Conformal Geometry, Representation Theory and Linear Fields

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1. Referent: Prof. Dr. Hermann Karcher

2. Referent: Prof. Dr. Jens Franke

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Introduction

The aim of this dissertation is to uncover and explore the intimate relation between three different branches of mathematics: (A) local differential geometry on conformal manifolds, in particular conformally invariant linear and bilinear differential operators, (B) homomorphisms and coproducts between induced modules in representation theory, and (C) interactions of particles and fields in classical relativistic linear field theories for electromagnetism and gravity in conformal Minkowski space.

The Möbius group $G = \mathbf{O}\,(p+1, n-p+1)$ of conformal diffeomorphisms acts transitively upon the conformal sphere $S^{n-p,p}$ of signature $(n-p, p)$; hence the sphere occurs as homogeneous space $S^{n-p,p} = G/P$ with a (parabolic) stabilizer group P. Representations W of the Möbius group enter into all three contexts in a fundamental way. In conformal geometry (A), the vector bundle associated to a G-representation W carries a canonical covariant derivative (induced by the normal Cartan connection); on the conformal sphere, the covariant derivative is flat and the space of parallel sections is isomorphic to W. This is interpreted in representation theory (B) as a canonical surjection from the G-module induced by W^* as a P-representation onto W^* using Frobenius reciprocity. The physical significance (C) of such representations W was discovered by Penrose, see [PR84], who called elements in W *twistors*. He identified W as the kernel of an overdetermined conformally invariant operator, called a *twistor operator*. In linear field theories twistors play the role of generalizations of electric charges such as gravitational masses, and they play a key role in this dissertation.

A second manifestation of the above relationship between conformal geometry, representation theory and field theory, is the fact that variations of the deRham complex of exterior derivatives between alternating forms occur in all three of them: (A) associated to any G-representation W is a locally exact complex of conformally invariant differential operators between sections of homogeneous bundles over $S^{n-p,p} = G/P$, see Eastwood Rice [ER87] or theorem 5.12, (B) any G-representation W is subject to the parabolic Bernstein Gelfand Gelfand resolution in terms of modules induced by P-representations, see Lepowsky [Lep77] or theorem 6.59, and (C) in $n = 4$ dimensions with Lorentzian signature $(3, 1)$ any such space W characterizes a linear field theory based upon gauges, potentials and fields, see [PR84] or section 5.1. For the trivial representation $W = \mathbb{R}$ we recover electromagnetism since Maxwell's equations are based upon the deRham complex. The complexes occurring in (A), (B) and (C) are named after Bernstein, Gelfand and Gelfand, and are called BGG complexes for short. The first operator in such a complex is the twistor operator.

The main result of this dissertation is a new construction of the BGG complexes using a projection S defined on twisted deRham complexes by an explicit finite power series. This projection induces a homotopy equivalence between the twisted deRham complex and the corresponding BGG complex, and has the additional benefit of revealing a new feature: not only linear operators, but bilinear operators, generalizing the wedge product on the deRham complex, are manifest at the interface of these three topics. In conformal geometry (A), the projection S translates the wedge product on alternating forms into bilinear differential pairings on the BGG complexes. These pairings and operators in the BGG complex are subject to a general Leibniz rule and hence induce a cup product on the BGG cohomology, see theorem 5.13. In algebraic terms (B), if $\mathfrak{g}$ is a semisimple Lie algebra and $\mathfrak{p} \subset \mathfrak{g}$ a parabolic subalgebra, then S is an explicit $\mathfrak{g}$-projection on the $\mathfrak{g}$-modules induced by $\Lambda^k(\mathfrak{g}/\mathfrak{p}) \otimes W^*$, see corollary 6.55. The image of the projection S is isomorphic to the $\mathfrak{g}$-module induced by the relative cohomology space $H^k((\mathfrak{g}/\mathfrak{p})^*, W^*)$. The map S defines a homotopy equivalence between the deRham resolution twisted by W^* and the parabolic BGG resolution of W^*, see theorem 6.59. The wedge coproduct induces a new coproduct on the BGG resolution which satisfies a general co-Leibniz rule, see theorem 6.61. In linear field theories (C) the *helicity* of a field theory is a number reflecting a particular *polarization property* of plane wave solutions of that theory. For example electromagnetic waves have helicity one and gravitational waves are expected to have helicity two. The bilinear differential pairings provide a wide-reaching generalization of helicity raising and lowering between solutions in different conformally invariant linear field theories, which leads, for example, to general constructions of conserved quantities, see the applications in section 5.4.

