EXTREMAL KÄHLER METRICS
ON PROJECTIVE BUNDLES OVER A CURVE

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Abstract. Let $M = P(E)$ be the complex manifold underlying the total space of the projectivization of a holomorphic vector bundle $E \to \Sigma$ over a compact complex curve Σ of genus ≥ 2. Building on ideas of Fujiki [28], we prove that M admits a Kähler metric of constant scalar curvature if and only if E is polystable. We also address the more general existence problem of extremal Kähler metrics on such bundles and prove that the splitting of E as a direct sum of stable subbundles is necessary and sufficient condition for the existence of extremal Kähler metrics in Kähler classes sufficiently far from the boundary of the Kähler cone. The methods used to prove the above results apply to a wider class of manifolds, called *rigid toric bundles over a semisimple base*, which are fibrations associated to a principal torus bundle over a product of constant scalar curvature Kähler manifolds with fibres isomorphic to a given toric Kähler variety. We discuss various ramifications of our approach to this class of manifolds.

Introduction

Extremal Kähler metrics were first introduced and studied by E. Calabi in [13, 14]. Let (M, J) denote a connected compact complex manifold of complex dimension m. A Kähler metric g on (M, J), with Kähler form $\omega = g(J\cdot, \cdot)$, is *extremal* if it is a critical point of the functional $g \mapsto \int_M s_g^2 \frac{\omega_g^m}{m!}$, where g runs over the set of all Kähler metrics on (M, J) within a fixed Kähler class $\Omega = [\omega]$, and s_g denotes the scalar curvature of g. As shown in [13], g is extremal if and only if the symplectic gradient $K := \mathrm{grad}_\omega s_g = J\,\mathrm{grad}_g s_g$ of s_g is a Killing vector field (i.e., $\mathcal{L}_K g = 0$) or, equivalently, a (real) holomorphic vector field (i.e., $\mathcal{L}_K J = 0$). Extremal Kähler metrics include Kähler metrics of constant scalar curvature (CSC Kähler metrics) in particular Kähler–Einstein metrics. Clearly, if the identity component $\mathrm{Aut}_0(M, J)$ of the automorphism group of (M, J) is reduced to $\{\mathrm{Id}\}$, i.e., if (M, J) has no non-trivial holomorphic vector fields, any extremal Kähler metric is CSC, whereas a CSC Kähler metric is Kähler–Einstein if and only if Ω is a multiple of the (real) first Chern class $c_1(M, J)$. In this paper, except for Theorem 1 below, we will be mainly concerned with extremal Kähler metrics of *non-constant* scalar curvature.

The *Lichnerowicz–Matsushima theorem* provides an obstruction to the existence of CSC Kähler metrics on (M, J) in terms of the structure of $\mathrm{Aut}_0(M, J)$, which must be *reductive* whenever (M, J) admits a CSC Kähler metric; in particular, for any CSC Kähler metric g, the identity component $\mathrm{Isom}_0(M, g)$ of the group of isometries of (M, g) is a *maximal* compact subgroup of $\mathrm{Aut}_0(M, J)$ [55, 49]. The latter fact remains true for any extremal Kähler metric (although $\mathrm{Aut}_0(M, J)$ is then no longer reductive in general) and is again an obstruction to the existence of extremal Kähler metrics [14, 48]. Another well-known obstruction to the existence of CSC Kähler metrics within a given class Ω involves the *Futaki character* [30, 14], of which a symplectic version, as developed in [47], will be used in this paper (cf. Lemma 2). Furthermore, it is now known that

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extremal Kähler metrics within a fixed Kähler class Ω are *unique* up to the action of the *reduced automorphism group*[1] $\widetilde{\mathrm{Aut}}_0(M,J)$ [10, 15, 20, 52].

It was suggested by S.-T. Yau [75] that a complete obstruction to the existence of extremal Kähler metrics in the Kähler class $\Omega = 2\pi c_1(L)$ on a projective manifold (M,J) polarized by an ample holomorphic line bundle L should be expressed in terms of *stability* of the pair (M,L). The currently accepted notion of stability is the K-*(poly)stability* introduced by G. Tian [69, 70] and refined by S. K. Donaldson [21]. The *Yau–Tian–Donaldson conjecture* can then be stated as follows. *A polarized projective manifold* (M,L) *admits a CSC Kähler metric if and only if it is K-polystable.* This conjecture is still open, but the implication 'CSC $\Rightarrow$ K-polystable' in the conjecture is now well-established, thanks to work by S. K. Donaldson [20], X. Chen–G. Tian [15], J. Stoppa [64], and T. Mabuchi [53, 54]. In order to account for extremal Kähler metrics of non-constant scalar curvature, G. Székelyhidi introduced [67, 66] the notion of *relative* K-(poly)stability with respect to a maximal torus of the automorphism group of the pair (M,L) — which is the same as the reduced automorphism group $\widetilde{\mathrm{Aut}}_0(M,J)$ — and the similar implication 'extremal $\Rightarrow$ relatively K-polystable' was recently established by G. Székelyhidi–J. Stoppa [65].

The Yau–Tian–Donaldson conjecture was inspired by and can be regarded as a non-linear counterpart of the well-known equivalence for a holomorphic vector bundle over a compact Kähler manifold (M,J,ω) to be *polystable* with respect to ω on the one hand and to admit a *hermitian–Einstein metric* [40, 74, 18] on the other. In the case when (M,J) is a Riemann surface this is the celebrated theorem of Narasimhan and Seshadri [56], which, in the geometric formulation given in [9, 17, 28] can be stated as follows. *Let E be a holomorphic vector bundle over a compact Riemann surface Σ. Then, E is polystable if and only if it admits a hermitian metric whose Chern connection is projectively-flat.*

This paper is mainly concerned with the existence of extremal Kähler metrics on *ruled manifolds* (M,J), which are total spaces of projective fibre bundles $P(E)$ with E being a holomorphic vector bundle over a compact Kähler manifold (S,J_S,ω_S) of constant scalar curvature. Notice that this class of complex manifolds includes most explicitly known examples so far of extremal Kähler manifolds of non-constant scalar curvature, starting with the first examples given by E. Calabi in [13]. For complex manifolds of this type one expects stability properties of (M,J) to be reflected in the stability of the vector bundle E. In fact, such a link was established by J. Ross–R. Thomas, with sharper results when the base is a compact Riemann surface of genus at least 1, [61, Thm. 5.12 and Thm. 5.13] (cf. also Remark 2 below.) This suggests that the existence of extremal Kähler metrics on $P(E)$ could be directly linked to the stability of the underlying bundle E. First evidence of such a direct link goes back to the work of Burns–deBartolomeis [12], and many partial results in this direction are now known, see e.g. the works of N. Koiso–Y. Sakane [42, 43], A. Fujiki [28], D. Guan [33], C. R. LeBrun [44], C. Tønnesen-Friedman [71], Y.-J. Hong [36, 37], A. Hwang–M. Singer [39], Y. Rollin–M. Singer [60], J. Ross–R. Thomas [61, 62], G. Székelyhidi [67, 66], and our previous work [5]. However, to the best of our knowledge, a complete understanding of the precise relation is yet to come.

While we principally focus on projective bundles, for which sharper results can be obtained (Theorems 1, 2, 4 below), the techniques and a number of results presented in this paper actually address a much wider class of manifolds, called *rigid toric bundles*

[1] $\widetilde{\mathrm{Aut}}_0(M,J)$ is the unique *linear algebraic subgroup* of $\mathrm{Aut}_0(M,J)$ such that the quotient $\mathrm{Aut}_0(M,J)/\widetilde{\mathrm{Aut}}_0(M,J)$ is a torus, namely the Albanese torus of (M,J) [27]; its Lie algebra is the space of (real) holomorphic vector fields whose zero-set is non-empty [27, 40, 45, 32].

over a semisimple base. These were introduced in our previous paper [4]; we recall their main features in Sect. 2 and prove a general existence theorem (Theorem 3) in Sect. 3.

The simplest situation considered in this paper is the case of a projective bundle over a *curve* (i.e., a compact Riemann surface). In this case, the existence problem for CSC Kähler metrics can be resolved.

Theorem 1. *Let $(M, J) = P(E)$ be a holomorphic projective bundle over a compact complex curve of genus ≥ 2. Then (M, J) admits a CSC Kähler metric in some (and hence any) Kähler class if and only if the underlying holomorphic vector bundle E is polystable.*

Remark 1. The 'if' part follows from the theorem of Narasimhan and Seshadri: if E is a polystable bundle of rank m over a compact curve (of any genus), then E admits a hermitian–Einstein metric which in turn defines a flat $PU(m)$-structure on $P(E)$ and, therefore, a family of locally-symmetric CSC Kähler exhausting the Kähler cone of $P(E)$, see e.g. [40], [28]. Note also that in the case when $P(E)$ fibres over $\mathbb{C}P^1$, E splits as a direct sum of line bundles, and the conclusion of Theorem 1 still holds by the Lichnerowicz–Matsushima theorem, see e.g. [5, Prop. 3].

Remark 2. On all manifolds considered in Theorem 1, rational Kähler classes form a dense subset in the Kähler cone. By the LeBrun–Simanca stability theorem [46, Thm. A] and Lemma 3 below it is then sufficient to consider the existence problem only for an integral Kähler class (or polarization). In this setting, it was shown by Ross–Thomas that any projective bundle $M = P(E)$ over a compact complex curve of genus ≥ 1 is K-polystable (with respect to some polarization) if and only if E is polystable [61, Thm. 5.13]. In view of this theorem, the "only if" part of Theorem 1 can therefore be alternatively recovered — for any genus ≥ 1 — as a consequence of recent papers by T. Mabuchi [53, 54].

By the de Rham decomposition theorem, an equivalent differential geometric formulation of Theorem 1 is that any CSC Kähler metric on (M, J) must be locally symmetric (see [28, Lemma 8] and [44]). It is in this form that we are going to achieve our proof of Theorem 1, building on the work of A. Fujiki [28]. In fact, [28] already proves Theorem 1 in the case when the underlying bundle E is simple, modulo the uniqueness of CSC Kähler metrics, which is now known [15, 20, 52].

In view of this, the main technical difficulty in proving Theorem 1 is related to the existence of automorphisms on $(M, J) = P(E) \to \Sigma$. The way we proceed is by fixing a maximal torus $\mathbb{T}$ (of dimension ℓ) in the identity component $\mathrm{Aut}_0(M, J)$ of the automorphism group, and showing that it induces a decomposition of $E = \bigoplus_{i=0}^{\ell} E_i$ as a direct sum of $\ell + 1$ indecomposable subbundles E_i, such that $\mathbb{T}$ acts by scalar multiplication on each E_i (see Lemma 1 below). By computing the Futaki invariant of the S^1 generators of $\mathbb{T}$, we show that the slopes of E_i must be all equal, should a CSC Kähler metric exist on $P(E)$ (see Lemma 3 below).[2] Then, following the proof of [28, Thm. 3], we consider small analytic deformations $E_i(t)$ of $E_i = E_i(0)$ with $E_i(t)$ being stable bundles for $t \neq 0$. This induces a $\mathbb{T}$-invariant Kuranishi family $(M, J_t) \cong P(E(t))$, where $E(t) = \bigoplus_{i=0}^{\ell} E_i(t)$, with (M, J) being the central fibre (M, J_0). We then generalize in Lemma 4 the stability-under-deformations results of [45, 46, 29], by using the crucial fact that our family is invariant under a fixed *maximal* torus. This allows us to show that any CSC (or more generally extremal) Kähler metric ω_0 on (M, J_0) can be included into a smooth family ω_t of extremal Kähler metrics on (M, J_t). As $E(t)$ is polystable

[2] For rational Kähler classes, this conclusion can be alternatively reached by combining [61, Thm. 5.3] and [23].

for $t \neq 0$, the corresponding extremal Kähler metric ω_t must be locally symmetric, by the uniqueness results [15, 20, 51]. This implies that ω_0 is locally symmetric too, and we conclude as in [28, Lemma 8].

We next consider the more general problem of existence of extremal Kähler metrics on the manifold $(M, J) = P(E) \to \Sigma$. Notice that the deformation argument explained above is not specific to the CSC case, but also yields that any extremal Kähler metric ω_0 on $(M, J) = P(E) \to \Sigma$ can be realized as a smooth limit (as $t \to 0$) of extremal Kähler metrics ω_t on $(M, J_t) = P(E(t))$, where $E(t) = \bigoplus_{i=0}^{\ell} E_i(t)$ with $E_i(t)$ being stable (and thus projectively-flat and indecomposable) bundles over Σ for $t \neq 0$, and where ℓ is the dimension of a maximal torus $\mathbb{T}$ in the identity component of the group of isometries of ω_0. Unlike the CSC case (where $E_i(t)$ must all have the same slope and therefore $E(t)$ is polystable), the existence problem for extremal Kähler metrics on the manifolds (M, J_t) is not solved in general. Our main working conjecture is that such a metric ω_t must always be *compatible* with the bundle structure (in a sense made precise in Sect. 2 below). As we observe in §3.4, if this conjecture were true it would imply that the initial bundle E must also split as a direct sum of stable subbundles (and that ω_0 must be compatible too). We are thus led to believe in the following general statement.

Conjecture 1. *A projective bundle $(M, J) = P(E)$ over a compact curve of genus ≥ 2 admits an extremal Kähler metric in some Kähler class if and only if E decomposes as a direct sum of stable subbundles.*

Remark 3. This conjecture turns out to be true in the case when E is of rank 2 and Σ is a curve of any genus, cf. [7] for an overview.

A partial answer to Conjecture 1 is given by the following result which deals with Kähler classes far enough from the boundary of the Kähler cone (noting that $H^2(M, \mathbb{R})$ is 2-dimensional so the Kähler cone up to positive scale is just a ray).

Theorem 2. *Let $p\colon P(E) \to \Sigma$ be a holomorphic projective bundle over a compact complex curve Σ of genus ≥ 2 and $[\omega_\Sigma]$ be a primitive Kähler class on Σ. Then the following are equivalent*:

- *there exists $k_0 \geq 0$ such that for any $k > k_0$ the Kähler class $\Omega_k = 2\pi c_1(\mathcal{O}(1)_E) + kp^*[\omega_\Sigma]$ on $(M, J) = P(E)$ admits an extremal Kähler metric*;
- *E splits as a direct sum of stable subbundles.*

In the case when E decomposes as the sum of at most two indecomposable subbundles,[3] *Conjecture* 1 *is true.*

The proof of Theorem 2, given in §3.4, will be deduced from a general existence theorem established in the much broader framework of *rigid* and *semisimple* toric bundles introduced in [4], whose main features are recalled in Sect. 2 below. As explained in Remark 7, this class of manifolds is closely related to the class of *multiplicity-free* manifolds recently discussed in Donaldson's paper [25]. Our most general existence result can be stated as follows.

Theorem 3. *Let (g, ω) be a compatible Kähler metric on M, where M is a rigid semisimple toric bundle over a CSC local product Kähler manifold (S, g_S, ω_S) with fibres isomorphic to a toric Kähler manifold (W, ω_W, g_W), as defined in Sect. 2. Suppose, moreover, that the fibre W admits a* compatible *extremal Kähler metric. Then, for any $k \gg 0$, the Kähler class $\Omega_k = [\omega] + kp^*[\omega_S]$ admits a compatible extremal Kähler metric.*

[3] This is equivalent to requiring that the automorphism group of $P(E)$ has a maximal torus of dimension ≤ 1.

The terms of this statement, in particular the concept of a *compatible* metric, are introduced in Sect. 2. The proof, given in Sect. 3, uses in a crucial way the stability under small perturbations of existence of compatible extremal metric (Proposition 2) which constitutes the delicate technical part of the paper. Another important consequence of Proposition 2 is the general openness theorem given by Corollary 1.

A non-trivial assumption in the hypotheses of Theorem 3 above is the existence of *compatible* extremal Kähler metric on the (toric) fibre W. This is solved when $W \cong \mathbb{C}P^r$ and $M = P(E)$ with E being holomorphic vector bundle of rank $r+1$, which is the sum of $\ell+1$ projectively-flat hermitian bundles, as a consequence of the fact that the Fubini–Study metric on $\mathbb{C}P^r$ admits a non-trivial hamiltonian 2-form of order $\ell \leq r$ (cf.[3]). We thus derive in §3.3 the following existence result.

Theorem 4. *Let $p\colon P(E) \to S$ be a holomorphic projective bundle over a compact Kähler manifold (S, J_S, ω_S). Suppose that (S, J_S, ω_S) is covered by the product of constant scalar curvature Kähler manifolds (S_j, ω_j), $j \in \mathcal{I}$ (a finite index set), and $E = \bigoplus_{i=0}^{\ell} E_i$ is the direct sum of projectively-flat hermitian bundles. Suppose further that for each i $c_1(E_i)/\mathrm{rk}(E_i) - c_1(E_0)/\mathrm{rk}(E_0)$ pulls back to $\sum_{j\in\mathcal{I}} p_{ji}[\omega_j]$ on $\prod_{j\in\mathcal{I}} S_j$ (for some constants p_{ji}). Then there exists $k_0 \geq 0$ such that for any $k > k_0$ the Kähler class $\Omega_k = 2\pi c_1(\mathcal{O}(1)_E) + kp^*[\omega_S]$ admits a compatible extremal Kähler metric.*

Remark 4. Theorem 4 is closely related to the results of Y.-J. Hong in [36, 37] who proves, under a technical assumption on the automorphism group of S, that for any hermitian–Einstein (i.e., polystable) bundle E over a CSC Kähler manifold S, the Kähler class $\Omega_k = 2\pi c_1(\mathcal{O}(1)_E) + kp^*[\omega_S]$ for $k \gg 0$ admits a compatible CSC Kähler metric if and only if the corresponding Futaki invariant $\mathfrak{F}_{\Omega_k}$ vanishes. However, in the case when E is not simple (i.e., has automorphisms other than multiples of identity) the condition $\mathfrak{F}_{\Omega_k} \equiv 0$ is not in general satisfied for these classes, see [5, §3.4 & 4.2] for specific examples. Thus, studying the existence of extremal rather than CSC Kähler metrics in Ω_k is essential. Another useful remark is that although the hypothesis in Theorem 4 that E is the sum of *projectively-flat* hermitian bundles over S is rather restrictive when S is not a curve, the above result suggests a general setting for the existence of extremal Kähler metrics in $\Omega_k = 2\pi c_1(\mathcal{O}(1)_E) + kp^*[\omega_S]$ for $k \gg 0$, where E is a direct sum of stable bundles (with not necessarily equal slopes) over a CSC Kähler base S.

In the concluding Sect. 4, we develop further our approach by extending the leading conjectures [21, 67] about existence of extremal Kähler metrics on toric varieties to the more general context of *compatible* Kähler metrics that we consider in this paper. Thus motivated, we explore in a greater detail examples when M is a projective plane bundle over a compact complex curve Σ. We show that when the genus of Σ is greater than 1, Kähler classes close to the boundary of the Kähler cone of M do not admit any extremal Kähler metric. In Appendix A, we introduce the notion of a *compatible extremal almost Kähler metric* (the existence of which is conjecturally equivalent to the existence of a genuine extremal Kähler metric) and show that if the genus of Σ is 0 or 1, then *any* Kähler class on M admits an explicit compatible extremal almost Kähler metric.

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1. Proof of Theorem 1

As we have already noted in Remark 1, the 'if' part of the theorem is well-known. So we deal with the 'only if' part.

