[\mathfrak{h},\mathfrak{m}] \subseteq \mathfrak{m}$, *then* $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{a},\mathfrak{b}) = \mathfrak{c}_{\mathfrak{h}}(\mathfrak{a},\mathfrak{b}) \oplus \big(\mathfrak{c}_{\mathfrak{g}}(\mathfrak{a}) \cap \mathfrak{m}\big)$.

A Lie algebra $\mathfrak{g}$ is *reductive* if it has a faithful semisimple representation (see Appendix A.1). This holds in particular if $\mathfrak{g}$ is nonabelian with irreducible adjoint representation (*i.e.*, $\mathfrak{g}$ has no proper nontrivial ideals); then $\mathfrak{g}$ is said to be *simple*. More generally, the adjoint representation of $\mathfrak{g}$ is faithful and semisimple if and only if $\mathfrak{g}$ is *semisimple*, *i.e.*, a direct sum of simple ideals. Thus any semisimple Lie algebra $\mathfrak{g}$ is reductive with $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$.

2.2. Filtered and graded Lie algebras.

Definition 2.2. A $\mathbb{Z}$-*graded vector space* is a vector space V equipped with a $\mathbb{Z}$-*grading*, *i.e.*, a direct sum decomposition $V = \bigoplus_{k\in\mathbb{Z}} V_k$. A *filtration* of a vector space V is a family $V^{(k)} : k \in \mathbb{Z}$ of subspaces of V such that $i \leqslant j \;\Rightarrow\; V^{(i)} \subseteq V^{(j)}$.

Let $V^+ := \bigcup_{k\in\mathbb{Z}} V^{(k)}$ and $V_- := \bigcap_{k\in\mathbb{Z}} V^{(k)}$. If $V^+ = V$ and $V_- = 0$, we say V is a *filtered vector space* and refer to $gr(V) := \bigoplus_{k\in\mathbb{Z}} V^{(k)}/V^{(k-1)}$ as the *associated graded vector space*.

Definition 2.3 (See *e.g.* [8])**.** A *$\mathbb{Z}$-graded Lie algebra* is a Lie algebra $\mathfrak{g}$ equipped with a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_k$ such that $\forall\, i,j \in \mathbb{Z}$, $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$.

A *filtration* of a Lie algebra $\mathfrak{g}$ is a filtration $\mathfrak{f}^{(k)} : k \in \mathbb{Z}$ of the underlying vector space such that $\forall\, i,j \in \mathbb{Z}$, $[\mathfrak{f}^{(i)}, \mathfrak{f}^{(j)}] \subseteq \mathfrak{f}^{(i+j)}$. If $\mathfrak{g}_k := \mathfrak{f}^{(k)}/\mathfrak{f}^{(k-1)}$, this induces a Lie algebra structure on $gr_{\mathfrak{f}}(\mathfrak{g}) := \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_k$. If $\mathfrak{f}^+ = \mathfrak{g}$ and $\mathfrak{f}_- = 0$, we say $\mathfrak{g}$ is a *filtered Lie algebra* with *associated graded Lie algebra* $gr_{\mathfrak{f}}(\mathfrak{g})$.

Proposition 2.4. *Let $\mathfrak{f}^{(-1)} := \mathfrak{n} \leqslant \mathfrak{g}$, $\mathfrak{f}^{(0)} := \mathfrak{p} \leqslant \mathfrak{n}_{\mathfrak{g}}(\mathfrak{n})$, and for $j > 0$ inductively define $\mathfrak{f}^{(-j-1)} = [\mathfrak{n}, \mathfrak{f}^{(-j)}]$ and $\mathfrak{f}^{(j)} = \mathfrak{c}_{\mathfrak{g}}(\mathfrak{n}, \mathfrak{f}^{(j-1)})$. Then for all $i,j \in \mathbb{Z}$, $[\mathfrak{f}^{(i)}, \mathfrak{f}^{(j)}] \subseteq \mathfrak{f}^{(i+j)}$ and if $\mathfrak{f}^{(-1)} \subseteq \mathfrak{f}^{(0)}$ (i.e., $\mathfrak{n} \trianglelefteq \mathfrak{p}$) then $\mathfrak{f}^{(k)} : k \in \mathbb{Z}$ is a filtration of $\mathfrak{g}$.*

Proof. By construction $[\mathfrak{f}^{(-1)}, \mathfrak{f}^{(j)}] \subseteq \mathfrak{f}^{(j-1)}$ for all $j \in \mathbb{Z}$, and $\mathfrak{f}^{(j-1)} \subseteq \mathfrak{f}^{(j)}$ for $j \neq 0$. Next, by Jacobi, for any $i > 0$, $[\mathfrak{f}^{(-i-1)}, \mathfrak{f}^{(j)}] = [[\mathfrak{f}^{(-1)}, \mathfrak{f}^{(-i)}], \mathfrak{f}^{(j)}] \subseteq [\mathfrak{f}^{(-1)}, [\mathfrak{f}^{(-i)}, \mathfrak{f}^{(j)}]] + [\mathfrak{f}^{(-i)}, \mathfrak{f}^{(j-1)}]$ for all $j \in \mathbb{Z}$, so induction on i shows that $[\mathfrak{f}^{(-i)}, \mathfrak{f}^{(j)}] \subseteq \mathfrak{f}^{(-i+j)}$ for $i > 0$ and $j \in \mathbb{Z}$.

We now show $[\mathfrak{f}^{(i)}, \mathfrak{f}^{(j)}] \subseteq \mathfrak{f}^{(i+j)}$ for $i,j \geqslant 0$ by induction on $i+j$: $[\mathfrak{f}^{(0)}, \mathfrak{f}^{(0)}] \subseteq \mathfrak{f}^{(0)}$ since $\mathfrak{f}^{(0)} = \mathfrak{p}$ is a subalgebra, and for $i + j > 0$, Jacobi implies $[[\mathfrak{f}^{(i)}, \mathfrak{f}^{(j)}], \mathfrak{n}] \subseteq [\mathfrak{f}^{(i)}, \mathfrak{f}^{(j-1)}] + [\mathfrak{f}^{(i-1)}, \mathfrak{f}^{(j)}] \subseteq \mathfrak{f}^{(i+j-1)}$ (inductively), *i.e.*, $[\mathfrak{f}^{(i)}, \mathfrak{f}^{(j)}] \subseteq \mathfrak{c}_{\mathfrak{g}}(\mathfrak{n}, \mathfrak{f}^{(i+j-1)}) = \mathfrak{f}^{(i+j)}$. □

We call this the filtration of $\mathfrak{g}$ *induced by* $\mathfrak{n} \trianglelefteq \mathfrak{p}$, or *by* $\mathfrak{n}$ if $\mathfrak{p} = \mathfrak{n}_{\mathfrak{g}}(\mathfrak{n})$. Its negative part is the *lower central series* of $\mathfrak{n}$, and if $\mathfrak{f}^{(-k)} = 0$ for sufficiently large k, we say $\mathfrak{n}$ is *nilpotent.*

Definition 2.5. Let $\mathfrak{g}, \mathfrak{f}^{(k)} : k \in \mathbb{Z}$ be a filtered Lie algebra. A *filtration* of a representation $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ of $\mathfrak{g}$ is a filtration $V^{(k)} : k \in \mathbb{Z}$ of V such that $\forall\, i,j \in \mathbb{Z}$, $\mathfrak{f}^{(i)} \cdot V^{(j)} := \rho(\mathfrak{f}^{(i)}, V^{(j)}) \subseteq V^{(i+j)}$. This induces a representation $\overline{\rho}$ of $gr_{\mathfrak{f}}(\mathfrak{g})$ on $gr(V) := \bigoplus_{k\in\mathbb{Z}} V_k$, where $V_k = V^{(k)}/V^{(k-1)}$, such that $\forall\, i,j \in \mathbb{Z}$, $\overline{\rho}(\mathfrak{g}_i, V_j) \subseteq V_{i+j}$. If V is filtered, we call V (or ρ) a *filtered representation* of $\mathfrak{g}$ with *associated graded representation* $gr(V)$ (or $\overline{\rho}$).

2.3. Engel's theorem and nilpotence.

Theorem 2.6 (Engel)**.** *Let $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a finite dimensional representation of a Lie algebra $\mathfrak{g}$ and $\mathfrak{n} \trianglelefteq \mathfrak{g}$. Then the following are equivalent*:
(1) *$\rho(x)$ is nilpotent for all $x \in \mathfrak{n}$*;
(2) *$\mathfrak{n}$ acts trivially on any irreducible subquotient of ρ*;
(3) *V is a filtered representation for the filtration of $\mathfrak{g}$ induced by $\mathfrak{n}$.*

Proof. The key is the following well-known lemma due to Engel; we omit the proof.

Lemma 2.7. *Let $\mathfrak{u} \leqslant \mathfrak{gl}(W)$ be a Lie subalgebra with σ nilpotent for all $\sigma \in \mathfrak{u}$; then if W is nonzero, $\exists w \in W$ nonzero such that for all $\sigma \in \mathfrak{u}$, $\sigma(w) = 0$.*

(1)⇒(2). Let $\rho'\colon \mathfrak{g} \to \mathfrak{gl}(W)$ be an irreducible subquotient of ρ and let $W' = \{w \in W \mid \rho'(\mathfrak{n})(w) = 0\}$. Since $\mathfrak{n} \trianglelefteq \mathfrak{g}$, W' is $\mathfrak{g}$-invariant. Since $\mathfrak{n}$ acts by nilpotent endomorphisms, Lemma 2.7 implies W' is nonzero; hence $W' = W$.

(2)⇒(3). This is an easy induction on $dim\, V$: if $V = 0$, we are done; otherwise V has a nontrivial irreducible $\mathfrak{g}$-invariant subspace $U \leqslant V$, which is in the kernel of $\rho(\mathfrak{n})$ by (2). By induction, V/U by is filtered by some $V^{(j)}/U : j \in \mathbb{Z}$. If k denotes the largest integer with $V^{(k)} = U$, we may redefine $V^{(j)} = 0$ for $j < k$ to make V into a filtered representation.

(3)⇒(1). Given such a filtration $V^{(j)} : j \in \mathbb{Z}$, we may assume $V^{(0)} = V$ and $V^{(j)} = 0$ for $j < -k$. Now any $x \in \mathfrak{n}$ satisfies $\rho(x)^{k+1} = 0$. □

If any of these conditions hold, $\mathfrak{n}$ is called a *nilpotency ideal* [2] for ρ; it follows from (3) that $\rho(\mathfrak{n})$ is a nilpotent Lie algebra. By (2), $\mathfrak{g}$ has a largest nilpotency ideal $\mathfrak{nil}_\rho(\mathfrak{g})$ for ρ, namely the intersection of the kernels of the simple subquotients of ρ. Thus $ker\, \rho \leqslant \mathfrak{nil}_\rho(\mathfrak{g})$ and equality holds if ρ is semisimple. Hence $\mathfrak{nil}_\rho(\mathfrak{g}) = 0$ if ρ is faithful and semisimple.

Henceforth, we assume $\mathfrak{g}$ is finite dimensional. An ideal $\mathfrak{n} \trianglelefteq \mathfrak{g}$ is a nilpotency ideal for the adjoint representation ad of $\mathfrak{g}$ if and only if it is nilpotent, and so $\mathfrak{nil}_{ad}(\mathfrak{g})$ is the largest nilpotent ideal of $\mathfrak{g}$, often called the *nilradical.*

Definition 2.8. The *nilpotent radical* $\mathfrak{nil}(\mathfrak{g}) \trianglelefteq \mathfrak{g}$ is the intersection of its largest nilpotency ideals, or equivalently, the intersection of the kernels of the simple representations of $\mathfrak{g}$.

Since $\mathfrak{nil}(\mathfrak{g}) \leqslant \mathfrak{nil}_{ad}(\mathfrak{g})$, it is a nilpotent ideal, and it is the intersection of the kernels of finitely many simple representations of $\mathfrak{g}$, hence the kernel of a semisimple representation of $\mathfrak{g}$. Thus $\mathfrak{g}/\,\mathfrak{nil}(\mathfrak{g})$ is reductive, and $\mathfrak{g}$ is reductive if and only if $\mathfrak{nil}(\mathfrak{g}) = 0$. The Lie algebra $\mathfrak{g}$ is a filtered Lie algebra via the *canonical filtration* $\mathfrak{g}^{(k)} : k \in \mathbb{Z}$ induced by $\mathfrak{nil}(\mathfrak{g}) \trianglelefteq \mathfrak{g}$, and with respect to the canonical filtration, any (finite dimensional) representation of $\mathfrak{g}$ is a filtered representation by Engel's theorem.

Definition 2.9. The *nilpotent cone* of a Lie algebra $\mathfrak{g}$ is $\mathcal{N}(\mathfrak{g}) = \{x \in \mathfrak{g} \mid \rho(x) \text{ is nilpotent for any representation } \rho \text{ of } \mathfrak{g}\}$. A *nil subalgebra* $\mathfrak{n} \leqslant \mathfrak{g}$ is a Lie subalgebra with $\mathfrak{n} \subseteq \mathcal{N}(\mathfrak{g})$.

Remarks 2.10. If $f\colon \mathfrak{h} \to \mathfrak{g}$ is a Lie algebra homomorphism, then any representation ρ of $\mathfrak{g}$ induces a representation $\rho \circ f$ of h, so $f(\mathcal{N}(\mathfrak{h})) \subseteq \mathcal{N}(\mathfrak{g})$. Clearly $x \in \mathcal{N}(\mathfrak{g})$ if and only if x is nilpotent in any *semisimple* representation of $\mathfrak{g}$. Thus $\mathcal{N}(\mathfrak{g})$ is the inverse image of the nilpotent cone in $\mathfrak{g}/\,\mathfrak{nil}(\mathfrak{g})$, which is reductive. A nil subalgebra $\mathfrak{n} \leqslant \mathfrak{g}$ is a nilpotency ideal for the adjoint representation of $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{n})$ on $\mathfrak{g}$, hence nilpotent.

Example 2A. Let V be a vector space with $\dim_{\mathbb{F}} V = n+1$ as in Example 1A. Then $\mathfrak{gl}(V)$ is a Lie algebra whose nilpotent cone $\mathcal{N}(\mathfrak{gl}(V))$ consists of the nilpotent endomorphisms of V. A subalgebra $\mathfrak{n}$ of $\mathfrak{gl}(V)$ is therefore a nil subalgebra iff there is a filtration of V such that $\mathfrak{n}$ acts trivially on the associated graded representation. With respect to a basis adapted to such a filtration, the elements of $\mathfrak{n}$ are strictly upper triangular. In particular, $\mathfrak{gl}(V)$ is reductive, and its derived algebra is the subalgebra $\mathfrak{sl}(V) = [\mathfrak{gl}(V), \mathfrak{gl}(V)]$ of traceless endomorphisms. Thus a nil subalgebra of $\mathfrak{gl}(V)$ is a nilpotent subalgebra of $\mathfrak{sl}(V)$.

Example 2B. Let U, Q_U be as in Example 1B, and let B_U be the associated symmetric bilinear form of signature $(n+k, n)$ on U. Then $\mathfrak{so}(U, Q_U) = \{A \in \mathfrak{gl}(U) \mid B_U(Au_1, u_2)+B_U(u_1, Au_2)\}$ is a Lie subalgebra of $\mathfrak{gl}(U)$, and $\mathcal{N}(\mathfrak{so}(U, Q_U))$ again consists of the elements of $\mathfrak{so}(U, Q_U)$ which are nilpotent endomorphisms of U. If $\mathfrak{n}$ is a nonzero nil subalgebra of $\mathfrak{so}(U, Q_U)$ then $\mathfrak{n} \cdot U$ is nontrivial, hence so is its intersection with $\mathfrak{c}_U(\mathfrak{n}, 0) = \bigcap_{A \in \mathfrak{n}} \ker A$ (because $\mathfrak{n}$ acts trivially on any irreducible summand of $\mathfrak{n} \cdot U$). The intersection $(\mathfrak{n} \cdot U) \cap \mathfrak{c}_U(\mathfrak{n}, 0)$ is isotropic, hence contains a 1-dimensional isotropic subspace π_1. Applying the same argument inductively to $\pi_1^\perp/\pi_1$, we obtain a filtration $0 \leqslant \pi_1 \leqslant \pi_2 \leqslant \cdots \leqslant \pi_n \leqslant \pi_n^\perp \leqslant \cdots \leqslant \pi_2^\perp \leqslant \pi_1^\perp \leqslant U$, with $\dim \pi_j = j$, preserved by $\mathfrak{n}$, and $\mathfrak{n}$ acts trivially on the associated graded representation.

2.4. Invariant forms and trace-forms. We summarize basic properties of invariant symmetric bilinear forms in Appendix A.3.

Definition 2.11. An invariant (symmetric bilinear) form on a filtered Lie algebra $\mathfrak{g}$ is *compatible* with the filtration iff $\mathfrak{f}^{(j-1)} \subseteq (\mathfrak{f}^{(-j)})^\perp$ for all $j \in \mathbb{Z}$. The restriction of a compatible invariant form to $\mathfrak{f}^{(j)} \times \mathfrak{f}^{(-j)}$ descends to a pairing $\mathfrak{g}_j \times \mathfrak{g}_{-j} \to \mathbb{F}$ and hence induces a invariant form on $gr_{\mathfrak{f}}(\mathfrak{g})$, called the *associated graded invariant form.*

Proposition 2.12. *If* $\mathfrak{f}^{(j-1)} = (\mathfrak{f}^{(-j)})^\perp$ *for all* $j \in \mathbb{Z}$*, the associated graded invariant form on* $gr_{\mathfrak{f}}(\mathfrak{g})$ *is nondegenerate, i.e.,* $\mathfrak{g}_j{}^\perp = \bigoplus_{k \neq -j} \mathfrak{g}_k$.

Proof. x is in $gr_{\mathfrak{f}}(\mathfrak{g})^\perp$ if and only if its homogeneous components are. Now for $x \in \mathfrak{g}_j$, we have $x \in gr_{\mathfrak{f}}(\mathfrak{g})^\perp$ if and only if x is orthogonal to $\mathfrak{g}_{-j}$, *i.e.*, any lift $\tilde{x}$ to $\mathfrak{f}^{(j)}$ is in $(\mathfrak{f}^{(-j)})^\perp$. On the other hand $x = 0$ if and only if the lift is in $\mathfrak{f}^{(j-1)}$. □

Definition 2.13. The *trace form* on $\mathfrak{g}$ associated to a representation $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$, is the invariant form $(x, y) \mapsto tr(\rho(x)\rho(y))$.

Proposition 2.14. *Let $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a filtered representation of a filtered Lie algebra. Then the trace form of ρ is compatible with the filtration $\mathfrak{f}^{(j)} : j \in \mathbb{Z}$ of $\mathfrak{g}$, and the associated graded invariant form on $\mathrm{gr}_{\mathfrak{f}}(\mathfrak{g})$ is the trace form of the induced representation $\overline{\rho}$.*

Proof. If $x \in \mathfrak{f}^{(j-1)}$ and $y \in \mathfrak{f}^{(-j)}$ then $\rho(x)\rho(y)$ maps $V^{(k)}$ to $V^{(k-1)}$ for all $k \in \mathbb{Z}$, hence is nilpotent. Thus $\rho(x)\rho(y)$ is trace-free, and hence x and y are orthogonal.

To compute the associated graded form on $x \in \mathfrak{g}_j$ and $y \in \mathfrak{g}_{-j}$, choose lifts $\tilde{x} \in \mathfrak{f}^{(j)}$, $\tilde{y} \in \mathfrak{f}^{(-j)}$ and a splitting $V \cong gr(V)$ of the filtration of V. Then the trace of $\rho(\tilde{x})\rho(\tilde{y})$ may be computed by restricting and projecting onto V_k, for each $k \in \mathbb{Z}$, computing the trace, and summing over k, which yields the trace of $\overline{\rho}(x)\overline{\rho}(y)$. □

Proposition 2.15. *Let $\langle\cdot,\cdot\rangle$ be the trace form of $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$.*

(1) *$\mathfrak{nil}_\rho(\mathfrak{g}) \leqslant \mathfrak{g}^\perp$ and the induced invariant form on $\mathfrak{g}/\mathfrak{nil}_\rho(\mathfrak{g})$ is a trace form.*
(2) *If $\langle\cdot,\cdot\rangle$ is nondegenerate, then $\mathfrak{nil}_\rho(\mathfrak{g}) = 0$.*
(3) *For a Lie homomorphism $f\colon \mathfrak{h} \to \mathfrak{g}$, $\langle\cdot,\cdot\rangle$ pulls back to a trace form associated to $\rho \circ f$.*
(4) *If $\mathfrak{h} \leqslant \mathfrak{g}$ is a subalgebra and $\rho(\mathfrak{h})$ is nilpotent, then $\mathfrak{h} \leqslant \mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})^\perp$.*

Proof. (1) By Engel's theorem, V is a filtered representation for the filtration of $\mathfrak{g}$ induced by $\mathfrak{nil}_\rho(\mathfrak{g})$; hence the trace-form is compatible and $\mathfrak{nil}_\rho(\mathfrak{g}) \leqslant \mathfrak{g}^\perp$.

(2) is immediate from (1) and (3) is obvious.

(4) Pull back (*i.e.*, restrict) $\langle\cdot,\cdot\rangle$ to $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$; since $\mathfrak{h}$ is a nilpotency ideal for the restriction of ρ to $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$, $\mathfrak{h} \subseteq \mathfrak{nil}_\rho(\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})) \subseteq \mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})^\perp \cap \mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) \subseteq \mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})^\perp$. □

Corollary 2.16. *Let $\langle\cdot,\cdot\rangle$ be a trace form on $\mathfrak{g}$.*

(1) *$\mathfrak{nil}(\mathfrak{g}) \leqslant \mathfrak{g}^\perp$ and the induced invariant form on $\mathfrak{g}/\mathfrak{nil}(\mathfrak{g})$ is a trace form.*
(2) *If $\langle\cdot,\cdot\rangle$ is nondegenerate, then $\mathfrak{g}$ is reductive.*
(3) *For a Lie homomorphism $f\colon \mathfrak{h} \to \mathfrak{g}$, $\langle\cdot,\cdot\rangle$ pulls back to a trace form on $\mathfrak{h}$.*
(4) *If $\mathfrak{h} \leqslant \mathfrak{g}$ is a nil subalgebra, then $\mathfrak{h} \leqslant \mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})^\perp$.*

2.5. Cartan criteria for nilpotency and reductive Lie algebras. In this subsection, we assume the underlying field $\mathbb{F}$ has characteristic zero (hence is perfect). Then we have the following straightforward characterization of the nilpotent cone $\mathcal{N}(\mathfrak{g})$.