The motivation to study linear field theories other than the electromagnetic theory stems from the quest for a better understanding of gravitational waves. This phenomenon is the gravitational analogue of electromagnetic waves—the latter have a satisfactory description in terms of Maxwell's equations. One way to study gravitational interactions is to set up a linear field theory with fields subject to equations analogous to Maxwell's equations and with a force law which predicts the motion of gravitational particles within these fields. Einstein's theory of gravity is nonlinear and linearizations of the Einstein equation are obvious candidates for a linear theory of gravity. The naive linearization of the Einstein equation on the level of the metric, with Minkowski background, fails to be conformally invariant. Instead we take the electromagnetic theory as a starting point for a development of general linear field theories. The guiding principle will be conformal invariance of the theory: gravitational waves are expected to travel with the speed of light and the light cone geometry only determines a conformal structure.

A linear field theory describing gravitational interaction deals with the following physical phenomenon: motion of gravitational masses produces fields which travel with the speed of light and these fields influence the motion of gravitational masses. In particular accelerated sources will produce gravitational wave solutions. Any G-representation W leads to such a field theory, but the expected polarization properties of plane gravitational waves rule out most W leaving only three theories due to Fierz, Bach and Penrose, which we will recall in some detail, see sections 5.5, 5.6, 5.7. The mentioned generalizations of helicity raising and lowering allow us to suggest the beginning of a classical theory of motion i.e. a theory describing the interaction of particles and fields in arbitrary field theories with conformally flat background: the elements in W play the role of generalized charges or gravitational masses in the corresponding linear field theory. We will demonstrate how an arbitrary worldline with an associated gravitational mass (i.e. an element in W, a twistor) gives rise to a conserved distributional source, see paragraph 5.6. Moreover we can solve the general field equations with that right hand side: the Lienard Wiechert fields in electromagnetism are solutions of Maxwell's equations with a distributional source coming from an electrically charged point-like worldline. We will construct general Lienard Wiechert fields solving the general field equations with a distributional source representing a gravitational mass associated to an arbitrary worldline, see paragraph 5.11. In electromagnetism a straight worldline induces the static Coulomb field, see paragraph 2.15 and accelerated worldlines induce radiation fields, i.e. fields which fall of like $1/r$ where r is the (luminosity) distance to the source, see paragraph 5.10. Hence the general Lienard Wiechert fields provide solutions which model the phenomenon that motion of gravitational masses produces gravitational waves that travel with the speed of light. On the other hand the Lorentz force law in electromagnetism predicts the force which acts on an electrically charged test particle moving in an electromagnetic field, see paragraph 2.14. This force is proportional to the electric test charge. Similarly the gravitational mass (i.e. the element in W) of a test particle allows to define a force from a general field which we suggest to be the force of the field acting on the test particle, see paragraph 5.7. This force law models the phenomenon that gravitational fields influence the motion of gravitational test particles. We claim that the above is the beginning of a theory of motion for gravitational particles influencing each other through their gravitational fields. Notice that pointlike particles as sources in the theory obtained by a linearization of the Einstein equation are only conserved, if they follow a straight line, see Stephani [Ste91] p. 89 or remark 4.27. This prevents a theory of motion of various particles influencing each other in a linearized Einstein theory, since already their individual conservation laws forces them to follow straight lines.