Let $(M,J) = P(E)$, where $E \to \Sigma$ is a holomorphic vector bundle of rank m over a compact curve Σ of genus ≥ 2. We want to prove that E is polystable if (M,J) admits a CSC Kähler metric ω. Also, by Remark 1, we will be primarily concerned with the case when the connected component of the identity $\mathrm{Aut}_0(M,J)$ of the automorphisms group of (M,J) is not trivial. Note that, as the normal bundle to the fibres of $P(E) \to \Sigma$ is trivial and the base is of genus ≥ 2, the group $\mathrm{Aut}_0(M,J)$ reduces to $H^0(\Sigma, PGL(E))$, the group of fibre-preserving automorphisms of E, with Lie algebra $\mathfrak{h}(M,J) \cong H^0(\Sigma, \mathfrak{sl}(E))$. As any holomorphic vector field in $\mathfrak{h}(M,J)$ has zeros, the Lichnerowicz–Matsushima theorem [49, 55] implies $\mathfrak{h}(M,J) = \mathfrak{i}(M,g) \oplus J\mathfrak{i}(M,g)$, where $\mathfrak{i}(M,g)$ is the Lie algebra of Killing vector fields of (M,J,ω). Thus, $\mathrm{Aut}_0(M,J) \neq \{\mathrm{Id}\}$ iff $\mathfrak{h}(M,J) \neq \{0\}$ iff $\mathfrak{i}(M,g) \neq \{0\}$. We will fix from now on a maximal torus $\mathbb{T}$ (of dimension ℓ) in the connected component of the group of isometries of (M,J,ω). Note that $\mathbb{T}$ is a maximal torus in $\mathrm{Aut}_0(M,J)$ too, by the Lichnerowicz–Matsushima theorem cited above.

We will complete the proof in three steps, using several lemmas.

We start with the following elementary but useful observation which allows us to relate a maximal torus $\mathbb{T} \subset \mathrm{Aut}_0(M,J)$ with the structure of E.

Lemma 1. *Let $(M,J) = P(E) \to S$ be a projective bundle over a compact complex manifold S, and suppose that the group $H^0(S, PGL(E))$ of fibre-preserving automorphisms of (M,J) contains a circle S^1. Then E decomposes as a direct sum $E = \bigoplus_{i=0}^{\ell} E_i$ of subbundles E_i with $\ell \geq 1$, such that S^1 acts on each factor E_i by a scalar multiplication.*

In particular, any maximal torus $\mathbb{T} \subset H^0(S, PGL(E))$ arises from a splitting as above, with E_i indecomposable and $\ell = \dim(\mathbb{T})$.

Proof. Any S^1 in $H^0(S, PGL(E))$ defines a $\mathbb{C}^\times$ holomorphic action on (M,J), generated by an element $\Theta \in \mathfrak{h}(M,J) \cong H^0(S, \mathfrak{sl}(E))$. For any $x \in S$, $\exp(t\Theta(x))$, $t \in \mathbb{C}$ generates a $\mathbb{C}^\times$ subgroup of $SL(E_x)$ and so $\Theta(x)$ must be diagonalizable. The coefficients of the characteristic polynomial of $\Theta(x)$ are holomorphic functions of $x \in S$, and therefore are constants. It then follows that Θ gives rise to a direct sum decomposition $E = \bigoplus_{i=0}^{\ell} E_i$ where E_i correspond to the eigenspaces of Θ at each fibre.

The second part of the lemma follows easily. □

Because of this result and the discussion preceding it, we consider the decomposition $E = \bigoplus_{i=0}^{\ell} E_i$ as a direct sum of indecomposable subbundles over Σ, corresponding to a fixed, maximal ℓ-dimensional torus $\mathbb{T}$ in the connected component of the isometry group of (g,J,ω). We note that the isometric action of $\mathbb{T}$ is *hamiltonian* as $\mathbb{T}$ has fixed points (on any fibre).

Our second step is to understand the condition that the Futaki invariant [30], with respect to the Kähler class $\Omega = [\omega]$ on (M,J), restricted to the generators of $\mathbb{T}$ is zero. Hodge theory implies that any (real) holomorphic vector field with zeros on a compact Kähler $2m$-manifold (M,J,ω,g) can be written as $X = \mathrm{grad}_\omega f - J\mathrm{grad}_\omega h$, where $f + ih$ is a complex-valued smooth function on M of zero integral (with respect to the volume form ω^m), called the *holomorphy potential* of X, and where $\mathrm{grad}_\omega f$ stands for the hamiltonian vector field associated to a smooth function f via ω. Then the (real) Futaki invariant associates to X the real number

$$\mathfrak{F}_\omega(X) = \int_M f\,Scal_g\ \omega^m,$$

where $Scal_g$ is the scalar curvature of g. Futaki shows [30] that $\mathfrak{F}_\omega(X)$ is independent of the choice of ω within a fixed Kähler class Ω, and that (trivially) $\mathfrak{F}_\omega(X) = 0$ if Ω contains a CSC Kähler metric. A related observation will be useful to us: with a fixed symplectic form ω, the Futaki invariant is independent of the choice of a compatible integrable almost complex structure within a path component.

Lemma 2. *Let J_t be a smooth family of integrable almost-complex structures compatible with a fixed symplectic form ω, which are invariant under a compact group G of symplectomorphisms acting in a hamiltonian way on the compact symplectic manifold (M, ω). Denote by $\mathfrak{g}_\omega \subset C^\infty(M)$ the finite dimensional vector space of smooth functions f such that $X = \mathrm{grad}_\omega f \in \mathfrak{g}$, where $\mathfrak{g}$ denotes the Lie algebra of G.*[4] *Then the L^2-orthogonal projection of the scalar curvature $Scal_{g_t}$ of (J_t, ω, g_t) to $\mathfrak{g}_\omega$ is independent of t.*

Proof. By definition, any $f \in \mathfrak{g}_\omega$ defines a vector field $X = \mathrm{grad}_\omega f$ which is in $\mathfrak{g}$, and is therefore Killing with respect to any of the Kähler metrics $g_t = \omega(\cdot, J_t\cdot)$. To prove our claim, we have to show that $\int_M f Scal_{g_t}\omega^m$ is independent of t. Using the standard variational formula for scalar curvature (see e.g. [11, Thm. 1.174]), we compute

$$\frac{d}{dt}Scal_{g_t} = \Delta(\mathrm{tr}_{g_t}h) + \delta\delta h - g_t(r, h) = \delta\delta h, \tag{1}$$

where h denotes $\frac{d}{dt}g_t$, while Δ, δ and r are the riemannian laplacian, the codifferential and the Ricci tensor of g_t, respectively. Note that to get the last equality, we have used the fact that h is J_t-anti-invariant (as all the J_t's are compatible with ω) while the metric and the Ricci tensor are J_t-invariant (on any Kähler manifold). Integrating against f, we obtain

$$\frac{d}{dt}\Big(\int_M f Scal_{g_t}\omega^m\Big) = \int_M (\delta\delta h) f\omega^m = \int_M g_t(h, Ddf)\omega^m,$$

where D is the Levi–Civita connection of g_t; however, as f is a Killing potential with respect to the Kähler metric (g_t, J_t), it follows that Ddf is J_t-invariant, and therefore $\int_M f Scal_{g_t}\omega^m$ is independent of t. □

Remark 5. One can extend Lemma 2 to any smooth family of (not necessarily integrable) G-invariant almost complex structures J_t compatible with ω. Then, as shown in [47], the L^2-projection to $\mathfrak{g}_\omega$ of the *hermitian scalar curvature* of the almost Kähler metric (ω, J_t) (see Appendix A for a precise definition) is independent of t. This gives rise to a *symplectic* Futaki invariant associated to a compact subgroup G of the group of hamiltonian symplectomorphisms of (M, ω).

Lemma 2 will be used in conjunction with the Narasimhan–Ramanan approximation theorem (see [57, Prop. 2.6] and [58, Prop. 4.1]), which implies that any holomorphic vector bundle E over a compact curve Σ of genus ≥ 2 can be included in an analytic family of vector bundles $E_t,\ t \in D_\varepsilon$ (where $D_\varepsilon = \{t \in \mathbb{C},\ |t| < \varepsilon\}$) over Σ, such that $E_0 := E$ and E_t is stable for $t \neq 0$. Such a family will be referred to in the sequel as a *small stable deformation* of E.

Lemma 3. *Suppose that the vector bundle $E = U \oplus V \to \Sigma$ splits as a direct sum of two subbundles, U and V. Consider the holomorphic S^1-action on $(M, J) = P(E)$, induced by fibrewise scalar multiplication by $\exp(i\theta)$ on V, and let $X \in \mathfrak{h}(M, J)$ be the (real) holomorphic vector field generating this action. Then the Futaki invariant of X with respect to some (and therefore any) Kähler class Ω on (M, J) vanishes if and only if U and V have the same slope.*

Proof. We take some Kähler form ω on $(M, J) = P(E)$ and, by averaging it over S^1, we assume that ω is S^1-invariant. As the S^1-action has fixed points, the corresponding real

[4] We will tacitly identify throughout the Lie algebra $\mathfrak{g}$ of a group G acting effectively on M with the Lie algebra of vector fields generated by the elements of $\mathfrak{g}$.

vector field X is J-holomorphic and ω-hamiltonian, i.e., $X = \mathrm{grad}_\omega f$ for some smooth function f with $\int_M f\omega^m = 0$.

We now consider small stable deformations $U_t, V_t,\ t \in D_\varepsilon$ of U and V, and put $E_t = U_t \oplus V_t$. Considering the projective bundle $P(E_t)$, we obtain a non-singular Kuranishi family (M, J_t) with $J_0 = J$. By the Kodaira stability theorem (see e.g. [41]) one can find a smooth family of Kähler metrics (ω_t, J_t) with $\omega_0 = \omega$. Using the vanishing of the Dolbeault groups $H^{2,0}(M, J_t) = H^{0,2}(M, J_t) = 0$, Hodge theory implies that by decreasing the initial ε if necessary, we can assume $[\omega_t] = [\omega]$ in $H^2_{dR}(M)$. Note that any J_t is S^1-invariant so, by averaging over S^1, we can also assume that ω_t is S^1-invariant. Applying the equivariant Moser lemma, one can find S^1-equivariant diffeomorphisms, Φ_t, such that $\Phi_t^*\omega_t = \omega$. Considering the pullback of J_t by Φ_t, the upshot of this construction is that we have found a smooth family of integrable complex structures J_t such that: (1) each J_t is compatible with the fixed symplectic form ω and is S^1-invariant; (2) $J_0 = J$; (3) for $t \neq 0$, the complex manifold (M, J_t) is equivariantly biholomorphic to $P(U_t \oplus V_t) \to \Sigma$ with U_t and V_t stable (and therefore projectively-flat) hermitian bundles.

If U and V have equal slopes, then $E_t = U_t \oplus V_t$ becomes polystable for $t \neq 0$, and (M, J_t) has a CSC Kähler metric in each Kähler class. It follows that the Futaki invariant of X on (M, J_t, ω) is zero for $t \neq 0$.

Conversely, if U and V have different slopes, it is shown in [5, §3.2] that the Futaki invariant of X is different from zero for any Kähler class on (M, J_t), $t \neq 0$.

We conclude using Lemma 2. □

This lemma shows that all the factors in the decomposition $E = \bigoplus_{i=0}^{\ell} E_i$ must have equal slope, should a CSC Kähler metric exists. As in the proof of Lemma 3, we consider small stable deformations $E_i(t)$ of E_i and our assumption for the slopes insures that $E(t) = \bigoplus_{i=0}^{\ell} E_i(t)$ is polystable for $t \neq 0$; furthermore, by acting with $\mathbb{T}$-equivariant diffeomorphisms, we obtain a smooth family of $\mathbb{T}$-invariant complex structures J_t compatible with ω, such that for $t \neq 0$, the complex manifold (M, J_t) has a locally-symmetric CSC Kähler metric in each Kähler class; by the uniqueness of the extremal Kähler metrics modulo automorphisms [15, 52], *any* extremal Kähler metric on (M, J_t) is locally-symmetric when $t \neq 0$. The third step in the proof of Theorem 1 is then to show that the initial CSC Kähler metric (J_0, ω) must be locally symmetric too. This follows from the next technical result which generalizes arguments of [46, 29] and fits with the general approach to the deformation theory of CSC Kähler metrics given in [68, Thm. 2].

Lemma 4. *Let J_t be a smooth family of integrable almost-complex structures compatible with a symplectic form ω on a compact manifold M, which are invariant under a torus $\mathbb{T}$ of hamiltonian symplectomorphisms of (M, ω). Suppose, moreover, that (J_0, ω) define an extremal Kähler metric and that $\mathbb{T}$ is a maximal torus in the reduced automorphism group of (M, J_0). Then there exists a smooth family of extremal Kähler metrics (J_t, ω_t, g_t), defined for sufficiently small t, such that $\omega_0 = \omega$ and $[\omega_t] = [\omega]$ in $H^2_{dR}(M)$.*

Proof. Recall that [32, 45] on any compact Kähler manifold (M, J), the *reduced* automorphism group, $\widetilde{\mathrm{Aut}}_0(M, J)$, is the identity component of the kernel of the natural group homomorphism from $\mathrm{Aut}_0(M, J)$ to the *Albanese torus* of (M, J); it is also the connected closed subgroup of $\mathrm{Aut}_0(M, J)$, whose Lie algebra $\mathfrak{h}_0(M, J) \subset \mathfrak{h}(M, J)$ is the ideal of holomorphic vector fields with zeros.

We denote by $\mathfrak{t}$ the Lie algebra of $\mathbb{T}$ and by $\mathfrak{h}$ (resp. $\mathfrak{h}_0$) the Lie algebra of the complex automorphism group (resp. reduced automorphism group) of the central fibre (M, J_0). As $\mathbb{T}$ acts in a hamiltonian way, we have $\mathfrak{t} \subset \mathfrak{h}_0$. By assumption, $\mathfrak{t}$ is a maximal abelian subalgebra of $\mathfrak{i}_0(M, g_0) = \mathfrak{i}(M, g_0) \cap \mathfrak{h}_0$, where $\mathfrak{i}(M, g_0)$ is the Lie algebra of Killing vector fields of (M, J_0, ω, g_0).

As in the Lemma 2 above, we let $\mathfrak{t}_\omega \subset C^\infty(M)$ be the finite dimensional space of smooth functions which are hamiltonians of elements of $\mathfrak{t}$. As the Kähler metric (J_0, ω, g_0) is extremal (by assumption), its scalar curvature $Scal_{g_0}$ is hamiltonian of a Killing vector field $X = \mathrm{grad}_\omega(Scal_{g_0}) \in \mathfrak{i}_0(M, g_0)$. Clearly, such a vector field is central, so $X \in \mathfrak{t}$ (by the maximality of $\mathfrak{t}$) and therefore $Scal_{g_0} \in \mathfrak{t}_\omega$.

For any $\mathbb{T}$-invariant Kähler metric $(\tilde{J}, \tilde{\omega}, \tilde{g})$ on M, we denote by $\mathfrak{t}_{\tilde{\omega}}$ the corresponding space of Killing potentials of elements of $\mathfrak{t}$ (noting that any $X \in \mathfrak{t}$ has zeros, so that $\mathbb{T}$ belongs to the reduced automorphism group of $(M, \tilde{J})$), and by $\Pi_{\tilde{\omega}}$ the L^2-orthogonal projection of smooth function to $\mathfrak{t}_{\tilde{\omega}}$, with respect to the volume form $\tilde{\omega}^m$. Obviously, if the scalar curvature $Scal_{\tilde{g}}$ of $\tilde{g}$ belongs to $\mathfrak{t}_{\tilde{\omega}}$, then $\tilde{g}$ is extremal.

Following [46], let $C^\infty_\perp(M)^{\mathbb{T}}$ denote the Fréchet space of $\mathbb{T}$-invariant smooth functions on M, which are L^2-orthogonal (with respect to the volume form ω^m) to $\mathfrak{t}_\omega$, and let $\mathcal{U}$ be an open set in $\mathbb{R} \times C^\infty_\perp(M)^{\mathbb{T}}$ of elements (t, f) such that $\omega + dd^c_t f$ is Kähler with respect to J_t (here d^c_t denotes the d^c-differential corresponding to J_t). We then consider the map $\Psi \colon \mathcal{U} \to \mathbb{R} \times C^\infty_\perp(M)^{\mathbb{T}}$, defined by

$$\Psi(t, f) = \Big(t, (\mathrm{Id} - \Pi_\omega) \circ (\mathrm{Id} - \Pi_{\tilde{\omega}})(Scal_{\tilde{g}})\Big),$$

where $\tilde{\omega} := \omega + dd^c_t f$ and $Scal_{\tilde{g}}$ is the scalar curvature of the Kähler metric $\tilde{g}$ defined by $(J_t, \tilde{\omega})$. One can check that this map is C^1 and compute (as in [45], by also using (1)) that its differential at $(0, 0) \in \mathcal{U}$ is

$$(T_{(0,0)}\Psi)(t, f) = (t, t\delta\delta h - 2\delta\delta(Ddf)^-),$$

where D and δ are respectively the Levi–Civita connection and the codifferential of g_0, $h = \big(\frac{dg_t}{dt}\big)_{t=0}$ and $(Ddf)^-$ denotes the J_0-anti-invariant part of Ddf. Note that $L(f) := \delta\delta((Ddf)^-)$ is a fourth order (formally) self-adjoint $\mathbb{T}$-invariant elliptic linear operator (known also as the *Lichnerowicz* operator, see e.g. [32]). When acting on smooth functions, L annihilates $\mathfrak{t}_\omega$ (because any Killing potential f satisfies $(Ddf)^- = 0$). It then follows that L leaves $C^\infty_\perp(M)^{\mathbb{T}}$ invariant and, by standard elliptic theory, we obtain an L^2-orthogonal splitting $C^\infty_\perp(M)^{\mathbb{T}} = \ker(L) \oplus \mathrm{im}(L)$. However, any smooth $\mathbb{T}$-invariant function f in $\ker(L)$ gives rise to a Killing field $X = \mathrm{grad}_\omega f$ in the centralizer of $\mathfrak{t} \subset \mathfrak{i}_0(M, g_0)$. As $\mathfrak{t}$ is a maximal abelian subalgebra of $\mathfrak{i}_0(M, g_0)$ we must have $X \in \mathfrak{t}$, i.e., $f \in \mathfrak{t}_\omega$. It follows that the kernel of L restricted to $C^\infty_\perp(M)^{\mathbb{T}}$ is trivial, and therefore L is an isomorphism of the Fréchet space $C^\infty_\perp(M)^{\mathbb{T}}$.

This understood, we are in position to apply standard arguments, using the implicit function theorem for the extension of Ψ to the Sobolev completion $L^{2,k}_\perp(M)^{\mathbb{T}}$ (with $k \gg 1$) of $C^\infty_\perp(M)^{\mathbb{T}}$, together with the regularity result for extremal Kähler metrics, precisely as in [45, 46, 29]. We thus obtain a family (t, ω_t) of smooth, $\mathbb{T}$-invariant extremal Kähler metrics (J_t, ω_t) (defined for t in a small interval about 0) which converge to the initial extremal Kähler metric (J_0, ω) (in any Sobolev space $L^{2,k}(M)$, $k \gg 1$, and hence, by the Sobolev embedding, in $C^\infty(M)$). $\square$

The uniqueness argument thus also applies at $t = 0$, and the initial metric is locally symmetric. We can now conclude the proof of Theorem 1 by a standard argument using the de Rham decomposition theorem (see [28, Lemma 8] and [44]). This realizes the fundamental group of Σ as a discrete subgroup group of isometries of the hermitian symmetric space $\mathbb{C}P^{m-1} \times \mathbb{H}$ and thus defines a projectively flat structure on $P(E) \to \Sigma$.