Proposition 2.17. *Let $\mathfrak{g}$ be a Lie algebra. Then $x \in \mathcal{N}(\mathfrak{g})$ if and only if $x \in [\mathfrak{g},\mathfrak{g}]$ and $ad(x)$ is nilpotent. In particular $\mathfrak{nil}(\mathfrak{g}) = \mathfrak{nil}_{ad}(\mathfrak{g}) \cap [\mathfrak{g},\mathfrak{g}]$.*

Proof. If $x \in \mathcal{N}(\mathfrak{g})$ then $ad\, x$ is nilpotent and x is trivial in any 1-dimensional representation. Conversely, if $x \in [\mathfrak{g},\mathfrak{g}]$ with $ad(x)$ nilpotent, then $\rho(x)$ is nilpotent in any semisimple representation $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ by Proposition A.3. Hence $x \in \mathcal{N}(\mathfrak{g})$. □

Lemma 2.18. *Suppose $\mathfrak{gl}(V) = \mathfrak{h} \oplus \mathfrak{m}$ with $\mathfrak{h} \leqslant \mathfrak{g}$, $\mathfrak{m} \subseteq \mathfrak{h}^\perp$ and $[\mathfrak{h},\mathfrak{m}] \subseteq \mathfrak{m}$. For subspaces $\mathfrak{b} \subseteq \mathfrak{a} \subseteq \mathfrak{h}$, let $\mathfrak{u} = \mathfrak{c}_{\mathfrak{h}}(\mathfrak{a},\mathfrak{b})$. Then any element of $\mathfrak{u} \cap \mathfrak{u}^\perp \subseteq \mathfrak{gl}(V)$ is nilpotent.*

Proof (*see* [2, 17]). By Lemma 2.1, $\mathfrak{c}_{\mathfrak{gl}(V)}(\mathfrak{a},\mathfrak{b}) = \mathfrak{u} \oplus (\mathfrak{c}_{\mathfrak{gl}(V)}(\mathfrak{a}) \cap \mathfrak{m})$, and $\mathfrak{m} \subseteq \mathfrak{h}^\perp$ so $\mathfrak{u}^\perp \cap \mathfrak{h} \subseteq \mathfrak{c}_{\mathfrak{gl}(V)}(\mathfrak{a},\mathfrak{b})^\perp$. Hence it suffices to prove the result for $\mathfrak{m} = 0$.

Let $x = x_s + x_n$ be the Jordan decomposition of $x \in \mathfrak{u} \leqslant \mathfrak{gl}(V)$, let $\mathbb{F}^c$ be a splitting field for x_s, let $\mathcal{S} \subseteq \mathbb{F}^c$ be the set of eigenvalues ${x_s}^c$, and let $f\colon \mathbb{F}^c \to \mathbb{Q}$ be a $\mathbb{Q}$-linear form on $\mathbb{F}^c$. Define $y \in \mathfrak{gl}(V^c)$ to be scalar multiplication by $f(\lambda)$ on the λ-eigenspace of x_s for all $\lambda \in \mathcal{S}$. Then by Lemma A.1, $ad\, y$ is a polynomial with no constant term in $ad\, {x_s}^c$, hence in $ad\, x^c$, so that $y \in \mathfrak{u}^c$. If also $x \in \mathfrak{u}^\perp$, then $0 = tr(x^c y) = \sum_{\lambda\in\mathcal{S}} m(\lambda)\lambda f(\lambda)$, where $m(\lambda) \in \mathbb{Z}^+$ is the multiplicity of λ. Applying f, we obtain $f(\lambda) = 0$ for all $\mathbb{Q}$-linear forms f and all $\lambda \in \mathcal{S}$. Hence $x_s = 0$ and x is nilpotent. □

Proposition 2.19 (Cartan's criterion)**.** *Suppose $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation. Then, with respect to the induced trace form, $\mathfrak{g}^\perp \cap [\mathfrak{g},\mathfrak{g}] \leqslant \mathfrak{nil}_\rho(\mathfrak{g})$.*

Proof. Take $\mathfrak{b} = \rho(\mathfrak{g}^\perp)$, $\mathfrak{a} = \rho(\mathfrak{g})$ and $\mathfrak{m} = 0$ in Lemma 2.18, so that $\mathfrak{u} = \mathfrak{c}_{\mathfrak{gl}(V)}(\rho(\mathfrak{g}), \rho(\mathfrak{g}^\perp))$ (and hence $\rho(\mathfrak{g}^\perp) \trianglelefteq \mathfrak{u}$). Since $\mathfrak{u} \leqslant \mathfrak{c}_{\mathfrak{gl}(V)}(\rho(\mathfrak{g}), \rho(\mathfrak{g})^\perp)$, $\rho([\mathfrak{g},\mathfrak{g}]) = [\rho(\mathfrak{g}), \rho(\mathfrak{g})] \leqslant \mathfrak{u}^\perp$. Hence $\rho(\mathfrak{g}^\perp \cap [\mathfrak{g},\mathfrak{g}]) \leqslant \mathfrak{u} \cap \mathfrak{u}^\perp$, and so $\mathfrak{g}^\perp \cap [\mathfrak{g},\mathfrak{g}] \trianglelefteq \mathfrak{g}$ is a nilpotency ideal for ρ. □

Theorem 2.20. *A finite dimensional Lie algebra $\mathfrak{g}$ over a field of characteristic zero is reductive if and only if it admits a nondegenerate trace form. Then $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g},\mathfrak{g}]$, the sum is orthogonal, and $[\mathfrak{g},\mathfrak{g}] \cong ad(\mathfrak{g}) = \mathfrak{c}_{\mathfrak{der}(\mathfrak{g})}(\mathfrak{z}(\mathfrak{g})) \cong \mathfrak{der}([\mathfrak{g},\mathfrak{g}])$ is semisimple.*

Proof. If $\mathfrak{g}$ is reductive, it has a faithful semisimple representation ρ, and $\mathfrak{g}^\perp \cap [\mathfrak{g},\mathfrak{g}] = 0$ with respect to the induced trace form by Proposition 2.19, so $\mathfrak{g}^\perp \subseteq [\mathfrak{g},\mathfrak{g}]^\perp = \mathfrak{c}_{\mathfrak{g}}(\mathfrak{g},\mathfrak{g}^\perp) = \mathfrak{z}(\mathfrak{g})$ by Proposition A.4 (1). Now $[\mathfrak{g},\mathfrak{g}]^\perp = \mathfrak{z}(\mathfrak{g}) \subseteq \mathfrak{nil}_{ad}(\mathfrak{g})$ has trivial intersection with $[\mathfrak{g},\mathfrak{g}]$ by Proposition 2.17, so the trace form is nondegenerate on $[\mathfrak{g},\mathfrak{g}]$. Since $\mathfrak{g}^\perp \leqslant \mathfrak{z}(\mathfrak{g})$, we may add a representation of $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ to ρ to make the trace form nondegenerate. Conversely, the existence of a nondegenerate trace form implies $\mathfrak{g}$ is reductive by Corollary 2.16.

Since $\mathfrak{nil}_{ad}(\mathfrak{g}) \cap [\mathfrak{g},\mathfrak{g}] = 0$, Dieudonné's famous observation [12] (see Proposition A.5) shows that $[\mathfrak{g},\mathfrak{g}]$ is semisimple. Thus $[\mathfrak{g},\mathfrak{g}] \cong ad(\mathfrak{g})$ is a nondegenerate ideal in $\mathfrak{der}(\mathfrak{g}) \leqslant \mathfrak{gl}(\mathfrak{g})$, and hence $\mathfrak{h} := ad(\mathfrak{g})^\perp \cap \mathfrak{der}(\mathfrak{g})$ is a complementary ideal in $\mathfrak{der}(\mathfrak{g})$. Now for any $D \in \mathfrak{h}$ and $x \in \mathfrak{g}$, $0 = [D, ad(x)] = ad(D(x))$, so $D(x) \in \mathfrak{z}(\mathfrak{g})$ and hence D vanishes on $[\mathfrak{g},\mathfrak{g}]$; thus $ad(\mathfrak{g}) = \mathfrak{c}_{\mathfrak{der}(\mathfrak{g})}(\mathfrak{z}(\mathfrak{g})) \cong \mathfrak{der}([\mathfrak{g},\mathfrak{g}])$. □

Definition 2.21. An *admissible form* on a Lie algebra $\mathfrak{g}$ is a trace form with $\mathfrak{g}^\perp = \mathfrak{nil}(\mathfrak{g})$.

Admissible forms always exist: just pull back a nondegenerate trace form on the reductive quotient $\mathfrak{g}/\mathfrak{nil}(\mathfrak{g})$. They can be used to construct linear subspaces of $\mathcal{N}(\mathfrak{g})$.

Proposition 2.22. *Let $\mathfrak{b} \subseteq \mathfrak{a} \subseteq \mathfrak{g}$ be subspaces of a Lie algebra $\mathfrak{g}$ which contain $\mathfrak{nil}(\mathfrak{g})$ and let $\mathfrak{u} = \mathfrak{c}_{\mathfrak{g}}(\mathfrak{a},\mathfrak{b})$. Then for any admissible form on $\mathfrak{g}$, $\mathfrak{u} \cap \mathfrak{u}^\perp \subseteq \mathcal{N}(\mathfrak{g})$.*

Proof. Any admissible form is induced by a semisimple representation $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$. Now $\mathfrak{gl}(V) = \rho(\mathfrak{g}) \oplus \rho(\mathfrak{g})^\perp$ and $\rho(\mathfrak{u}) = \mathfrak{c}_{\rho(\mathfrak{g})}(\rho(\mathfrak{a}), \rho(\mathfrak{b}))$, so for any $x \in \mathfrak{u} \cap \mathfrak{u}^\perp$, $\rho(x)$ is nilpotent by Lemma 2.18, and so $ad(\rho(x))$ is nilpotent on $\rho(\mathfrak{g})$. Since $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{g},\mathfrak{g}^\perp) \subseteq \mathfrak{u}$, $\mathfrak{u}^\perp \subseteq [\mathfrak{g},\mathfrak{g}]$ by Proposition A.4 (1) and so $\rho(x) \subseteq [\rho(\mathfrak{g}), \rho(\mathfrak{g})]$. Thus $\rho(x) \in \mathcal{N}(\rho(\mathfrak{g}))$ by Proposition 2.17, and so $x \in \mathcal{N}(\mathfrak{g})$. □

3. Parabolic subalgebras

3.1. General definition. Henceforth, the characteristic of underlying field $\mathbb{F}$ will be zero.

Definition 3.1. A subalgebra $\mathfrak{p}$ of a Lie algebra $\mathfrak{g}$ is *parabolic in* $\mathfrak{g}$ if it contains the normalizer $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{m})$ of a maximal nil subalgebra $\mathfrak{m} \subseteq \mathcal{N}(\mathfrak{g})$ of $\mathfrak{g}$.

Any Lie algebra $\mathfrak{g}$ is a parabolic subalgebra of itself, and more generally, if $\mathfrak{p} \leqslant \mathfrak{q} \leqslant \mathfrak{g}$ are subalgebras such that $\mathfrak{p}$ is parabolic in $\mathfrak{g}$ then $\mathfrak{q}$ is parabolic in $\mathfrak{g}$. At the other extreme, the minimal parabolic subalgebras of $\mathfrak{g}$ are the normalizers of its maximal nil subalgebras.

Proposition 3.2. *Let $\mathfrak{p}$ be a subalgebra of $\mathfrak{g}$ such that $\mathfrak{nil}(\mathfrak{p})$ contains $\mathfrak{nil}(\mathfrak{g})$. Then $\mathfrak{p}$ is parabolic in $\mathfrak{g}$ if and only if $\mathfrak{p}/\mathfrak{nil}(\mathfrak{g})$ is parabolic in $\mathfrak{g}/\mathfrak{nil}(\mathfrak{g})$.*

Proof. Since $\mathfrak{nil}(\mathfrak{g})$ is an ideal in $\mathfrak{g}$ and $\mathcal{N}(\mathfrak{g}/\mathfrak{nil}(\mathfrak{g})) = \mathcal{N}(\mathfrak{g})/\mathfrak{nil}(\mathfrak{g})$, $\mathfrak{m}$ is a maximal nil subalgebra of $\mathfrak{g}$ if and only if $\mathfrak{m}$ contains $\mathfrak{nil}(\mathfrak{g})$ and $\mathfrak{m}/\mathfrak{nil}(\mathfrak{g})$ is a maximal nil subalgebra of $\mathfrak{g}/\mathfrak{nil}(\mathfrak{g})$; now $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{m})/\mathfrak{nil}(\mathfrak{g}) = \mathfrak{n}_{\mathfrak{g}/\mathfrak{nil}(\mathfrak{g})}(\mathfrak{m}/\mathfrak{nil}(\mathfrak{g}))$ and the result follows. □

For $\mathfrak{q} \leqslant \mathfrak{g}$, define $\mathfrak{nil}_{\mathfrak{g}}(\mathfrak{q}) := \mathfrak{nil}_{ad_{\mathfrak{g}}}(\mathfrak{q}) \cap [\mathfrak{g},\mathfrak{g}]$, the largest ideal of $\mathfrak{q}$ contained in $\mathcal{N}(\mathfrak{g})$.

Proposition 3.3. *Suppose $\mathfrak{nil}(\mathfrak{g}) \leqslant \mathfrak{m} \leqslant \mathfrak{g}$ and let $\mathfrak{q} = \mathfrak{n}_{\mathfrak{g}}(\mathfrak{m})$. Then for any admissible form on $\mathfrak{g}$,* (1) $\mathfrak{m} = \mathfrak{nil}_{\mathfrak{g}}(\mathfrak{q}) \implies \mathfrak{m} = \mathfrak{q} \cap \mathfrak{q}^\perp$, *and* (2) $\mathfrak{m} = \mathfrak{q} \cap \mathfrak{q}^\perp \implies \mathfrak{m} = \mathfrak{q}^\perp$.

Proof (*see* [2, 23, 27]). Since $\mathfrak{g}^\perp \leqslant \mathfrak{m} \trianglelefteq \mathfrak{q} \leqslant \mathfrak{g}$, $[\mathfrak{q}, \mathfrak{q}^\perp] \leqslant \mathfrak{q}^\perp$ and $\mathfrak{q} \cap \mathfrak{q}^\perp$ is an ideal in $\mathfrak{q}$, which is contained in $\mathcal{N}(\mathfrak{g})$ by Proposition 2.22. If $\mathfrak{m} \subseteq \mathcal{N}(\mathfrak{g})$ then $\mathfrak{m} \subseteq \mathfrak{q} \cap \mathfrak{q}^\perp$ by Corollary 2.16 (4), and (1) follows. Now $\mathfrak{q}$ acts on $\mathfrak{q}^\perp/\mathfrak{m}$ with $\mathfrak{m}$ as a nilpotency ideal. However, the $\mathfrak{m}$-action has kernel $(\mathfrak{q} \cap \mathfrak{q}^\perp)/\mathfrak{m} = 0$. Hence $\mathfrak{q}^\perp = \mathfrak{m}$ by Engel's theorem. □

This result leads to the following equivalences, cf. [2, 4, 7, 15].

Theorem 3.4. *For a subalgebra* $\mathfrak{p} \leqslant \mathfrak{g}$ *with* $\mathfrak{nil}(\mathfrak{g}) \leqslant \mathfrak{nil}(\mathfrak{p})$, *the following are equivalent.*

(1) *There is an admissible form on* $\mathfrak{g}$ *such that* $\mathfrak{p}^\perp$ *is a nilpotent subalgebra of* $[\mathfrak{g}, \mathfrak{g}]$.
(2) *There is an admissible form on* $\mathfrak{g}$ *such that* $\mathfrak{p}^\perp$ *is a nil subalgebra of* $\mathfrak{g}$.
(3) $\mathfrak{p}$ *is parabolic in* $\mathfrak{g}$.
(4) *For any admissible form on* $\mathfrak{g}$, $\mathfrak{p}^\perp \subseteq \mathfrak{p}$ *and* $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{p}) = \mathfrak{p}$.
(5) $\mathfrak{p} = \mathfrak{n}_{\mathfrak{g}}(\mathfrak{nil}_{\mathfrak{g}}(\mathfrak{p}))$.
(6) *For any admissible form on* $\mathfrak{g}$, $\mathfrak{p}^\perp = \mathfrak{nil}_{\mathfrak{g}}(\mathfrak{p}) = \mathfrak{nil}(\mathfrak{p})$.
(7) $\dim \mathfrak{g} - \dim \mathfrak{p} = \dim \mathfrak{nil}(\mathfrak{p}) - \dim \mathfrak{nil}(\mathfrak{g})$.

Proof. By Corollary 2.16, $\mathfrak{nil}(\mathfrak{p}) \leqslant \mathfrak{nil}_{\mathfrak{g}}(\mathfrak{p}) \leqslant \mathfrak{p}^\perp$ for any admissible form on $\mathfrak{g}$; also, since $\mathfrak{g}^\perp \leqslant \mathfrak{p} \leqslant \mathfrak{g}$, $\mathfrak{p} \subseteq \mathfrak{n}_{\mathfrak{g}}(\mathfrak{p}) = \mathfrak{n}_{\mathfrak{g}}(\mathfrak{p}^\perp) = [\mathfrak{p}, \mathfrak{p}^\perp]^\perp$ by Proposition A.4 (1). In particular (6) and (7) are equivalent, since $\dim \mathfrak{p}^\perp = \dim \mathfrak{g} - \dim \mathfrak{p} + \dim \mathfrak{g}^\perp$ and $\mathfrak{g}^\perp = \mathfrak{nil}(\mathfrak{g})$.

(1)⇔(2) (cf. [4]). Since $\mathfrak{p}^\perp \leqslant \mathfrak{g}$ and $\mathfrak{p} \leqslant \mathfrak{n}_{\mathfrak{g}}(\mathfrak{p}^\perp)$, Proposition 2.4 yields subspaces $\mathfrak{p}^{(j)}$ of $\mathfrak{g}$ with $\mathfrak{p}^{(-1)} = \mathfrak{p}^\perp$, $\mathfrak{p}^{(0)} = \mathfrak{p}$ and $[\mathfrak{p}^\perp, \mathfrak{p}^{(j)}] \leqslant \mathfrak{p}^{(j-1)}$ for all $j \in \mathbb{Z}$. Now $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{p}^\perp, (\mathfrak{p}^{(-j)})^\perp) = [\mathfrak{p}^\perp, \mathfrak{p}^{(-j)}]^\perp$, so induction on j shows that $\mathfrak{p}^{(j-1)} = (\mathfrak{p}^{(-j)})^\perp$ for $j > 0$, and hence $\bigcup_{k\geqslant 0} \mathfrak{p}^{(k)} = \big(\bigcap_{k<0} \mathfrak{p}^{(k)}\big)^\perp = \{0\}^\perp = \mathfrak{g}$, since $\mathfrak{p}^\perp$ is nilpotent. Hence $\mathfrak{p}^\perp$ is an $ad_{\mathfrak{g}}$-nilpotent subalgebra of $[\mathfrak{g}, \mathfrak{g}]$, *i.e.*, a nil subalgebra of $\mathfrak{g}$ by Proposition 2.17. The converse is immediate.

(2)⇔(3). $\mathfrak{p}^\perp$ is a nil subalgebra of $\mathfrak{g}$ iff $\mathfrak{p}^\perp \subseteq \mathfrak{m}$ iff $\mathfrak{m}^\perp \subseteq \mathfrak{p}$ for a maximal nil subalgebra $\mathfrak{m}$, and by Proposition 3.3, $\mathfrak{m}^\perp = \mathfrak{n}_{\mathfrak{g}}(\mathfrak{m})$.

(1–3)⇒(4). Corollary 2.16 (4) implies $\mathfrak{p}^\perp \subseteq \mathfrak{n}_{\mathfrak{g}}(\mathfrak{p}^\perp)^\perp$, hence $\mathfrak{p}^\perp \subseteq \mathfrak{n}_{\mathfrak{g}}(\mathfrak{p}^\perp) \subseteq \mathfrak{p}^{\perp\perp} = \mathfrak{p}$.

(4)⇒(5). Since $\mathfrak{p}^\perp \subseteq \mathfrak{p} = \mathfrak{n}_{\mathfrak{g}}(\mathfrak{p}^\perp)$, $\mathfrak{p}^\perp = \mathfrak{p} \cap \mathfrak{p}^\perp \trianglelefteq \mathfrak{p}$ and is nil by Proposition 2.22, so $\mathfrak{p}^\perp \leqslant \mathfrak{nil}_{\mathfrak{g}}(\mathfrak{p})$; hence equality holds and $\mathfrak{p} = \mathfrak{n}_{\mathfrak{g}}(\mathfrak{nil}_{\mathfrak{g}}(\mathfrak{p}))$.

(5)⇒(6) By Proposition 3.3, $\mathfrak{nil}_{\mathfrak{g}}(\mathfrak{p}) = \mathfrak{p}^\perp$, and hence $\mathfrak{p} = \mathfrak{n}_{\mathfrak{g}}(\mathfrak{p}^\perp) = [\mathfrak{p}, \mathfrak{p}^\perp]^\perp$, so $\mathfrak{p}^\perp \subseteq [\mathfrak{p}, \mathfrak{p}^\perp] + \mathfrak{g}^\perp \leqslant \mathfrak{nil}(\mathfrak{p})$ since $[\mathfrak{p}, \mathfrak{nil}_{\mathfrak{g}}(\mathfrak{p})] \subseteq \mathfrak{nil}(\mathfrak{p})$.