Invariant differential operators are of interest in differential geometry: the Bianchi identity of the curvature tensor or the Codazzi equation of the second fundamental form of a hypersurface are examples where differential operators occur in integrability conditions. The first analytic question on a manifold with conformal structure is which differential operators are intrinsically defined. A classification of linear conformally invariant first order operators has been given by Fegan, Hitchin, Gauduchon [Feg76, Hit80, Gau91], see theorem 4.36. Second order operators were classified by Branson [Bra96, Bra98], see the examples 4.47, 4.48 and proposition 4.51. Some of these operators are twistor operators, i.e. a section in the kernel of such an operator is the curved analogue of an element in a G-representation W, see propositions 4.17, 4.24. Motivated by conservation of energy momentum in relativity, we will explain, in a geometric context, how twistors give rise to new conserved properties along conformal geodesics using bilinear differential pairings along curves, see propositions 4.19, 4.22, 4.25. We extend the theory of first and second order conformally invariant operators by characterizing conformally invariant bilinear differential pairings by algebraic data and construct some new classes of simple examples, see 4.39, 4.42, 4.52 ff. First and second order operators and pairings are sufficient to do helicity raising and lowering between electromagnetism and the three linear theories of gravity as indicated in sections 5.5, 5.6, 5.7. Indeed it was these examples of conformally invariant bilinear pairings and Leibniz rules which led us to conjecture the general existence of differential pairings imitating the wedge product between forms.

In this dissertation we will explore the new product structure in BGG complexes on homogeneous parabolic geometries, i.e. on homogeneous spaces G/P where G is a semisimple Lie group and P is a parabolic stabilizer group. For an extension of our result to curved parabolic geometries using Cartan connections see the article by Calderbank and the author [CD99]. Parabolic geometry includes for example conformal geometry in $n \geq 3$ dimensions, Möbius geometry in $n = 2$ dimensions, projective geometry, Graßmannian geometry and Cauchy Riemann geometry. The programme of parabolic invariant theory was initiated by Fefferman and Graham [Fef79, FG85]. We will focus on G-equivariant differential operators and pairings on G/P acting on sections of homogeneous bundles. Let $\mathfrak{p} \subset \mathfrak{g}$ denote the Lie algebras of $P \subset G$. The intimate relation between geometry and algebra is given in this context by the fact that equivariant differential operators on G/P correspond to $\mathfrak{g}$-equivariant homomorphisms between Verma modules induced by $\mathfrak{p}$-representations. First versions of this observation go back to Bernstein Gelfand Gelfand [BGG75] and Eastwood Rice [ER87] and were fully discovered independently by Baston Eastwood [BE89], Collingwood Shelton [CS90] and Soergel [Soe90]. The study of homomorphisms between Verma modules was initiated by Verma [Ver68] and continued by Bernstein Gelfand Gelfand [BGG71] and Lepowsky [Lep77]. In interesting special cases such as the conformal case all Verma module homomorphisms are known due to Boe Collingwood [BC85], but in general a complete classification has not been achieved. The resolution of a $\mathfrak{g}$-representation in terms of induced parabolic Verma modules due to Bernstein Gelfand Gelfand [BGG71] and Lepowsky [Lep77] provides us with a complex of homomorphisms between Verma modules, which are called standard homomorphisms. The corresponding sequence of invariant differential operators is a locally exact complex. The problem with these standard homomorphisms constructed by Verma and Bernstein Gelfand Gelfand is that they were only explicit in special cases, for instance if the corresponding highest weights are related by a simple root reflection. In a sequence of pioneering papers [Bas90, Bas91], Baston introduced a number of general methods to construct invariant differential operators in conformal geometry and a related class of parabolic geometries, which he called almost hermitian symmetric structures. In the general parabolic setting Čap, Slovák and Souček constructed in [CSS99] the standard operators by an inductive process. In this context our main result is a new construction of the Bernstein Gelfand Gelfand resolution in terms of an explicit finite power series of simple endomorphisms. The applied method permits us to define the additional structure of a coproduct on the resolution. The corresponding bilinear differential product on the sequence of differential operators satisfies a Leibniz rule.