2. Rigid toric bundles and the generalized Calabi construction

In this section, we recall the notions of *rigid* and *semisimple* isometric hamiltonian torus actions on compact Kähler manifolds, as well the *generalized Calabi construction* of *compatible* Kähler metrics on such manifolds [4]. Compatible Kähler metrics on rigid toric bundles over a semisimple base provide a framework to search for extremal Kähler metrics which parallels (and extends) the theory of toric extremal Kähler metrics developed in [21, 22, 24]. We shall apply the construction to projective bundles of the form $P(E_0 \oplus \cdots \oplus E_\ell) \to S$, where E_i is a projectively-flat hermitian bundle over a Kähler manifold (S, ω_S), and prove the existence of compatible extremal Kähler metrics in classes sufficiently far from the boundary of the Kähler cone, cf. Theorems 3 and 4.

2.1. **Rigid and semisimple torus actions.**
Most of the material in this section is taken from [4, Sect. 2] to which we refer the Reader for further details.

Definition 1. Let (M, g, J, ω) be a connected Kähler $2m$-manifold with an effective isometric hamiltonian action of an ℓ-torus $\mathbb{T}$ with momentum map $z\colon M \to \mathfrak{t}^*$. We say the action is *rigid* if for all $x \in M$, $R_x^* g$ depends only on $z(x)$, where $R_x\colon \mathbb{T} \to \mathbb{T}\cdot x \subset M$ is the orbit map.

Equivalently, the action is rigid if, for any two generators X_ξ, X_η (with $\xi, \eta \in \mathfrak{t}$) the smooth function $g(X_\xi, X_\eta)$ is constant on the levels of the momentum map z.

The standard m-torus action on a toric Kähler $2m$-manifold is rigid. If M is compact, as we shall assume henceforth, many features of toric Kähler manifolds are shared by rigid torus actions. In particular:

- the image of M under the momentum map z is a *Delzant* polytope $\Delta \subset \mathfrak{t}^*$ [4, Prop. 4];
- the points in the interior Δ^0 are the regular values of z;
- the complex (stable) quotient $\hat{S}$ of M by the complexified torus $\mathbb{T}^c$ (which is *a priori* a $2(m-\ell)$-dimensional complex orbifold) is smooth [4, Prop. 5];
- $M^0 := z^{-1}(\Delta^0)$ is a principal $\mathbb{T}^c$ bundle over $\hat{S}$.

The Delzant construction [16], associates to Δ (and hence M) a smooth compact symplectic 2ℓ-manifold (V, ω_V), with a toric $\mathbb{T}$-action, and there are complex structures J_V making V into a toric variety under the action of the complexified torus $\mathbb{T}^c$.

The inverse image under z of the union of the codimension one faces of Δ is a (reducible) subvariety Ξ of M such that the blow-up of M along Ξ is $\mathbb{T}^c$-equivariantly biholomorphic to the toric bundle $\hat{M} := M^0 \times_{\mathbb{T}^c} V \to \hat{S}$. In simple examples, Ξ is a divisor and $\hat{M}$ is biholomorphic to M. In general, M, like $\hat{M}$, is a $\mathbb{T}^c$-equivariant compactification of the principal $\mathbb{T}^c$-bundle M^0 over $\hat{S}$; we shall therefore abuse language and refer to M as a *rigid toric bundle.*

The general form of M will be used in applications to projective bundles, but only in the special case that the (rigid) torus action is also "semisimple" in the following sense.

Definition 2. An isometric hamiltonian torus action on a Kähler manifold (M, g, J, ω) with momentum map z and complex (stable) quotient[5] $\hat{S}$ is *semisimple* if at any regular value of z, the z-derivative of the family $\omega_{\hat{S}}(z)$ of Kähler forms (induced by the symplectic quotient at momentum level z) is parallel and diagonalizable with respect to $\omega_{\hat{S}}(z)$.

This definition essentially implies that the universal cover of $\hat{S}$ is a Kähler product $\prod_{j\in\mathcal{I}}(S_j, \omega_j)$ ($\mathcal{I}$ finite) such that the pullback of $\omega_{\hat{S}}(z)$ is homothetic to ω_j on each factor by functions $f_j(z)$. The distributions on $\hat{S}$ induced by this product structure are simultaneous eigenspaces of $\omega_{\hat{S}}(z)$, and in the rigid case the functions f_j are affine in z.

[5]This quotient is well-defined within a connected component of the regular values of z.

In this case each component of Ξ with codimension $d+1>1$ in M corresponds to a $\mathbb{C}P^d$ factor (with constant holomorphic sectional curvature) in the Kähler product.

Conversely, these features lead to a general construction of rigid toric bundles with semisimple torus actions. Let $\mathbb{T}=\mathfrak{t}/2\pi\Lambda$ be a real (compact) ℓ-torus, with Lie algebra $\mathfrak{t}$, let Δ be a Delzant polytope in the dual space $\mathfrak{t}^*$, with codimension one faces $F_b : b\in\mathcal{B}$ and primitive inward normals $u_b\in\mathfrak{t}$, let $(V,J_V,\omega_V,\mathbb{T})$ be a toric Kähler 2ℓ-manifold, with momentum polytope Δ, and let S be compact Kähler manifold whose universal cover is a Kähler product $\prod_{a\in\mathcal{A}}(S_a,\omega_a)$ (with finite index set $\mathcal{A}$ disjoint from $\mathcal{B}$).

For each $b\in\mathcal{B}$, fix an integer $d_b\geq 0$, and let $\hat S$ be the total space of a fibre product (for $b\in\mathcal{B}$) of flat unitary $\mathbb{C}P^{d_b}$-bundles over S. The universal cover of $\hat S$ is then a Kähler product $\prod_{j\in\mathcal{I}}(S_j,\omega_j)$, where $\mathcal{I}=\mathcal{A}\cup\{b\in\mathcal{B} : d_b>0\}$ and for each $b\in\mathcal{B}$, $S_b=\mathbb{C}P^{d_b}$ with Fubini–Study metric ω_b normalized so that $[\omega_b]=2\pi c_1(\mathcal{O}_{\mathbb{C}P^{d_b}}(1))$ (i.e., ω_b has scalar curvature $2d_b(d_b+1)$, or equivalently, holomorphic sectional curvature 2).

Let $\hat P$ be a principal $\mathbb{T}$-bundle over $\hat S$, such that $-2\pi c_1(\hat P)\in H^2(\hat S,\mathfrak{t})$, is *diagonalizable* with respect to the local product structure of $\hat S$, i.e., is of the form $\sum_{j\in\mathcal{I}}[\omega_j]\otimes p_j=\sum_{a\in\mathcal{A}}[\omega_a]\otimes p_a+\sum_{b\in\mathcal{B}}[\omega_b]\otimes u_b$, where the p_j are (constant) elements of $\mathfrak{t}$ with $p_b=u_b$ for $b\in\mathcal{B}$. Let $\hat M=\hat P\times_{\mathbb{T}}V$ be the associated toric bundle over $\hat S$.

It remains to describe the blow-down map $\hat M\to M$, which is characterized by its behaviour over each fibre of $\hat S\to S$. Each such fibre is biholomorphic to $S'=\prod_{b\in\mathcal{B}}\mathbb{C}P^{d_b}$, on which $\hat P$ restricts to a principal $\mathbb{T}$-bundle P' with $c_1(P')=\sum_{b\in\mathcal{B}}c_1(\mathcal{O}_{\mathbb{C}P^{d_b}}(-1))\otimes u_b$. The blow-down W of $\hat W:=P'\times_{\mathbb{T}}V$ over S' may be described as a *restricted toric quotient* [4]. For this, recall that the Delzant construction realizes V as a symplectic quotient of $\mathbb{C}^{\mathcal{B}}:=\prod_{b\in\mathcal{B}}\mathbb{C}$ by a subgroup G of the diagonal torus action; now replace $\mathbb{C}^{\mathcal{B}}$ with $\prod_{b\in\mathcal{B}}\mathbb{C}^{d_b+1}$, which has a diagonal action of the same torus (by scalar multiplication on each factor), hence also an action of G. The symplectic quotient by G is the blow-down W. We refer to [4, §1.6] for details.[6]

Definition 3. A blow-down of $\hat M=\hat P\times_{\mathbb{T}}V\to\hat S$ is a (locally trivial) fibre bundle M over S which fits into a $\mathbb{T}$-equivariant commutative diagram

$$\begin{tikzcd} \hat M=\hat P\times_{\mathbb{T}}V \arrow[r] \arrow[d] & \hat S \arrow[d] \\ M \arrow[r] & S, \end{tikzcd}$$

and is $\mathbb{T}$-equivariantly biholomorphic to $\hat W\to W$ over each fibre of $\hat S\to S$.

A complex manifold $(M,J,\mathbb{T})$ obtained in this way will be called a *rigid toric bundle over a semisimple base*.

In [4, §2.5], this blow-down construction was discussed in terms of the universal covers of $\hat M\to M$ and $\hat S\to S$, the latter cover being a trivial S' bundle. More generally, if P is a principal $\mathbb{T}$-bundle on S whose first Chern class satisfies $-2\pi c_1(P)=\sum_{a\in\mathcal{A}}[\omega_a]\otimes p_a$, then $\hat M=\hat P\times_{\mathbb{T}}V\cong P\times_{\mathbb{T}}\hat W$ and so $M=P\times_{\mathbb{T}}W$ satisfies Definition 3.

2.2. **Projective bundles as rigid toric bundles.** The constructions of the previous paragraph will now be specialized to the prominent case of projective bundles, when the Delzant polytope Δ is a simplex in $\mathfrak{t}^*\cong\mathbb{R}^\ell$, with codimension one faces F_j, $j\in\mathcal{B}=\{0,1,\dots\ell\}$. The associated complex toric variety V is then $\mathbb{C}P^\ell$ and $\hat M$ is $\mathbb{T}^c$-equivariantly biholomorphic to a $\mathbb{C}P^\ell$-bundle over $\hat S$; since $\hat M$ comes from a *principal* $\mathbb{T}^c$-bundle, it has the form $P(\mathcal{L}_0\oplus\cdots\oplus\mathcal{L}_\ell)\to\hat S$, where $\mathcal{L}_i$ are hermitian holomorphic line bundles (the $\mathbb{T}^c$ action is induced by scalar multiplication on $\mathcal{L}_i$).

[6]A simple illustration of this construction is $W=\mathbb{C}P^2$ seen as a fibrewise S^1-equivariant blow-down of the first Hirzebruch surface $\hat W=P(\mathcal{O}\oplus\mathcal{O}(-1))\to\mathbb{C}P^1$.

The blow-down of $\hat{M}$ is encoded by the realization of $\hat{S}$ as a fibre product of flat projective unitary $\mathbb{C}P^{d_b}$-bundles over a Kähler manifold S. We shall only consider flat projective bundles of the form $P(E)$, where E is a projectively-flat hermitian vector bundle over S (the obstruction to the existence of such an E is a torsion element of $H^2(S, \mathcal{O}^*)$; in particular, it always exists if S is a Riemann surface). We thus write $\hat{S} = P(E_0) \times_S \cdots \times_S P(E_\ell) \to S$, where each $E_i \to S$ is a projectively-flat hermitian bundle of rank $d_i + 1$, and we assume that $c_1(E_i)/(d_i + 1) - c_1(E_0)/(d_0 + 1)$ pulls back to $\sum_{a\in\mathcal{A}} p_{ia}[\omega_a]$ on the covering space $\prod_{a\in\mathcal{A}}(S_a, \omega_a)$ of S.

In this case, $\hat{M} = P\big(\mathcal{O}(-1)_{E_0} \oplus \cdots \oplus \mathcal{O}(-1)_{E_\ell}\big) \to \hat{S}$, where $\mathcal{O}(-1)_{E_i}$ is the (fibrewise) tautological line bundle over $P(E_i) \to S$ — trivial over the other factors of $\hat{S}$ — whereas $M = P\big(E_0 \oplus \cdots \oplus E_\ell\big) \to S$: in each fibre over S, the blow-down is biholomorphic to the map from the total space of $P\big(\bigoplus_{j=0}^{\ell} \mathcal{O}(-1)_{\mathbb{C}^{d_j+1}}\big) \to \prod_{j=0}^{\ell} \mathbb{C}P^{d_j+1}$ to $\mathbb{C}P^d$, given by a direct sum decomposition $\mathbb{C}^{d+1} = \bigoplus_{j=0}^{\ell} \mathbb{C}^{d_j+1}$, cf., [4].

2.3. **The generalized Calabi construction.** A compact Kähler manifold with rigid semisimple isometric hamiltonian action of an ℓ-torus $\mathbb{T}$ is biholomorphic to a rigid toric bundle (M, J) over a semisimple base. According to [4, Thm. 2], on any such rigid toric bundles, the *compatible* Kähler metrics (g, ω) (for which the ℓ-torus action is rigid, semisimple, isometric and hamiltonian) may be described using a *generalized Calabi construction* with three main ingredients — only two if there is no blow-down.

The first ingredient is the choice of a compatible $\mathbb{T}$-invariant Kähler metric g_V on the symplectic toric manifold $(V, \omega_V, \mathbb{T})$ with Delzant polytope Δ. The description of such metrics is well known (see e.g. [1, 2, 22, 34]). On the union V^0 of the generic $\mathbb{T}$ orbits (the inverse image of the interior Δ^0 under the momentum map $z\colon V \to \Delta \subset \mathfrak{t}^*$) there are "angular coordinates" $t\colon V^0 \to \mathfrak{t}/2\pi\Lambda$, defined up to an additive constant, with $dt(JX_\xi) = 0$ and $dt(X_\xi) = \xi$ (the orthogonal distribution to the $\mathbb{T}$ orbits is integrable). The action-angle coordinates (z, t) identify each tangent space with $\mathfrak{t} \oplus \mathfrak{t}^*$ and the symplectic form on V is $\omega_V = \langle dz \wedge dt\rangle$, where the angle brackets denote contraction of $\mathfrak{t}$ and $\mathfrak{t}^*$. Compatible Kähler metrics may then be written in the form

$$g_V = \langle dz, \mathbf{G}, dz\rangle + \langle dt, \mathbf{H}, dt\rangle, \tag{2}$$

where $\mathbf{G}$ is a positive definite $S^2\mathfrak{t}$-valued function on Δ^0, $\mathbf{H}$ is its inverse in $S^2\mathfrak{t}^*$ and $\langle\cdot,\cdot,\cdot\rangle$ denotes the pointwise contraction $\mathfrak{t}^* \times S^2\mathfrak{t} \times \mathfrak{t}^* \to \mathbb{R}$ or the dual contraction. The corresponding almost complex structure is defined by

$$J dt = -\langle \mathbf{G}, dz\rangle \tag{3}$$

from which it follows that J is integrable if and only if $\mathbf{G}$ is the hessian of a function U (the *symplectic potential*) on Δ^0 [34].

In order for such a metric on V^0 to compactify on V, U must satisfy boundary conditions [2, 4, 22], which we describe following [4, Prop. 1]. For any face $F \subset \Delta$, let $\mathfrak{t}_F \subset \mathfrak{t}$ denote the vector subspace spanned by inward normals $u_i \in \mathfrak{t}$ to codimension one faces of Δ containing F; the codimension of $\mathfrak{t}_F$ is dimension of F, and the annihilator ${\mathfrak{t}_F}^0 \subset \mathfrak{t}^*$ is isomorphic to $(\mathfrak{t}/\mathfrak{t}_F)^*$. *A smooth function U on Δ^0 corresponds to a $\mathbb{T}$-invariant, ω_V-compatible Kähler metric g_V via* (2) *if and only if the $S^2\mathfrak{t}^*$-valued function $\mathbf{H} = \mathrm{Hess}(U)^{-1}$ on Δ^0 satisfies the following conditions*:

- [smoothness] $\mathbf{H}$ is the restriction to Δ^0 of a smooth $S^2\mathfrak{t}^*$-valued function on Δ;
- [boundary values] for z in a codimension one face $F_i \subset \Delta$ with inward normal u_i,

$$\mathbf{H}_z(u_i, \cdot) = 0 \qquad \text{and} \qquad (d\mathbf{H})_z(u_i, u_i) = 2u_i, \tag{4}$$

where the differential $d\mathbf{H}$ is viewed as a smooth $S^2\mathfrak{t}^* \otimes \mathfrak{t}$-valued function on Δ;

• [positivity] for any point z in the interior of a face $F \subseteq \Delta$, $\mathbf{H}_z(\cdot,\cdot)$ is positive definite when viewed as a smooth function with values in $S^2(\mathfrak{t}/\mathfrak{t}_F)^*$.

We denote by $\mathcal{S}(\Delta)$ the space of all symplectic potentials U on Δ satisfying these conditions: any $U \in \mathcal{S}(\Delta)$ defines a $\mathbb{T}$-invariant, ω_V-compatible Kähler metric on V.

The second ingredient, nontrivial only if "blow-downs" are present, is to transfer the metric on V to the restricted toric quotient variety W, which is birational to $\hat{W} = P' \times_{\mathbb{T}} V$, where P' is a $\mathbb{T}$-principal bundle over $\prod_{b\in\mathcal{B}} \mathbb{C}P^{d_b}$. For this we choose a connection 1-form θ' on P', with curvature $d\theta' = \sum_{b\in\mathcal{B}} \omega_b \otimes u_b$, where ω_b is the (normalized) Fubini–Study metric on $\mathbb{C}P^{d_b}$ of scalar curvature $2d_b(d_b+1)$, and $u_b \in \mathfrak{t}$ is the inward normal to the codimension one face $F_b \subset \Delta$ (on which $\langle u_b, z\rangle + c_b = 0$). We still denote by $\theta' \in \Omega^1(W^0, \mathfrak{t})$ the induced 1-form on the open dense subset $W^0 := P' \times_{\mathbb{T}} V^0$ of $\hat{W}$. The Kähler structure on W^0 is then defined by

$$\begin{aligned} g_W &= \sum_{b\in\mathcal{B}} (\langle u_b, z\rangle + c_b) g_b + \langle dz, \mathbf{G}, dz\rangle + \langle \theta', \mathbf{H}, \theta'\rangle, \\ \omega_W &= \sum_{b\in\mathcal{B}} (\langle u_b, z\rangle + c_b)\omega_b + \langle dz \wedge \theta'\rangle, \qquad d\theta' = \sum_{b\in\mathcal{B}} \omega_b \otimes u_b, \end{aligned} \tag{5}$$

with $\mathbf{G} = \mathrm{Hess}(U) = \mathbf{H}^{-1}$. By [4, Prop. 2], (g_W, ω_W) extends to a smooth, $\mathbb{T}$-invariant, Kähler structure on W.

The third and final ingredient builds on the second by a similar construction of a Kähler structure on $M^0 = \hat{P} \times_{\mathbb{T}} V^0$, using a connection on $\hat{P}$, with curvature (covered by) $\sum_{a\in\mathcal{A}} \omega_a \otimes p_a + \sum_{b\in\mathcal{B}} \omega_b \otimes u_b$. The restriction of this connection to each fibre of $\hat{S} \to S$ is then isomorphic to (P', θ') over $\prod_{b\in\mathcal{B}} \mathbb{C}P^{d_b}$. We denote by $\hat{\theta} \in \Omega^1(M^0, \mathfrak{t})$ the induced connection 1-form on $M^0 = \hat{P} \times_{\mathbb{T}} V^0$, on which we define a Kähler structure

$$\begin{aligned} g &= \sum_{j\in\mathcal{I}} (\langle p_j, z\rangle + c_j) g_j + \langle dz, \mathbf{G}, dz\rangle + \langle \hat{\theta}, \mathbf{H}, \hat{\theta}\rangle, \\ \omega &= \sum_{j\in\mathcal{I}} (\langle p_j, z\rangle + c_j)\omega_j + \langle dz \wedge \hat{\theta}\rangle, \qquad d\hat{\theta} = \sum_{j\in\mathcal{I}} \omega_j \otimes p_j, \end{aligned} \tag{6}$$

where:

• $\mathbf{G} = \mathrm{Hess}(U) = \mathbf{H}^{-1}$, where U is the symplectic potential of the chosen toric Kähler structure g_V on V;

• for each $b \in \mathcal{B}$, $p_b = u_b$ and the real number c_b is such that $\langle p_b, z\rangle + c_b = 0$ on the codimension one face F_b;

• for each $a \in \mathcal{A}$, $\langle p_a, z\rangle + c_a$ is positive on Δ.