(6)⇒(1). Immediate, since $\mathfrak{nil}(\mathfrak{p}) \subseteq [\mathfrak{p}, \mathfrak{p}]$. □

Proposition 3.5. *Suppose* $\mathfrak{p} \leqslant \mathfrak{q} \leqslant \mathfrak{g}$ *are subalgebras with* $\mathfrak{q}$ *parabolic in* $\mathfrak{g}$. *Then* $\mathfrak{p}$ *is parabolic in* $\mathfrak{q}$ *if and only if it is parabolic in* $\mathfrak{g}$.

Proof. Fix an admissible form on $\mathfrak{g}$; its restriction to $\mathfrak{q}$ is admissible, since $\mathfrak{q}^\perp = \mathfrak{nil}(\mathfrak{q})$. Since $\mathfrak{p} \subseteq \mathfrak{q} \subseteq \mathfrak{g}$, we have $\mathfrak{g}^\perp \subseteq \mathfrak{q}^\perp \subseteq \mathfrak{p}^\perp$, *i.e.*, $\mathfrak{nil}(\mathfrak{g}) \subseteq \mathfrak{nil}(\mathfrak{q}) \subseteq \mathfrak{p}^\perp \cap \mathfrak{q}$.

If $\mathfrak{p}$ is parabolic in $\mathfrak{q}$, $\mathfrak{q}^\perp = \mathfrak{nil}(\mathfrak{q}) \subseteq \mathfrak{p}$ and $\mathfrak{nil}(\mathfrak{p}) = \mathfrak{p}^\perp \cap \mathfrak{q}$. It follows that $\mathfrak{nil}(\mathfrak{g}) \subseteq \mathfrak{nil}(\mathfrak{p})$ and $\mathfrak{p}^\perp \subseteq \mathfrak{q}^{\perp\perp} = \mathfrak{q}$, hence $\mathfrak{nil}(\mathfrak{p}) = \mathfrak{p}^\perp$, as required. Conversely if $\mathfrak{p}$ is parabolic in $\mathfrak{g}$, then $\mathfrak{p}^\perp = \mathfrak{nil}(\mathfrak{p})$ contains $\mathfrak{nil}(\mathfrak{q})$, and since $\mathfrak{p}^\perp \subseteq \mathfrak{p} \subseteq \mathfrak{q}$, $\mathfrak{p}$ is parabolic in $\mathfrak{q}$. □

Corollary 3.6. *Let* $\mathfrak{p} \leqslant \mathfrak{q} \leqslant \mathfrak{g}$ *with* $\mathfrak{g}$ *reductive, and* $\mathfrak{q}$ *parabolic in* $\mathfrak{g}$. *Then* $\mathfrak{p}$ *is parabolic in* $\mathfrak{g}$ *if and only if* $\mathfrak{nil}(\mathfrak{p})$ *contains* $\mathfrak{nil}(\mathfrak{q})$ *and* $\mathfrak{p}/\mathfrak{nil}(\mathfrak{q})$ *is parabolic in* $\mathfrak{q}/\mathfrak{nil}(\mathfrak{q})$.

We refer to $\mathfrak{q}_0 = \mathfrak{q}/\mathfrak{nil}(\mathfrak{q})$ as the (reductive) *Levi quotient* of $\mathfrak{q}$.

3.2. Grading elements and splittings.

Definition 3.7. A *grading element* for a Lie algebra $\mathfrak{p}$ is an element $\chi \in gr(\mathfrak{p})$ (the associated graded algebra of the canonical filtration) with $[\chi, x] = jx$ for all $j \in \mathbb{Z}$ and $x \in \mathfrak{p}_j$. A (reductive) *Levi subalgebra* is a Lie subalgebra of $\mathfrak{p}$ complementary to $\mathfrak{nil}(\mathfrak{p})$.

A grading element χ exists if and only if the derivation D of $gr(\mathfrak{p})$ defined by $Dx = jx$ (for all $j \in \mathbb{Z}$ and $x \in \mathfrak{p}_j$) is inner; χ then belongs to the centre of $\mathfrak{p}_0$, and is unique modulo the centre of $gr(\mathfrak{p})$. If $\mathfrak{p}$ is reductive, *i.e.*, $\mathfrak{nil}(\mathfrak{p}) = 0$, then $0 \in \mathfrak{p} = \mathfrak{p}_0$ is a grading element.

Proposition 3.8 (see [5, 7])**.** *If $\mathfrak{p}$ has a grading element $\chi \in \mathfrak{p}_0$, then the following sets are in canonical* $\exp(\mathfrak{nil}(\mathfrak{p}))$*-equivariant bijection*:
- *lifts of χ to $\mathfrak{p}$*;
- *splittings $\mathfrak{p} \cong gr(\mathfrak{p})$ of the canonical filtration of $\mathfrak{p}$*;
- *Levi subalgebras of $\mathfrak{p}$.*

Furthermore, the action of $\exp(\mathfrak{nil}(\mathfrak{p}))$ *is free and transitive.*

Proof. A lift of χ to $\mathfrak{p}$ determines a splitting $gr(\mathfrak{p}) \cong \mathfrak{p}$ via its eigenspace decomposition. For any such splitting, the image of $\mathfrak{p}_0$ is a Levi subalgebra of $\mathfrak{p}$, and any Levi subalgebra of $\mathfrak{p}$ contains a unique lift of χ. Since $\mathfrak{p}_0 = \mathfrak{p}/\,\mathfrak{nil}(\mathfrak{p})$, lifts of χ form an affine space modelled on $\mathfrak{nil}(\mathfrak{p})$, which is nilpotent, so $\exp(\mathfrak{nil}(\mathfrak{p}))$ acts freely and transitively on the lifts. □

For a parabolic subalgebra $\mathfrak{p} \leqslant \mathfrak{g}$, we let $\mathfrak{p}^{(j)}$ be the filtration of $\mathfrak{g}$ induced by $\mathfrak{nil}(\mathfrak{p}) \trianglelefteq \mathfrak{p} \leqslant \mathfrak{g}$ (cf. Theorem 3.4), and denote the associated graded Lie algebra by $gr_{\mathfrak{p}}(\mathfrak{g})$.

Proposition 3.9. *Let $\mathfrak{p}$ be parabolic in a reductive Lie algebra $\mathfrak{g}$. Then the associated graded algebra $gr_{\mathfrak{p}}(\mathfrak{g})$ is reductive, and its centre is contained in $\mathfrak{p}_0 = \mathfrak{p}/\,\mathfrak{nil}(\mathfrak{p})$.*

Proof. Since $\mathfrak{g}$ is reductive, it admits a nondegenerate trace form, induced by a representation $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$. Since $\mathfrak{p}^{\perp} = \mathfrak{nil}(\mathfrak{p})$, it acts nilpotently on V, and hence induces a filtration on V with $V^{(j)} = V$ for $j \geqslant 0$ (say) and $V^{(j-1)} = \rho(\mathfrak{p}^{\perp}, V^{(j)})$ for $j \leqslant 0$. This makes V into a filtered representation of $\mathfrak{g}$ (with the filtration by $\mathfrak{p}^{(j)} : j \in \mathbb{Z}$).

We have already observed that $\mathfrak{p}^{(j-1)} = (\mathfrak{p}^{(-j)})^{\perp}$ for $j > 0$, hence writing $k = 1 - j$ and taking perps, we have $(\mathfrak{p}^{(-k)})^{\perp} = \mathfrak{p}^{(k-1)}$ for $k \leqslant 0$. The induced trace form on $gr_{\mathfrak{p}}(\mathfrak{g})$ is therefore nondegenerate, and hence $gr_{\mathfrak{p}}(\mathfrak{g})$ is reductive.

For any $x \in \mathfrak{z}(gr_{\mathfrak{p}}(\mathfrak{g}))$, its homogeneous components are also in the centre. Now for a homogeneous central element $x \in \mathfrak{p}_j := \mathfrak{p}^{(j)}/\mathfrak{p}^{(j-1)}$, any lift centralizes $\mathfrak{nil}(\mathfrak{p})$, so it must lie in $\mathfrak{p}$, *i.e.*, $j \leqslant 0$. However, since $\mathfrak{p}^{(k)} = [\mathfrak{nil}(\mathfrak{p}), \mathfrak{p}^{(k+1)}]$ for $k < 0$, we must also have $j \geqslant 0$, *i.e.*, the centre of $gr_{\mathfrak{p}}(\mathfrak{g})$ is contained in $\mathfrak{p}_0$. □

Corollary 3.10. *A parabolic subalgebra $\mathfrak{p}$ of a reductive Lie algebra $\mathfrak{g}$ has a unique grading element $\chi \in \mathfrak{z}(\mathfrak{p}_0) \cap [gr_{\mathfrak{p}}(\mathfrak{g}), gr_{\mathfrak{p}}(\mathfrak{g})]$.*

Indeed, the derivation D of $gr_{\mathfrak{p}}(\mathfrak{g})$ defined by $Dx = jx$ for $x \in \mathfrak{g}_j$ vanishes on the centre of $gr_{\mathfrak{p}}(\mathfrak{g})$ and preserves its semisimple complement. It follows that D is an inner derivation, *i.e.*, $D = ad\,\chi$ for $\chi \in \mathfrak{z}(\mathfrak{p}_0)$ which is determined uniquely by requiring it is in the complement $[gr_{\mathfrak{p}}(\mathfrak{g}), gr_{\mathfrak{p}}(\mathfrak{g})]$ to centre of $gr_{\mathfrak{p}}(\mathfrak{g})$.

3.3. Pairs of parabolic subalgebras.

Definition 3.11 (see [10])**.** Parabolic subalgebras $\mathfrak{p}, \mathfrak{q}$ of a reductive Lie algebra $\mathfrak{g}$ are said to be *costandard* if $\mathfrak{p} \cap \mathfrak{q}$ is parabolic in $\mathfrak{g}$.

Proposition 3.12. *For parabolics $\mathfrak{p}, \mathfrak{q} \leqslant \mathfrak{g}$, with $\mathfrak{g}$ reductive, the following are equivalent*:
1. *$\mathfrak{p}$ and $\mathfrak{q}$ are costandard*;
2. *$\mathfrak{p} \cap \mathfrak{q}$ contains a minimal parabolic subalgebra of $\mathfrak{g}$*;
3. *for some* (*hence any*) *admissible form on $\mathfrak{g}$, $\mathfrak{p}^{\perp} \leqslant \mathfrak{q}$* (*or equivalently, $\mathfrak{q}^{\perp} \leqslant \mathfrak{p}$*).

Proof. The first two conditions are manifestly equivalent. Now for any admissible form on $\mathfrak{g}$, $(\mathfrak{p} \cap \mathfrak{q})^{\perp} = \mathfrak{p}^{\perp} + \mathfrak{q}^{\perp}$. Thus (1) and (2) imply (3). For the converse, it suffices to show that (3) implies $\mathfrak{p}^{\perp} + \mathfrak{q}^{\perp}$ is nilpotent. For this, let $\mathfrak{r}^{(-i)} = \sum_{j \geqslant 0, k \geqslant 0, i = j+k} \mathfrak{p}^{(-i)} \cap \mathfrak{q}^{(-j)}$. Then $\mathfrak{r}^{(-1)} = \mathfrak{p}^{\perp} + \mathfrak{q}^{\perp}$, and $[\mathfrak{r}^{(-1)}, \mathfrak{r}^{(-i)}] \subseteq \mathfrak{r}^{(-i-1)}$, so $\mathfrak{r}^{(-1)}$ is nilpotent. □

Proposition 3.13. *If $\mathfrak{p}^1, \dots \mathfrak{p}^k \leqslant \mathfrak{g}$ are parabolic and pairwise costandard in a reductive Lie algebra $\mathfrak{g}$, then $\mathfrak{p}^1 \cap \cdots \cap \mathfrak{p}^k$ is parabolic in $\mathfrak{g}$.*

Proof. Induction on k, with $k = 1$ being trivial: if $\mathfrak{p}^1 \cap \cdots \cap \mathfrak{p}^{k-1}$ is parabolic, and $\mathfrak{p}^k$ is costandard with each $\mathfrak{p}^i$, then $(\mathfrak{p}^k)^\perp \subseteq \mathfrak{p}^i$ for all $i = 1, \dots k-1$, hence $(\mathfrak{p}^k)^\perp \subseteq \mathfrak{p}^1 \cap \cdots \cap \mathfrak{p}^{k-1}$, which proves the result by Proposition 3.12. □

Definition 3.14. Parabolic subalgebras $\mathfrak{p}, \widehat{\mathfrak{p}}$ of a reductive Lie algebra $\mathfrak{g}$ are said to be *opposite* if (for some, hence any, admissible form) $\mathfrak{p} + \widehat{\mathfrak{p}}^\perp = \mathfrak{g}$ (*i.e.*, $\mathfrak{p}^\perp \cap \widehat{\mathfrak{p}} = 0$) and $\mathfrak{p}^\perp + \widehat{\mathfrak{p}} = \mathfrak{g}$ (*i.e.*, $\mathfrak{p} \cap \widehat{\mathfrak{p}}^\perp = 0$); in other words, $\mathfrak{g} = \mathfrak{p} \oplus \widehat{\mathfrak{p}}^\perp$ (*i.e.*, $\mathfrak{p} \cap \widehat{\mathfrak{p}}$ is a Levi subalgebra of $\widehat{\mathfrak{p}}$) which means equivalently that $\mathfrak{g} = \mathfrak{p}^\perp \oplus \widehat{\mathfrak{p}}$ (*i.e.*, $\mathfrak{p} \cap \widehat{\mathfrak{p}}$ is a Levi subalgebra of $\mathfrak{p}$).

Remarks 3.15. If $\mathfrak{p} \leqslant \mathfrak{g}$ is a parabolic subalgebra, then the set of parabolic subalgebras $\widehat{\mathfrak{p}}$ opposite to $\mathfrak{p}$ is an $\exp(\mathfrak{nil}(\mathfrak{p}))$-torsor (a *G-torsor* for a group G is a simply transitive G-set); Proposition 3.8 yields canonical isomorphisms between the following $\exp(\mathfrak{nil}(\mathfrak{p}))$-torsors:
- lifts of the grading element $\chi \in \mathfrak{p}_0$ to $\mathfrak{p} = \mathfrak{p}^{(0)}$;
- splittings $gr_{\mathfrak{p}}(\mathfrak{g}) \cong \mathfrak{g}$;
- parabolic subalgebras $\widehat{\mathfrak{p}}$ opposite to $\mathfrak{p}$ in $\mathfrak{g}$;
- Levi subalgebras of $\mathfrak{p}$.

An (algebraic) *Weyl structure* for $\mathfrak{p}$ is an element of this (*i.e.*, any of these) torsor(s). An opposite pair $\mathfrak{p}, \widehat{\mathfrak{p}} \leqslant \mathfrak{g}$ gives a vector space direct sum decomposition of $\mathfrak{g}$ into subalgebras $\mathfrak{nil}(\mathfrak{p}) \oplus \mathfrak{p} \cap \widehat{\mathfrak{p}} \oplus \mathfrak{nil}(\widehat{\mathfrak{p}})$, where $\mathfrak{p} \cap \widehat{\mathfrak{p}}$ is a Levi subalgebra of both $\mathfrak{p}$ and $\widehat{\mathfrak{p}}$, hence is orthogonal to both $\mathfrak{nil}(\mathfrak{p})$ and $\mathfrak{nil}(\widehat{\mathfrak{p}})$ with respect to any admissible form on $\mathfrak{g}$. Thus any admissible form on $\mathfrak{g}$ restricts to an admissible form on $\mathfrak{p} \cap \widehat{\mathfrak{p}}$ and a duality between $\mathfrak{nil}(\mathfrak{p})$ and $\mathfrak{nil}(\widehat{\mathfrak{p}})$.

Proposition 3.16. *Let* $\mathfrak{p}$ *and* $\mathfrak{q}$ *be parabolic in* $\mathfrak{g}$*. Then* $\mathfrak{r} := \mathfrak{p} \cap \mathfrak{q} + \mathfrak{nil}(\mathfrak{q})$ *is parabolic in* $\mathfrak{g}$ *with* $\mathfrak{nil}(\mathfrak{r}) = \mathfrak{nil}(\mathfrak{p}) \cap \mathfrak{q} + \mathfrak{nil}(\mathfrak{q})$.

Proof. It suffices to prove that $\mathfrak{r}/\,\mathfrak{nil}(\mathfrak{q})$ is parabolic in $\mathfrak{q}_0 := \mathfrak{q}/\,\mathfrak{nil}(\mathfrak{q})$. Introduce an admissible form on $\mathfrak{g}$ and the induced admissible form on $\mathfrak{q}_0$. First note that $(\mathfrak{p} \cap \mathfrak{q} + \mathfrak{q}^\perp)^\perp = (\mathfrak{p} \cap \mathfrak{q})^\perp \cap \mathfrak{q} = (\mathfrak{p}^\perp + \mathfrak{q}^\perp) \cap \mathfrak{q}$. Since $\mathfrak{q}^\perp \leqslant \mathfrak{q}$, this gives $\mathfrak{r}^\perp = \mathfrak{p}^\perp \cap \mathfrak{q} + \mathfrak{q}^\perp$. Then $(\mathfrak{r}/\,\mathfrak{nil}(\mathfrak{q}))^\perp = \mathfrak{r}^\perp/\,\mathfrak{nil}(\mathfrak{q})$ is contained in $\mathfrak{r}/\,\mathfrak{nil}(\mathfrak{q})$ and in $[\mathfrak{q}_0, \mathfrak{q}_0]$, since $\mathfrak{p}^\perp = [\mathfrak{p}, \mathfrak{p}^\perp] + \mathfrak{nil}(\mathfrak{g})$. Now $\mathfrak{r}^\perp/\,\mathfrak{nil}(\mathfrak{q})$ is nilpotent since $\mathfrak{p}^\perp = \mathfrak{nil}(\mathfrak{p})$ is nilpotent. □

Proposition 3.17. *Let* $\mathfrak{p}, \mathfrak{q}$ *be parabolic subalgebras of a reductive Lie algebra* $\mathfrak{g}$*. Then the following are canonically isomorphic* $\exp(\mathfrak{nil}(\mathfrak{p}) \cap \mathfrak{nil}(\mathfrak{q}))$*-torsors*:
- *lifts* $\xi_{\mathfrak{p}}$ *and* $\xi_{\mathfrak{q}}$ *of the grading elements of* $\mathfrak{p}$ *and* $\mathfrak{q}$ *with* $[\xi_{\mathfrak{p}}, \xi_{\mathfrak{q}}] = 0$ (*thus* $\xi_{\mathfrak{p}}, \xi_{\mathfrak{q}} \in \mathfrak{p} \cap \mathfrak{q}$);
- *Levi subalgebras of* $\mathfrak{p}$ *in* $\mathfrak{p} \cap \mathfrak{q}$;
- *Levi subalgebras of* $\mathfrak{q}$ *in* $\mathfrak{p} \cap \mathfrak{q}$.

Proof. Let $\mathfrak{r} = \mathfrak{p} \cap \mathfrak{q} + \mathfrak{nil}(\mathfrak{q})$. Then $\mathfrak{r}$ is a parabolic subalgebra of $\mathfrak{g}$ contained in $\mathfrak{q}$, so that $\mathfrak{r}/\,\mathfrak{nil}(\mathfrak{q})$ is a parabolic subalgebra of $\mathfrak{q}_0 := \mathfrak{q}/\,\mathfrak{nil}(\mathfrak{q})$. The grading element $\chi_{\mathfrak{q}}$ of $\mathfrak{q}$ lies in the centre of $\mathfrak{q}/\,\mathfrak{nil}(\mathfrak{q})$ and hence in $\mathfrak{r}/\,\mathfrak{nil}(\mathfrak{q})$. Consequently the lifts of $\chi_{\mathfrak{q}}$ belong to an affine subspace of $\mathfrak{r}$ modelled on $\mathfrak{nil}(\mathfrak{q})$, which therefore meets $\mathfrak{p} \cap \mathfrak{q}$ in an affine subspace modelled on $\mathfrak{p} \cap \mathfrak{nil}(\mathfrak{q})$. Mutatis mutandis, there is an affine subspace of lifts of the grading element of $\mathfrak{p}$ to $\mathfrak{p} \cap \mathfrak{q}$, modelled on $\mathfrak{nil}(\mathfrak{p}) \cap \mathfrak{nil}(\mathfrak{q})$.

Now if $\xi_{\mathfrak{p}}$ and $\xi_{\mathfrak{q}}$ are lifts of grading elements of $\mathfrak{p}$ and $\mathfrak{q}$ to $\mathfrak{p} \cap \mathfrak{q}$, then $[\xi_{\mathfrak{p}}, \xi_{\mathfrak{q}}]$ belongs to $\mathfrak{nil}(\mathfrak{p}) \cap \mathfrak{nil}(\mathfrak{q})$, on which $ad\,\xi_{\mathfrak{p}}$ and $ad\,\xi_{\mathfrak{q}}$ are invertible. Hence the equation $[\xi_{\mathfrak{p}}, \xi_{\mathfrak{q}}] = 0$ uniquely determines either lift to $\mathfrak{p} \cap \mathfrak{q}$ from the other, and the compatible lifts form an affine space modelled on $\mathfrak{nil}(\mathfrak{p}) \cap \mathfrak{nil}(\mathfrak{q})$, hence a torsor for $\exp(\mathfrak{nil}(\mathfrak{p}) \cap \mathfrak{nil}(\mathfrak{q}))$. □

3.4. Minimal Levi subalgebras and maximal anisotropic subalgebras. We refer to Levi subalgebras of parabolic subalgebras of $\mathfrak{g}$ as Levi subalgebras of $\mathfrak{g}$. In particular a *minimal Levi subalgebra* of $\mathfrak{g}$ is a Levi subalgebra of a minimal parabolic subalgebra $\mathfrak{b} \leqslant \mathfrak{g}$.

Corollary 3.18. *If* $\mathfrak{p}, \mathfrak{q} \leqslant \mathfrak{g}$ *are parabolic,* $\mathfrak{p} \cap \mathfrak{q}$ *contains a minimal Levi subalgebra of* $\mathfrak{g}$.