The structure of this dissertation is as follows. The first chapter brings together dimensional analysis in physics and geometry. The following scalars: a length scale, an electric charge, an inertial mass, or a volume density have different distinctive *dimensions*. Therefore we will not treat all of them as real numbers, but as elements in different one dimensional vector spaces, see definition 1.1. A choice of a *unit* corresponds to the choice of a basis vector in such a one dimensional space. The dimensional analysis, well known to physicists, carries over to geometry when a preferred length scale is not available. It allows to define conformal structures invariantly, see definition 2.1, it determines the order of differential operators and pairings, and it provides a first hint towards the physical interpretation of a given tensor—see the examples in section 1.2. In section 1.3 we recall the deRham complex its wedge product and Leibniz rule. The structure of this complex lies at the heart of this dissertation since our main result proved in the last chapter 6 is an application and generalization of it. The last section 1.6 applies densities to differential geometry, i.e. defines a covariant derivative on these line bundles. These so called *Weyl derivatives* not only provide a geometric interpretation of the first Maxwell equation, but also become fundamental in conformal geometry on manifolds.

Chapter 2 defines conformal structures on affine spaces and smooth manifolds and relates it to electromagnetism and general relativity. It contains in section 2.2 an introduction to

Minkowski's relativistic formulation of electromagnetism emphasizing its conformally invariant aspects such as the electromagnetic constitutive relation in vacuum. In section 2.3 we describe how Weyl derivatives of the density bundle parameterize the covariant derivatives of the tangent bundle on conformal manifolds. A differential geometric construction involving a choice of Weyl derivative is called conformally invariant, if it does not depend upon that choice. In particular for constructions involving higher derivatives, like the curvature tensor, we present, in section 2.6, a convenient way to study the dependence on the Weyl derivative. In section 2.7 we recall the first few integrability conditions of the conformal curvature tensor. These Bianchi identities motivate the field equations for geometric theories of gravity, such as Einstein's or Bach's theory.

Chapter 3 provides a link between the change of Weyl derivative and the Lie algebra $\mathfrak{g}$ of vector fields leaving the conformal affine structure invariant. In section 3.3 we identify this Möbius algebra as a linear Lie algebra $\mathfrak{g} = \mathfrak{so}(p+1, n-p+1)$. This allows to use subspaces of the tensor algebra of $\mathbb{R}^{p+1,n-p+1}$ as representations W for $\mathfrak{g}$. Elements of W are called *twistors* and they induce *twistor fields* on the affine conformal space, i.e. functions with values in the space of *coinvariants* of W. This inclusion, twistors $\mapsto$ twistor fields, is the beginning of the conformal Bernstein Gelfand Gelfand complex. In chapter 4 we will construct invariant differential operators annihilating these twistor fields for $W = \Lambda^{k+1}\mathbb{R}^{p+1,n-p+1}$. Hence such a twistor operator is the next map in the BGG complex.

Chapter 4 splits into two parts. The first three sections 4.1, 4.2 and 4.3 deal with conformal invariants along curves. The invariance of the lightlike acceleration suggests a classical law of the interaction of an electromagnetic field with a light ray. This gives a classical model for the light-light interactions predicted by quantum electrodynamics, although these interactions are expected to be very weak and so it remains a great challenge to observe them in the laboratory. Motivated by energy momentum conservation in general relativity we present in section 4.3 a geometric interpretation of twistor fields: they lead to conserved properties along conformal geodesics (circles) on arbitrary conformal manifolds. In conformal affine space this observation has the reinterpretation that a conformal geodesic (perhaps with an additional parallel tensor attached to it) induces a natural twistor up to scale. The second part, sections 4.4 and 4.5, focuses on differential invariants up to order two for sections of natural bundles on conformal manifolds. The main definition is a 2-jet operator 4.44 in terms of a Weyl derivative. This operator transforms under a linear change of Weyl derivative in a purely algebraic way, hence any second order conformal invariant known from the affine space translates into an invariant on an arbitrary conformal manifold. This allows to characterize linear and bilinear conformal invariants algebraically. We will give some general examples of operators and pairings which were motivated from the study of linear field theories for gravity, which are special cases of BGG complexes including their wedge product structure.