Clearly, (6) extends to a smooth tensor on $\hat{M}$ and by [4, Thm. 2], it is the pullback of a smooth metric on the blow-down M. This is obvious when $\hat{S} \to S$ is the trivial S'-bundle (for instance, when $\hat{M}$ is simply connected), since the connection $\hat{\theta}$ on $\hat{P} \to \hat{S} = S' \times S$ is then isomorphic to the sum of (pulled back) connections θ on $P \to S$ and θ' on $P' \to S'$ with curvatures $\sum_{a\in\mathcal{A}} \omega_a \otimes p_a$ and $\sum_{b\in\mathcal{B}} \omega_b \otimes u_b$ respectively. Hence $M \cong P \times_{\mathbb{T}} W$ and $M^0 \cong P \times_{\mathbb{T}} W^0$ where $W^0 = P' \times_{\mathbb{T}} V^0$. It follows that the metric (g, ω) (6) restricts to (g_W, ω_W) (5) on each W^0 fibre; the latter compactifies smoothly on W, and $\langle p_a, z\rangle + c_a$ are strictly positive on M, so (6) defines a Kähler structure on M. To handle the general case, one can consider the universal covers of $\hat{M}$ and M and use the previous argument, noting that the smooth extension of the metric is a local property; a direct argument in the case of the projective bundles described in §2.2 can be given along the lines of [5, §1.3]. This completes the *generalized Calabi construction* according to [4].

Assuming that the metrics (g_j, ω_j), the connection 1-form $\hat{\theta}$, the polytope Δ and the constants (p_j, c_j) are all fixed, (6) defines a family of Kähler metrics parametrized by symplectic potentials $U \in \mathcal{S}(\Delta)$ (or, equivalently, by toric Kähler metrics on $(V, \omega_V, \mathbb{T})$). We note that for this family, the symplectic 2-form ω remains unchanged, so we obtain a family of $\mathbb{T}$-invariant ω-compatible Kähler metrics corresponding to *different* complex structures. However, any two such complex structures are biholomorphic, under a $\mathbb{T}$-equivariant diffeomorphism in the identity component: this is well-known [2, 21] in the case of a symplectic toric manifolds (i.e., on $(V, \omega_V, \mathbb{T})$), and the same argument holds (fibrewise) on W and M (see [5, §1.4]). The pullbacks of the symplectic form ω under such diffeomorphisms belong to a common Kähler class Ω on a fixed complex manifold (M, J) (we can take J to be the complex structure on M introduced in Definition 3: it corresponds to the *standard* symplectic potential U_0, see [2, 34]).

Definition 4. A *compatible Kähler metric* (g, ω) on a rigid toric bundle M over a semisimple base S is one arising via the generalized Calabi construction (and hence given explicitly by (6) on M^0 for a symplectic potential $U \in \mathcal{S}(\Delta)$). The corresponding Kähler classes on (M, J) are called *compatible Kähler classes.*

We shall further assume that the metrics (g_j, ω_j) are fixed and have constant scalar curvature $Scal_j$ (with $Scal_b = 2d_b(d_b + 1)$ for $b \in \mathcal{B}$),[7] and that Δ and p_j are fixed. Recall that for $b \in \mathcal{B}$, the constants c_b are also fixed by requiring $\langle u_b, z\rangle + c_b = 0$ on the codimension one face $F_b \subset \Delta$. The real constants c_a, for $a \in \mathcal{A}$, can vary (on a given manifold (M, J)) and they parametrize the compatible Kähler classes.

2.4. **The isometry Lie algebra.** For a compact Kähler manifold (M, g), we denote by $\mathfrak{i}_0(M, g)$ the Lie algebra of all Killing vector fields with zeros; this is equivalently the Lie algebra of all hamiltonian Killing vector fields.

Lemma 5. *Let (g, ω) be a compatible Kähler metric on M, where the stable complex quotient $\hat{S}$ is equipped with the local product Kähler metric $(g_{\hat{S}}, \omega_{\hat{S}})$ covered by $\prod_{j\in\mathcal{I}}(S_j, \omega_j)$. Denote by $\hat{p}\colon M^0 \to \hat{S}$ the principal $\mathbb{T}^c$-fibre structure of the regular part M^0 of $\mathbb{T}$ action on M. Let $\mathfrak{z}(\mathbb{T}, g)$ be the centralizer in $\mathfrak{i}_0(M, g)$ of the ℓ-torus $\mathbb{T}$.*

Then the vector space $\mathfrak{z}(\mathbb{T}, g)$ is the direct sum of a lift of $\mathfrak{i}_0(\hat{S}, g_{\hat{S}})$ and the Lie algebra $\mathfrak{t} \subset \mathfrak{i}_0(M, g)$ of $\mathbb{T}$ in such a way that the natural homomorphism $\hat{p}_*\colon \mathfrak{z}(\mathbb{T}, g) \to \mathfrak{i}_0(S, g_S)$ is a surjection.

In the case $\ell = 1$, this result is proven in [5, Prop. 3] and the proof there generalizes easily; for the convenience of the Reader, we here supply details for arbitrary ℓ.

Proof. Denote by $K = \mathrm{grad}_\omega z \in C^\infty(M, TM) \otimes \mathfrak{t}^*$ the family of hamiltonian Killing vector fields generated by $\mathbb{T}$: thus, the span of K realizes the Lie algebra $\mathfrak{t}$ of $\mathbb{T}$ as a subalgebra of $\mathfrak{i}_0(M, g)$.

Let X be a holomorphic vector field on $\hat{S}$ which is hamiltonian with respect to $\omega_{\hat{S}}$; then the projection X_j of X onto the distribution $\mathcal{H}_j$ (induced by TS_j on the universal cover $\prod_{j\in\mathcal{I}} S_j$ of S) is a Killing vector field with zeros, so $\iota_{X_j}\omega_{\hat{S}} = -df_j$ for some function f_j (with integral zero). Thus $\sum_{j\in\mathcal{I}} f_j p_j$ is a family of hamiltonians for X with respect to the family of symplectic forms covered by $\sum_{j\in\mathcal{I}} \omega_j \otimes p_j$: since this is the curvature $d\hat{\theta}$ of the connection on M^0, X lifts to a holomorphic vector field $\tilde{X} = X_H + \sum_{j\in\mathcal{I}} f_j \langle p_j, K\rangle$ on M^0, which is hamiltonian with potential $\sum_{j\in\mathcal{I}}(\langle p_j, z\rangle + c_j) f_j$ and commutes with the components of K. (Here X_H is the horizontal lift to M^0 with respect to $\hat{\theta}$.) As the

[7] Presumably, the Kähler metrics (g_j, ω_j) must be CSC in order to obtain an extremal Kähler metric (g, ω) as above. We do not prove this here, but this fact has been established for $\ell = 1$ in [6, Prop. 14].

metric g extends to M and $\tilde{X}$ is Killing with respect to g, it extends to M too (note that $M \setminus M^0$ has codimension ≥ 2). It is not difficult to see that $\tilde{X}$ has zeros on M (in fact, if $s_0 \in \hat{S}$ is a zero of X then $\tilde{X} - \sum_{j\in\mathcal{I}} f_j(s_0)\langle p_j, K\rangle$ vanishes on M^0) so that $\tilde{X}$ is an element of $\mathfrak{i}_0(M,g)$. Of course, this shows that the Killing potential $\sum_{j\in\mathcal{I}}(\langle p_j, z\rangle + c_j) f_j$ extends as a smooth function on M.

Conversely, any $\tilde{X} \in \mathfrak{z}(\mathbb{T}, g)$ is a $\mathbb{T}^c$-invariant holomorphic vector field, so its restriction to M^0 is projectable to a holomorphic vector field $X \in \mathfrak{h}_0(\hat{S})$. This allows to reverse the above arguments: for $\tilde{X} = X_H + f\langle p, K\rangle + hJ\langle q, K\rangle$ (where $p, q \in \mathfrak{t}$ and $f, g \in C^\infty(\hat{S})$) be Killing with respect to the metric (6), we must have $q = 0$ and X be Killing with respect to $g_{\hat{S}}$. Such a vector field maps to zero iff it is a constant multiple of a component of K. This gives a projection to $\mathfrak{i}_0(\hat{S}, g_{\hat{S}})$ splitting the inclusion just defined. □

This is the main ingredient in the proof of the following result.

Proposition 1. *Let (g,ω) be a compatible Kähler metric on M, where the factors (S_j, ω_j) in the covering $\prod_{j\in\mathcal{I}} S_j$ of the stable quotient $\hat{S}$ have constant scalar curvature. Then g is invariant under a maximal torus G of the reduced automorphism group $\widetilde{\mathrm{Aut}}_0(M, J)$.*

Proof. Let G be a maximal torus in the group of hamiltonian isometries $\mathrm{Isom}_0(M, g)$, containing the ℓ-torus $\mathbb{T}$. By Lemma 5, G is the product of a maximal torus in the group of hamiltonian isometries $\mathrm{Isom}_0(\hat{S}, g_{\hat{S}})$ and the ℓ-torus $\mathbb{T}$. Denote by $\mathfrak{g} \subset \mathfrak{i}_0(M, g)$ the corresponding Lie algebra. We are going to show that $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} + J\mathfrak{g}$ is a maximal abelian subalgebra of $\mathfrak{h}_0(M, J)$.

As in the proof of Lemma 5, we consider natural homomorphism $\hat{p}_* \colon \mathfrak{z}(\mathbb{T}, J) \to \mathfrak{h}_0(\hat{S})$ from the centralizer $\mathfrak{z}(\mathbb{T}, J)$ of $\mathbb{T}$ in $\mathfrak{h}_0(M, J)$ to $\mathfrak{h}_0(\hat{S})$. The proof of Lemma 5 shows that the restriction of $\hat{p}_*$ to $\mathfrak{z}(\mathbb{T}, g)$ is surjective onto $\mathfrak{i}_0(\hat{S}, g_{\hat{S}})$.

By assumption, the induced Kähler metric $(g_{\hat{S}}, \omega_{\hat{S}})$ on $\hat{S}$ is of constant scalar curvature, so by the Lichnerowicz–Matsushima theorem [49, 55], $\mathfrak{h}_0(\hat{S})$ is the complexification of $\mathfrak{i}_0(\hat{S}, g_{\hat{S}})$. It follows that $\hat{p}_* \colon \mathfrak{z}(\mathbb{T}, J) \to \mathfrak{h}_0(S)$ is also surjective. As $\mathfrak{g} \subset \mathfrak{z}(\mathbb{T}, g)$ is a maximal abelian subalgebra, its projection to $\mathfrak{i}_0(S, g_S)$ must also be a maximal abelian subalgebra, so is then the image $\hat{p}_*(\mathfrak{g}^{\mathbb{C}}) \subset \mathfrak{h}_0(\hat{S})$ (by using the Lichnerowicz–Matsushima theorem again). It follows that $\mathfrak{g}^{\mathbb{C}} \subset \mathfrak{h}_0(M, J)$ is maximal abelian iff $\mathfrak{g}^{\mathbb{C}} \cap \mathfrak{h}_{\hat{S}}(\hat{M})$ is a maximal abelian subalgebra of the complex algebra of fibre-preserving holomorphic vector fields $\mathfrak{h}_{\hat{S}}(\hat{M})$. But the fibre V is a toric variety under $\mathbb{T}$, so $\mathfrak{g}^{\mathbb{C}} \cap \mathfrak{h}_{\hat{S}}(\hat{M}) = \mathfrak{t}^{\mathbb{C}} = \mathfrak{t} + J\mathfrak{t}$, which is clearly a maximal abelian subalgebra of $\mathfrak{h}(V, J_V)$ and hence also of $\mathfrak{h}_{\hat{S}}(\hat{M})$. □

2.5. **The extremal vector field.** For convenience, we will introduce at places a basis of $\mathfrak{t}$ (resp. of $\mathfrak{t}^*$), for example by taking ℓ generators of the lattice Λ (where $\mathbb{T} = \mathfrak{t}/2\pi\Lambda$). This identifies the vector space $\mathfrak{t}$ with $\mathbb{R}^\ell$ (and $\mathfrak{t}^*$ with $(\mathbb{R}^\ell)^*$), and fixes a basis of Poisson commuting hamiltonian Killing fields $K_1, \ldots, K_\ell$ in K. Thus, a $S^2\mathfrak{t}^*$-valued function $\mathbf{H}$ on Δ can be seen as an $\ell \times \ell$-matrix of functions $(H_{rs}) = \mathbf{H}$ on Δ. Similarly, we write $z = (z_1, \ldots, z_\ell)$ for the momentum coordinates with respect to $K_1, \ldots, K_\ell$.

An important technical feature of the Kähler metrics given by the generalized Calabi construction (6) is the simple expression of their scalar curvature in terms of the geometry of (V, g_V) and $(\hat{S}, g_{\hat{S}})$ (see e.g. [3, p. 380]):

$$Scal_g = \sum_{j\in\mathcal{I}} \frac{Scal_j}{\langle p_j, z\rangle + c_j} - \frac{1}{p(z)} \sum_{r,s=1}^{\ell} \frac{\partial^2}{\partial z_r \partial z_s}(p(z) H_{rs}), \tag{7}$$

where $p(z) = \prod_{j\in\mathcal{I}}(\langle p_j, z\rangle + c_j)^{d_j}$. This formula generalizes the expression obtained by Abreu [1] in the toric case (when $\hat{S}$ is a point).

Another immediate observation is that the volume form $\mathrm{Vol}_\omega = \omega^m$ is given by

$$\omega^m = p(z)\Big(\omega_{\hat S}^d \wedge \langle dz \wedge \hat\theta\rangle^{\wedge \ell}\Big) = p(z)\Big(\bigwedge_j \omega_j^{\wedge d_j}\Big) \wedge \langle dz \wedge \hat\theta\rangle^{\wedge \ell}, \tag{8}$$

where $\sum_{j\in\mathcal{I}} d_j = d = m-\ell$. It follows that integrals over M of functions of z (pullbacks from Δ) are given by integrals on Δ with respect to the volume form $p(z)\,dv$, where dv is the (constant) euclidean volume form on $\mathfrak{t}^*$, obtained by wedging any generators of the lattice Λ.

We now recall the definition [31] of the *extremal vector field* of a compact Kähler manifold (M,J,g,ω). Let G be a maximal connected compact subgroup of the reduced automorphism group $\widetilde{\mathrm{Aut}}_0(M,J)$.[8] Following [31], the *extremal vector field* of a G-invariant Kähler metric (g,J,ω) on M is the Killing vector field whose Killing potential is the L^2-projection of the scalar curvature $Scal_g$ of g to the space $\mathfrak{g}_\omega$ of all Killing potentials (with respect to g) of elements of the Lie algebra $\mathfrak{g}$. As shown by Futaki and Mabuchi [31], this definition is independent of the choice of a G-invariant Kähler metric within the given Kähler class $\Omega = [\omega]$ on (M,J). The extremal vector field is necessarily in the centre of $\mathfrak{g}$, so it can also be defined by taking G to be a maximal torus in $\widetilde{\mathrm{Aut}}_0(M,J)$. This remark is useful for the Kähler metrics in (6), as we have seen in Proposition 1 that they are automatically invariant under such a torus G. In this case, by Lemma 5, $\mathfrak{g}_\omega$ is the direct sum of $\mathfrak{t}_\omega$ (which in turn is identified with the space of affine functions of z) and a subspace of Killing potentials of zero integral of lifts of Killing vector fields on $(\hat S, g_{\hat S})$. We have shown in the proof of Lemma 5 that the latter potentials are all of the form $\sum_j(\langle p_j,z\rangle + c_j)f_j$ where f_j is a function on $\hat S$ of zero integral with respect to $\omega^d_{\hat S}$. As the scalar curvature of a compatible metric is a function of z only (assuming $Scal_j$ are constant—see (7)) it follows from (8) that the L^2-projection of $Scal_g$ to $\mathfrak{g}_\omega$ lies in $\mathfrak{t}_\omega$. This shows that the extremal vector field is in $\mathfrak{t}$ and the projection of $Scal_g$ orthogonal to the Killing potentials of g takes the form:

$$Scal_g^{\perp} = \langle A, z\rangle + B + Scal_g,$$

where

$$\begin{cases} \sum_s \alpha_s A_s + \alpha B + 2\beta &= 0,\\ \sum_s \alpha_{rs} A_s + \alpha_r B + 2\beta_r &= 0, \end{cases} \tag{9}$$

with $\alpha = \int_\Delta p(z)dv, \qquad \alpha_r = \int_\Delta z_r p(z)dv, \qquad \alpha_{rs} = \int_\Delta z_r z_s p(z) dv,$

$$\beta = \frac12 \int_\Delta Scal_g p(z)dv = \int_{\partial\Delta} p(z)d\sigma + \frac12\int_\Delta \Big(\sum_j \frac{Scal_j}{\langle p_j, z\rangle + c_j}\Big)p(z)dv,$$

$$\beta_r = \frac12 \int_\Delta Scal_g z_r p(z)dv = \int_{\partial\Delta} z_r p(z)d\sigma + \frac12\int_\Delta \Big(\sum_j \frac{Scal_j}{\langle p_j, z\rangle + c_j}\Big)z_r p(z)dv.$$

Here $d\sigma$ is the $(\ell-1)$-form on $\partial\Delta$ with $u_i \wedge d\sigma = -dv$ on the face F_i with normal u_i. These formulae are immediate once one applies the divergence theorem and the boundary conditions (4) for $\mathbf{H}$, noting that the normals are inward normals, which introduces a sign compared to the usual formulation of the divergence theorem.

The extremal vector field of (M,g,J,ω) is $-\langle A, K\rangle$, where $K \in C^\infty(M,TM)\otimes \mathfrak{t}^*$ is the generator of the $\mathbb{T}$ action.

[8] By a well-known result of Calabi [14], any extremal Kähler metric is invariant under such a G.

3. Stability of the extremal equation and Theorems 2–4

3.1. Stability of solutions of the extremal equation under small perturbation. It follows from the considerations in §2.5 that on a given manifold M of the type we consider, finding a *compatible* extremal Kähler metric (g, ω) of the form (6) reduces to solving the equation (for a unknown symplectic potential $U \in \mathcal{S}(\Delta)$)

$$\langle A, z\rangle + B + \sum_{j\in\mathcal{I}} \frac{Scal_j}{c_j + \langle p_j, z\rangle} - \frac{1}{p(z)} \sum_{r,s} \frac{\partial^2}{\partial z_r \partial z_s}\big(p(z) H_{rs}\big) = 0, \tag{10}$$

where

- $(H_{rs}) = \mathbf{H} = (\mathrm{Hess}(U))^{-1}$;
- $(c_j, p_j, Scal_j)$ are fixed constants;
- $p(z) = \prod_{j\in\mathcal{I}}(c_j + \langle p_j, z\rangle)^{d_j}$ is strictly positive on Δ^0 but vanishes on the blow-down faces F_b, $b \in \mathcal{B}$;
- A and B are expressed in terms of $(c_j, p_j, Scal_j)$ by (9).

Recall from §2.3 that the real constants c_a $(a \in \mathcal{A})$ parametrize compatible Kähler classes on a given manifold M. A general result of LeBrun–Simanca [46] affirms that Kähler classes admitting extremal Kähler metric form an open subset of the Kähler cone. We want to obtain a relative version of this result, by showing that *compatible* Kähler classes which admit a *compatible* extremal Kähler metric is an open condition on the parameters c_a.