Indeed, $\mathfrak{p}$ contains a minimal parabolic subalgebras $\mathfrak{b}$ and by Proposition 3.17 there is a minimal Levi subalgebra in $\mathfrak{b} \cap \mathfrak{q} \leqslant \mathfrak{p} \cap \mathfrak{q}$.

Proposition 3.19. *If $\mathfrak{h}$ is a Levi subalgebra of $\mathfrak{g}$ then $\mathcal{N}(\mathfrak{h}) = \mathcal{N}(\mathfrak{g}) \cap \mathfrak{h}$, and if $\mathfrak{h}$ is a minimal Levi subalgebra, every $x \in \mathfrak{h}$ is semisimple in (the adjoint representation of) $\mathfrak{g}$.*

Proof. Any admissible form on $\mathfrak{g}$ induces an admissible form on $\mathfrak{h}$ (see Remarks 3.15). Now by Corollary 2.16, any $x \in \mathcal{N}(\mathfrak{g}) \cap \mathfrak{h}$ is also in $\mathfrak{c}_{\mathfrak{g}}(x)^{\perp} \cap \mathfrak{h} \subseteq \mathfrak{z}(\mathfrak{h})^{\perp} \cap \mathfrak{h} = [\mathfrak{h}, \mathfrak{h}]$. Since x is $ad_{\mathfrak{h}}$-nilpotent, it is in $\mathcal{N}(\mathfrak{h})$, and the other inclusion is automatic. If $\mathfrak{h}$ is minimal, $\mathcal{N}(\mathfrak{h}) = 0$ (else $\mathfrak{h}$ would contain a proper parabolic subalgebra). Now $\mathfrak{h} = \mathfrak{c}_{\mathfrak{g}}(\mathfrak{z}(\mathfrak{h}))$ and hence $ad_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{c}_{\mathfrak{der}(\mathfrak{g})}(\mathfrak{z}(\mathfrak{h}))$, which is closed under Jordan decomposition by Proposition A.2. ☐

Definition 3.20. A subalgebra $\mathfrak{k}$ of a reductive Lie algebra $\mathfrak{g}$ is called *anisotropic*, *toral* or *ad-semisimple* if every element of $\mathfrak{k}$ is semisimple in (the adjoint representation of) $\mathfrak{g}$.

Over an algebraically closed field, anisotropic subalgebras are abelian [17, 21].

Proposition 3.21. *Let $\mathfrak{k}$ be an anisotropic subalgebra of $\mathfrak{g}$ containing a lift $\xi_{\mathfrak{p}}$ of the grading element of a parabolic $\mathfrak{p}$. Then $\xi_{\mathfrak{p}} \in \mathfrak{z}(\mathfrak{k})$, i.e., $\mathfrak{k}$ is in the Levi subalgebra $\mathfrak{c}_{\mathfrak{g}}(\xi_{\mathfrak{p}})$.*

Proof. Since $ad(\xi_{\mathfrak{p}})$ is semisimple (with integer eigenvalues), the invariant subspace $\mathfrak{k} \leqslant \mathfrak{g}$ is a direct sum of eigenspaces for $ad(\xi_{\mathfrak{p}})|_{\mathfrak{k}}$. Now if $[\xi_{\mathfrak{p}}, x] = jx$ for some $x \in \mathfrak{k}$ and $j \in \mathbb{Z}$, then $ad(x)(\xi_{\mathfrak{p}}) = -jx$ and $ad(x)^2(\xi_{\mathfrak{p}}) = 0$. Now $ad(x)$ is semisimple, so $x = 0$ or $j = 0$. ☐

Corollary 3.22. *Minimal Levi subalgebras $\mathfrak{h} \leqslant \mathfrak{g}$ are maximal anisotropic subalgebras.*

The converse does not necessarily hold unless the underlying field is algebraically closed, in which case minimal Levi subalgebras are abelian and called *Cartan subalgebras.* A minimal parabolic subalgebra $\mathfrak{b}$ with abelian Levi factor is solvable (*i.e.*, $[\mathfrak{b}, \mathfrak{b}]$ is nilpotent) and called a *Borel subalgebra.*

Let $\mathfrak{k}$ be a minimal Levi subalgebra of $\mathfrak{g}$, also called an *anisotropic kernel.* Since $\mathfrak{z}(\mathfrak{k})$ is anisotropic and abelian, there is a splitting field for its action on $\mathfrak{g}$, *i.e.*, a field extension $\mathbb{F}^c$ such that the adjoint action of $\mathfrak{z}(\mathfrak{k})$ on $\mathfrak{g}^c = \mathfrak{g} \otimes_{\mathbb{F}} \mathbb{F}^c$ is simultaneously diagonalizable. Let $\mathfrak{a}$ be the subspace of $\mathfrak{z}(\mathfrak{k})$ whose elements have all eigenvalues in $\mathbb{F}$.

Definition 3.23. For $\alpha \in \mathfrak{a}^*$, let $\mathfrak{g}_{\alpha} = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{a}\}$. If $\alpha \neq 0$ and $\mathfrak{g}_{\alpha} \neq 0$, we say that $\alpha \in \mathfrak{a}^*$ is a (restricted) *root* of $\mathfrak{g}$, and call $\mathfrak{g}_{\alpha}$ the *root space* of α. The *root lattice* Λ_r of $(\mathfrak{g}, \mathfrak{k})$ is the (free) $\mathbb{Z}$-submodule of $\mathfrak{a}^*$ generated by the set Φ of roots. Elements of its dual $\Lambda_{cw} = \{\xi \in \mathfrak{a} \mid \alpha(\xi) \in \mathbb{Z} \text{ for all } \alpha \in \Phi\}$ are called *coweights.*

Since $\mathfrak{k}$ is the centralizer of $\mathfrak{z}(\mathfrak{k})$, $\mathfrak{g}$ has a root space decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_{\alpha},$$

with $\mathfrak{g}_{\alpha}^* \cong \mathfrak{g}_{-\alpha}$ (using any admissible form on $\mathfrak{g}$). The kernels of the roots have intersection $\mathfrak{z}(\mathfrak{g})$ in $\mathfrak{a} = \mathfrak{z}(\mathfrak{g}) \oplus (\mathfrak{a} \cap [\mathfrak{g}, \mathfrak{g}])$, hence span its annihilator $\mathfrak{z}(\mathfrak{g})^{\circ} \cong (\mathfrak{a} \cap [\mathfrak{g}, \mathfrak{g}])^*$. We shall also need the following basic fact concerning the existence of *coroots* $h_{\alpha} : \alpha \in \Phi$ in $\mathfrak{a}$.

Proposition 3.24. *For any $\alpha \in \Phi$, there is a unique $h_{\alpha} \in \mathfrak{a} \cap [\mathfrak{g}_{\alpha}, \mathfrak{g}_{-\alpha}]$ with $\alpha(h_{\alpha}) = 2$; furthermore, for any nonzero $x_{\alpha} \in \mathfrak{g}_{\alpha}$, there exists $y_{\alpha} \in \mathfrak{g}_{-\alpha}$ with $h_{\alpha} = [x_{\alpha}, y_{\alpha}]$.*

Proof (*see* [2, 17]). For any $x \in \mathfrak{g}_{\alpha}$ and $y \in \mathfrak{g}_{-\alpha}$, if $[x, y] \in \ker \alpha \subseteq \mathfrak{a}$ then $x, y, [x, y]$ span a subalgebra $\mathfrak{s} \leqslant \mathfrak{g}$ with $[x, y] \in [\mathfrak{s}, \mathfrak{s}] \cap \mathfrak{z}(\mathfrak{s}) \subseteq \mathcal{N}(\mathfrak{s})$, so $[x, y] \in [\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{k}$ is nilpotent and semisimple, hence zero. For $x = x_{\alpha} \neq 0$, $[x_{\alpha}, \mathfrak{g}_{-\alpha}] \cap \mathfrak{a} = \mathfrak{c}_{\mathfrak{g}}(x_{\alpha})^{\perp} \cap \mathfrak{a} = (\ker \alpha)^{\perp} \cap \mathfrak{a}$ so there is a unique $h_{\alpha} \in \mathfrak{a}$ with $\alpha(h_{\alpha}) = 2$ and $[x_{\alpha}, y_{\alpha}] = h_{\alpha}$ for some $y_{\alpha} \in \mathfrak{g}_{-\alpha}$. ☐

For any parabolic subalgebra $\mathfrak{p}$ containing $\mathfrak{k}$, the unique lift $\xi_{\mathfrak{p}}$ of its grading element to $\mathfrak{z}(\mathfrak{k})$ acts on $\mathfrak{g}$ with integer eigenvalues, so $\xi_{\mathfrak{p}} \in \Lambda_{cw}$. For any $\xi \in \Lambda_{cw}$ and $j \in \mathbb{Z}$, let $\Phi_{\xi}^{j} = \{\alpha \in \Phi \mid \alpha(\xi) = k\}$ and $\Phi_{\xi}^{\pm} = \{\alpha \in \Phi \mid \pm\alpha(\xi) \in \mathbb{Z}^+\}$. Then:

- $\Phi_{\xi}^{\pm}$ and Φ_{ξ}^{0} are (relatively) additively closed in Φ, with $\Phi = \Phi_{\xi}^{-} \sqcup \Phi_{\xi}^{0} \sqcup \Phi_{\xi}^{+}$;

- $\mathfrak{p}_\xi := \mathfrak{k} \oplus \bigoplus_{\alpha\in\Phi\setminus\Phi_\xi^+} \mathfrak{g}_\alpha$ is a parabolic subalgebra with nilpotent radical $\mathfrak{p}_\xi^\perp = \bigoplus_{\alpha\in\Phi_\xi^-} \mathfrak{g}_\alpha$;
- for any parabolic $\mathfrak{p} \supseteq \mathfrak{k}$, $\{\xi \in \Lambda_{cw} \mid \mathfrak{p}_\xi = \mathfrak{p}\}$ is additively closed and contains $\xi_\mathfrak{p}$.

Remark 3.25. It may be illuminating to compare the above theory with standard approaches to the theory of real reductive Lie algebras [16, 19, 25, 38], in which the main ingredient is a Cartan decomposition of $\mathfrak{g}$, *i.e.*, a symmetric decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ (into the $+1$ and -1 eigenspaces of an involution) such that $\mathfrak{h}$ is a maximal compact subalgebra. Then $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{m}$ and $\mathfrak{k} = (\mathfrak{k} \cap \mathfrak{h}) \oplus \mathfrak{a}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$. It follows that $\mathfrak{a}$ is the "split part" $\mathfrak{t} \cap \mathfrak{m}$ of a "maximally split" Cartan subalgebra $\mathfrak{t}$, where $\mathfrak{t} \cap \mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{k} \cap \mathfrak{h}$. The (restricted) roots in $\mathfrak{a}^*$ are restrictions of roots in $\mathfrak{t}^*$. We shall refer to $\mathfrak{a} \leqslant \mathfrak{k}$ in general as a *split Cartan subalgebra* of $\mathfrak{g}$.

Example 3A. Let V be as in Examples 1A–2A. The parabolic subalgebras of $\mathfrak{gl}(V)$ are the stabilizers of flags in the incidence system Γ^V of proper nontrivial subspaces $W \leqslant V$. Elements W_1, W_2 are incident if and only if their stabilizers (maximal proper parabolics) are costandard. A minimal parabolic subalgebra is the stabilizer of full flag, *i.e.*, a Borel subalgebra, while a minimal Levi subalgebra $\mathfrak{k}$ is the stabilizer of a direct sum decomposition of V into one dimensional subspaces, *i.e.*, a Cartan subalgebra of $\mathfrak{gl}(V)$. If $\mathbb{F} = \mathbb{R}$ and $V = \mathbb{R}^n$, then $\mathfrak{so}_n \oplus \mathbb{R}\, id$ is a maximal anisotropic subalgebra of $\mathfrak{gl}_n(\mathbb{R})$, but not a minimal Levi subalgebra.

Example 3B. Let U, Q_U be as in Examples 1B and 2B. The parabolic subalgebras of $\mathfrak{so}(U, Q_U)$ are stabilizers of flags in the incidence system Γ^{U,Q_U}, with incident flags corresponding to costandard parabolics. A minimal parabolic subalgebra is the stabilizer of a full flag $\pi_1 \leqslant \pi_2 \leqslant \cdots \leqslant \pi_n$ of isotropic subspaces, while a minimal Levi subalgebra $\mathfrak{k}$ is the stabilizer of an orthogonal decomposition $U = \big(\bigoplus \mathcal{R}_\mathfrak{k}\big) \oplus W_\mathfrak{k}$ where Q_U is positive definite on $W_\mathfrak{k}$, $\dim W_\mathfrak{k} = k$, and the n elements of $\mathcal{R}_\mathfrak{k}$ have signature $(1, 1)$. If $W_\mathfrak{k} \cong \mathbb{R}^k$ and $\bigoplus \mathcal{R}_\mathfrak{k} \cong \mathbb{R}^{n,n}$, then $\mathfrak{k} \cong \mathfrak{so}_k \oplus \mathfrak{a} \subseteq \mathfrak{so}_{n+k,n}$ where $\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{so}_{n,n}$. Thus $\mathfrak{so}_{n+k} \oplus \mathfrak{so}_n \subseteq \mathfrak{so}_{n+k,n}$ is maximal anisotropic, but not a minimal Levi subalgebra.

3.5. Minimal parabolic subalgebras and lowest weight representations. Let $\mathfrak{b}$ be a minimal parabolic subalgebra containing a minimal Levi subalgebra $\mathfrak{k}$. The corresponding Weyl structure $\xi_\mathfrak{b}$ induces a grading $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{b}_j$ such that $\mathfrak{b}$ is the nonpositive part, with $\mathfrak{b}_0 = \mathfrak{k}$ and $\mathfrak{b}_j = \bigoplus_{\alpha\in\Phi^j} \mathfrak{g}_\alpha$ for $j \neq 0$ (where $\Phi^j = \Phi^j_{\xi_\mathfrak{b}}$). Thus $\Phi^0 = \varnothing$ and Φ is the disjoint union of *positive roots* $\alpha \in \Phi^+$ and negative roots $\alpha \in \Phi^-$. For $\lambda, \mu \in \mathfrak{a}^*$, we write $\lambda \geqslant \mu$ ($\lambda > \mu$) if $\lambda - \mu$ is a (nonzero) sum of positive roots.

Definition 3.26. Let $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation and $\mathfrak{k}$ a minimal Levi subalgebra with split Cartan subalgebra $\mathfrak{a}$. For $\lambda \in \mathfrak{a}^*$, the λ-*weight space* is the simultaneous eigenspace $V_\lambda = \{v \in V \mid \forall\, h \in \mathfrak{a},\ \rho(h)v = \lambda(h)v\}$, which is a representation of $\mathfrak{k}$. If $V_\lambda \neq 0$, λ is called a *weight* of V. Let $\mathfrak{b}$ be a minimal parabolic containing $\mathfrak{k}$. A *lowest weight vector* with *lowest weight* λ is a vector $v \in V_\lambda$ with $\rho(\mathfrak{nil}(\mathfrak{b}), v) = 0$. If V is generated by a lowest weight vector, it is called a *lowest weight representation*.

Proposition 3.27. *For $\mathfrak{k} \subseteq \mathfrak{b} \subseteq \mathfrak{g}$ as above, let V be a lowest weight representation with lowest weight λ. Then V is a direct sum of weight spaces with weights $\mu \geqslant \lambda$, the lowest weight λ is unique, and $V_\lambda = \rho(\mathfrak{b}, v) = \rho(\mathfrak{k}, v)$ for any lowest weight vector v.*

Proof. Let v be a lowest weight vector and let W be the span of elements of the form $\rho(y_1)\cdots\rho(y_k)v$ with each y_j either in $\mathfrak{k}$ or $\mathfrak{g}_\alpha$ with $\alpha \in \Phi^+$. A standard inductive argument using $\rho(x)\rho(y) = \rho([x, y]) + \rho(y)\rho(x)$ shows (with $x \in \mathfrak{b}_{-1}$) that W is $\rho(\mathfrak{g})$-invariant, hence $W = V$, and (with $x \in \mathfrak{a}$) that W is a sum of weight spaces V_μ with $\mu \geqslant \lambda$, where equality only holds if all y_j belong to $\mathfrak{k}$. □

Theorem 3.28. *Let $\mathfrak{b}$ be a minimal parabolic subalgebra of $\mathfrak{g}$ containing a minimal Levi subalgebra $\mathfrak{k}$, so that $\Phi = \Phi^+ \sqcup \Phi^- = \bigsqcup_{j\in\mathbb{Z}} \Phi^j$ and $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{b}_j$.*

(1) *Any parabolic* $\mathfrak{q}$ *containing* $\mathfrak{b}$ *is the stabilizer of a* 1*-dimensional lowest weight space* $L_{\mathfrak{q}} = V_{\mathfrak{q}}$ *(for* $\mathfrak{b}$*) in a lowest weight representation* $V_{\mathfrak{q}}$ *of* $\mathfrak{g}$*, and* $\lambda(h_\alpha) = 0$ *if* $\alpha(\xi_{\mathfrak{q}}) = 0$*.*
(2) *For any* $\alpha \in \Phi^1$*,* $\mathfrak{g}_\alpha$ *is irreducible for* $\mathfrak{k}$*, and there is a maximal proper parabolic subalgebra* $\mathfrak{q}^\alpha$ *such that for all* $\beta \in \Phi^1$*,* $\beta(\xi^\alpha) = \delta_{\alpha\beta}$*, where* $\xi^\alpha = \xi_{\mathfrak{q}_\alpha}$*. If* $L_{\mathfrak{q}^\alpha} \subseteq V_{\mathfrak{q}^\alpha}$ *as in (1), then the lowest weight is a negative multiple of* $\lambda^\alpha \in \mathfrak{a}^*$ *with* $\lambda^\alpha(h_\beta) = \delta_{\alpha\beta}$*.*
(3) Φ^1 *is a basis for* Λ_r*, and parabolic subalgebras containing* $\mathfrak{b}$ *are in bijection with subsets* J *of* Φ^1*, where the parabolic* $\mathfrak{q}_J \leqslant \mathfrak{g}$ *corresponding to* J *is* $\bigcap_{\alpha\in J} \mathfrak{q}^\alpha$ *and* $\xi_{\mathfrak{q}_J} = \sum_{\alpha\in J} \xi^\alpha$*.*

Proof (*see* [2, 17, 29]). (1) Let $L_{\mathfrak{q}} := \wedge^d \mathfrak{nil}(\mathfrak{q}) \leqslant \wedge^d \mathfrak{g}$ with $d = \dim \mathfrak{nil}(\mathfrak{q})$ so $\dim L_{\mathfrak{q}} = 1$, and let $V_{\mathfrak{q}} \subseteq \wedge^d \mathfrak{g}$ be the $\mathfrak{g}$-submodule generated by $L_{\mathfrak{q}}$. Since $L_{\mathfrak{q}} \subseteq V_{\mathfrak{q}}$ has stabilizer $\mathfrak{q}$, $\mathfrak{nil}(\mathfrak{b}) \leqslant [\mathfrak{q},\mathfrak{q}]$ acts trivially and $L_{\mathfrak{q}}$ is a lowest weight space for $\mathfrak{b}$. If $\alpha(\xi_{\mathfrak{q}}) = 0$ then $\mathfrak{g}_{\pm\alpha} \subseteq \mathfrak{q}$ and hence $h_\alpha \in [\mathfrak{q},\mathfrak{q}]$.

(2)–(3) Since $\widehat{\mathfrak{b}} = \sum_{j\in\mathbb{N}} \mathfrak{b}_j$ is a parabolic (opposite to $\mathfrak{b}$), $\mathfrak{b}_1$ generates $\mathfrak{b}_+ = \mathfrak{nil}(\widehat{\mathfrak{b}}) = \sum_{j\in\mathbb{Z}^+} \mathfrak{b}_j = \sum_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$—hence Φ^1 spans Λ_r and for any proper (parabolic) subalgebra $\mathfrak{q} \leqslant \mathfrak{g}$ containing $\mathfrak{b}$, $\mathfrak{q} \cap \mathfrak{b}_1$ is a proper $\mathfrak{k}$-invariant subspace of $\mathfrak{b}_1$.

If $\mathfrak{s}$ is a maximal proper $\mathfrak{k}$-invariant subspace of $\mathfrak{b}_1$, and $\mathfrak{n}$ is the (nilpotent) Lie subalgebra of $\mathfrak{b}_+$ generated by $\mathfrak{s}$, then an inductive argument shows that $[\mathfrak{b},\mathfrak{n}] \subseteq \mathfrak{b}\oplus\mathfrak{n}$, so that $\mathfrak{q} := \mathfrak{b}\oplus\mathfrak{n}$ is parabolic in $\mathfrak{g}$ with $\mathfrak{q}\cap\mathfrak{b}_1 = \mathfrak{s}$. The positive eigenspaces of $\xi_{\mathfrak{q}}$ meet $\mathfrak{b}_1$ in an irreducible complement to $\mathfrak{s}$, which must be a root space $\mathfrak{g}_\alpha$ with $\alpha \in \Phi^1$. Hence all such $\mathfrak{s}$ have the form $\mathfrak{s} = \bigoplus\{\mathfrak{g}_\beta \mid \beta \in \Phi^1, \beta \neq \alpha\}$ and the corresponding $\mathfrak{q} = \mathfrak{q}^\alpha$ has $\beta(\xi_{\mathfrak{q}}) = \delta_{\alpha\beta}$ for all $\beta \in \Phi^1$. The rest of the theorem follows straightforwardly. □

Thus Φ^1 is a *basis of simple roots*, *i.e.*, a $\mathbb{Z}$-basis for Λ_r with respect to which any $\alpha \in \Phi \subseteq \Lambda_r$ either has all coefficients nonnegative, or all coefficients nonpositive. The corresponding *fundamental coweights* $\xi^\alpha : \alpha \in \Phi^1$ form a basis for $\mathfrak{a}\cap[\mathfrak{g},\mathfrak{g}]$ so that $\mathrm{rank}\,\Lambda_r = \dim(\mathfrak{a}\cap[\mathfrak{g},\mathfrak{g}])$. The weights $\lambda^\alpha \in (\mathfrak{a}\cap[\mathfrak{g},\mathfrak{g}])^*$ with $\lambda^\alpha(h_\beta) = \delta_{\alpha\beta}$ are also uniquely determined, and called the *fundamental weights*.