Chapter 5 begins with a general analysis of the ingredients and principles a linear field theory like electromagnetism should follow. In sections 5.2 and 5.3 we demonstrate how differential pairings and Leibniz rules support Penrose's idea to view twistors as generalized charges. In section 5.4 we use our main result (which we will prove in chapter 6) to define the notion of a conformally invariant linear field theory. In the following three sections 5.5, 5.6 and 5.7 we investigate three particular field theories in connection with linear gravity.

We will calculate the relevant pairings here in terms of a Weyl derivative. This happens in advance of the main result to motivate and to engender a better understanding of pairings.

Chapter 6 develops the Bernstein Gelfand Gelfand resolution in the context of homogeneous parabolic geometry. In section 3.4 we already observed that the conformal sphere is an example of parabolic geometry. In the first section 6.1 we define a jet operator encoding the information of all derivatives of a section of a homogeneous bundle at a point into a linear form on an induced module, also called a *Verma module*. Invariant linear differential operators and bilinear pairings are then in one to one correspondence with homomorphisms and coproducts between Verma modules. In section 6.2 we fix the notion of a parabolic subalgebra in the way we will use it in the next section 6.3 on finite dimensional relative Lie algebra homology. The operators of the BGG complex act between sections of homogeneous bundles associated to these relative Lie algebra homology spaces. In section 6.4 we rewrite the deRham complex in terms of homomorphisms between Verma modules. The key observations are made in section 6.5 where we present a variation of Kostant's Hodge theory of Lie algebra cohomology. This allows to define a projection S on the Verma modules induced by the cochains which projects onto a submodule isomorphic to the Verma module induced by the cohomology. This projection S translates the deRham homomorphism to the BGG homomorphisms and the wedge coproduct to a BGG coproduct.

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Chapter 1

Densities and dimensional analysis

This chapter, although elementary and simple, illuminates the role played by scalar densities: we will bring together scale invariance in analysis, physics and geometry. To each number w there is a one dimensional vector space L^w of elements with geometric dimension $(\mathit{length})^w$. These spaces will allow us to define conformal metrics intrinsically (see definition 2.1) and to specify domain and target of conformally invariant differential operators properly (see chapter 4 on differential conformal invariants). Here densities enable us to develop a simple geometric dimensional analysis of arbitrary tensors. One novel feature is a discussion of the exemplary correspondence between physical dimensions and geometric dimensions: scalars of some particular physical dimension will be identified with elements in a L^w. We will indicate how the Sobolev number in functional analysis also relates to L^w. Volume densities L^{-n} allow to treat integration on manifolds globally without orientability assumptions. Particular applications are natural adjoints of differential operators, distributions and integral theorems, see paragraphs 1.18, 1.22 and 1.39. Adjoints, distributions and integral theorems clarify fundamental constructions in physical field theories such as Maxwell's equations 2.12 and sources from pointlike charges, see paragraph 5.6. We clarify the distinction between integration over oriented and cooriented submanifolds in section 1.4, which allows to distinguish between electric and magnetic charge, see definitions 2.19 and 2.21. We will recall the divergence theorem 1.39 on manifolds with boundaries since in full generality it is less widely appreciated than the Stokes' theorem, but is more natural since it avoids orientability assumptions. On manifolds the density bundles can be equipped with a Weyl derivative which illustrates the differential geometric aspect of densities and will be pursued later in the context of conformal geometry. We view the mentioned concepts of densities, dimensional analysis and Weyl derivatives as being as fundamental as tangent vectors or linear connections.