We will state and prove our stability result in a slightly more general setting, by considering (10) as a family of differential operators on $\mathcal{S}(\Delta)$, parametrized by $\lambda \in \{(c_a, p_a, Scal_a),\ a \in \mathcal{A}\}$ (thus λ takes values in a $(2+\ell)|\mathcal{A}|$-dimensional euclidean vector space). For any λ such that $\langle p_a, z\rangle + c_a > 0$ on Δ, we consider

$$P_\lambda(U) = \langle A_\lambda, z\rangle + B_\lambda + \sum_{j\in\mathcal{I}} \frac{Scal_j}{c_j + \langle p_j, z\rangle} - \frac{1}{p_\lambda p_0} \sum_{r,s=1}^{\ell} \frac{\partial^2}{\partial z_r \partial z_s}\big(p_\lambda p_0 H_{rs}\big), \tag{11}$$

where $(H_{rs}) = \mathrm{Hess}(U)^{-1}$, $p_\lambda(z) = \prod_{a\in\mathcal{A}}(\langle p_a, z\rangle + c_a)^{d_a}$, $p_0(z) = \prod_{b\in\mathcal{B}}(\langle u_b, z\rangle + c_b)^{d_b}$, and A_λ, B_λ are introduced by (9). The central result of this section is the following one.

Proposition 2. *Let (g_0, ω_0) be a compatible extremal Kähler on M, with symplectic potential U_0 and parameters $\lambda_0 = (c_a^0, p_a^0, Scal_a^0)$, $a \in \mathcal{A}$. Then there exists $\varepsilon > 0$ such that for any λ with $|\lambda - \lambda_0| < \varepsilon$ there exists a symplectic potential $U_\lambda \in \mathcal{S}(\Delta)$ such that $P_\lambda(U_\lambda) = 0$ on Δ^0.*

The proof of this proposition has several steps and will occupy the rest of this section.

It is not immediately clear from (11) that P_λ is a well-defined differential operator: in the presence of blow-downs, the terms $\frac{Scal_b}{c_b + \langle p_b, z\rangle}$ and $\frac{1}{p_0(z)}$ become degenerate on the boundary of Δ.[9] Of course, for $\lambda = \lambda_0$ we know from (10) that $P_{\lambda_0}(U) = Scal_g^\perp$ where g is the compatible metric on M corresponding to U, and $Scal_g^\perp$ is the L^2-projection of the scalar curvature to the space of functions orthogonal to the Killing potentials of g. However, for generic values of λ the data $(c_a, p_a, Scal_a)$ are not longer associated with a compatible Kähler class on a smooth manifold: for this to be true p_a and $Scal_a$ must satisfy *integrality* conditions. To overcome this technical difficulty, we are going to rewrite our equation on the smooth compact manifold W. (Note that for $b \in \mathcal{B}$, $p_b = u_b, c_b, Scal_b = 2d_b(d_b + 1)$ are fixed in our construction.)

[9] This does not affect the principal part of P_λ, which is concentrated in the scalar curvature $Scal_V = -\sum_{r,s} \frac{\partial^2}{\partial z_r \partial z_s} H_{rs}$ of the induced Kähler metric g_V on V [1], and is manifestly independent of λ.

Recall from §2.3 that any symplectic potential $U \in \mathcal{S}(\Delta)$ introduces a compatible Kähler metric (g_W, ω_W) on the manifold W obtained by blowing down $\hat{W} = P' \times_{\mathbb{T}} V$. Thus, (W, g_W, ω_W) itself is obtained by the generalized Calabi construction with S being a point.

By a well-known result of G. W. Schwarz [63], the space $C^\infty(V)^{\mathbb{T}}$ of $\mathbb{T}$-invariant smooth functions on the toric symplectic manifold $(V, \omega_V, \mathbb{T})$ is identified with the space of pullbacks (via the momentum map z) of smooth functions $C^\infty(\Delta)$ on Δ; similarly, the space of smooth $\mathbb{T}$-invariant functions on W (resp. on M) which are constant on the inverse images of the momentum map z is identified with the space $C^\infty(\Delta)$. We will use implicitly these identification throughout. Occasionally, when we want to emphasize the dependence of this identification on z, we will denote these isomorphisms by S_z. With this convention, we have

Lemma 6. *Let $U \in \mathcal{S}(\Delta)$ be a symplectic potential of a compatible Kähler metric g_V on $(V, \omega_V, \mathbb{T})$ and (g_W, ω_W) be the corresponding compatible Kähler metric on W. Then, for any λ such that $\langle p_a, z\rangle + c_a > 0$ on Δ,*

$$\begin{aligned} P_\lambda(U) =& \langle A_\lambda, z\rangle + B_\lambda + \sum_{a\in\mathcal{A}} \frac{Scal_a}{c_a + \langle p_a, z\rangle} + Scal_W \\ & - \frac{1}{p_\lambda(z)} \sum_{r,s=1}^{\ell} \Big(\Big(\frac{\partial^2 p_\lambda}{\partial z_r \partial z_s}\Big)(z) g_W(K_r, K_s)\Big) \\ & + \frac{2}{p_\lambda(z)} \sum_{r=1}^{\ell} \Big(\Big(\frac{\partial p_\lambda}{\partial z_r}\Big)(z) \Delta_W z_r \Big), \end{aligned}$$

where $Scal_W$ and Δ_W respectively denote the scalar curvature and the riemannian laplacian of g_W, and $dz_r = -\omega_W(K_r, \cdot)$.

Proof. We work on the open dense subset $W^0 = P' \times_{\mathbb{T}} V^0$ where the compatible metric (g_W, ω_W) takes the explicit form (5). The formula (7) for the scalar curvature of the compatible metric g_W then specifies to

$$Scal_W = \sum_{b\in\mathcal{B}} \frac{Scal_b}{\langle p_b, z\rangle + c_b} - \frac{1}{p_0(z)} \sum_{r,s=1}^{\ell} \frac{\partial^2}{\partial z_r \partial z_s}(p_0(z) H_{rs}).$$

Still using the explicit form (5) of the Kähler structure, we calculate that for the pullback to W of a smooth function $f(z)$ on Δ

$$\begin{aligned} dd^c_W f &= d\Big(\sum_{r,s=1}^{\ell} \frac{\partial f}{\partial z_s} H_{rs} \theta'_r\Big) \\ &= \sum_{k,r,s=1}^{\ell} \frac{\partial}{\partial z_k}\Big(\frac{\partial f}{\partial z_s} H_{rs}\Big) dz_k \wedge \theta'_r + \sum_{b\in\mathcal{B}} \Big(\sum_{r,s=1}^{\ell} \frac{\partial f}{\partial z_s} H_{rs} p_{br}\Big)\omega_b, \end{aligned} \tag{12}$$

where the decompositions $\theta' = (\theta'_1, \dots, \theta'_\ell)$ and $p_b = (p_{b1}, \dots, p_{b\ell})$ are with respect to the chosen basis of $\mathfrak{t}$ and $\mathfrak{t}^*$. Wedging with ω_W, we obtain the following expression for the laplacian

$$\Delta_W f = -\frac{1}{p_0(z)} \sum_{r,s=1}^{\ell} \frac{\partial}{\partial z_r}\Big(p_0(z) \frac{\partial f}{\partial z_s} H_{rs}\Big). \tag{13}$$

Specifying (13) to $f = z_r$ and putting the above formulae back in (11) implies the lemma. ∎

Note that $\frac{1}{p_\lambda(z)}$ and $\frac{Scal_a}{c_a+\langle p_a,z\rangle}$ pull back to smooth functions on W for λ such that $c_a + \langle p_a, z\rangle > 0$ on Δ, and A_λ and B_λ are well-defined and depend smoothly on λ (at least for λ close to λ_0). Thus, Lemma 6 implies that P_λ is a fully non-linear fourth order differential operator which depends smoothly on λ (for λ sufficiently close to λ_0). It follows that $P_\lambda(U) \in C^\infty(\Delta)$ for any $U \in \mathcal{S}(\Delta)$.

Our problem is formulated in terms of compatible Kähler metrics on V (or, equivalently, on W and M) with respect to a fixed symplectic form ω_V (resp. ω_W and ω). This introduces the space of symplectic potentials $\mathcal{S}(\Delta)$ where we have to work with smooth functions on Δ^0 which have a prescribed boundary behaviour on $\partial\Delta$. Our lack of understanding of the convergence in this space (with respect to suitable Sobolev norms) leads us to make an additional technical step and reformulate our initial problem as an existence result on a suitable subspace of the space $\mathcal{M}_\Omega(M)^G \cong \{f \in C_0^\infty(M)^G : \omega_0 + dd^c f > 0\}$ of G-invariant Kähler metrics in the Kähler class of (g_0, J_0, ω_0), where $C_0^\infty(M)^G$ denotes the space of G-invariant smooth functions on M of zero integral with respect to ω_0^m (thus $\mathcal{M}_\Omega(M)^G$ is viewed as an open set in $C_0^\infty(M)^G$ with respect to $||\cdot||_{C^2}$). Once this interpretation is achieved, we will apply the implicit function theorem along the lines of the proof of Lemma 4.

First of all, note that the Frechét space $C^\infty(\Delta)$ pulls back via z to a closed subspace in $C^\infty(V)^{\mathbb{T}}$, $C^\infty(W)^T$ and $C^\infty(M)^G$, where T (resp. G) is a maximal torus in $\widetilde{\mathrm{Aut}}_0(W)$ (resp. $\widetilde{\mathrm{Aut}}_0(M)$) containing $\mathbb{T}$, as in Proposition 1: this follows easily from the description of the Lie algebras of T and G given in Lemma 5. Furthermore, by (8), the corresponding *normalized* subspaces of functions with zero integral for the measures $p_\lambda(z)p_0(z)\mathrm{Vol}_{\omega_V^0}$, $p_\lambda(z)\mathrm{Vol}_{\omega_W^0}$ and Vol_{ω_0}, respectively, are identified with the space $C_0^\infty(\Delta)$ of smooth functions of zero integral with respect to the volume form $d\mu_0 = p_{\lambda_0}(z)p_0(z)dv$ on Δ^0: this normalization will be used throughout.

Secondly, to adopt the classical point of view of Kähler metrics within a given Kähler class on a fixed complex manifold, we consider the Fréchet space $\mathcal{M}_\Omega(V)^{\mathbb{T}} \cong \{f \in C_0^\infty(\Delta) : \omega_V^0 + dd_V^c f > 0\}$ of $\mathbb{T}$-invariant Kähler metrics in the Kähler class $\Omega = [\omega_V^0]$, where the complex structure on V (resp. on W and M) is determined (and will be fixed throughout) by the initial compatible metric (g_0, ω_0); similarly, we introduce the spaces $\mathcal{M}_\Omega(W)^T$ and $\mathcal{M}_\Omega(M)^G$ of Kähler metrics in the given Kähler class which are invariant under a maximal torus (see Proposition 1). These three spaces are interrelated by the generalized Calabi construction as follows.

Lemma 7. *Let $\tilde\omega_V = \omega_V^0 + dd_V^c f$ be a Kähler metric in $\mathcal{M}_\Omega(V)^{\mathbb{T}}$. Then $\tilde\omega_W = \omega_W^0 + dd_W^c f$ and $\tilde\omega = \omega_0 + dd_M^c f$ define Kähler metrics in $\mathcal{M}_\Omega(W)^T$ and $\mathcal{M}_\Omega(M)^G$ respectively, such that $\tilde\omega_V, \tilde\omega_W$ and $\tilde\omega$ are linked by the generalized Calabi construction on M, with respect to the data $(\Delta, \hat S, \hat\theta, \omega_j)$ of the initial metric ω_0, but with momentum co-ordinate $\tilde z = z + d_V^c f(K)$.*

Proof. A direct calculation based on the expressions of $dd_V^c f, dd_W^c f$ and $dd_M^c f$, see (12); we leave the details to the Reader. ☐

Lemma 7 allows us to introduce subspaces of *compatible* Kähler metrics $\mathcal{M}_\Omega^{\mathrm{comp}}(W) = \mathcal{M}_\Omega(W)^T \cap C_0^\infty(\Delta)$ and $\mathcal{M}_\Omega^{\mathrm{comp}}(M) = \mathcal{M}_\Omega(M)^G \cap C_0^\infty(\Delta)$ (within a fixed Kähler class Ω) and identify each of them with the space $\mathcal{M}_\Omega(V)^{\mathbb{T}}$. The correspondence which associates to any $\tilde\omega_W \in \mathcal{M}_\Omega^{\mathrm{comp}}(W)$ (resp. $\tilde\omega \in \mathcal{M}_\Omega^{\mathrm{comp}}(M)$) the corresponding symplectic potential $\tilde U \in \mathcal{S}(\Delta)$[10] allows us to reformulate our existence problem on the space

[10]For a metric $\tilde\omega_V = \omega_V^0 + dd_V^c f \in \mathcal{M}^{\mathbb{T}}(V)$ the corresponding symplectic potential $\tilde U$ is linked to f by a Legendre transform [2, 34]; this is true fibrewise for metrics in $\mathcal{M}_\Omega^{\mathrm{comp}}(W)$ and $\mathcal{M}_\Omega^{\mathrm{comp}}(M)$.

$\mathcal{M}^{\rm comp}_{\Omega}(W)$ as follows: for any λ sufficiently close to λ_0 (so that A_λ, B_λ are well-defined and $\langle p_a, z\rangle + c_a > 0$ on Δ), we consider the family of differential operators on $\mathcal{M}_{\Omega}(W)^T$

$$\begin{aligned} Q_\lambda(\tilde\omega_W) = \frac{p_\lambda(\tilde z)}{p_{\lambda_0}(\tilde z)}\Big[&\langle A_\lambda, \tilde z\rangle + B_\lambda + \sum_{j\in\mathcal{I}} \frac{Scal_j}{c_j + \langle p_j, \tilde z\rangle} \\ &+ \widetilde{Scal}_W - \frac{1}{p_\lambda(\tilde z)}\sum_{r,s}\Big(\Big(\frac{\partial^2 p_\lambda}{\partial z_r \partial z_s}\Big)(\tilde z)\tilde g_W(K_r, K_s)\Big) \\ &+ \frac{2}{p_\lambda(\tilde z)}\sum_r \Big(\Big(\frac{\partial p_\lambda}{\partial z_r}\Big)(\tilde z)\tilde\Delta_W(\tilde z_r)\Big)\Big], \end{aligned} \tag{14}$$

where $\tilde z = z + d^c_W f(K)$ is the momentum map of $\mathbb{T}$ with respect to the Kähler form $\tilde\omega_W = \omega^0_W + dd^c_W f$ of the Kähler metric $\tilde g_W$, and $\widetilde{Scal}_W$ (resp. $\tilde\Delta_W$) denote the scalar curvature (resp. laplacian) of $\tilde g_W$. Thus, by Lemmas 6 and 7, *any Kähler metric* $\tilde\omega_W \in \mathcal{M}^{\rm comp}_{\Omega}(W)$ *for which* $Q_\lambda(\tilde\omega_W) = 0$ *gives rise to a symplectic potential* $\tilde U \in \mathcal{S}(\Delta)$ *solving* $P_\lambda(\tilde U) = 0$.

The positive factor $\frac{p_\lambda(\tilde z)}{p_{\lambda_0}(\tilde z)}$ in front of Q_λ is introduced so that *for any compatible metric* $\tilde\omega_W \in \mathcal{M}^{\rm comp}_{\Omega}(W)$, *the function* $S_{\tilde z}(Q_\lambda(\tilde\omega_W))$ *is* L^2*-orthogonal with respect to the measure* $d\mu_0 = p_{\lambda_0}p_0 dv$ *on* Δ *to the space of affine functions on* $\mathfrak{t}^*$, where, we recall, $S_{\tilde z}$ denotes the identification of $\mathbb{T}$-invariant smooth functions on W which are constant on the inverse images of $\tilde z$ (equivalently of z) with pullbacks via $\tilde z$ of smooth functions on Δ. Indeed, by Lemma 6, $p_{\lambda_0}(\tilde z)p_0(\tilde z)Q_\lambda(\tilde\omega_W) = P_\lambda(\tilde U)p_\lambda(\tilde z)p_0(\tilde z)$, so integrating the r.h.s. of (11) by parts and using (4) we get

$$\begin{aligned} \int_\Delta P_\lambda(U) f(z) p_\lambda(z) p_0(z) dv = &-\int_\Delta \langle \mathbf{H}, \mathrm{Hess}(f)\rangle p_\lambda(z)p_0(z)dv \\ &+ \int_\Delta \Big(\langle A, z\rangle + B + \sum_{j\in\mathcal{I}}\frac{Scal_j}{c_j + \langle p_j, z\rangle}\Big) f(z)p_\lambda(z)p_0(z)dv \\ &+ 2\int_{\partial\Delta} f(z)p_\lambda(z)p_0(z)d\sigma, \end{aligned}$$

which holds for any smooth function $f(z)$. When f is affine, the first term on the r.h.s is clearly zero, while by the definition (9) of A_λ and B_λ the sum of the two other terms is zero too; our claim then follows by Lemma 7 and the expression (8) for the volume form of the compatible metric $\tilde\omega_W$.

Let Π_0 denote the orthogonal L^2-projection of $C^\infty(\Delta)$ to the finite dimensional subspace of affine functions of $\mathfrak{t}^*$ with respect to the measure $d\mu_0 = p_{\lambda_0}p_0 dv$ on Δ, and $C^\infty_\perp(\Delta)$ be the kernel of Π_0. We then consider the map $\Psi\colon \mathcal{U} \to \mathbb{R}^{(2+\ell)|\mathcal{A}|} \times C^\infty_\perp(\Delta)$, defined in a small neighbourhood $\mathcal{U}$ of $(\lambda_0, 0) \in \mathbb{R}^{(2+\ell)|\mathcal{A}|} \times C^\infty_\perp(\Delta)$ by

$$\Psi(\lambda, f) = \Big(\lambda, (\mathrm{Id} - \Pi_0)(S_z(Q_\lambda(\tilde\omega_W))\Big),$$

where $\tilde\omega_W = \omega^0_W + dd^c_W f$ is a compatible metric on $\mathcal{M}^{\rm comp}_{\Omega}(W)$. Note that if f has sufficiently small C^1-norm, the equation $(\mathrm{Id} - \Pi_0)\circ(S_z(Q_\lambda(\tilde\omega_W)) = 0$ is satisfied if and only if $Q_\lambda(\tilde\omega_W) = 0$: this follows from the fact that $\Pi_0 \circ S_{\tilde z}\circ S_z^{-1}$ defines a continuous family of linear endomorphisms of the finite dimensional space of affine functions on $\mathfrak{t}^*$, with the identity corresponding to $\tilde\omega_W = \omega^0_W$; thus $\Pi_0\circ S_{\tilde z}\circ S_z^{-1}\circ\Pi_0$ is invertible for $\tilde\omega_W$ close to ω^0_W, and hence (by using that $\Pi_0(S_{\tilde z}(Q(\tilde\omega_W)) = 0$) we get

$$\Pi_0\circ S_{\tilde z}\circ S_z^{-1}\circ(\mathrm{Id} - \Pi_0)\Big(S_z(Q_\lambda(\tilde\omega_W)\Big) = -\Pi_0\circ S_{\tilde z}\circ S_z^{-1}\circ\Pi_0\Big(S_z(Q_\lambda(\tilde\omega_W))\Big)$$

which is zero iff $S_z(Q_\lambda(\tilde\omega_W) = 0$ i.e., $Q_\lambda(\tilde\omega_W) = 0$.

By the discussion above, we are in position to complete the proof of Proposition 2 by applying the inverse function theorem to the extension of Ψ to suitable Sobolev spaces, together with elliptic regularity (as in [46], see also the proof of Lemma 4) in order to find a family $\tilde{\omega}_W^\lambda = \omega_W^0 + dd^c_W f_\lambda$ of smooth compatible metrics satisfying $\Psi(\lambda, f_\lambda) = (\lambda, 0)$ for $|\lambda - \lambda_0| < \varepsilon$.