Conversely, any basis of simple roots $\Psi \subseteq \Phi$ determines a unique element ξ of Λ_{cw} with $\alpha(\xi) = 1$ for all $\alpha \in \Psi$, and then $\mathfrak{p}_\xi$ is a minimal parabolic subalgebra of $\mathfrak{g}$ with $\Psi = \Phi^1$.

4. Global theory and parabolic buildings

We now have sufficient information to adopt a more global perspective on parabolic subalgebras of a reductive Lie algebra $\mathfrak{g}$. Let G be a connected algebraic group with Lie algebra $\mathfrak{g}$ (over a field $\mathbb{F}$ of characteristic zero), so that the centre $Z(G)$ has Lie algebra $\mathfrak{z}(\mathfrak{g})$, and $G/Z(G) \cong G^{ss} \leqslant \mathrm{Aut}(\mathfrak{g})$ is the identity component of the automorphism group, called the *adjoint group* of $\mathfrak{g}$; G^{ss} is generated by $\exp(\mathrm{ad}\,x)$ for $x \in \mathcal{N}(\mathfrak{g})$ (these generate a connected normal subgroup of $\mathrm{Aut}(\mathfrak{g})$ whose Lie algebra meets every simple component of $\mathfrak{g}$ nontrivially). For a subspace $\mathfrak{s} \subseteq \mathfrak{g}$ we let $N_G(\mathfrak{s}) \leqslant G$ be the stabilizer of $\mathfrak{s}$ and $C_G(\mathfrak{s})$ the kernel of the action of $N_G(\mathfrak{s})$ on $\mathfrak{s}$. These subgroups have Lie algebras $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{s})$ and $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{s})$ respectively.

4.1. Homogeneity and generalized flag varieties. Via its adjoint group, G acts on the set $\mathscr{P}^{\mathfrak{g}}$ of parabolic subalgebras $\mathfrak{q}$ of $\mathfrak{g}$, and its orbits are called *generalized flag varieties.* Since any such $\mathfrak{q}$ is self-normalizing ($\mathfrak{n}_{\mathfrak{g}}(\mathfrak{q}) = \mathfrak{q}$), its stabilizer $N_G(\mathfrak{q})$ has Lie algebra $\mathfrak{q}$.

Proposition 4.1. *Any generalized flag variety embeds into the projective space of a lowest weight representation of* $\mathfrak{g}$*.*

Proof. Use Theorem 3.28 (1): for any $g \in G$, the infinitesimal stabilizer of $g\cdot L_{\mathfrak{q}} = L_{g\cdot\mathfrak{q}}$ is $g\cdot\mathfrak{q}$ and hence the adjoint orbit of $\mathfrak{q}$ is isomorphic to the G-orbit of $L_{\mathfrak{q}} \in P(V_{\mathfrak{q}})$. □

Proposition 4.2. *For* $\alpha \in \Phi$*, let* $h_\alpha = [x_\alpha, y_\alpha] \in \mathfrak{a}$ *with* $\alpha(h_\alpha) = 2$*,* $x_\alpha \in \mathfrak{g}_\alpha$ *and* $y_\alpha \in \mathfrak{g}_{-\alpha}$*.*
(1) *The automorphism* $g = \exp(\mathrm{ad}(x_\alpha))\exp(\mathrm{ad}(-y_\alpha))\exp(\mathrm{ad}(x_\alpha))$ *of* $\mathfrak{g}$ *preserves* $\mathfrak{a}\cap[\mathfrak{g},\mathfrak{g}]$*, sending* h *to* $g\cdot h = h - \alpha(h)h_\alpha$*;*
(2) *For all* $\alpha,\beta \in \Phi$*,* $\beta(h_\alpha) \in \mathbb{Z}$ *and* $\sigma_\alpha(\beta) := \beta - \beta(h_\alpha)\alpha \in \Phi$*.*

Finally, if $\alpha_1, \dots \alpha_k$ is a basis of simple roots, and Φ_+ the corresponding set of positive roots, then for any $i \in \{1, \dots k\}$, σ_{α_i} permutes $\Phi_+ \setminus span\{\alpha_i\}$.

Proof (*see* [2, 17]). (1) The automorphism $g = \exp(ad(x_\alpha))\exp(ad(-y_\alpha))\exp(ad(x_\alpha))$ restricts to the identity on $ker\,\alpha$ and sends h_α to $-h_\alpha$. It thus sends any $h \in \mathfrak{a}$ to $h - \alpha(h)h_\alpha \in \mathfrak{a}$ as required.

(2) For any $\alpha \in \Phi$, such a triple $x_\alpha, y_\alpha, h_\alpha$ exists and spans an $\mathfrak{sl}_2$ subalgebra of $\mathfrak{g}$. Since $\beta(g \cdot h) = \beta(h - \alpha(h)h_\alpha) = (\beta - \beta(h_\alpha)\alpha)(h) = \sigma_\alpha(\beta)(h)$, and $g \cdot [h, z] = [g \cdot h, g \cdot z]$, g restricts to an isomorphism $\mathfrak{g}_\beta \cong \mathfrak{g}_{\sigma_\alpha^{-1}(\beta)} = \mathfrak{g}_{\sigma_\alpha(\beta)}$. Hence if $\beta \in \Phi$, $\sigma_\alpha(\beta) \in \Phi$. The $\mathfrak{sl}_2$ relations $ad\, h_\alpha \circ ad\, x_\alpha = ad\, x_\alpha \circ (ad\, h_\alpha + 2)$ and $ad\, h_\alpha \circ ad\, y_\alpha = ad\, y_\alpha \circ (ad\, h_\alpha - 2)$ then show that the eigenvalue $\sigma_\alpha(\beta)(h_\alpha) = -\beta(h_\alpha)$ differs from $\beta(h_\alpha)$ by $2k \in 2\mathbb{Z}$, *i.e.*, $\beta(h_\alpha) \in \mathbb{Z}$.

The last part is standard: any $\alpha = \sum_{j=1}^k n_j\alpha_j \in \Phi_+ \setminus span\{\alpha_i\}$ has $n_j > 0$ for some $j \neq i$, as does $\sigma_{\alpha_i}(\alpha) = \alpha - \alpha(h_i)\alpha_i$; hence $\sigma_{\alpha_i}(\alpha)$ is positive. □

This result shows that Φ is a *root system* in Λ_r, and we refer to σ_α, for $\alpha \in \Phi$, as a *root reflection*. The system need not be "reduced": if α is a root, then there may be integer multiples of α other than $\pm\alpha$ which are roots; however, $\sigma_{m\alpha} = \sigma_\alpha$ for any $m \in \mathbb{Z} \setminus \{0\}$. There are irreducible nonreduced systems denoted BC_n in addition to the Dynkin classification [17].

Proposition 4.3. *$N_G(\mathfrak{k})$ acts transitively on minimal parabolic subalgebras containing $\mathfrak{k}$.*

Proof. To show any two minimal parabolic subalgebras $\mathfrak{b}, \mathfrak{c}$ containing $\mathfrak{k}$ are conjugate by an element of $N_G(\mathfrak{k})$, let us say a root α is "shared" if the multiples α which are positive for $\mathfrak{b}$ are also positive for $\mathfrak{c}$; otherwise, they are negative for $\mathfrak{c}$ and we say α is "unshared". We now use complete induction on the number of unshared roots.

If there are none then $\mathfrak{b} = \mathfrak{c}$, otherwise there is a root space $\mathfrak{g}_{\alpha_i}$ in $\mathfrak{b}_1 \cap \mathfrak{c}$, so α_i (and its multiples) is unshared. By Proposition 4.2, there exists $g \in N_G(\mathfrak{a})$ such that $g \cdot \mathfrak{b}$ has positive roots $\sigma_{\alpha_i}(\Phi_+)$, where Φ_+ is the set of positive roots for $\mathfrak{b}$; now $g \cdot \mathfrak{b}$ and $\mathfrak{c}$ have fewer unshared roots, hence are conjugate under $N_G(\mathfrak{k})$. □

Note that the proof shows more: if we consider $\mathfrak{c}$ to be fixed, then the element of $N_G(\mathfrak{a})$ needed to conjugate $\mathfrak{b}$ to $\mathfrak{c}$ is obtained by an iterative application of simple root reflections.

4.2. The global Weyl group and parabolic incidence system. Let $\mathscr{B}^{\mathfrak{g}}$ be the set of all minimal parabolic subalgebras of $\mathfrak{g}$, let $\mathscr{W}^{\mathfrak{g}}$ be the set of all "BK-pairs" $(\mathfrak{b}, \mathfrak{k})$ where $\mathfrak{b}$ is a minimal parabolic subalgebra of $\mathfrak{g}$ and $\mathfrak{k}$ is a Levi subalgebra of $\mathfrak{b}$, and let $\mathscr{A}^{\mathfrak{g}}$ be the set of minimal Levi subalgebras in $\mathfrak{g}$. Given $\mathfrak{k} \in \mathscr{A}^{\mathfrak{g}}$, let $\mathfrak{a} \leqslant \mathfrak{z}(\mathfrak{k})$ be the corresponding split Cartan subalgebra, so that $\mathfrak{k} = \mathfrak{c}_{\mathfrak{g}}(\mathfrak{a}) = \mathfrak{n}_{\mathfrak{g}}(\mathfrak{a})$ is the Lie algebra of $K := C_G(\mathfrak{a})$.

Theorem 4.4. *G acts transitively on $\mathscr{W}^{\mathfrak{g}}$, hence also on $\mathscr{B}^{\mathfrak{g}}$ and $\mathscr{A}^{\mathfrak{g}}$. The stabilizer of $(\mathfrak{b}, \mathfrak{k}) \in \mathscr{W}^{\mathfrak{g}}$ is $K = C_G(\mathfrak{a})$.*

Proof. Any two minimal parabolic subalgebras $\mathfrak{b}, \mathfrak{c} \leqslant \mathfrak{g}$ share a (minimal) Levi subalgebra $\mathfrak{k}$, hence are conjugate by an element of $N_G(\mathfrak{k}) \leqslant G$. On the other hand, the set of such $\mathfrak{k}$ contained in a given $\mathfrak{b}$ is a torsor for $\exp(\mathfrak{nil}(\mathfrak{b})) \leqslant G$. Now suppose that $g \in G$ fixes $(\mathfrak{b}, \mathfrak{k})$ and let $\lambda_1, \dots \lambda_k \in (\mathfrak{a} \cap [\mathfrak{g}, \mathfrak{g}])^*$ be the fundamental weights corresponding to $\mathfrak{b}$. For each j there is an irreducible lowest weight representation V_j whose lowest weight is a positive multiple of λ_j. Since the action of g on V_j sends weight spaces to weight spaces and lowest weight vectors to lowest weight vectors, the induced action on $\mathfrak{a} \cap [\mathfrak{g}, \mathfrak{g}]$ fixes λ_j. Since $\mathfrak{a} = \mathfrak{z}(\mathfrak{g}) \oplus (\mathfrak{a} \cap [\mathfrak{g}, \mathfrak{g}])$ and the fundamental weights span $(\mathfrak{a} \cap [\mathfrak{g}, \mathfrak{g}])^*$, $g \in C_G(\mathfrak{a})$. □

Proposition 4.5. *Let $\mathcal{Q}$ be an adjoint orbit of maximal* (*proper*) *parabolic subalgebras of $\mathfrak{g}$. Then for any minimal parabolic subalgebra $\mathfrak{b}$, there is a unique $\mathfrak{q} \in \mathcal{Q}$ with $\mathfrak{b} \leqslant \mathfrak{q}$.*

Proof. Fix a Levi subalgebra $\mathfrak{k} \leqslant \mathfrak{b}$. Since G acts transitively on minimal parabolic subalgebras, for any $\mathfrak{q}' \in \mathcal{Q}$, there exists $g \in G$ with $g \cdot \mathfrak{b} \subseteq \mathfrak{q}'$, so $\mathfrak{b} \subseteq \mathfrak{q} := g^{-1} \cdot \mathfrak{q}' \in \mathcal{Q}$. Suppose now that $\mathfrak{q}, \mathfrak{q}' = g \cdot \mathfrak{q} \in \mathcal{Q}$ both contain $\mathfrak{b}$; then $\mathfrak{q}$ contains both $\mathfrak{b}$ and $g^{-1} \cdot \mathfrak{b}$, which are therefore conjugate by an element of $N_G(\mathfrak{q})$ (since $N_G(\mathfrak{q})$ acts transitively on minimal parabolic subalgebras of $\mathfrak{q}/\mathfrak{q}^\perp$). Thus we may assume $\mathfrak{q}' = g\cdot\mathfrak{q}$ with $g \in N_G(\mathfrak{b})$, and fix a (minimal) Levi subalgebra $\mathfrak{k}$ of $\mathfrak{b}$. Now since $\mathfrak{k}$ and $g^{-1}\cdot\mathfrak{k}$ are both Levi subalgebras of $\mathfrak{b}$, they are related by an element of $\exp(\mathfrak{b}^\perp) \subseteq N_G(\mathfrak{b}) \subseteq N_G(\mathfrak{q})$ so we may assume $g \in N_G(\mathfrak{k}) \cap N_G(\mathfrak{b}) = C_G(\mathfrak{a})$ by Theorem 4.4. Hence $\mathfrak{q}' = \mathfrak{q}$. □

Thus costandard maximal parabolic subalgebras in the same adjoint orbit are equal.

We summarize the development so far with a double fibration of G-homogeneous spaces

$$\begin{tikzcd} & \mathscr{W}^{\mathfrak{g}} \arrow[dl] \arrow[dr] & \\ \mathscr{A}^{\mathfrak{g}} & & \mathscr{B}^{\mathfrak{g}} \end{tikzcd}$$

in which the fibre over $\mathfrak{b} \in \mathscr{B}^{\mathfrak{g}}$ is an $\exp(\mathfrak{nil}(\mathfrak{b}))$-torsor (the Weyl structures for $\mathscr{B}^{\mathfrak{g}}$). If we choose a basepoint $(\mathfrak{b},\mathfrak{k}) \in \mathscr{W}^{\mathfrak{g}}$, then the orbit-stabilizer theorem provides isomorphisms $\mathscr{W}^{\mathfrak{g}} \cong G/K$, $\mathscr{B}^{\mathfrak{g}} \cong G/N_G(\mathfrak{b})$ and $\mathscr{A}^{\mathfrak{g}} \cong G/N_G(\mathfrak{k})$. The fibre of $\mathscr{W}^{\mathfrak{g}}$ over $\mathfrak{k} \in \mathscr{A}^{\mathfrak{g}}$ is a torsor for the *local Weyl group* $W_{\mathfrak{k}}(\mathfrak{g}) := N_G(\mathfrak{a})/C_G(\mathfrak{a}) = N_G(\mathfrak{k})/K$.

Definition 4.6. The *global Weyl group* $W(\mathfrak{g})$ of $\mathfrak{g}$ is the automorphism group $Aut_G(\mathscr{W}^{\mathfrak{g}})$ of the *Weyl space* $\mathscr{W}^{\mathfrak{g}}$ of BK-pairs $(\mathfrak{b},\mathfrak{k})$, *i.e.*, $W(\mathfrak{g})$ is the set of bijections $\mathscr{W}^{\mathfrak{g}} \to \mathscr{W}^{\mathfrak{g}}$ commuting with the G-action.

We shall write the G-action on $\mathscr{W}^{\mathfrak{g}}$ on the left, and the $W(\mathfrak{g})$ action on the right.

Proposition 4.7. *The Weyl space $\mathscr{W}^{\mathfrak{g}}$ is a principal $W(\mathfrak{g})$-bundle over $\mathscr{A}^{\mathfrak{g}}$, i.e., the action of $W(\mathfrak{g})$ on $\mathscr{W}^{\mathfrak{g}}$ is fibre-preserving, and each fibre is a $(W_{\mathfrak{k}}(\mathfrak{g}), W(\mathfrak{g}))$-bitorsor. In particular, any basepoint $(\mathfrak{b},\mathfrak{k}) \in \mathscr{W}^{\mathfrak{g}}$ yields an isomorphism between $W_{\mathfrak{k}}(\mathfrak{g})$ and $W(\mathfrak{g})$.*

Proof. Note that $W(\mathfrak{g})$ preserves the fibration of $\mathscr{W}^{\mathfrak{g}}$ over $\mathscr{A}^{\mathfrak{g}}$ and thus induces a right action of $W(\mathfrak{g})$ on $\mathscr{A}^{\mathfrak{g}}$ commuting with G. However, any $\mathfrak{k} \in \mathscr{A}^{\mathfrak{g}}$ is the Lie algebra of its stabilizer $N_G(\mathfrak{k})$ in G, and for any $w \in W(\mathfrak{g})$, $N_G(\mathfrak{k}w) = N_G(\mathfrak{k})$, so $\mathfrak{k}w = \mathfrak{k}$. Thus the induced action of $W(\mathfrak{g})$ on $\mathscr{A}^{\mathfrak{g}}$ is trivial, *i.e.*, the action of $W(\mathfrak{g})$ on $\mathscr{W}^{\mathfrak{g}}$ is fibre-preserving. Since the G-action is transitive, the $W(\mathfrak{g})$ action is free, and so any $w \in W$ is uniquely determined by what it does to a base-point $(\mathfrak{b},\mathfrak{k}) \in \mathscr{W}^{\mathfrak{g}}{}_{\mathfrak{k}}$. If also $(\mathfrak{c},\mathfrak{k}) \in \mathscr{W}^{\mathfrak{g}}{}_{\mathfrak{k}}$, there is an automorphism sending $g\cdot(\mathfrak{b},\mathfrak{k})$ to $g\cdot(\mathfrak{c},\mathfrak{k})$ for all $g \in G$, because the effective quotient $W_{\mathfrak{k}}(\mathfrak{g}) = N_G(\mathfrak{k})/K$ acts freely on $\mathscr{W}^{\mathfrak{g}}{}_{\mathfrak{k}}$. Hence $W(\mathfrak{g})$ also acts transitively on fibres. □

Remark 4.8. Another description of $W(\mathfrak{g})$ uses a natural groupoid structure on $\mathscr{A}^{\mathfrak{g}}$: the set of morphisms from $\mathfrak{k}$ to $\mathfrak{k}'$ in $\mathscr{A}^{\mathfrak{g}}$ is $N_G(\mathfrak{k},\mathfrak{k}')/K$, *i.e.*, the set of elements of G which conjugate $\mathfrak{k}$ to $\mathfrak{k}'$, modulo the right action of K (or equivalently, the left action of K'). Since $Hom_{\mathscr{A}}(\mathfrak{k},\mathfrak{k}) = W_{\mathfrak{g}}(\mathfrak{k})$, we refer to $\mathscr{A}^{\mathfrak{g}}$, with this structure, as the *Weyl groupoid of* $\mathfrak{g}$.

Now $\mathscr{W}^{\mathfrak{g}}$ determines a representation (or action) of the Weyl groupoid $\mathscr{A}^{\mathfrak{g}}$: any $[g] \in Hom_{\mathscr{A}}(\mathfrak{k},\mathfrak{k}') = N_G(\mathfrak{k},\mathfrak{k}')/K$ induces a function $\mathscr{W}^{\mathfrak{g}}{}_{\mathfrak{k}} \to \mathscr{W}^{\mathfrak{g}}{}_{\mathfrak{k}'}$ sending $(\mathfrak{b},\mathfrak{k})$ to $(g\cdot\mathfrak{b}, g\cdot\mathfrak{k}) = (g\cdot\mathfrak{b},\mathfrak{k}')$; these determine a functor $\mathscr{A}^{\mathfrak{g}} \to \mathbf{Set}$. Furthermore, on each fibre $\mathscr{W}^{\mathfrak{g}}{}_{\mathfrak{k}}$, the action of $W_{\mathfrak{g}}(\mathfrak{k})$ is free and transitive. Thus we may equip $\mathscr{W}^{\mathfrak{g}}$ with a trivial (2-connected) groupoid structure over $\mathscr{A}^{\mathfrak{g}}$, in which there is a unique morphism between any two points $(\mathfrak{b},\mathfrak{k})$ and $(\mathfrak{c},\mathfrak{k}')$, labelled by the unique element of $Hom_{\mathscr{A}}(\mathfrak{k},\mathfrak{k}') = N_G(\mathfrak{k},\mathfrak{k}')/K$ whose representatives send $\mathfrak{b}$ to $\mathfrak{c}$. In other words, $\mathscr{W}^{\mathfrak{g}}$ is a *universal groupoid cover* of $\mathscr{A}^{\mathfrak{g}}$, and $W(\mathfrak{g})$ is its group of "deck transformations".

In addition to its G-space structure, $\mathscr{P}^{\mathfrak{g}}$ has an incidence relation: $\mathfrak{p},\mathfrak{p}' \in \mathscr{P}^{\mathfrak{g}}$ are incident if they are costandard. This incidence structure of $\mathscr{P}^{\mathfrak{g}}$ is determined by the maximal proper parabolic subalgebras $\mathfrak{p}$. Let $\mathcal{I}_{\mathfrak{g}}$ be the set of adjoint orbits of such $\mathfrak{p}$.

Definition 4.9. The *parabolic incidence system* $\mathit{\Gamma}^{\mathfrak{g}}$ of $\mathfrak{g}$ is the set of all maximal proper parabolic subalgebras $\mathfrak{p} \leqslant \mathfrak{g}$, equipped with the incidence relation

$$\mathfrak{p}\text{—}\mathfrak{p}' \quad \text{iff} \quad \mathfrak{p} \text{ and } \mathfrak{p}' \text{ are costandard, } \textit{i.e.}, \ \mathfrak{p}\cap\mathfrak{p}' \text{ is parabolic in } \mathfrak{g}$$

and the type function $t_{\Gamma^{\mathfrak{g}}} : |\Gamma^{\mathfrak{g}}| \to \mathcal{I}_{\mathfrak{g}}$ sending $\mathfrak{p}$ to its adjoint orbit $G \cdot \mathfrak{p}$.

Theorem 4.10. *Let $\mathfrak{g}$ be a reductive Lie algebra. Then the map*

$$\Lambda \colon \bigsqcup_{J \subseteq \mathcal{I}_{\mathfrak{g}}} \mathcal{F}\Gamma^{\mathfrak{g}}(J) \to \mathscr{P}^{\mathfrak{g}};\ \sigma \mapsto \bigcap_{\mathfrak{p} \in \sigma} \mathfrak{p}$$

induces an incidence isomorphism over $P(\mathcal{I}_{\mathfrak{g}})$, where parabolics $\mathfrak{p}, \mathfrak{q} \in \mathscr{P}^{\mathfrak{g}}$ are incident if they are costandard, and the fibres of the type function $\mathscr{P}^{\mathfrak{g}} \to P(\mathcal{I}_{\mathfrak{g}})$ are adjoint orbits.

Proof. By Proposition 3.13, Λ is well-defined. For any $\mathfrak{q} \in \mathscr{P}^{\mathfrak{g}}$, after choosing a minimal parabolic subalgebra $\mathfrak{b}$ contained on $\mathfrak{q}$, Theorem 3.28 and Proposition 4.5 show that the maximal parabolic subalgebras containing $\mathfrak{q}$ have intersection $\mathfrak{q}$ and form a J-flag in $\mathcal{F}\Gamma^{\mathfrak{g}}(J)$ for some $J \in P(\mathcal{I}_{\mathfrak{g}})$. Now if $\mathfrak{q} = \mathfrak{p}^1 \cap \cdots \cap \mathfrak{p}^k$ and $g \in G$ then $g \cdot \mathfrak{q} = g \cdot \mathfrak{p}^1 \cap \cdots g \cdot \mathfrak{p}^k$, so the adjoint orbit of $\mathfrak{q}$ is in the image of $\mathcal{F}\Gamma^{\mathfrak{g}}(J)$ and we may label it by J. Distinct adjoint orbits have distinct labels by Proposition 4.5 and Λ intertwines incidence of flags with incidence of parabolic subalgebras. Hence it is an incidence isomorphism. □

Henceforth we label parabolic adjoint orbits (*i.e.*, generalized flag varieties) $\mathscr{P}^{\mathfrak{g}}(J)$ in $\mathfrak{g}$ by subsets J of $\mathcal{I}_{\mathfrak{g}}$ using this isomorphism.

Example 4A. Continuing Examples 1A–3A, we take $G = GL(V)$, and the adjoint group is $PGL(V)$. There is a bijection between minimal Levi subalgebras $\mathfrak{k} \leqslant \mathfrak{gl}(V)$ and $n+1$ element subsets $\mathcal{S}_{\mathfrak{k}} \subseteq P(V) = Gr_1(V)$ with $V = \bigoplus \mathcal{S}_{\mathfrak{k}}$: $\mathcal{S}_{\mathfrak{k}} = \{L \in P(V) \mid \mathfrak{k} \cdot L \subseteq L\}$ and $\mathfrak{k} = \bigcap_{L \in \mathcal{S}_{\mathfrak{k}}} \mathfrak{stab}_{\mathfrak{gl}(V)}(L)$, which is the Cartan subalgebra of diagonal matrices with respect to any basis representing the elements of $\mathcal{S}_{\mathfrak{k}}$, hence $\mathfrak{a} = \mathfrak{k}$. Thus $N_G(\mathfrak{k})$ is the subgroup of $GL(V)$ preserving the decomposition $V = \bigoplus \mathcal{S}_{\mathfrak{k}}$ and the local Weyl group $W_{\mathfrak{k}}(\mathfrak{gl}(V)) = N_G(\mathfrak{k})/C_G(\mathfrak{k})$ is canonically isomorphic to $Sym(\mathcal{S}_{\mathfrak{k}})$.

The minimal parabolic subalgebra $\mathfrak{b}$ stabilizing the full flag $0 = W_0 \leqslant W_1 \leqslant \cdots \leqslant W_n \leqslant W_{n+1} = V$ contains $\mathfrak{k}$ if and only if there is a bijection $j \mapsto L_j \colon \{1, \dots n+1\} \to \mathcal{S}_{\mathfrak{k}}$ such that $W_j = W_{j-1} \oplus L_j$, and hence $W_j = L_1 \oplus \cdots \oplus L_j$. Hence the fibre of $\mathscr{W}^{\mathfrak{g}}$ over $\mathfrak{k} \in \mathscr{A}^{\mathfrak{g}}$ may be identified canonically with the set of bijections $\{1, \dots n+1\} \to \mathcal{S}_{\mathfrak{k}}$, and the global Weyl group $W(\mathfrak{g})$ with $Sym_{n+1} = Sym(\{1, \dots n+1\})$. Note that for each $j \in \{1, \dots n\}$ the transposition $\sigma_j = (j\ j+1)$ sends $\mathfrak{b}$ to the stabilizer of the full flag $\tilde{W}_1 \leqslant \cdots \leqslant \tilde{W}_n$ with $\tilde{W}_i = W_i$ for $i \neq j$ and $\tilde{W}_j = W_{j-1} \oplus L_{j+1}$. These transpositions generate Sym_{n+1}.

A maximal (proper) parabolic subalgebra contains $\mathfrak{k}$ if and only if the subspace it stabilizes is a sum of elements of $\mathcal{S}_{\mathfrak{k}}$. Hence the incidence system of such parabolic subalgebras is isomorphic to the incidence system of proper nontrivial subsets of $\mathcal{S}_{\mathfrak{k}}$, cf. Example 1A.

Example 4B. Continuing Examples 1B–3B, we take G to be the adjoint group $SO_0(U, Q_U)$. Minimal Levi subalgebras $\mathfrak{k}$ now correspond to orthogonal direct sum decompositions $U = (\bigoplus \mathcal{R}_{\mathfrak{k}}) \oplus W_{\mathfrak{k}}$ where Q_U is positive definite on $W_{\mathfrak{k}}$, $\dim W_{\mathfrak{k}} = k$, and $\mathcal{R}_{\mathfrak{k}}$ is a set of n signature $(1,1)$ subspaces of U. Again $N_G(\mathfrak{k}) = N_G(\mathfrak{a})$ is the subgroup of $SO_0(U, Q_U)$ preserving this decomposition of U, and the local Weyl group $W_{\mathfrak{k}}(\mathfrak{so}(U, Q_U))) = N_G(\mathfrak{k})/K$ is isomorphic to $Sym(\mathcal{R}_{\mathfrak{k}}) \ltimes {\mathbb{Z}_2}^{\mathcal{S}_{\mathfrak{k}}}$, where any representative of $f \in {\mathbb{Z}_2}^{\mathcal{S}_{\mathfrak{k}}}$ preserves any $R \in \mathcal{R}_{\mathfrak{k}}$, and if $f(R) = 0$, it preserves the two isotropic lines in R; otherwise, it swaps them.

The minimal parabolic subalgebra $\mathfrak{b}$ stabilizing the full flag $\pi_1 \leqslant \cdots \leqslant \pi_n$ of isotropic subspaces of U contains $\mathfrak{k}$ if and only if for each $j \in \{1, \dots n\}$ there is an isotropic line $L_j \leqslant R \in \mathcal{R}_{\mathfrak{k}}$ such that $\pi_j = \pi_{j-1} \oplus L_j$ (where $\pi_0 = 0$). The fibre of $\mathscr{W}^{\mathfrak{g}}$ over $\mathfrak{k} \in \mathscr{A}^{\mathfrak{g}}$ is isomorphic to the space of such maps, which identifies the global Weyl group with $Sym_n \ltimes {\mathbb{Z}_2}^n$. The canonical generators are $\rho_j = (j\ j+1)$ for $j \in \{1, \dots n-1\}$, while $\rho_n \in {\mathbb{Z}_2}^n$ with $\rho_n(j) = \delta_{jn}$. Thus for any $j \in \{1, \dots n\}$, ρ_j sends $\mathfrak{b}$ to the stabilizer of $\tilde{\pi}_1 \leqslant \cdots \leqslant \tilde{\pi}_n$ where $\tilde{\pi}_i = \pi_i$ for $i \neq j$, $\tilde{\pi}_j = \pi_{j-1} \oplus L_{j+1}$ for $j \neq n$, and otherwise, if $j = n$, $\tilde{\pi}_n = \pi_{n-1} \oplus \tilde{L}_n$ where $\tilde{L}_n$ is isotropic with $L_n \oplus \tilde{L}_n \in \mathcal{R}_{\mathfrak{k}}$.

Let $\mathcal{R}$ be a set of n disjoint two element sets. A subset of $\bigcup \mathcal{R}$ is *admissible* if it contains at most one element from each two element set. The nonempty admissible subsets form an incidence system $\Gamma^{\mathcal{R},\pm}$ over $\mathcal{I}_n$, with the number of elements as the type, and incidence by containment. The incidence system of maximal parabolic subalgebras of $\mathfrak{so}(U, Q_U)$ containing $\mathfrak{k}$ is isomorphic to $\Gamma^{\mathcal{R},\pm}$; this is the discrete model promised in Example 1B.

In general, the preceding analysis of parabolic subalgebras can be used to show that $\mathscr{P}^{\mathfrak{g}}$ is a Tits building [1, 14, 36] with apartment complex $\mathscr{A}^{\mathfrak{g}}$. We find it more convenient to do this in the framework of chamber systems [3, 30, 39], initiated by Tits in [37].

4.3. Chamber systems and Coxeter groups.

Definition 4.11 ([30, 39])**.** A *chamber system* over $\mathcal{I}$ is a graph Δ with an edge labelling $\lambda\colon E_\Delta \to \mathcal{I}$ such that for each $i \in \mathcal{I}$, the *i-adjacency* relation $b \overset{i}{—} c$ (*i.e.*, $b = c$ or $\{b, c\}$ is an edge with label i) is an equivalence relation. A *chamber subsystem* is a subgraph with the induced edge labelling, and a *chamber morphism* $\varphi\colon \Delta \to \Delta'$ over $\mathcal{I}$ is morphism of edge labelled graphs, *i.e.*, a map on vertices such that $b \overset{i}{—} c$ implies $\varphi(b) \overset{i}{—} \varphi(c)$.

The vertices of a chamber system are called *chambers*, and the equivalence class of a chamber c under i-adjacency is called its *i-panel* $p_i(c)$. We shall often blur the distinction between Δ and its underlying set $|\Delta|$ of chambers. For an i-panel p and a chamber subsystem A, we say $p \in A$ if $A \cap p$ is nonempty (*i.e.*, for some chamber $c \in A$, $p = p_i(c)$).

Proposition 4.12. *Let Γ be an incidence system over $\mathcal{I}$. Then the set of full flags in Γ can be made into a chamber system $\Delta = \mathcal{C}\Gamma$ via the relation $b \overset{i}{—} c$ iff their subflags of type $\mathcal{I} \setminus \{i\}$ are equal: $b \cap t_\Gamma^{-1}(\mathcal{I} \setminus \{i\}) = c \cap t_\Gamma^{-1}(\mathcal{I} \setminus \{i\})$.*

Definition 4.13. A *firm* i-panel is one containing at least two chambers; it is *thin* if it has exactly two, and *thick* if it has more than two. A firm, thin or thick chamber system is one whose i-panels (for all $i \in \mathcal{I}$) are all firm, thin or thick respectively.

In a thin chamber system Δ, for each $i \in \mathcal{I}$ there is a free involution of $|\Delta|$, which interchanges the two chambers in each i-panel. These involutions are not chamber automorphisms; they generate a right action on $|\Delta|$ of the free group over $\mathcal{I}$ with the property that $b \overset{i}{—} c$ iff $b = ci$ iff $bi = c$.

Definition 4.14. The *structure group* of a thin chamber system Δ is the subgroup of $Sym(|\Delta|)$ generated by the involutions $b \mapsto bi$ for $i \in \mathcal{I}$ (the effective quotient of the free group action generated by $\mathcal{I}$). Its group elements are the vertices of a chamber system with $w \overset{i}{—} w'$ iff $w' = wi$ (the *Cayley graph* of the presentation) and we denote the structure group (viewed in this way) by W_Δ, or by $\mathcal{I} \hookrightarrow W_\Delta$ (to indicate the generating set).

Proposition 4.15. *Let Δ be a thin chamber system.*

- *A permutation of $|\Delta|$ is a chamber morphism if and only if it commutes with W_Δ.*
- *Δ is connected if and only if W_Δ acts transitively. In this case the automorphism group $Aut(\Delta)$ acts freely and the following are equivalent:*
  1. *for any adjacent $b, c \in \Delta$, there is an automorphism of Δ interchanging b and c;*
  2. *Δ is homogeneous, i.e., $Aut(\Delta)$ acts transitively on (the chambers of) Δ;*
  3. *W_Δ acts freely on Δ;*
  4. *there is a unique function $\delta_\Delta\colon \Delta \times \Delta \to W_\Delta$ with $\delta_\Delta(b, c) = w$ iff $c = bw$.*

Thus a connected homogeneous thin chamber system Δ is an $(Aut(\Delta), W_\Delta)$-bitorsor, and, for fixed $b \in \Delta$, the map $\delta_\Delta(b, \cdot)\colon \Delta \to W_\Delta$ is an isomorphism of chamber systems.

4.4. Apartments and buildings. Henceforth, all chamber systems will be nonempty, firm and connected.

Definition 4.16. Let Δ be a chamber system and let $\mathcal{I} \hookrightarrow W$ be a group generated by involutions. An *apartment* in Δ is a (connected) homogeneous thin chamber subsystem A of Δ. A *W-distance* on Δ is a map $\delta_\Delta\colon \Delta \times \Delta \to W$ such that for any $b, c, c' \in \Delta$, $\delta_\Delta(b,c) = \delta_\Delta(c,b)^{-1}$ and $c' \overset{i}{—} c$ in Δ implies $\delta_\Delta(b,c') \overset{i}{—} \delta_\Delta(b,c)$ in W. An apartment A is *compatible* with a W-distance δ_Δ if $W_A \cong W$ with $\delta_A = \delta_\Delta|_A$. We say (Δ, δ_Δ) is a *building* (of type W) if for any $b, c \in \Delta$ there is a compatible apartment A containing b and c.

A thin building (Δ, δ_Δ) is just a homogeneous thin chamber system with its natural function $\delta_\Delta\colon \Delta \times \Delta \to W = W_\Delta$. Strictly speaking, we should require that W is a *Coxeter group* (as it will be in our examples). In the literature, the notion of a building is usually defined either purely in terms of δ_Δ (with no mention of apartments) or in terms of apartments (often using the language of simplicial complexes rather than chamber systems). We now relate our hybrid approach to the latter.

The idea is that we can define $\delta_\Delta(b,c) = \delta_A(b,c)$ for an apartment A containing b and c, provided that the apartments we admit give the same answer, and yield a W-distance.

Lemma 4.17. *Let Δ a chamber system and let $\varphi\colon A_1 \to A_2$ be an isomorphism between apartments A_1 and A_2 fixing $b \in A_1 \cap A_2$. Then:*

(1) *φ fixes $A_1 \cap A_2$ pointwise if and only if for any $c \in A_1 \cap A_2$, $\delta_{A_1}(b,c) = \delta_{A_2}(b,c)$;*

(2) *for any $c_1 \in A_1$ and $c_2 \in A_2$ with $c_1 \overset{i}{—} c_2$, i.e., $p := p_i(c_2) = p_i(c_2)$, we have that $\delta_{A_1}(b,c_1) \overset{i}{—} \delta_{A_2}(b,c_2)$ if and only if φ sends $p \cap A_1$ to $p \cap A_2$.*

Proof. (1) If $c = bw \in A_1$ then $\varphi(c) = bw$ in A_2, which is c iff $\delta_{A_2}(b,c) = w = \delta_{A_2}(b,c)$.

(2) If $w = \delta_{A_1}(b,c_1)$, then φ sends $p \cap A_1 = \{c_1, c_1 i\}$ to $\{c', c'i\} \subseteq A_2$ for the unique $c' \in A_2$ with $\delta_{A_2}(b,c') = w$. This is $p \cap A_2 = \{c_2, c_2 i\}$ if and only if $\delta_{A_1}(b,c_1) \overset{i}{—} \delta_{A_2}(b,c_2)$. □

In (2), we say that A_1 and A_2 *share* the i-panel p, that $p \in A_1$ and $p \in A_2$, and that an isomorphism φ sending $p \cap A_1$ to $p \cap A_2$ *preserves* p.

Definition 4.18. An *apartment complex* $\mathscr{A}$ in Δ is a set of apartments such that any two chambers in Δ belong to a common apartment in $\mathscr{A}$. We say $\mathscr{A}$ is *regular* iff for any two intersecting apartments $A_1, A_2 \in \mathscr{A}$, there is a chamber isomorphism $A_1 \to A_2$ fixing $A_1 \cap A_2$ pointwise and preserving any i-panel they share.

Proposition 4.19. *A chamber system Δ is a building with respect to some $\delta_\Delta\colon \Delta \times \Delta \to W$ if and only if it admits a regular apartment complex $\mathscr{A}$ (whose apartments have type W), in which case there is a unique δ_Δ such that the apartments in $\mathscr{A}$ are compatible.*

Proof. If (Δ, δ_Δ) is a building then the set $\mathscr{A}$ of compatible apartments form an apartment complex, while given an apartment complex $\mathscr{A}$, there is at most one δ_Δ with $\delta_\Delta|_A = \delta_A$ for all $A \in \mathscr{A}$. The equivalence now follows easily from Lemma 4.17. □

The chamber isomorphisms $A_1 \to A_2$ often arise as restrictions of automorphisms of Δ.

Definition 4.20. A group G of chamber automorphisms of a chamber system Δ preserving an apartment complex $\mathscr{A}$ is said to be *strongly transitive* if it acts transitively on $\mathscr{W} := \{(a, A) \mid a \in A \in \mathscr{A}\}$. In this situation $\mathscr{A}$ is regular if and only if

(R) for any chamber $b \in \Delta$ and any i-panel or chamber p, $Stab_G(b) \cap Stab_G(p)$ acts transitively on $\{A \in \mathscr{A} : b, p \in A\}$.

Indeed condition (R) implies regularity by Lemma 4.17. Conversely, if $\mathscr{A}$ is regular, then for any $A_1, A_2 \in \mathscr{A}$ and any $b \in A_1 \cap A_2$, the unique isomorphism $A_1 \to A_2$ fixing b is the restriction of some $g \in G$ by strong transitivity, so condition (R) holds.

We can now summarize what we have established in the homogeneous case.

Theorem 4.21 (Abstract Bruhat Decomposition)**.** *Suppose Δ is a chamber system over $\mathcal{I} \hookrightarrow W$ with apartment complex $\mathscr{A}$ and a strongly transitive group G of automorphisms satisfying condition* (R)*. Then Δ is a building, where the apartments in $\mathscr{A}$ are compatible with a W-distance δ_Δ which induces a bijection $G\backslash(\Delta \times \Delta) \to W$.*

Proof. Since $\mathscr{A}$ is a regular apartment complex, it determines a W-distance $\delta_\Delta \colon \Delta \times \Delta \to W$ which agrees with δ_A on any apartment $A \in \mathscr{A}$. Thus δ_Δ is surjective and G-invariant. Furthermore, for any $w \in W$, G acts transitively on pairs (b, c) with $\delta_\Delta(b, c) = w$ (if (b', c') is another such pair, then after choosing apartments $A, A' \in \mathscr{A}$ containing b, c and b', c' respectively, the element of G sending (b, A) to (b', A') also sends c to c'). Hence the induced surjection $G\backslash(\Delta \times \Delta) \to W$ is also injective. □

4.5. Parabolic buildings and the Bruhat decomposition. We now apply Proposition 4.12 to the parabolic incidence system $\Gamma^{\mathfrak{g}}$, to obtain a chamber system whose chambers are the full flags of $\Gamma^{\mathfrak{g}}$, which we may identify with the minimal parabolic subalgebras of $\mathfrak{g}$, and hence denote $\mathscr{B}^{\mathfrak{g}}$; two such subalgebras $\mathfrak{b}, \mathfrak{c}$ are then i-adjacent (for $i \in \mathcal{I}_{\mathfrak{g}}$) iff there is a (necessarily unique) parabolic subalgebra of type $\mathcal{I}_{\mathfrak{g}} \setminus \{i\}$ containing both $\mathfrak{b}$ and $\mathfrak{c}$.

Theorem 4.22. *The chamber system $\mathscr{B}^{\mathfrak{g}}$ of minimal parabolic subalgebras is a building over the global Weyl group $W(\mathfrak{g})$, which is a Coxeter group, and there is a canonical bijection between $G\backslash(\mathscr{B}^{\mathfrak{g}} \times \mathscr{B}^{\mathfrak{g}})$ and $W(\mathfrak{g})$.*

Proof. For any minimal Levi subalgebra $\mathfrak{k} \in \mathscr{A}^{\mathfrak{g}}$, the subgraph of $\mathscr{B}^{\mathfrak{g}}$ given by the minimal parabolic subalgebras containing $\mathfrak{k}$ is a thin chamber system of type $W(\mathfrak{g})$ whose automorphism group is the local Weyl group $N_G(\mathfrak{k})/C_G(\mathfrak{k})$ (whose restricted root system action implies $W(\mathfrak{g})$ is a Coxeter group). Any two minimal parabolic subalgebras belong to such an apartment by Corollary 3.18, which also shows that the apartment complex of type $W(\mathfrak{g})$ defined by $\mathscr{A}^{\mathfrak{g}}$ satisfies (R). The result now follows from Theorem 4.21. □

We refer to $\mathscr{B}^{\mathfrak{g}}$ as the *parabolic building* of $\mathfrak{g}$. If we fix a minimal parabolic subalgebra $\mathfrak{b}$ with stabilizer $B = N_G(\mathfrak{b})$ then $\mathscr{B}^{\mathfrak{g}} \cong G/B$ (as a G-space) and $G\backslash(\mathscr{B}^{\mathfrak{g}} \times \mathscr{B}^{\mathfrak{g}}) \cong G\backslash(G/B \times G/B) \cong B\backslash G/B$, the set of double cosets of B in G.