Let us first introduce the functional spaces we will work on. Recall that $C^\infty(\Delta)$ is seen as a (closed) Fréchet subspace of the space of T-invariant smooth functions on W (resp. G-invariant smooth functions on M) which are constant on the inverse images of the momentum map z for the sub-torus $\mathbb{T}$. It follows from the description of the Lie algebra of T (resp. G) given in Lemma 5 that $C^\infty_\perp(\Delta)$ is precisely the intersection of $C^\infty(\Delta)$ with the space $C^\infty_\perp(W)^T$ of T-invariant smooth functions on W which are L^2-orthogonal with respect to $p_{\lambda_0}\mathrm{Vol}_{\omega_W^0}$ to Killing potentials of g_W^0 (resp. the space $C^\infty_\perp(M)^G$ of G-invariant smooth functions on M which are L^2-orthogonal with respect to Vol_{ω_0} to Killing potentials of g_0). We let $L^{2,k}_\perp(W,\Delta)$ (resp. $L^{2,k}_\perp(M,\Delta)$) be the closure of $C^\infty_\perp(\Delta)$ with respect to the Sobolev norm $||\cdot||_2^k$ on W for the measure $p_{\lambda_0}(z)\mathrm{Vol}_{\omega_W^0}$ and riemannian metric g_W^0 (resp. the Sobolev norm $||\cdot||_2^k$ on M with respect to Vol_{ω_0} and g_0). For $k \gg 1$, the Sobolev embedding $L^{2,k+4}_\perp(W,\Delta) \subset C^3_\perp(\Delta)$ allows us to extend the differential operator Ψ to a C^1-map from a neighbourhood of $(\lambda_0, 0) \in \mathbb{R}^{(2+\ell)|\mathcal{A}|} \times L^{2,k+4}_\perp(W,\Delta)$ into $L^{2,k}_\perp(W,\Delta)$, such that $\Psi(\lambda_0, 0) = 0$; furthermore, as the principal part of Q_λ is concentrated in the term $\widetilde{Scal_W}$, one can see that Ψ is a fourth order quasi-elliptic operator [46].

The following lemma now suffices to apply the inverse function theorem.

Lemma 8. *Let $T_0\colon C^\infty_\perp(\Delta) \to C^\infty_\perp(\Delta)$ be the linearization at $\omega_W^0 \in \mathcal{M}_\Omega^{\mathrm{comp}}(W)$ of Q_{λ_0}. Then T_0 is an isomorphism of Fréchet spaces.*

Proof. Let (g_0, J_0, ω_0) be the compatible extremal Kähler metric on M corresponding to the initial value $\lambda = \lambda_0$. For any function $f \in C^\infty_\perp(\Delta)$ we consider the compatible Kähler metric $\tilde{g}$ on M, with Kähler form $\tilde{\omega} = \omega_0 + dd^c_M f$ and the compatible Kähler metric $\tilde{g}_W$ on W with Kähler form $\tilde{\omega}_W = \omega_W^0 + dd^c_W f$. We saw already in §2.5 that for $\lambda = \lambda_0$, $Q_{\lambda_0}(\tilde{\omega}_W) = P_{\lambda_0}(\tilde{U}) = Scal^\perp_{\tilde{g}}$, where $\tilde{U}$ and $Scal^\perp_{\tilde{g}}$ are the symplectic potential and normalized scalar curvature of $\tilde{g}$. It then follows from [32, 45] that the linearization T_0 of Q_{λ_0} (at ω_W^0) is equal to -2 times the Lichnerowicz operator L of (g_0, ω_0) acting on the space of pullbacks (via z) of functions in $C^\infty_\perp(\Delta)$. We have already observed in the proof of Lemma 4 that L is an isomorphism when restricted to the space $C^\infty_\perp(M)^G$ of G-invariant smooth functions L^2-orthogonal to Killing potentials of g_0. The main point here is to refine this by showing that L is an isomorphism when restricted to subspace $C^\infty_\perp(\Delta)$, the only missing piece being the surjectivity.

Suppose for a contradiction that $L\colon C^\infty_\perp(\Delta) \to C^\infty_\perp(\Delta)$ is not surjective. Considering the extension of L to an operator between the Sobolev spaces $L^{2,4}_\perp(M,\Delta) \to L^2_\perp(M,\Delta)$ (by elliptic theory L is a closed operator), our assumption is then equivalent to the existence of a non-zero function $u \in L^2_\perp(M,\Delta)$ such that, for any $\phi \in C^\infty_\perp(\Delta)$, $L(\phi)$ is L^2 orthogonal to u. As any sequence of functions converging in $L^2(M)$ has a point-wise converging subsequence, $u = u(z)$ is (the pullback to M of) a L^2-function on Δ, and using (8) we have

$$\int_M L(\phi) u \omega_0^m = \int_{\Delta^0} L(\phi) u(z) p(z) dv = 0. \tag{15}$$

We claim that (15) implies

$$\int_M L(f)u\ \omega_0^m = 0 \tag{16}$$

for any $f \in C^\infty_\perp(M)^G$. This would be a contradiction because L extends to an isomorphism between the closures $L^{2,4}_\perp(M)^G$ and $L^2_\perp(M)^G$ of $C^\infty_\perp(M)^G$ in the corresponding Sobolev spaces on M.

It is enough to establish (16) by integrating on $M^0 = z^{-1}(\Delta^0)$ (which is the complement of the union of submanifolds of real codimension at least 2).

The Lichnerowicz operator L has the following general equivalent expression [32, 45]

$$L(f) = \frac{1}{2}\Delta^2_{g_0} f + g_0(dd^c f, \rho_{g_0}) + \frac{1}{2} g_0(df, dScal_{g_0}), \tag{17}$$

where ρ_{g_0} is the Ricci form of (g_0, J_0) and Δ_{g_0} is its laplacian. We will use the specific form (6) of g_0 to express the r.h.s of the above equality in terms of the geometry of (V, g^0_V) and $(\hat{S}, g_{\hat{S}})$.

Any G-invariant (and hence $\mathbb{T}$-invariant) smooth function f on M can be written as a function $f_s(z)$ of the momentum map z depending smoothly on $s \in \hat{S}$ (as the pullback of f to $\hat{M}$ is smooth, f_s is a smooth function on Δ, not only on Δ^0). Similarly, for any $z \in \Delta^0$, $f_z(s) = f(z, s)$ will denote the corresponding smooth function on $\hat{S}$.

Using [4, Prop. 7] and the specific form (6) of g_0, it is straightforward to check that on M^0 we have

$$\begin{aligned}
dd^c f = & \sum_{k,r,t=1}^{\ell} \frac{\partial}{\partial z_k}\Big(\frac{\partial f}{\partial z_t} H_{rt}\Big) dz_k \wedge \hat{\theta}_r + \sum_{j\in\mathcal{I}} \Big(\sum_{r,t=1}^{\ell} \frac{\partial f}{\partial z_t} H_{rt} p_{jr}\Big)\omega_j \\
& + \sum_{r=1}^{\ell} \Big(d_{\hat{S}} \Big(\sum_{s=1}^{\ell} \frac{\partial f}{\partial z_t} H_{rt}\Big) \wedge \hat{\theta}_r + d^c_{\hat{S}} \Big(\sum_{t=1}^{\ell} \frac{\partial f}{\partial z_t} H_{rt}\Big) \wedge J\hat{\theta}_r \Big) \\
& + d_{\hat{S}} d^c_{\hat{S}} f_z \\
\Delta_{g_0} f = & \Delta_{\hat{S},z} f_z + \Delta_{g_0} f_s; \\
\rho_{g_0} = & \sum_{j\in\mathcal{I}} \rho_j - \sum_{k,r,t=1}^{\ell} \frac{\partial}{\partial z_k}\Big(\frac{1}{2p(z)} \frac{\partial (p(z) H_{tr})}{\partial z_t}\Big) dz_k \wedge \hat{\theta}_r \\
& - \frac{1}{2p(z)} \sum_{j\in\mathcal{I}} \Big(\sum_{r,t=1}^{\ell} \frac{\partial (p(z) H_{rt})}{\partial z_t} p_{jr}\Big)\omega_j, \\
Scal_{g_0} = & Scal_{\hat{S},z} - \frac{1}{p(z)} \sum_{r,t=1}^{\ell} \frac{\partial^2}{\partial z_r \partial z_t}(p(z) H_{rt}),
\end{aligned}$$

where

- $p(z) = \prod_{j\in\mathcal{I}} (\langle p_j, z\rangle + c_j)^{d_j}$;
- $\hat{\theta} = (\hat{\theta}_1, \ldots, \hat{\theta}_\ell)$ and $p_j = (p_{j1}, \ldots, p_{j\ell})$ with respect to the chosen basis of $\mathfrak{t}$;
- $d_{\hat{S}}$ and $d^c_{\hat{S}}$ are the differential and the d^c-operator acting on functions and forms on $\hat{S}$;
- (g_j, ω_j) are the product CSC Kähler factors of the Kähler metric $(g_{\hat{S}}, \omega_{\hat{S}})$, with respective Ricci forms ρ_j and laplacians Δ_{g_j};
- $g_{\hat{S},z} = \sum_{j\in\mathcal{I}} (\langle p_j, z\rangle + c_j) g_j$ is the quotient Kähler metric on $\hat{S}$ at z, and $\omega_{\hat{S},z}$, $Scal_{\hat{S},z}$ and $\Delta_{\hat{S},z}$ denote its Kähler form, scalar curvature and laplacian, respectively;

Substituting back in (17), we obtain

$$\begin{aligned} L(f) =& L(f_s) + L_{\hat{S},z}(f_z) + \Delta_{\hat{S},z}\big((\Delta_{g_0} f_s)_z\big) + \Delta_{g_0}\big((\Delta_{\hat{S},z} f_z)_s\big) \\ &+ \sum_{j\in\mathcal{I}} R_j(z)\Delta_{g_j}(f_z), \end{aligned}$$

where $L_{\hat{S},z}$ is the Lichnerowicz operator of $g_{\hat{S},z}$, and $R_j(z)$ are coefficients (that can be found explicitly from the above formulae) depending only on z, and such that $p(z)R_j(z)$ are smooth on Δ.

If we integrate the above expression for $L(f)$ against $u(z)$ (by using (8)) we get that $\int_M L(f)u\ \omega_0^m$ is a non-zero constant multiple of

$$\begin{aligned} &\int_{\hat{S}}\Big(\int_{\Delta^0} L(f_s)u(z)p(z)dv\Big)\omega_{\hat{S}}^d + \int_{\hat{S}}\Big(\int_{\Delta^0} \Delta_{g_0}\big((\Delta_{\hat{S},z} f_z)_s\big)u(z)p(z)dv\Big)\omega_{\hat{S}}^d \\ &+ \int_{\Delta^0}\Big(\int_{\hat{S}} L_{\hat{S},z}(f_z)\omega_{\hat{S},z}^d\Big)u(z)dv + \int_{\Delta^0}\Big(\int_{\hat{S}} \Delta_{\hat{S},z}\big((\Delta_{g_0} f_s)_z\big)\omega_{\hat{S},z}^d\Big)u(z)dv \\ &+ \sum_{j\in\mathcal{I}}\int_{\Delta^0}\Big(\int_{\hat{S}} \Delta_{g_j}(f_z)\ \omega_{\hat{S}}^d\Big)p(z)R_j(z)u(z)dv. \end{aligned}$$

To see that all the terms vanish, note that the first term is zero by (15); the third and fourth terms are zero because $L_{\hat{S},z}$ and $\Delta_{\hat{S},z}$ are self-adjoint (with respect to $\omega_{\hat{S},z}$) and therefore their images are L^2-orthogonal to constants on $\hat{S}$. The fifth term is also zero because $\Delta_{g_j}(f)$ is L^2-orthogonal to constants on $\hat{S}$ with respect to $\omega_{\hat{S}}$: this follows easily from the local product structure of $g_{\hat{S}}$. For the second term one uses that Δ_{g_0} defines a self-adjoint operator on $C^\infty(\Delta)$ with respect to the measure $p(z)dv$: thus, for any smooth function $\phi(z)$ on Δ,

$$\int_{\hat{S}}\Big(\int_{\Delta^0} \Delta_{g_0}\big((\Delta_{\hat{S},z} f_z)_s\big)\phi(z)p(z)dv\Big)\omega_{\hat{S}}^d = \int_{\Delta^0}\Big(\int_{\hat{S}} \Delta_{\hat{S},z}(f_z)\omega_{\hat{S},z}^d\Big)(\Delta_{g_0}\phi)dv = 0$$

because $\Delta_{\hat{S},z} f_z$ is L^2-orthogonal to constants on $\hat{S}$; as u is in the closure in L^2 of pullbacks of smooth functions on Δ, the second term vanishes too.

This concludes the proof of the lemma. □

An immediate consequence of Proposition 2 is the following

Corollary 1. *The existence of a compatible extremal Kähler metric is an open condition on the set of admissible Kähler classes on M.*

Proof. As we have already observed, the admissible Kähler classes are parametrized by the real constants c_a for $a \in \mathcal{A}$. We thus apply Proposition 2 by taking $\lambda = (c_a, p_a^0, Scal_a^0)$. □

3.2. **Proof of Theorem 3.** To deduce Theorem 3 from Proposition 2, we observe that the differential operators (11) satisfy $P_{t\lambda} = P_\lambda$ for any real number $t \neq 0$.

On any Kähler manifold (M, g, ω) obtained by the generalized Calabi construction with data $\lambda = (c_a, p_a, Scal_a)$, we can consider the sequence of differential operators P_{λ_k} where $\lambda_k = (c_a + k, p_a, Scal_a)$. The differential operator P_{λ_k} is the same as $P_{\lambda_k/k}$ and $\frac{1}{k}\lambda_k$ converges when $k \to \infty$ to the data corresponding to the extremal Kähler metric equation for a compatible Kähler metrics on W. Theorem 3 follows at once.

Remark 6. As any invariant Kähler metric on a toric manifold is compatible, Theorem 3 implies the existence of (compatible) extremal metrics on a rigid semisimple toric bundles M over a CSC locally product Kähler manifold, in the case when there are no blow-downs and $W = V$ is a toric extremal Kähler manifold.

Remark 7. An interesting class of rigid toric bundles comes from the *multiplicity-free manifolds* recently discussed in [25]. A typical example is obtained by taking a compact connected semisimple Lie group G and a maximal torus $\mathbb{T} \subset G$ with Lie algebra $\mathfrak{t}$; if we pick a positive Weyl chamber $\mathfrak{t}_+ \subset \mathfrak{t}$ (and identify $\mathfrak{t}$ with its dual space $\mathfrak{t}^*$ via the Killing form), for any Delzant polytope Δ contained in the interior of $\mathfrak{t}_+$, one can consider the manifold $M = G \times_{\mathbb{T}} V \xrightarrow{p} S = G/\mathbb{T}$, where V is the toric manifold with Delzant polytope Δ. Note that G is a principal $\mathbb{T}$-bundle over the flag manifold $S = G/\mathbb{T}$ with a connection 1-form $\theta \in \Omega^1(G, \mathfrak{t})$ whose curvature $\omega(z) = \langle d\theta, z\rangle$ defines a family of symplectic forms on S (the Kirillov–Kostant–Souriau forms); identifying $S \cong G^c/B$, where B is a Borel subgroup of the complexification G^c of G, each $\omega(z)$ defines a homogeneous Kähler metric $g(z)$ on the complex manifold S (which is therefore of constant scalar curvature); the Ricci form ω_S of $\omega(z)$ is independent of z, giving rise to the normal (Kähler–Einstein) metric g_S on S. Now, for any toric Kähler metric on V, corresponding to a symplectic potential $U \in \mathcal{S}(\Delta)$, one considers the Kähler metric on M

$$g = p^*(g(z) + kg_S) + \langle dz, \mathbf{G}, dz\rangle + \langle \theta, \mathbf{H}, \theta\rangle, \quad \omega = p^*(\omega(z) + k\omega_S) + \langle dz \wedge d\theta\rangle,$$

where $\mathbf{G} = \mathrm{Hess}(U)$, $\mathbf{H} = \mathbf{G}^{-1}$, $z \in \Delta$ and $k > 0$. In this case, $G \to S = G/\mathbb{T}$ is not necessarily a *diagonalizable* principal $\mathbb{T}$-bundle over $S = G/\mathbb{T}$ (in other words, $M = G \times_{\mathbb{T}} V \to S = G/\mathbb{T}$ is a rigid but not in general semisimple toric bundle). However, most parts of the discussion in Sect. 2 do extend to this case too (see also [3]), with some obvious modifications. The key points are that (a) the volume form of $g(z) + kg_S$ is a multiple $p(z)$ (depending only on z) of Vol_{g_S}: this allows to extend the curvature computations (see [3, Prop. 7]) and formula (8) to this case, (b) for any $z \in \Delta$, $g(z)+kg_S$ is a CSC Kähler metric on S: this allows to extend the results in §2.4, and (c) there is a similar formula to (7) for the scalar curvature of g, found by Raza [59], which allows to reduce the extremal equation for the Kähler metrics in the above form to (10) with p_a being essentially the positive roots of G, $c_a = k$ and $Scal_a$ positive constants. Proposition 2 and its corollaries (Corollary 1 and Theorem 3) extend to this setting too. We thus get both openness and existence of extremal Kähler metrics of the above form when V is an extremal toric Kähler variety and $k \gg 0$.

3.3. **Proof of Theorem 4.** Theorem 4 (from the introduction) may be derived as the special case of Theorem 3, in which $V = \mathbb{C}P^\ell$ and $W = \mathbb{C}P^r$, $r \geq \ell \geq 1$ and $M = P(E_0 \oplus \cdots \oplus E_\ell) \to S$ (see §2.2). It follows from the general theory of hamiltonian 2-forms [3, 4] that any Fubini–Study metric on $\mathbb{C}P^r$ admits a rigid semisimple isometric action of an ℓ-dimensional torus $\mathbb{T}$, for any $1 \leq \ell \leq r$ (see in particular [3, Prop. 17] and [4, Thm. 5]): thus, $W = \mathbb{C}P^r$ admits a compatible extremal Kähler metric.

Let ω be a compatible Kähler on M; as the fibre is $\mathbb{C}P^r$, by re-scaling, we can assume without loss of generality that $[\omega] = 2\pi c_1(\mathcal{O}(1)_E) + p^*\alpha$, where α is a cohomology class on S. The form (6) of ω and the assumption on the first Chern classes $c_1(E_i)$ imply that α is diagonal with respect to the product structure of S, in the sense that it pulls back to the covering product space as $\alpha = \sum_{a\in\mathcal{A}} q_a[\omega_a]$ for some real constants q_a. Therefore, $\Omega_k = 2\pi c_1(\mathcal{O}(1)_E) + kp^*[\omega_S] = [\omega] + \sum_{a\in\mathcal{A}}(k - q_a)p^*[\omega_a]$. If we choose q with $q > q_a$, then $\tilde{\omega} = \omega + \sum_{a\in\mathcal{A}}(q - q_a)p^*\omega_a$ is clearly a compatible Kähler metric too. Thus, $\Omega_k = [\tilde{\omega}] + (k - q)p^*[\omega_S]$ with $[\tilde{\omega}]$ compatible, and we conclude using Theorem 3.

3.4. **Proof of Theorem 2.** The sufficiency of the condition for the existence of an extremal Kähler metric follows from Theorem 4.

For the necessity, suppose (g, ω) is an extremal Kähler metric in $\Omega_k = 2\pi c_1(\mathcal{O}(1)_E) + kp^*[\omega_\Sigma]$ on $(M, J) = P(E_0 \oplus \ldots \oplus E_\ell) \to \Sigma$, where E_i are indecomposable holomorphic vector bundles over a compact curve Σ of genus $\mathbf{g} \geq 2$. We can assume without loss of generality that ω_Σ is the Kähler form of a constant curvature metric on Σ and, by virtue of Theorem 1, that the scalar curvature of g is not constant. In particular, $\ell \geq 1$.