Proposition 4.23. *There is a unique and involutive map $\mathrm{op}_{\mathfrak{g}} \colon \mathcal{I}_{\mathfrak{g}} \to \mathcal{I}_{\mathfrak{g}}$ (i.e., ${\mathrm{op}_{\mathfrak{g}}}^2 = id_{\mathcal{I}_{\mathfrak{g}}}$) such that for any $J \subseteq \mathcal{I}_{\mathfrak{g}}$ and any parabolic subalgebra $\mathfrak{p}$ of type J, its opposite parabolic subalgebras have type $\mathrm{op}_{\mathfrak{g}}(J)$.*

Proof. For any parabolic subalgebra $\mathfrak{p}$, the parabolic subalgebras opposite to $\mathfrak{p}$ belong to a single adjoint orbit by Remarks 3.15. Furthermore, if $\widehat{\mathfrak{p}}$ is opposite to $\mathfrak{p}$ then $g \cdot \widehat{\mathfrak{p}}$ is opposite to $g \cdot \mathfrak{p}$, and if $\mathfrak{p}'$ is costandard with $\mathfrak{p}$ then by Corollary 5.5, there exists $\widehat{\mathfrak{p}}'$ opposite to $\mathfrak{p}'$ and costandard with $\widehat{\mathfrak{p}}$; it follows that $\widehat{\mathfrak{p}} \cap \widehat{\mathfrak{p}}'$ is opposite to $\mathfrak{p} \cap \mathfrak{p}'$. Hence the opposition involution of adjoint orbits is determined uniquely by the opposition relation on maximal parabolic subalgebras. □

Definition 4.24. The map $\mathrm{op}_{\mathfrak{g}}$ is called the *duality involution* of $\mathfrak{g}$.

5. Parabolic projection and geometric configurations

5.1. Parabolic projection. Let $\mathfrak{q}$ be parabolic in $\mathfrak{g}$, let $\mathfrak{q}_0 = \mathfrak{q}/\mathfrak{nil}(\mathfrak{q})$ and let $Q = N_G(\mathfrak{q}) \subseteq G$. By Corollary 3.6, the inverse image of any parabolic in $\mathfrak{q}_0$ is a parabolic subalgebra of $\mathfrak{g}$ contained in $\mathfrak{q}$. By Proposition 3.16, this process has a right inverse.

Definition 5.1. Let $\mathfrak{q}$ be a parabolic subalgebra of a reductive Lie algebra $\mathfrak{g}$. Then *parabolic projection* $\Pi_{\mathfrak{q}} \colon \mathscr{P}^{\mathfrak{g}} \to \mathscr{P}^{\mathfrak{q}_0}$ is defined by

$$\Pi_{\mathfrak{q}}(\mathfrak{p}) = (\mathfrak{p} \cap \mathfrak{q} + \mathfrak{nil}(\mathfrak{q}))/\mathfrak{nil}(\mathfrak{q}).$$

The following property is immediate from the definition and the special case $\mathfrak{p} \subseteq \mathfrak{p}'$.

Proposition 5.2. *If $\mathfrak{p}, \mathfrak{p}' \leqslant \mathfrak{g}$ are costandard parabolic subalgebras then $\Pi_{\mathfrak{q}}(\mathfrak{p} \cap \mathfrak{p}') \subseteq \Pi_{\mathfrak{q}}(\mathfrak{p}) \cap \Pi_{\mathfrak{q}}(\mathfrak{p}')$. Thus $\Pi_{\mathfrak{q}}$ preserves incidence.*

By Theorem 4.10, any maximal (proper) parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{q}$ is the intersection with $\mathfrak{q}$ of a maximal parabolic subalgebra of $\mathfrak{g}$ whose adjoint orbit in $\mathcal{I}_{\mathfrak{g}}$ is uniquely determined by the adjoint orbit $Q \cdot \mathfrak{p}$. There is thus a natural inclusion $\iota_{\mathfrak{q}} : \mathcal{I}_{\mathfrak{q}_0} \to \mathcal{I}_{\mathfrak{g}}$ whose image is the complement of the type $J_{\mathfrak{q}} \in P(\mathcal{I}_{\mathfrak{g}})$ of $\mathfrak{q}$ in $\mathfrak{g}$. More generally, if $\mathfrak{p} \in \mathscr{P}^{\mathfrak{q}_0}$ has type $J \in P(\mathcal{I}_{\mathfrak{q}_0})$ then its inverse image in $\mathfrak{q}$ has type $\iota_{\mathfrak{q}}(J) \cup J_{\mathfrak{q}} \in P(\mathcal{I}_{\mathfrak{g}})$.

Parabolic projection has a more complicated behaviour in general, although on the set $\mathscr{P}^{\mathfrak{g}}_{\mathfrak{q}\text{-co}}$ of parabolic subalgebras of $\mathfrak{g}$ costandard with $\mathfrak{q}$, the behaviour is straightforward.

Proposition 5.3. *The restriction of $\Pi_{\mathfrak{q}}$ to $\mathscr{P}^{\mathfrak{g}}_{\mathfrak{q}\text{-co}}$ sends parabolic subalgebras $\mathfrak{p}$ of type $J \in P(\mathcal{I}_{\mathfrak{g}})$ to parabolic subalgebras of type $\iota_{\mathfrak{q}}^{-1}(J) \in P(\mathcal{I}_{\mathfrak{q}_0})$.*

Proof. If $\mathfrak{p}$ is costandard with $\mathfrak{q}$ then $\Pi_{\mathfrak{q}}(\mathfrak{p}) = \mathfrak{p} \cap \mathfrak{q} / \mathfrak{nil}(\mathfrak{q})$. Since $\mathfrak{p} \cap \mathfrak{q}$ is in the adjoint orbit $J \cup J_{\mathfrak{q}}$, its projection to $\mathfrak{q}_0$ is in $\iota_{\mathfrak{q}}^{-1}(J \setminus J_{\mathfrak{q}}) = \iota_{\mathfrak{q}}^{-1}(J)$. □

Parabolic subalgebras of $\mathfrak{g}$ are rarely costandard with $\mathfrak{q}$; generically, they are weakly opposite to $\mathfrak{q}$ in the following sense.

Definition 5.4 ([26])**.** Parabolic subalgebras $\mathfrak{p}, \mathfrak{q} \leqslant \mathfrak{g}$ are said to be *weakly opposite* if $\mathfrak{p} + \mathfrak{q} = \mathfrak{g}$ (equivalently $\mathfrak{nil}(\mathfrak{p}) \cap \mathfrak{nil}(\mathfrak{q}) = 0$).

The weakly opposite case is also amenable to analysis.

Lemma 5.5. *Parabolic subalgebras $\mathfrak{p}$ and $\mathfrak{q}$ of $\mathfrak{g}$ are weakly opposite if and only if there is a parabolic subalgebra $\widehat{\mathfrak{p}}$ opposite to $\mathfrak{p}$ and costandard with $\mathfrak{q}$.*

Proof. If $\widehat{\mathfrak{p}}$ is opposite to $\mathfrak{p}$ and costandard with $\mathfrak{q}$ then $\mathfrak{nil}(\mathfrak{q}) \leqslant \widehat{\mathfrak{p}}$ so $\mathfrak{nil}(\mathfrak{p}) \cap \mathfrak{nil}(\mathfrak{q}) = 0$. Conversely, if $\mathfrak{nil}(\mathfrak{p}) \cap \mathfrak{nil}(\mathfrak{q}) = 0$, we may take $\widehat{\mathfrak{p}}$ opposite to $\mathfrak{p}$ using a Weyl structure $\xi_{\mathfrak{p}}$ in $\mathfrak{p} \cap \mathfrak{q}$: $\mathfrak{nil}(\mathfrak{q})$ then has nonnegative eigenvalues for $\xi_{\mathfrak{p}}$, hence lies in $\widehat{\mathfrak{p}}$. □

Lemma 5.6. *Let $\mathfrak{p}, \widehat{\mathfrak{p}}$ be parabolic in $\mathfrak{g}$ with $\widehat{\mathfrak{p}}$ opposite to $\mathfrak{p}$ and costandard with $\mathfrak{q}$. Then $\Pi_{\mathfrak{q}}(\widehat{\mathfrak{p}})$ is opposite to $\Pi_{\mathfrak{q}}(\mathfrak{p})$ in $\mathfrak{q}_0$.*

Proof. Since $\Pi_{\mathfrak{q}}(\widehat{\mathfrak{p}}) = \widehat{\mathfrak{p}} \cap \mathfrak{q} / \mathfrak{nil}(\mathfrak{q})$, $\mathfrak{nil}(\Pi_{\mathfrak{q}}(\widehat{\mathfrak{p}})) = (\mathfrak{nil}(\widehat{\mathfrak{p}}) + \mathfrak{nil}(\mathfrak{q}))/\mathfrak{nil}(\mathfrak{q})$. As $\mathfrak{p} + \mathfrak{nil}(\widehat{\mathfrak{p}}) = \mathfrak{g}$ with $\mathfrak{nil}(\widehat{\mathfrak{p}}) \subseteq \mathfrak{q}$, we have $\mathfrak{p} \cap \mathfrak{q} + \mathfrak{nil}(\widehat{\mathfrak{p}}) = \mathfrak{q}$ and hence $(\mathfrak{p} \cap \mathfrak{q} + \mathfrak{nil}(\mathfrak{q})) + (\mathfrak{nil}(\widehat{\mathfrak{p}}) + \mathfrak{nil}(\mathfrak{q})) = \mathfrak{q}$. It follows that $\Pi_{\mathfrak{q}}(\mathfrak{p}) + \mathfrak{nil}(\Pi_{\mathfrak{q}}(\widehat{\mathfrak{p}})) = \mathfrak{q}_0$.

It remains to show that $\Pi_{\mathfrak{q}}(\widehat{\mathfrak{p}}) + \mathfrak{nil}(\Pi_{\mathfrak{q}}(\mathfrak{p})) = \mathfrak{q}_0$, *i.e.*, $\widehat{\mathfrak{p}} \cap \mathfrak{q} + \mathfrak{nil}(\mathfrak{p}) \cap \mathfrak{q} = \mathfrak{q}$ (since $\mathfrak{nil}(\mathfrak{q}) \subseteq \widehat{\mathfrak{p}} \cap \mathfrak{q}$). For this, we use Corollary 3.18 to introduce a minimal Levi subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ contained in the parabolic subalgebras $\mathfrak{p}$ and $\widehat{\mathfrak{p}} \cap \mathfrak{q}$. In particular, $\mathfrak{k} \subseteq \mathfrak{p} \cap \widehat{\mathfrak{p}}$ and so the root space decomposition of $\mathfrak{g}$ associated to $\mathfrak{k}$ refines the direct sum decomposition $\mathfrak{g} = \widehat{\mathfrak{p}} \oplus \mathfrak{nil}(\mathfrak{p})$. Since $\mathfrak{q}$ contains $\mathfrak{k}$, it is a sum of root spaces, so $\mathfrak{q} = \mathfrak{g} \cap \mathfrak{q} = \widehat{\mathfrak{p}} \cap \mathfrak{q} \oplus \mathfrak{nil}(\mathfrak{p}) \cap \mathfrak{q}$. □

Let $\mathscr{P}^{\mathfrak{g}}_{\mathfrak{q}\text{-op}}$ be the set of all parabolic subalgebras of $\mathfrak{g}$ which are weakly opposite to $\mathfrak{q}$.

Proposition 5.7. *The restriction of $\Pi_{\mathfrak{q}}$ to $\mathscr{P}^{\mathfrak{g}}_{\mathfrak{q}\text{-op}}$ sends parabolic subalgebras $\mathfrak{p}$ of type $J \in P(\mathcal{I}_{\mathfrak{g}})$ to parabolic subalgebras of type $(\mathrm{op}_{\mathfrak{g}} \circ \iota_{\mathfrak{q}} \circ \mathrm{op}_{\mathfrak{q}_0})^{-1}(J) \in P(\mathcal{I}_{\mathfrak{q}_0})$.*

Proof. This is immediate from Corollary 5.5 and Lemma 5.6. □

Minimal parabolic subalgebras also have straightforward projections.

Proposition 5.8. *If $\mathfrak{b} \in \mathscr{B}^{\mathfrak{g}}$ then $\Pi_{\mathfrak{q}}(\mathfrak{b}) \in \mathscr{B}^{\mathfrak{q}_0}$. Furthermore, if $\mathfrak{b}_0 \in \mathscr{B}^{\mathfrak{q}_0}$ then there exists $\mathfrak{b} \in \mathscr{B}^{\mathfrak{g}}$ weakly opposite to $\mathfrak{q}$ with $\Pi_{\mathfrak{q}}(\mathfrak{b}) = \mathfrak{b}_0$.*

Proof. Using Corollary 3.18, let $\mathfrak{k}$ be a minimal Levi subalgebra contained in $\mathfrak{b}$ and $\mathfrak{q}$, and let $\widehat{\mathfrak{q}}$ be the opposite subalgebra to $\mathfrak{q}$ containing $\mathfrak{k}$. Then the root space decomposition of $\mathfrak{k}$ refines the direct sum decomposition $\mathfrak{g} = \mathfrak{nil}(\mathfrak{q}) \oplus \mathfrak{q} \cap \widehat{\mathfrak{q}} \oplus \mathfrak{nil}(\widehat{\mathfrak{q}})$, and so $\mathfrak{b}$ splits across this decomposition. For any root α with $\mathfrak{g}_\alpha \subseteq \mathfrak{q} \cap \widehat{\mathfrak{q}}$, precisely one of $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ is contained in $\mathfrak{b}$, so that $\mathfrak{b} \cap \mathfrak{q} \cap \widehat{\mathfrak{q}}$ is minimal in $\mathfrak{q} \cap \widehat{\mathfrak{q}}$. This is an isomorph of $\Pi_\mathfrak{q}(\mathfrak{b})$ in $\mathfrak{q}_0$.

For the second part, let $\mathfrak{b}_0 = \mathfrak{c}/\mathfrak{nil}(\mathfrak{q})$, where $\mathfrak{c}$ is a minimal parabolic subalgebra of $\mathfrak{g}$. Let $\widehat{\mathfrak{c}}$ be a minimal parabolic subalgebra of $\mathfrak{g}$ opposite to $\mathfrak{c}$. Then $\widehat{\mathfrak{c}}$ is weakly opposite to $\mathfrak{q}$. Since both $\Pi_\mathfrak{q}(\mathfrak{c})$ and $\Pi_\mathfrak{q}(\widehat{\mathfrak{c}})$ are minimal parabolic subalgebras of $\mathfrak{q}$, there exists $q \in Q$ such that

$$\mathfrak{c} = q \cdot (\widehat{\mathfrak{c}} \cap \mathfrak{q} + \mathfrak{nil}(\mathfrak{q})) = (q \cdot \widehat{\mathfrak{c}}) \cap \mathfrak{q} + \mathfrak{nil}(\mathfrak{q});$$

whence $\mathfrak{b} := q \cdot \widehat{\mathfrak{c}}$ is weakly opposite to $\mathfrak{q}$ with $\Pi_\mathfrak{q}(\mathfrak{b}) = \mathfrak{b}_0$. □

Proposition 5.9. *If $\mathfrak{b}, \mathfrak{c} \in \mathscr{B}^\mathfrak{g}$ are both weakly opposite to $\mathfrak{q}$, then $\mathfrak{b} \in Q \cdot \mathfrak{c}$.*

Proof. Using Corollary 5.5, let $\widehat{\mathfrak{b}}$ and $\widehat{\mathfrak{c}}$ be respectively minimal parabolic subalgebras of $\mathfrak{g}$ opposite to $\mathfrak{b}$ and $\mathfrak{c}$, and costandard with $\mathfrak{q}$. Then $\widehat{\mathfrak{b}}/\mathfrak{nil}(\mathfrak{q})$ and $\widehat{\mathfrak{c}}/\mathfrak{nil}(\mathfrak{q})$ are minimal parabolic subalgebras of $\mathfrak{q}_0$, so there exists $q \in Q$ such that $\widehat{\mathfrak{c}} = q \cdot \widehat{\mathfrak{b}}$. It follows that $q^{-1} \cdot \mathfrak{c}$ is opposite to $\widehat{\mathfrak{b}}$, hence conjugate to $\mathfrak{b}$ by an element $q' \in \exp(\mathfrak{nil}(\widehat{\mathfrak{b}}))$ by Remarks 3.15. Since $\mathfrak{nil}(\widehat{\mathfrak{b}}) \subseteq \mathfrak{q}$, $\mathfrak{b}$ and $\mathfrak{c}$ are in the same Q-orbit. □

Corollary 5.10. *Any two parabolic subalgebras of $\mathfrak{g}$ in the same adjoint orbit and weakly opposite to $\mathfrak{q}$ are in the same Q-orbit.*

Proof. This follows from Theorem 4.10 and Propositions 4.5 and 5.9. □

Let $\nu_\mathfrak{q} := \mathrm{op}_\mathfrak{g} \circ \iota_\mathfrak{q} \circ \mathrm{op}_{\mathfrak{q}_0} : \mathcal{I}_{\mathfrak{q}_0} \to \mathcal{I}_\mathfrak{g}$, and observe that the direct and inverse image operations in induced by $\nu_\mathfrak{q}$ may be used to pull back or push forward presheaves from $\mathcal{I}_\mathfrak{g}$ to $\mathcal{I}_{\mathfrak{q}_0}$ or vice versa. In particular, $\nu_\mathfrak{q}^* \mathscr{P}^\mathfrak{g}_{\mathfrak{q}\text{-op}}(J) := \mathscr{P}^\mathfrak{g}_{\mathfrak{q}\text{-op}}(\nu_\mathfrak{q}(J))$ is isomorphic to the flag complex of the incidence system $\nu_\mathfrak{q}^* \Gamma^\mathfrak{g}_{\mathfrak{q}\text{-op}} := \bigsqcup_{i \in \mathcal{I}_{\mathfrak{q}_0}} \mathscr{P}^\mathfrak{g}_{\mathfrak{q}\text{-op}}(\nu_\mathfrak{q}(i))$ (with the obvious type function), while $\nu_{\mathfrak{q}*} \mathscr{P}^{\mathfrak{q}_0}(J) := \mathscr{P}^{\mathfrak{q}_0}(\nu_\mathfrak{q}^{-1}(J))$ defines a chamber system $\nu_{\mathfrak{q}*}\mathscr{B}^{\mathfrak{q}_0}$ over $\mathcal{I}_g$ with vertices $\mathscr{P}^{\mathfrak{q}_0}(\mathcal{I}_{\mathfrak{q}_0})$. These operations are adjoint functors, so that morphisms from $\nu_\mathfrak{q}^* \mathscr{P}^\mathfrak{g}_{\mathfrak{q}\text{-op}}$ to $\mathscr{P}^{\mathfrak{q}_0}$ over $\mathcal{I}_{\mathfrak{q}_0}$ correspond bijectively to morphisms from $\mathscr{P}^\mathfrak{g}_{\mathfrak{q}\text{-op}}$ to $\nu_{\mathfrak{q}*}\mathscr{P}^{\mathfrak{q}_0}$. Parabolic projection is such a morphism.

Theorem 5.11. $\Pi_\mathfrak{q}$ *defines an incidence morphism $\psi_\mathfrak{q} : \nu_\mathfrak{q}^* \Gamma^\mathfrak{g}_{\mathfrak{q}\text{-op}} \to \Gamma^{\mathfrak{q}_0}$ and a chamber morphism $\varphi_\mathfrak{q} : \mathscr{B}^\mathfrak{g}_{\mathfrak{q}\text{-op}} \to \nu_{\mathfrak{q}*}\mathscr{B}^{\mathfrak{q}_0}$.*

5.2. Geometric configurations and their projections. We end this paper by returning full circle to Section 1 and the motivation from incidence geometry. We have noted in Proposition 4.12 that any incidence system Γ has an associated chamber system $\Delta = \mathcal{C}\Gamma$ whose chambers are the full flags of Γ. In fact $\mathcal{C}$ is functorial and has an adjoint [31].

Definition 5.12. Let Δ be a chamber system over $\mathcal{I}$. For $i \in \mathcal{I}$, an *i-coresidue* of Δ is a connected component of the graph obtained from Δ by removing all edges with label i.

Proposition 5.13. *For a chamber system Δ over $\mathcal{I}$, let $\mathcal{E}\Delta$ be the graph with type function $t_{\mathcal{E}\Delta} : |\mathcal{E}\Delta| \to \mathcal{I}$ whose vertices of type i are the i-coresidues v of Δ with edges v—w if and only if v and w contain a common chamber. Then $\mathcal{E}\Delta$ is an incidence system.*

This is immediate: if two i-coresidues contain a common chamber, they are equal.

If $\Delta = \mathcal{C}\Gamma$ is the chamber system of full flags of a incidence system Γ then all the full flags in an i-coresidue have the same element of type i; the map assigning this element to the i-coresidue is in fact an incidence morphism $\mathcal{E}\mathcal{C}\Gamma \to \Gamma$ (the co-unit of the adjunction), and is an isomorphism if and only if Γ is:

- *flag regular*, *i.e.*, any element belongs to a full flag; and
- *residually connected*, *i.e.*, for any $i \in \mathcal{I}$, any two full flags containing a common element of type i have the same i-coresidue.

The parabolic incidence system $\Gamma^{\mathfrak{g}}$ has both properties: identifying full flags with minimal parabolics using Theorem 4.10, we first note that any maximal (proper) parabolic contains a minimal parabolic, which implies flag regularity. Now if minimal parabolic $\mathfrak{b},\mathfrak{c}\subseteq\mathfrak{q}$ for a maximal parabolic $\mathfrak{q}$, then $\mathfrak{b}/\operatorname{\mathfrak{nil}}(\mathfrak{q})$ and $\mathfrak{c}/\operatorname{\mathfrak{nil}}(\mathfrak{q})$ are chambers in $\mathscr{B}^{\mathfrak{q}_0}$, which is connected. Thus $\mathfrak{b},\mathfrak{c}$ have the same $G\cdot\mathfrak{q}$-coresidue, and so $\Gamma^{\mathfrak{g}}$ is residually connected.

If Δ^W is a homogeneous thin chamber system over $\mathcal{I}$ with structure group W generated by $\mathcal{I}\hookrightarrow W$, then its i-coresidues are the orbits of the subgroup of W generated by $\mathcal{I}\setminus\{i\}$. If $W=W(\mathfrak{g})$ then the image of an injective chamber morphism $\Delta^{W(\mathfrak{g})}\to\mathscr{B}^{\mathfrak{g}}$ is an apartment, and such a labelled apartment thus determines an incidence morphism $\Gamma^{W(\mathfrak{g})}\to\Gamma^{\mathfrak{g}}$.

Definition 5.14. Let $\mathfrak{g}$ be a reductive Lie algebra. A *geometric configuration* in $\Gamma^{\mathfrak{g}}$ with *combinatorial type* (or *combinatorics*) Γ is an incidence morphism $\Gamma\to\Gamma^{\mathfrak{g}}$. A *standard configuration* in $\Gamma^{\mathfrak{g}}$ is an incidence morphism $\psi\colon\Gamma^{W(\mathfrak{g})}\to\Gamma^{\mathfrak{g}}$ corresponding to a labelled apartment $\Delta^{W(\mathfrak{g})}\hookrightarrow\mathscr{B}^{\mathfrak{g}}$.

Standard configurations are rather special, but can be used to obtain more general configurations by parabolic projection. Suppose then that $\mathfrak{q}$ is parabolic in $\mathfrak{g}$, define $\nu_{\mathfrak{q}}:\mathcal{I}_{\mathfrak{q}_0}\to\mathcal{I}_{\mathfrak{g}}$ as in the previous section, and let $\psi\colon\Gamma^{W(\mathfrak{g})}\to\Gamma^{\mathfrak{g}}$ be a standard configuration weakly opposite to $\mathfrak{q}$, *i.e.*, with image in $\Gamma^{\mathfrak{g}}_{\mathfrak{q}\text{-op}}$. Then ψ immediately defines a configuration $\nu_{\mathfrak{q}}^*\psi\colon\nu_{\mathfrak{q}}^*\Gamma^{W(\mathfrak{g})}\to\nu_{\mathfrak{q}}^*\Gamma^{\mathfrak{g}}_{\mathfrak{q}\text{-op}}$ by restricting along $\nu_{\mathfrak{q}}$ to elements with types in $\mathcal{I}_{\mathfrak{q}_0}$.

Definition 5.15. For $\mathfrak{q}$ parabolic in $\mathfrak{g}$, the *parabolic projection* of a standard configuration ψ weakly opposite to $\mathfrak{q}$ is the geometric configuration $\psi_{\mathfrak{q}}\circ\nu_{\mathfrak{q}}^*\psi\colon\nu_{\mathfrak{q}}^*\Gamma^{W(\mathfrak{g})}\to\Gamma^{\mathfrak{q}_0}$.

Example 5A. Recall from Example 1A the incidence system $\Gamma^{\mathcal{S}}$ of proper nonempty subsets of an $n+1=\dim V$ element set $\mathcal{S}$. An injective incidence system morphism $\Gamma^{\mathcal{S}}\to\Gamma^{\mathfrak{gl}(V)}$ defines a *simplex* in $P(V)$—the complete configuration of all projective subspaces spanned by subsets of $n+1$ distinct points in $P(V)$; these are the standard configurations in $\Gamma^{\mathfrak{gl}(V)}$.

If $\mathfrak{q}$ is the infinitesimal stabilizer of a point $L\in P(V)$, then parabolic projection defines an incidence system morphism $\nu_{\mathfrak{q}}^*\Gamma^{\mathfrak{gl}(V)}_{\mathfrak{q}\text{-op}}\to\Gamma^{\mathfrak{gl}(V/L)}$, where $\Gamma^{\mathfrak{gl}(V/L)}$ is an incidence system over $\mathcal{I}_{n-1}$ with the natural inclusion of types sending j to $j+1$. Since linear subspaces of complementary dimensions are opposite, $\nu_{\mathfrak{q}}:\mathcal{I}_{n-1}\to\mathcal{I}_n$ is the composite $j\mapsto n-j\mapsto n-j+1\mapsto n+1-(n-j+1)=j$. Explicitly, if $W\leqslant V$ does not contain L then the infinitesimal stabilizer of $P(W)$ is weakly opposite to $\mathfrak{q}$ and is projected to (the infinitesimal stabilizer of) $P((W+L)/L)$. The image of a simplex weakly opposite to $\mathfrak{q}$ is thus a complete configuration of $n+1$ points in general position in $P(V/L)$. For example, the parabolic projection of a tetrahedron in $\mathbb{R}P^3$ is a quadrilateral (four points and six lines) in $\mathbb{R}P^2$.

Example 5B. Example 4B shows that for standard configurations in $\Gamma^{\mathfrak{so}(U,Q_U)}$, we may take the combinatorial type to be the incidence system $\Gamma^{\mathcal{R},\pm}$ over $\mathcal{I}_n$ of admissible subsets of $\bigcup\mathcal{R}$, where $\mathcal{R}$ is a set of n disjoint two element sets. This is isomorphic to the set of all faces of an *n-cross polytope* or *n-orthoplex* with the singleton subsets as vertices, or, dually, to the set of all faces of an *n-cube*, with the n-element subsets as vertices. A standard configuration maps the vertices of the n-cross to points in the quadric of isotropic 1-dimensional subspaces of U, such that the admissible subsets of vertices (corresponding to faces of the n-cross) span subspaces which are isotropic, *i.e.*, lie entirely in the quadric.

If $\mathfrak{q}$ is the infinitesimal stabilizer of a point L on the quadric, then (the infinitesimal stabilizer of) any isotropic subspace which does not contain L and is not contained in $L^\perp$ is weakly opposite to L (*i.e.*, to $\mathfrak{q}$). Its projection onto the quadric of isotropic lines in $L^\perp/L$ is $(W\cap L^\perp+L)/L$. For example if $U=\mathbb{R}^{4,3}$, then $L^\perp/L$ is isomorphic to $\mathbb{R}^{3,2}$, and parabolic projection maps lines and planes in a real 5-quadric Q^5 to points and lines in a 3-quadric Q^3. A 3-cross is an octahedron, and so parabolic projection constructs configurations of 12 points and 8 lines in Q^3. This has an interpretation in Lie circle geometry [9]: points in Q^3 parametrize oriented circles in $S^2\cong\mathbb{R}^2\cup\{\infty\}$, which have oriented contact when the line joining the points is isotropic (*i.e.*, lies in Q^3).

Given a geometric configuration $\nu_{\mathfrak{q}}^* \Gamma^{W(\mathfrak{g})} \to \Gamma^{\mathfrak{q}_0}$, it is natural to ask if and when it may be obtained by parabolic projection from a standard configuration in $\Gamma^{\mathfrak{g}}$. This is a lifting problem which is studied in [24] by comparing the moduli of such configurations to the moduli of those obtained by parabolic projection. Here a crucial role is played by the fundamental result that all apartments in $\mathscr{B}^{\mathfrak{g}}$ belong to the apartment complex $\mathscr{A}^{\mathfrak{g}}$, hence are all conjugate by G. We shall return to these ideas in subsequent work.

Appendix A. Standard Lie theory background

A.1. Semisimplicity and reducibility. Let $A \subseteq \mathit{End}_{\mathbb{F}}(V)$ for a finite dimensional vector space V over a field $\mathbb{F}$; then (V, A) is *simple* or *irreducible* if V has exactly two invariant subspaces $0, V \neq 0$, and *semisimple* or *completely reducible* if every A-invariant subspace of V has an A-invariant complement, *i.e.*, V has a direct sum decomposition into irreducibles. These notions depend only on the (associative) $\mathbb{F}$-subalgebra of $\mathit{End}_{\mathbb{F}}(V)$ generated by A, and extend to the case that V carries a representation ρ of a group, associative algebra or Lie algebra by taking A to be be the image of ρ in $\mathit{End}_{\mathbb{F}}(V)$.

For any field extension $\mathbb{F} \subseteq \mathbb{F}^c$ and any $\mathbb{F}$-vector space V, let $V^c := \mathbb{F}^c \otimes_{\mathbb{F}} V$ be the induced $\mathbb{F}^c$-vector space, which is functorial in V (any linear map $\alpha\colon V \to W$ induces $\alpha^c\colon V^c \to W^c$). If $\mathbb{F}$ is perfect (*i.e.*, any algebraic extension is separable) and (V, A) is semisimple, then so is (V^c, A^c). In particular, if $\alpha \in \mathit{End}_{\mathbb{F}}(V)$ then α (*i.e.*, $(V, \{\alpha\})$) is semisimple iff its minimal polynomial p_α has distinct irreducible factors iff α^c is diagonalizable (*i.e.*, V^c has a basis of eigenvectors for α^c) in a (separable) splitting field $\mathbb{F}^c$ for p_α.

Lemma A.1. *If $\alpha \in \mathit{End}_{\mathbb{F}}(V)$ is semisimple and α^c has eigenvalues $\mathcal{S} \subseteq \mathbb{F}^c$ (in a separable splitting field $\mathbb{F}^c$ for p_α) then $\mathrm{ad}\,\alpha \in \mathit{End}_{\mathbb{F}}(\mathfrak{gl}(V))$ is semisimple and $(\mathrm{ad}\,\alpha)^c = \mathrm{ad}(\alpha^c)$ has eigenvalues $\{\lambda - \mu \mid \lambda, \mu \in \mathcal{S}\}$. Furthermore if, for some additive group homomorphism $f\colon \mathbb{F}^c \to \mathbb{F}^c$, $\beta \in \mathit{End}_{\mathbb{F}^c}(V^c)$ is scalar multiplication by $f(\lambda)$ on the λ-eigenspace of α^c for all $\lambda \in \mathcal{S}$, then $\mathrm{ad}\,\beta$ is a polynomial in $\mathrm{ad}\,\alpha^c$ with no constant term.*

Proof (*see* [2, 17]). Let $V_\lambda : \lambda \in \mathcal{S}$ denote the eigenspaces of α^c in V^c. Clearly $(\mathit{ad}\,\alpha)^c = \mathit{ad}(\alpha^c)$ has eigenvalue $\lambda - \mu$ on the subspace consisting of all $\mathfrak{gl}(V^c)$ which map V_μ to V_λ and all other eigenspaces to zero. These subspaces span $\mathfrak{gl}(V^c)$ so $\mathit{ad}\,\alpha^c$, hence $\mathit{ad}\,\alpha$, is semisimple. Finally, $\mathit{ad}\,\beta = q(\mathit{ad}\,\alpha^c)$ where $q(\lambda - \mu) = f(\lambda) - f(\mu)$ for all $\lambda, \mu \in \mathcal{S}$. Since f is additive, such a polynomial q exists by Lagrange interpolation, and has $q(0) = 0$. □

A.2. Jordan decomposition. Let V, W be vector spaces over a perfect field $\mathbb{F}$. Any $\alpha \in \mathfrak{gl}(V)$ has a unique (additive) *Jordan decomposition* $\alpha = \alpha_s + \alpha_n$ such that α_s is semisimple, α_n is nilpotent (*i.e.*, ${\alpha_n}^k = 0$ for k sufficiently large) and $[\alpha_s, \alpha_n] = 0$. Further, α_s and α_n are polynomials in α with no constant term. The following properties of Jordan decomposition, for $\alpha \in \mathfrak{gl}(V)$ and $\beta \in \mathfrak{gl}(W)$, are straightforward and standard [2].

- If $\alpha \in \mathfrak{c}_{\mathfrak{gl}(V)}(A, B)$ for some $B \subseteq A \subseteq V$, then $\alpha_s, \alpha_n \in \mathfrak{c}_{\mathfrak{gl}(V)}(A, B)$.
- If $\phi\colon V \to W$ is a linear map with $\phi \circ \alpha = \beta \circ \phi$ then $\phi \circ \alpha_s = \beta_s \circ \phi$ and $\phi \circ \alpha_n = \beta_n \circ \phi$.
- In $\mathfrak{gl}(V \otimes W)$, $(\alpha \otimes 1 + 1 \otimes \beta)_s = \alpha_s \otimes 1 + 1 \otimes \beta_s$ and $(\alpha \otimes 1 + 1 \otimes \beta)_n = \alpha_n \otimes 1 + 1 \otimes \beta_n$.
- In $\mathfrak{gl}(V^*)$, $\alpha^{\mathrm{T}} = (\alpha_s)^{\mathrm{T}} + (\alpha_n)^{\mathrm{T}}$ is the Jordan decomposition of the transpose α^{T}.

Thus Jordan decomposition is preserved in tensor representations of $\mathfrak{gl}(V)$ and the first property extends to subspaces $B \subseteq A$ in such a tensor representation. In particular, if V has an algebra structure, and α is a derivation, so are α_s and α_n. Also recall that a linear map $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation iff for any $x \in \mathfrak{g}$, $\mathit{ad}(\rho(x)) \circ \rho = \rho \circ \mathit{ad}(x)\colon \mathfrak{g} \to \mathfrak{gl}(V)$.

Proposition A.2. *Let $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation and $x \in \mathfrak{g}$. Then $\mathit{ad}(x)_s, \mathit{ad}(x)_n \in \mathfrak{der}(\mathfrak{g})$ (the Lie algebra of derivations of $\mathfrak{g}$), $\mathit{ad}(\rho(x))_s = \mathit{ad}(\rho(x)_s)$, $\mathit{ad}(\rho(x))_n = \mathit{ad}(\rho(x)_n)$, $\mathit{ad}(\rho(x)_s) \circ \rho = \rho \circ \mathit{ad}(x)_s$ and $\mathit{ad}(\rho(x)_n) \circ \rho = \rho \circ \mathit{ad}(x)_n$.*

In particular, for $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ faithful and $x \in \mathfrak{g}$ with $\rho(x)_s = \rho(x_s)$ and $\rho(x)_n = \rho(x_n)$ for some $x_s, x_n \in \mathfrak{g}$, it follows that $\mathit{ad}(x_s) + \mathit{ad}(x_n)$ is the Jordan decomposition of $\mathit{ad}(x)$.

Conversely, the following is the key to the proof of the abstract Jordan decomposition (or preservation of Jordan decomposition) for semisimple Lie algebras [2, 17, 21].

Proposition A.3. *Let $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a semisimple representation over a field $\mathbb{F}$ of characteristic zero, and suppose $x = x_s + x_n$ where $\mathrm{ad}(x_s)$ is semisimple, $\mathrm{ad}(x_n)$ is nilpotent, $x_n \in [\mathfrak{g}, \mathfrak{g}]$ and $[x_s, x_n] = 0$. Then $\rho(x)_s = \rho(x_s)$ and $\rho(x)_n = \rho(x_n)$.*

Proof. Since $ad(x_s)$ is semisimple, $\rho(x_s)_n \in \mathfrak{c}_{\mathfrak{gl}(V)}(\rho(\mathfrak{g}))$ by Proposition A.2, hence the inverse image of its span is an ideal in $\mathfrak{g}$ acting nilpotently on V, so $\rho(x_s)_n = 0$ since $\mathfrak{nil}_\rho(\mathfrak{g}) = \ker \rho$; thus $\rho(x_s)$ is semisimple. Similarly, since $ad(x_n)$ is nilpotent, $\rho(x_n)_s \in \mathfrak{c}_{\mathfrak{gl}(V)}(\rho(\mathfrak{g}))$, hence $\rho(x_n)_s^c \in \mathfrak{c}_{\mathfrak{gl}(V^c)}(\rho(\mathfrak{g})^c)$ for any field extension $\mathbb{F} \subseteq \mathbb{F}^c$. Since $\rho(x_n)_s$ is a polynomial in $\rho(x_n)$, $\rho(x_n)_s^c$ preserves any $\mathfrak{g}$-invariant subspace of V^c; but also $\rho(x_n)_s = \rho(x_n) - \rho(x_n)_n$, where the first term is in $[\rho(\mathfrak{g}), \rho(\mathfrak{g})]$ and the second term is nilpotent, so $\rho(x_n)_s^c$ has vanishing trace on any such subspace. If $\mathbb{F}^c$ is the algebraic closure of $\mathbb{F}$, $\rho(x_n)_s^c$ then vanishes on any minimal invariant subspace of V^c. Since ρ is semisimple, $\rho(x_n)_s = 0$, *i.e.*, $\rho(x_n)$ is nilpotent. Now $[\rho(x_s), \rho(x_n)] = \rho([x_s, x_n]) = 0$, so the result follows. ☐

A.3. Invariant forms. An *invariant form* on a Lie algebra $\mathfrak{g}$ is a symmetric bilinear form $(x, y) \mapsto \langle x, y\rangle$ such that $\langle [z, x], y\rangle + \langle x, [z, y]\rangle = 0$ for all $x, y, z \in \mathfrak{g}$. For a subspace $\mathfrak{s} \subseteq \mathfrak{g}$, we define $\mathfrak{s}^\perp = \{x \in \mathfrak{g} \mid \forall y \in \mathfrak{s},\ \langle x, y\rangle = 0\}$. For $\mathfrak{s}, \mathfrak{t} \subseteq \mathfrak{g}$, $\mathfrak{s} \subseteq \mathfrak{t}^\perp$ iff $\mathfrak{t} \subseteq \mathfrak{s}^\perp$, and we say $\mathfrak{s}, \mathfrak{t}$ are *orthogonal*, written $\mathfrak{s} \perp \mathfrak{t}$. If $\mathfrak{s} \perp \mathfrak{s}$, we say $\mathfrak{s}$ is *isotropic*; for any subspace $\mathfrak{s}$, $\mathfrak{s} \cap \mathfrak{s}^\perp$ is isotropic. Invariance means that $(x, y, z) \mapsto \langle [x, y], z\rangle$ is a 3-form on $\mathfrak{g}$, and hence for any subspaces $\mathfrak{r}, \mathfrak{s}, \mathfrak{t} \subseteq \mathfrak{g}$, $\mathfrak{r} \perp [\mathfrak{s}, \mathfrak{t}]$ iff $\mathfrak{s} \perp [\mathfrak{t}, \mathfrak{r}]$ iff $\mathfrak{t} \perp [\mathfrak{r}, \mathfrak{s}]$. Thus $\mathfrak{r} \subseteq [\mathfrak{s}, \mathfrak{t}]^\perp$ iff $[\mathfrak{r}, \mathfrak{s}] \subseteq \mathfrak{t}^\perp$ iff $\mathfrak{r} \subseteq \mathfrak{c}_\mathfrak{g}(\mathfrak{s}, \mathfrak{t}^\perp)$, and so $\mathfrak{c}_\mathfrak{g}(\mathfrak{s}, \mathfrak{t}^\perp) = [\mathfrak{s}, \mathfrak{t}]^\perp = [\mathfrak{t}, \mathfrak{s}]^\perp = \mathfrak{c}_\mathfrak{g}(\mathfrak{t}, \mathfrak{s}^\perp)$.

Proposition A.4. *Let $\langle \cdot, \cdot\rangle$ be an invariant form on $\mathfrak{g}$ and let $\mathfrak{s}$ be a subspace of $\mathfrak{g}$.*

(1) *$\mathfrak{n}_\mathfrak{g}(\mathfrak{s}) \subseteq \mathfrak{c}_\mathfrak{g}(\mathfrak{s}, \mathfrak{s}^{\perp\perp}) = [\mathfrak{s}, \mathfrak{s}^\perp]^\perp = \mathfrak{n}_\mathfrak{g}(\mathfrak{s}^\perp)$, $\mathfrak{c}_\mathfrak{g}(\mathfrak{s}) \subseteq \mathfrak{c}_\mathfrak{g}(\mathfrak{s}, \mathfrak{g}^\perp) = [\mathfrak{s}, \mathfrak{g}]^\perp$ and $\mathfrak{z}(\mathfrak{g}) \subseteq \mathfrak{c}_\mathfrak{g}(\mathfrak{g}, \mathfrak{g}^\perp) = [\mathfrak{g}, \mathfrak{g}]^\perp$ with equality (in each containment) if $\langle \cdot, \cdot\rangle$ is nondegenerate.*

(2) *If $\mathfrak{s} \leqslant \mathfrak{g}$, then $[\mathfrak{s}, \mathfrak{s}^\perp] \subseteq \mathfrak{s}^\perp$, hence also $\mathfrak{s} \cap \mathfrak{s}^\perp \trianglelefteq \mathfrak{s}$; in particular, $\mathfrak{g}^\perp$ is an isotropic ideal in $\mathfrak{g}$ (which is zero iff $\langle \cdot, \cdot\rangle$ is nondegenerate).*

(3) *If $\mathfrak{s} \trianglelefteq \mathfrak{g}$, then $\mathfrak{s}^\perp \trianglelefteq \mathfrak{g}$, hence also $\mathfrak{s} \cap \mathfrak{s}^\perp \trianglelefteq \mathfrak{g}$, $[\mathfrak{s}, \mathfrak{s}^\perp] \trianglelefteq \mathfrak{g}$ and $\mathfrak{s} + \mathfrak{s}^\perp \trianglelefteq \mathfrak{g}$.*

(4) *$\mathfrak{g} \perp [\mathfrak{s}, \mathfrak{s}]$ if and only if $[\mathfrak{g}, \mathfrak{s}] \perp \mathfrak{s}$. In particular, if $\langle \cdot, \cdot\rangle$ is nondegenerate, any isotropic ideal in $\mathfrak{g}$ is abelian.*

Proposition A.5. *Suppose $\mathfrak{g}$ is a Lie algebra with a nondegenerate invariant form and no nontrivial abelian ideals. Then $\mathfrak{g}$ is semisimple.*

Proof (*see* [12]). We induct on the dimension of $\mathfrak{g}$. Let $\mathfrak{l}$ be a minimal nontrivial ideal in $\mathfrak{g}$; then $\mathfrak{l} \cap \mathfrak{l}^\perp$ is an isotropic ideal in $\mathfrak{g}$, hence abelian by Proposition A.4 (4). Since $\mathfrak{l}$ is nonabelian (hence simple) by assumption, $\mathfrak{l} \cap \mathfrak{l}^\perp = 0$. Hence $\mathfrak{g}$ is the orthogonal direct sum of $\mathfrak{l}$ and $\mathfrak{l}^\perp$, and the latter is semisimple by induction. ☐

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