We have seen in Lemma 1 that the ℓ-dimensional torus $\mathbb{T}$ acting by scalar multiplication on each E_i is maximal in the reduced automorphism group $\widetilde{\mathrm{Aut}}_0(M,J) \cong H^0(\Sigma, PGL(E))$. By a well-known result of Calabi [14] the identity component of the group of Kähler isometries of an extremal Kähler metric is a maximal compact subgroup of $\mathrm{Aut}_0(M,J)$, so we can assume without loss that (g,ω) is $\mathbb{T}$-invariant.

By considering small stable deformations $E_i(t)$ and applying Lemma 4, we can find a smooth family of extremal $\mathbb{T}$-invariant Kähler metrics (J_t, g_t, ω_t), converging to (J,ω) in any $C^k(M)$, such that $(M, J_t) \cong P(\bigoplus_{i=1}^{\ell} E_i(t))$, and $[\omega_t] = [\omega]$ in $H^2_{dR}(M)$. By the equivariant Moser lemma, we can assume without loss of generality that $\omega_t = \omega$.

It is not difficult to see that any Kähler class on (M, J_t) (for $t \neq 0$) is *compatible*: this follows from the fact that the cohomology $H^2(M) \cong H^{1,1}(M, J_t)$ is generated by any compatible Kähler class on (M, J_t) and the pullback $p^*[\omega_\Sigma]$. By Theorem 4 and the uniqueness of the extremal Kähler metrics up to automorphisms [15], for any $t \neq 0$ we can take $k \gg 0$ such that the extremal Kähler metric (g_t, ω) on (M, J_t) is compatible with respect to the rigid semisimple action of the maximal torus $\mathbb{T}$. Strictly speaking, Theorem 4 produces a lower bound k_0 for such k, depending on J_t. However, in our case $|\mathcal{A}| = 1$, the simplex Δ, the moment map z and the metric on Σ are fixed, and the parameter $\lambda = (c, p, Scal_\Sigma)$ defining the corresponding extremal equation (10) for a compatible metric on $(M, J_t, [\omega])$ is independent of t: indeed, the constants $p \in \mathfrak{t}$ and $c \in \mathbb{R}$ are determined by the first Chern classes $c_1(E_i)$ and the cohomology class $\Omega_k = [\omega] \in H^2_{dR}(M)$. Thus, the deformation argument used in §3.2 produces a lower bound k_0 independent of t, such that for any $k > k_0$ and $t \neq 0$, (g_t, ω) is an extremal Kähler metric in Ω_k with respect to which the maximal torus $\mathbb{T}$ acts in a rigid and semisimple way.

Take a regular value z_0 of the momentum map z associated to the hamiltonian action of $\mathbb{T}$ on (M,ω) and consider the family of Kähler quotient metrics $(\hat{g}_t, \hat{J}_t)$ on the symplectic quotient $\hat{S}$. By identifying the symplectic quotient with the stable quotient, we see that $(\hat{S}, \hat{J}_t) \cong P(E_0(t)) \times_\Sigma \cdots \times_\Sigma P(E_\ell(t)) \to \Sigma$ (see §2.2). As for $t \neq 0$ the action of $\mathbb{T}$ is rigid and semisimple and g_t is compatible, the quotient Kähler metric $(\hat{g}_t, \hat{J}_t)$ must be locally a product of CSC Kähler metrics. By the de Rham decomposition theorem $\hat{g}_t$ must be a locally-symmetric metric modelled on the hermitian-symmetric space $\mathbb{C}P^{d_0} \times \cdots \times \mathbb{C}P^{d_\ell} \times \mathbb{H}$, where $d_i + 1 = \mathrm{rk}(E_i)$ (so that $\mathbb{C}P^{d_i}$ is a point if $d_i = 1$) and $\mathbb{H}$ is the hyperbolic plane. By continuity, $(\hat{g}_0, \hat{J}_0)$ is a locally-symmetric Kähler metric on $\hat{S}$ of the same type. By the de Rham decomposition theorem and considering the form of the covering transformations we obtain representations $\rho_i \colon \pi_1(\Sigma) \to PU(d_i+1)$, and therefore E_i must be stable by the standard theory [56].

Finally, consider the case when $\ell = 1$, i.e., $(M,J) = P(E_0 \oplus E_1)$. If E_0 and E_1 have equal slopes, any extremal Kähler metric on (M,J) must be CSC by Lemma 3 and our claim follows from Theorem 1.

If the slopes of E_0 and E_1 are different and (g,ω) is some extremal Kähler metric on $(M,J) = P(E_0 \oplus E_1)$, Lemma 4 shows that ω is the smooth limit of extremal Kähler metrics ω_t on $(M, J_t) = P(E_0(t) \oplus E_1(t))$ where $E_i(t)$ are stable for $t \neq 0$. Now, [5, Thm. 1] affirms that for any $t \neq 0$, ω_t is compatible with respect to the natural S^1-action, so we conclude as above that E_0 and E_1 must be stable. □

4. Further observations

4.1. **Relative K-energy and the main conjecture.** Leaving aside the specific motivation of this paper to study projective bundles over a curve, the theory of rigid semisimple toric bundles which we reviewed in Sect. 2 extends the theory of extremal Kähler metrics on toric manifolds [21, 22, 24, 67, 76, 77] to this more general context.

To recast the leading conjectures [21, 67] in the toric case to this setting, recall from [21] that if we parametrize compatible Kähler metrics g by their symplectic potentials $U \in \mathcal{S}(\Delta)$, then the relative (Mabuchi–Guan–Simanca) K-energy $\mathcal{E}^{\Omega}$ on this space satisfies the functional equation

$$\begin{aligned}(d\mathcal{E}^{\Omega})_g(\dot U) &= \int_\Delta (Scal_g^{\perp})\dot U(z)p(z)dv\\ &= \int_\Delta\Big(\Big(\langle A, z\rangle + B + \sum_{j\in\mathcal{I}}\frac{Scal_j}{\langle p_j, z\rangle + c_j}\Big)p(z) - \frac{\partial^2}{\partial z_r\partial z_s}(p(z)H_{rs})\Big)\dot U(z)dv\\ &= 2\int_{\partial\Delta}\dot U(z)p(z)d\sigma + \int_\Delta\Big(\langle A, z\rangle + B + \sum_{j\in\mathcal{I}}\frac{Scal_j}{\langle p_j, z\rangle + c_j}\Big)\dot U(z)p(z)dv\\ &\quad - \int_\Delta\langle \mathbf{H}, \operatorname{Hess}\dot U(z)\rangle p(z)dv,\end{aligned}$$

where we have used (10) and integration by parts by taking into account (4). Following [21, 67, 76], let us introduce the linear functional

$$\mathcal{F}^{\Omega}(f) := \int_{\partial\Delta} f(z)p(z)d\sigma + \frac{1}{2}\int_\Delta\Big(\langle A, z\rangle + B + \sum_j\frac{Scal_j}{\langle p_j, z\rangle + c_j}\Big)f(z)p(z)dv. \tag{18}$$

The above calculation of $d\mathcal{E}^{\Omega}_g$ shows that $\mathcal{F}^{\Omega}(f) = 0$ if f is an affine function of z. Furthermore, using the fact that the derivative of $\log\det\mathbf{H}$ is $\operatorname{tr}\mathbf{H}^{-1}d\mathbf{H}$, we obtain the following generalization of Donaldson's formula for $\mathcal{E}^{\Omega}$:

$$\mathcal{E}^{\Omega}(U) = 2\mathcal{F}^{\Omega}(U) - \int_\Delta\Big(\log\det\operatorname{Hess}U(z)\Big)p(z)dv. \tag{19}$$

(In case of doubt about the convergence of the integrals, one can introduce a reference potential U_c and a relative version $\mathcal{E}^{\Omega}_{g_c}$ of $\mathcal{E}^{\Omega}$, but in fact, as Donaldson shows, the convexity of U ensures that the positive part of $\log\det\operatorname{Hess}U(z)$ is integrable, hence $-\log\det\operatorname{Hess}U(z)$ has a well defined integral in $(-\infty,\infty]$.)

According to [21, 67], the existence of a solution $U \in \mathcal{S}(\Delta)$ to (10) should be entirely governed by properties of the linear functional (18):

Conjecture 2. *Let Ω be a compatible class on M. Then the following conditions should be equivalent:*

1. *Ω admits an extremal Kähler metric.*
2. *Ω admits a compatible extremal Kähler metric (i.e.,* (10) *has a solution in $\mathcal{S}(\Delta)$).*
3. *$\mathcal{F}^{\Omega}(f) \geq 0$ for any piecewise linear convex function f on Δ, and is equal to zero if and only if f is affine.*[11]

Of course, by the proof of Theorem 2, Conjecture 2 would imply Conjecture 1.

Our formula (19) can be used to show as in [21, Prop. 7.1.3] that $\mathcal{F}^{\Omega}(f) \geq 0$ if the relative K-energy is bounded from below. However, according to Chen–Tian [15], the boundedness from below of $\mathcal{E}^{\Omega}$ is a necessary condition for the existence of an extremal Kähler metric.

[11] Generalizing computations in [21, 67, 76], one can show that the value of $\mathcal{F}^{\Omega}$ at a *rational* piecewise linear convex function computes the relative Futaki invariant introduced in [67] of a 'compatible' toric test configuration on (M, Ω); in general, one might need positivity of $\mathcal{F}^{\Omega}$ on a larger space of convex functions [21, Conjecture 7.2.2.] in order to solve (10) but in the case when $\ell = 2$ and the base $\hat S$ is a point Donaldson shows in [21] that the space of piecewise linear convex functions will do.

If Ω admits a *compatible* extremal Kähler metric with symplectic potential U and inverse hessian $\mathbf{H}$, one can use (10) and integration by parts (taking into account (4)) in order to re-write (18) as

$$\mathcal{F}^{\Omega}(f) = \int_{\Delta} \langle \mathbf{H}, \mathrm{Hess} f\rangle p(z) dv. \tag{20}$$

This formula makes sense for smooth functions $f(z)$, but can also be used to calculate $\mathcal{F}^{\Omega}(f)$ in distributional sense for any piecewise linear convex function as in [77]: using the fact that $\mathbf{H}$ is positive definite, we obtain the analogue of a result in [77], showing that the second statement of Conjecture 2 implies the third.

We thus have the following partial result.

Proposition 3. *If Ω admits an extremal Kähler metric then $\mathcal{F}^{\Omega}(f) \geq 0$ for any convex piecewise linear function. If Ω admits a compatible extremal metric then, furthermore, $\mathcal{F}^{\Omega}(f) = 0$ if and only if f is an affine function on Δ.*

Of course, the most difficult part of Conjecture 2 is to prove (3) $\Rightarrow$ (2). So far the Conjecture 2 has been fully established in the cases when $\ell = 1$ [5] and when M is a toric surface (i.e., $\ell = 2$ and $\hat{S}$ is a point) with vanishing extremal vector field [24].

4.2. **Computing $\mathcal{F}^{\Omega}$.** It is natural to consider (following Donaldson [21]) the space of $S^2\mathfrak{t}^*$-valued functions $\mathbf{H}$ on Δ satisfying just the boundary conditions (4). If such a function satisfies the (underdetermined, linear) equation (10), then formula (20) holds, and it can be used to compute the action of $\mathcal{F}^{\Omega}$ (in distributional sense) on piecewise linear functions.

Note that if a solution to (10) exists, then so do many because the double divergence is underdetermined.

If a solution $\mathbf{H}$ of (10) happens to be *positive definite* on each face of Δ, i.e., if it verifies the positivity condition in §2.3, then formulae (6) introduce an almost Kähler metric on M (see e.g. [4]) and one can show that (7) computes its hermitian scalar curvature (see Appendix A). Thus, positive definite solutions of (10) correspond to compatible extremal almost Kähler metrics. If such extremal almost Kähler metrics exist, it then follows from (20) (see [77] and Proposition 3 above) that the condition (3) of Conjecture 2 is verified. Thus, the existence of a positive definite solution $\mathbf{H}$ of (10) (and verifying the boundary conditions (4)) is conjecturally equivalent to the existence of a compatible extremal Kähler metric (corresponding to another positive definite function $\mathbf{H}^{\Omega}$ with inverse equal to the hessian of a function U_{Ω}). In fact, following [21], as $\log\det$ is strictly convex on positive definite matrices, the functional $\int_{\Delta}(\log\det \mathbf{H})p(z)dv$ is strictly convex on the space of positive definite solutions of (10), and therefore has at most one minimum $\mathbf{H}^{\Omega}$. Such a minimum would automatically have its inverse equal to the hessian of a function U_{Ω} (see [21]). Thus, $\mathbf{H}^{\Omega}$ would then give the extremal Kähler metric in the compatible Kähler class Ω.

Thus motivated, it is natural to wonder if on the manifolds we consider in this paper a (not necessarily positive definite) solution $\mathbf{H}$ of (10) exists, thus generalizing the extremal polynomial introduced in [5] on $M = P(E_0 \oplus E_1) \to S$ (in fact $\mathbf{P}(z) = p(z)\mathbf{H}(z)$ would be the precise generalization).

4.3. **Example: projective plane bundles over a curve.** We illustrate the above discussion by explicit calculations on the manifold $M = P(\mathcal{O} \oplus \mathcal{L}_1 \oplus \mathcal{L}_2) \to \Sigma$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are holomorphic line bundles over a compact complex curve Σ of genus $\mathbf{g}$. We put $p_i = \deg(\mathcal{L}_i)$ and assume without loss of generality that $p_2 \geq p_1 \geq 0$. Note that in the case $p_1 = p_2 = 0$, the vector bundle $E = \mathcal{O} \oplus \mathcal{L}_1 \oplus \mathcal{L}_2$ is polystable, and therefore the existence of extremal Kähler metrics is given by Theorem 1. The cases $p_1 = p_2 > 0$

and $p_2 > p_1 = 0$, on the other hand, are solved in [5]. We thus assume furthermore that $p_2 > p_1 > 0$.

To recast our example in the set up of Sect. 2, we take a riemannian metric g_Σ of constant scalar curvature $4(1-\mathbf{g})$ on Σ. To ease the notation, we put $C = 4(\mathbf{g}-1)$. Let z_i be the momentum map of the natural S^1-action by multiplication on $\mathcal{L}_i$. Thus, without loss of generality, for a compatible Kähler metric on M, the momentum coordinate $z = (z_1, z_2)$ takes values in the simplex $\Delta = \{(z_1, z_2) \in \mathbb{R}^2 \mid z_1 \geq 0, z_2 \geq 0, 1-z_1-z_2 \geq 0\}$ (which is the Delzant polytope of the fibre $\mathbb{C}P^2$ viewed as a toric variety).

It is shown in [5, App. A2] that in this case there are no extremal compatible Kähler metrics with a hamiltonian 2-form of order 2 while Theorem 2 does imply existence of compatible extremal Kähler metrics in small Kähler classes. Therefore, we do not have an *explicit* construction of these extremal Kähler metrics. Instead, we will now attempt to find explicit extremal almost Kähler metrics (see the preceding section and the Appendix below). We thus want to find a smooth matrix function $\mathbf{H}(z) = (H_{rs}(z))$ satisfying the boundary conditions (4) and which solves the linear equation (10). Motivated by the explicit form of such a matrix in the case when a hamiltonian 2-form does exist [3], we look for solutions of a 'polynomial' form $H_{rs} = \frac{P_{rs}}{(c+p_1z_1+p_2z_2)}$, where $P_{rs}(z)$ are fourth degree polynomials in z_1 and z_2, and the constant c is such that $c + p_1z_1 + p_2z_2 > 0$ on Δ, i.e., $c \in (0, \infty)$: recall that c parametrizes the compatible Kähler classes on M. The boundary conditions are then solved by

$$\begin{aligned}
P_{11} &= 2(c + p_1z_1 + p_2z_2)z_1(1 - z_1)\\
&\qquad + z_1^2(x_0z_2^2 + x_2(1 - z_1 - z_2)^2 + 2y_1z_2(1 - z_1 - z_2)),\\
P_{12} &= -2(c + p_1z_1 + p_2z_2)z_1z_2\\
&\qquad + z_1z_2(y_0(1 - z_1 - z_2)^2 - x_0z_1z_2 - (1 - z_1 - z_2)(y_1z_1 + y_2z_2)),\\
P_{22} &= 2(c + p_1z_1 + p_2z_2)z_2(1 - z_2)\\
&\qquad + z_2^2(x_0z_1^2 + x_1(1 - z_1 - z_2)^2 + 2y_2z_1(1 - z_1 - z_2)),
\end{aligned}$$

where $x_0, x_1, x_2, y_0, y_1, y_2$ are free parameters. The extremal condition (10) corresponds to the linear equations

$$\begin{aligned}
y_0 &= -x_1 - x_2 + v_0,\\
y_1 &= -x_0 - x_2 + v_1,\\
y_2 &= -x_0 - x_1 + v_2,
\end{aligned} \tag{21}$$

with

$$\begin{aligned}
v_0 &= \frac{-(12c+C+4p_1+4p_2)(5cp_1^2+p_1^3+5cp_1p_2+5p_1^2p_2+5cp_2^2+5p_1p_2^2+p_2^3))}{(2(50c^3+50c^2p_1+13cp_1^2+p_1^3+50c^2p_2+37cp_1p_2+5p_1^2p_2+13cp_2^2+5p_1p_2^2+p_2^3)}\\
v_1 &= \frac{-(12c+C+4p_1+4p_2)(15cp_1^2+3p_1^3-15cp_1p_2+3p_1^2p_2+5cp_2^2-3p_1p_2^2+p_2^3))}{(2(50c^3+50c^2p_1+13cp_1^2+p_1^3+50c^2p_2+37cp_1p_2+5p_1^2p_2+13cp_2^2+5p_1p_2^2+p_2^3)}\\
v_2 &= \frac{-(12c+C+4p_1+4p_2)(5cp_1^2+p_1^3-15cp_1p_2-3p_1^2p_2+15cp_2^2+3p_1p_2^2+3p_2^3))}{(2(50c^3+50c^2p_1+13cp_1^2+p_1^3+50c^2p_2+37cp_1p_2+5p_1^2p_2+13cp_2^2+5p_1p_2^2+p_2^3)}.
\end{aligned}$$

Thus, given a compatible Kähler class on M, we have a 3-parameter family of smooth 'polynomial' solutions $\mathbf{H}(z)$ to (10), which verify the boundary conditions (4) on Δ. Now, investigating the integrability condition (that $\mathbf{H}^{-1}$ be a hessian of a smooth function on Δ^0, see the previous section), we find out that it is equivalent to the following five algebraic equations on the parameters $(x_0, x_1, x_2, y_0, y_1.y_2)$

$$x_0 = y_1 + y_2, \qquad x_1 = y_2 + y_0, \qquad x_2 = y_0 + y_1, \tag{22}$$

$$\begin{aligned}
2(p_2 - p_1)y_0 + 2p_2y_1 - y_0y_1 = 0\\
2(p_1 - p_2)y_0 + 2p_1y_2 - y_0y_2 = 0.
\end{aligned} \tag{23}$$

The problem is over-determined, but there is a unique solution $\mathbf{H}_0^\Omega$ satisfying the linear system (22) (additionally to (21)): we compute that this solution is given by

$$\begin{aligned}
x_0 &= \tfrac{1}{10}(-2v_0 + 3v_1 + 3v_2),\\
x_1 &= \tfrac{1}{10}(3v_0 - 2v_1 + 3v_2),\\
x_2 &= \tfrac{1}{10}(3v_0 + 3v_1 - 2v_2),\\
y_0 &= \tfrac{1}{10}(4v_0 - v_1 - v_2),\\
y_1 &= \tfrac{1}{10}(-v_0 + 4v_1 - v_2),\\
y_2 &= \tfrac{1}{10}(-v_0 - v_1 + 4v_2).
\end{aligned}$$

Substituting back in (23), one sees that the full integrability conditions can be solved if $12c+C+4p_1+4p_2 = 0$ (a constraint that is never satisfied for $p_2 > p_1 \geq 1, C = 4(\mathbf{g}-1)$ and $c > 0$); this observation is consistent with the non-existence result in [5, App. A2].

We now investigate the positivity condition for our distinguished solution $\mathbf{H}_0^\Omega$ of (4) and (10). First of all, when $c \to \infty$, the v_i's tend to 0, so $\mathbf{H}_0^\Omega$ tends to the matrix associated to a Fubini–Study metric on $\mathbb{C}P^2$. It follows that $\mathbf{H}_0^\Omega$ becomes positive-definite on each face for sufficiently small Kähler classes, and therefore $\mathbf{H}_0^\Omega$ defines an *explicit* extremal (non-Kähler) almost Kähler metric in Ω (see Appendix A below). This is of course consistent (via Conjecture 2) with the existence of a (non-explicit) extremal Kähler metric in Ω, given by Theorem 2. Furthermore, if $\mathbf{g} = 0, 1$ (i.e., $C < 0$), a computer assisted verification shows that, in fact, $\mathbf{H}_0^\Omega$ is positive definite on each face of Δ for *all* Kähler classes. We thus obtain the following result.

Proposition 4. *Let $M = P(E) \xrightarrow{p} \Sigma$ with $E = \mathcal{O} \oplus \mathcal{L}_1 \oplus \mathcal{L}_2$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are holomorphic line bundles of degrees $1 \leq p_1 < p_2$ over a compact complex curve Σ of genus $\mathbf{g}$.*

If $\mathbf{g} = 0, 1$, then M admits a compatible extremal almost Kähler metric in any Kähler class $\Omega_k = 2\pi c_1(\mathcal{O}(1)_E) + kp^*[\omega_\Sigma]$. In particular, for every Kähler class on M the condition (3) *of Conjecture* 2 *is verified.*

If $\mathbf{g} \geq 2$, then the same conclusion holds for the Kähler classes $\Omega_k = 2\pi c_1(\mathcal{O}(1)_E) + kp^*[\omega_\Sigma]$ with $k \gg 0$, i.e., the Kähler classes sufficiently far from the boundary of the Kähler cone.

As speculated in the previous section, the explicit solution $\mathbf{H}_0^\Omega$ of (4) and (10) can be used to compute the action of the functional $\mathcal{F}^\Omega$ on piece-wise linear convex functions (by extending formula (20) in a distributional sense, after integrating by parts and using (4)). As a simple illustration of this, let us take a simple crease function f_a with crease along the segment $S_a = \{(t, a-t), 0 < t < a\}$ for some $a \in (0,1)$ (thus as $a \to 0$, the crease moves to the lower left corner of the simplex Δ). A normal of the crease is $u = (1,1)$ and one easily finds that

$$\begin{aligned}
\mathcal{F}^\Omega(f_a) &= \int_{S_a} H_0^\Omega(u,u)d\sigma\\
&= \int_0^a ((H_{11} + 2H_{12} + H_{22})(t, a-t))(c + p_1 t + p_2(a-t))\, dt,
\end{aligned} \tag{24}$$

where $d\sigma$ is the contraction of the euclidian volume dv on $\mathbb{R}^2$ by u. Note that the integrand (being a rational function of c with a non-vanishing denominator at $c = 0$), and hence the integral, is continuous near $c = 0$; for $c = 0$ the integral equals

$$\tfrac{1}{6}(1-a)a^3(-C + 2(p_1+p_2) + a(C + 4(p_1+p_2))),$$

which is clearly negative for $a \in (0,1)$ sufficiently small as long as $C = 4(\mathbf{g}-1) > 2(p_1+p_2)$. If we take $\mathbf{g} > 2$, such p_1 and p_2 do exist. By Proposition 3, this implies a non-existence result of extremal Kähler metrics when p_1 and p_2 satisfy the above inequality and c is small enough. (As a special case, for $p_1 = p_2$ we have recast the non-existence part of [5, Thm. 6].)

Proposition 5. *Let M be as in Proposition 4, with $\mathbf{g} > 2$ and p_1, p_2 satisfying $2(\mathbf{g}-1) > p_1+p_2$. Then the Kähler classes sufficiently close to the boundary of the Kähler cone do not admit any extremal Kähler metric.*

Appendix A. Compatible extremal almost Kähler metrics

In this appendix, we calculate the hermitian scalar curvature of a compatible almost Kähler metric and extend the notion of *extremal* Kähler metrics to the more general almost Kähler case.

Recall that on a general almost Kähler manifold (M^{2m}, g, J, ω), the *canonical hermitian connection* ∇ is defined by

$$\nabla_X Y = D_X Y - \frac{1}{2} J(D_X J)(Y), \tag{25}$$

where D is the Levi–Civita connection of g. Note that

$$g((D_X J)Y, Z) = \frac{1}{2} g(N(X,Y), JZ) \tag{26}$$

where $N(X,Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]$ is the Nijenhuis tensor of J. The *Ricci form*, ρ^∇, of ∇ represents $2\pi c_1(M, J)$ and its trace s^∇ (given by $2m\rho^\nabla \wedge \omega^{m-1} = s^\nabla \omega^m$) is called *hermitian scalar curvature* of (g, J, ω).

The hermitian scalar curvature plays an important role in a setting described by Donaldson [19] (see also [32]), in which s^∇ is identified with the momentum map of the action of the group $\mathrm{Ham}(M, \omega)$ of hamiltonian symplectomorphisms of a compact symplectic manifold (M, ω) on the (formal) Kähler Fréchet space of ω-compatible almost Kähler metrics $\mathcal{AK}_\omega$. It immediately follows from this formal picture [26, 47] that the critical points of the functional on $\mathcal{AK}_\omega$

$$g \longmapsto \int_M (s^\nabla)^2 \omega^m$$

are precisely the ω-compatible almost Kähler metrics for which $\mathrm{grad}_\omega s^\nabla$ is a Killing vector field. This provides a natural extension of the notion of an extremal Kähler metric to the more general almost Kähler context.

Definition 5. An almost Kähler metric (g, ω) for which $\mathrm{grad}_\omega s^\nabla$ is a Killing vector field is called *extremal*.

Now let M be a manifold obtained by the generalized Calabi construction of §2.3. In the notation of this section, for any $S^2\mathfrak{t}^*$-valued function $\mathbf{H}$ on Δ, satisfying the boundary and positivity conditions, formulae (6) introduce a pair (g, ω) of a smooth metric g and a symplectic form ω on M, such that the field of endomorphisms J defined by $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$ is an almost complex structure, i.e., (g, ω) is an almost Kähler structure on M.[12] We shall refer to such pairs (g, ω) as *compatible almost Kähler metrics* on M.

[12] It is easily seen as in [1] that J is integrable, i.e., (g, ω) defines a Kähler metric, if and only if $\mathbf{H}^{-1}$ is the hessian of a smooth function on Δ^0.

Lemma 9. *The hermitian scalar curvature s^∇ of a compatible almost Kähler metric corresponding to $\mathbf{H} = (H_{rs})$ is given by*

$$s^\nabla = \sum_{j\in\mathcal{I}} \frac{Scal_j}{c_j + \langle p_j, z\rangle} - \frac{1}{p(z)} \sum_{r,s=1}^{\ell} \frac{\partial^2}{\partial z_r \partial z_s}(p(z)H_{rs}).$$

Proof. The result is local and we work on the open dense subset M^0 where the ℓ-torus $\mathbb{T}$ acts freely. Recall that M^0 is a principal $\mathbb{T}^c$ bundle over $\hat{S}$. Let $\mathcal{V}$ be the foliation defined by the $\mathbb{T}^c$ fibres and $K_r = J\mathrm{grad}_g z_r$ be the Killing vector fields generating $\mathbb{T}$; then $T\mathcal{V}$ is spanned by K_r, JK_r at each point of M^0 and, by construction,

$$\mathcal{L}_{K_r} J = 0, \quad K_r^\flat = \sum_s H_{rs}\hat{\theta}_s, \quad JK_r^\flat = -dz_r. \tag{27}$$

In order to compute the hermitian Ricci tensor, we take a local non-vanishing *holomorphic* section $\Phi_{\hat{S}}$ of the anti-canonical bundle $\mathcal{K}_{\hat{S}}^{-1} = \wedge^{d,0}(\hat{S})$ of $\hat{S}$ (which pulls back to a $(d,0)$-form on M) and wedge it with the $(\ell,0)$-form $\Phi_{\mathcal{V}} = (K_1^\flat - \sqrt{-1}JK_1^\flat) \wedge \cdots \wedge (K_\ell^\flat - \sqrt{-1}JK_\ell^\flat)$. Thus, $\Phi = \Phi_{\hat{S}} \wedge \Phi_{\mathcal{V}}$ is a non-vanishing section of $\mathcal{K}_{M^0}^{-1}$ and the hermitian Ricci form ρ^∇ is then given by

$$\rho^\nabla = -d\mathfrak{Im}(\alpha),$$

where $\nabla\Phi = \alpha \otimes \Phi$.

Denote by $\mathcal{H}$ the g-orthogonal complement of $T\mathcal{V}$; the spaces $\mathcal{H}$ and $T\mathcal{V}$ then define the decomposition of TM^0 as the sum of horizontal and vertical spaces and, therefore

$$\nabla_X \Phi = (\nabla_X^{\mathcal{H}} \Phi_{\hat{S}}) \wedge \Phi_{\mathcal{V}} + \Phi_{\hat{S}} \wedge \nabla_X^{\mathcal{V}} \Phi_{\mathcal{V}}, \tag{28}$$

where $\nabla_X Y = \nabla_X^{\mathcal{H}} Y + \nabla_X^{\mathcal{V}} Y$ denotes the decomposition into horizontal and vertical parts.

Our first observation is that [3, Prop. 8] generalizes in the non-integrable case in the following sense: *The foliation $\mathcal{V}$ is totally-geodesic with respect to both the Levi–Civita and hermitian connections.* Indeed, with respect to the Levi–Civita connection D we have $\langle D_{K_r} K_s, X\rangle = \langle D_{JK_r} K_s, X\rangle = 0$ for any $X \in \mathcal{H}$; using $[K_r, JK_s] = 0$, our claim reduces to check that $\langle D_{JK_r} JK_s, X\rangle = 0$. We take X be the horizontal lift of a basic vector field and use the Koszul formula

$$\begin{aligned}
2\langle D_{JK_r} JK_s, X\rangle &= \mathcal{L}_{JK_r}\langle JK_s, X\rangle + \mathcal{L}_{JK_s}\langle JK_r, X\rangle - \mathcal{L}_X \langle JK_r, JK_s\rangle \\
&\quad + \langle [JK_r, JK_s], X\rangle + \langle \mathcal{L}_X JK_r, JK_s\rangle + \langle \mathcal{L}_X JK_s, JK_r\rangle \\
&= \langle \mathcal{L}_X JK_r, JK_s\rangle + \langle \mathcal{L}_X JK_s, JK_r\rangle \\
&= (\mathcal{L}_X g)(JK_r, JK_s) = (\mathcal{L}_X \hat{g}(z))(JK_r, JK_s) = 0,
\end{aligned}$$

where $(\hat{g} = \hat{g}(z), \hat{\omega} = \hat{\omega}(z))$ denote the Kähler quotient structure on $\hat{S}$ (also identified with the horizontal part of (g, ω)). Considering the hermitian connection ∇, by (25) and (26), our claim reduces to showing that $N(K_r, X)$ is horizontal for any $X \in \mathcal{H}$; using (26) and the fact that $\mathcal{V}$ is totally-geodesic with respect to D, we get $\langle N(K_r, X), JU\rangle = 2\langle (D_U J)(K_r), X\rangle = 0$, for any $U \in T\mathcal{V}$.

The observation that $\mathcal{V}$ is totally geodesic with respect to D shows that formulae (42)–(46) in [3, Prop. 9] hold true in the non-integrable case too, i.e., we have

$$\begin{aligned}
D_X Y &= D_X^{\mathcal{H}} Y - C(X, Y) \\
D_X U &= \langle C(X, \cdot), U\rangle + [X, U]^{\mathcal{V}} \\
D_U X &= [U, X]^{\mathcal{H}} + \langle C(X, \cdot), U\rangle \\
D_U V &= D_U^{\mathcal{V}} V,
\end{aligned} \tag{29}$$

where $X, Y \in \mathcal{H}$, $U, V \in T\mathcal{V}$ and $C(\cdot,\cdot)$ is the O'Neill tensor given by

$$2C(X,Y) = \sum_{r=1}^{\ell}\Big(\Omega_r(X,Y)K_r + \Omega_r(JX,Y)JK_r\Big)$$

with $\Omega_r = d\hat{\theta}_r = \sum_{j\in\mathcal{I}} p_{jr}\otimes\omega_j$. Using (25), it follows that the horizontal lift of $\nabla^{\hat{S}}$ coincides with the projections of both D and ∇ to horizontal vectors. In particular, for any horizontal lift X, $\nabla^{\mathcal{H}}_X\Phi_{\hat{S}} = \frac{1}{2}\Big((d_{\hat{S}} - \sqrt{-1}d^c_{\hat{S}})\log||\Phi_{\hat{S}}||^2_{\hat{g}}\Big)(X)\Phi_{\hat{S}}$. On the other hand, as K_r are Killing and $\mathcal{V}$ is totally geodesic, $\nabla^{\mathcal{V}}_X\Phi_{\mathcal{V}} = 0$, so that we get from (28)

$$\alpha(X) = \frac{1}{2}\Big((d_{\hat{S}} - \sqrt{-1}d^c_{\hat{S}})\log||\Phi_{\hat{S}}||^2_{\hat{g}}\Big)(X), \quad \forall X\in\mathcal{H}. \tag{30}$$

To compute $\alpha(U)$ for $U\in T\mathcal{V}$, consider first $\nabla_U\Phi_{\mathcal{V}}$. As $\mathcal{V}$ is totally geodesic, we can write $\nabla_U\Phi_{\mathcal{V}} = (a(U) - \sqrt{-1}b(U))\Phi_{\mathcal{V}}$. It follows from the very definition of $\Phi_{\mathcal{V}}$ (and the fact that $\mathrm{span}(K_1,\cdots,K_\ell)$ is ω-Lagrangian) that $\Phi_{\mathcal{V}}(K_1,K_2,\cdots,K_\ell) = \det g(K_r,K_s) = \det\mathbf{H}$, and therefore

$$(\nabla_U\Phi_{\mathcal{V}})(K_1,K_2,\cdots,K_\ell) = \Big(a(U) - \sqrt{-1}b(U)\Big)\det\mathbf{H}.$$

Using the definition of $\Phi_{\mathcal{V}}$ again, we obtain

$$b(U) = \mathrm{trace}(\mathbf{H}^{-1}\circ A_U), \quad (A_U)_{rs} = -\langle\nabla_U K_r, JK_s\rangle.$$

Using that K_r is Killing, (26) and (27) we further calculate

$$\begin{aligned}
(A_U)_{rs} &= -\langle D_U K_r, JK_s\rangle + \frac{1}{2}\langle(D_U J)(K_r), K_s\rangle\\
&= \frac{1}{2}\Big(dK_r^\flat(JK_s,U) - \frac{1}{2}\omega(N(K_r,K_s),U)\Big)\\
&= \frac{1}{2}\Big(\sum_{p,k}H_{rk,p}dz_p(JK_s)\hat{\theta}_k(U) - \frac{1}{2}\sum_k dz_k([JK_r,JK_s])\hat{\theta}_k(U)\Big)\\
&= \frac{1}{2}\Big(-\sum_{k,p}H_{rk,p}H_{ps}\hat{\theta}_k(U) - \frac{1}{2}\sum_k dz_k([JK_r,JK_s])\hat{\theta}_k(U)\Big)\\
&= -\frac{1}{4}\sum_{k,p}(H_{rk,p}H_{ps} + H_{sk,p}H_{pr})\hat{t}heta_k(U),
\end{aligned}$$

so that

$$b(U) = \sum_{r,s}H^{rs}(A_U)_{rs} = -\frac{1}{2}\sum_{r,k}H_{rk,r}\hat{\theta}_k(U). \tag{31}$$

Finally, in order to compute $\nabla_U\Phi_{\hat{S}}$, note that $\mathcal{L}_U\Phi_{\hat{S}} = 0$, and therefore

$$\begin{aligned}
(\nabla_U\Phi_{\hat{S}})(X_1,\cdots,X_d) = \sum_{k=1}^{d}\Big(&\frac{1}{2}\Phi_{\hat{S}}(X_1,\cdots,X_{k-1},J(D_UJ)(X_k),X_{k+1},\cdots,X_d)\\
&- \Phi_{\hat{S}}(X_1,\cdots,X_{k-1},(D^{\mathcal{H}}_{X_k}U),X_{k+1},\cdots,X_d)\Big),
\end{aligned}$$

where $X_k\in\mathcal{H}$. Now, using (29), we further specify

$$\begin{aligned}
&(D^{\mathcal{H}}_{X_k}U) = \frac{1}{2}\sum_{r=1}^{\ell}\sum_{j\in\mathcal{I}}\Big(K_r^\flat(U)p_{jr}(J_jX_k^j) - JK_r^\flat(U)p_{jr}X_k^j\Big),\\
&\Big((D_UJ)(X_k)\Big)^{\mathcal{H}} = 0,
\end{aligned}$$

where X_k^j (resp. J_j) denote the $g_{\hat{S}}$-orthogonal projection (resp. restriction) of X_k (resp. J) to the subspace $TS_j \subset T\hat{S}$ (recall that the universal cover of $(\hat{S}, g_{\hat{S}})$ is the Kähler product of (S_j, g_j, ω_j), so that the projection of TS_j to $T\hat{S}$ is a well-defined D-parallel subbundle of $T\hat{S}$). Using (27), and the expressions (30) and (31), we eventually find that

$$\begin{aligned}\mathfrak{Im}(\alpha) &= -\frac{1}{2}d^c_{\hat{S}}\log||\Phi_{\hat{S}}||^2_{\hat{g}} + \frac{1}{2}d^c\log p(z) + \frac{1}{2}\sum_{k,r}H_{kr,k}\hat{\theta}_r\\ &= -\frac{1}{2}d^c_{\hat{S}}\log||\Phi_{\hat{S}}||^2_{\hat{g}} + \frac{1}{2p(z)}\sum_{k,r}\Big((\frac{\partial p}{\partial z_k})H_{kr} + p(z)\frac{\partial H_{kr}}{\partial z_k}\Big)\hat{\theta}_r\\ \rho^\nabla &= \sum_{j\in\mathcal{I}}\rho_j - \sum_{i,r,k}\frac{\partial}{\partial z_k}\Big(\frac{1}{2p(z)}\frac{\partial(p(z)H_{ir})}{\partial z_i}\Big)dz_k\wedge\theta_r\\ &\quad - \frac{1}{2p(z)}\sum_{i,r}\frac{\partial(p(z)H_{ir})}{\partial z_i}\frac{\partial\hat{\omega}}{\partial z_r},\end{aligned}$$

where, we recall, ρ_j is the Ricci form of (S_j, g_j, ω_j), $\hat{\omega}(z) = \sum_{j\in\mathcal{I}}\Big(\sum_{r=1}^{\ell}(p_{jr}z_r + c_j)\omega_j\Big)$, and $p(z) = \prod_{j\in\mathcal{I}}\Big(\sum_{r=1}^{\ell}p_{jr}z_r + c_j\Big)^{d_j}$. The formula for s^∇ follows easily. $\square$

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