1.1 Linear algebra of densities

We begin this chapter with the linear algebra of densities and define a simple geometric dimensional analysis for tensors. From a representation theoretic view point the geometric dimension is characterized by the weight of the tensor representation restricted to the centre of the general linear group. The following explicit definition is taken from [Cal98b]:

Definition 1.1 Let V be a real n-dimensional vector space and $w \in \mathbb{R}$ a real number. A homogeneous map $\mu\colon \Lambda^n V - \{0\} \to \mathbb{R}$ with the property $\mu(\lambda\omega) = |\lambda|^{-w/n}\mu(\omega)$ for all $\lambda \in \mathbb{R} - \{0\}$ and $\omega \in \Lambda^n V - \{0\}$ is called a *density of weight w*.

The set of all densities of weight w forms a one dimensional vector space $L^w(V)$ or simply L^w. This vector space is oriented, since a nontrivial density takes either positive or negative values. The various vector spaces of densities satisfy canonical isomorphisms: $L^{w_1} \otimes L^{w_2} = L^{w_1+w_2}$, $L^{w*} = L^{-w}$ and $L^0 = \mathbb{R}$.

Remark 1.2 The vector space L^w can equally well be characterized as the representation $\mathbf{GL}(V) \to \mathbf{GL}(L^w) = \mathbb{R} - \{0\}$ given by $A \mapsto |\det A|^{w/n}$. Dilations (positive multiples of the identity) $A = \lambda id$ act by multiplication with λ^w on L^w. The multiples of the identity form the centre of $\mathbf{GL}(V)$ and w is also called the *central weight* of L^w.

Elements of L^1 may be thought of as scalars with dimensions of (*length*). General tensors of V, i.e. elements of $L^k \otimes V^i \otimes V^{*j}$ will be said to have central weight $w := k + i - j$ or dimensions of $(\textit{length})^{k+i-j}$, since dilations λid naturally act upon them by multiplication with λ^{k+i-j} (hence $w = k+i-j$ is sometimes called the *degree of homogeneity*). The central weight of the tensor product of two tensors is given by the sum of individual central weights.

Example 1.3 Real numbers $\mathbb{R} = L^0$ are weightless and dimensionless. Vectors of V have central weight $+1$ and describe translations of dimensions (*length*). A positive density of weight $-n$ assigns naturally a real number as volume to an n-dimensional parallelepiped and has dimension $(\textit{length})^{-n}$. The tensor product $L^{-n} \otimes \Lambda^n V$ is the weightless space of *pseudoscalars*. This one dimensional space naturally carries a norm given by $|\mu \otimes \omega| := |\mu(\omega)|$. The two orientations of V are in one to one correspondence with the two unit elements of $L^{-n} \otimes \Lambda^n V$.

Remark 1.4 This definition of weight is in agreement with Gauduchon's convention in [Gau91]. In the fundamental articles by Fegan [Feg76] and Hitchin [Hit80] the definition of weight differ from ours by a minus sign. However, in parts of the literature the definition of the weight is very different and those definitions of weight do not allow the satisfactory dimensional analysis as discussed in the next section. Eastwood in [Eas96] assigns k as weight to elements in $L^k \otimes V^i \otimes V^{*j}$. Similarly Bergmann [Ber76], Stephani [Ste91] and Weyl [Wey70] prefer to assign $-k/n$ as weight for $L^k \otimes V^i \otimes V^{*j}$.

1.2 Physical dimensions and fundamental constants

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Physical quantities are measurable aspects of physical objects. The value of a scalar quantity is the product of a *real number* and a chosen *unit* (e.g. a ruler has length 30cm). Independent of the unit is the *dimension* characterizing the type of the scalar quantity in question. From a basic set of dimensions like (*length*), (*time*), (*inertial mass*), (*electric charge*), (*gravitational mass*) and (*temperature*), one can generate derived dimensions as commutative products and powers of these basic dimensions like $(\textit{acceleration}) := (\textit{length})/(\textit{time})^2$.

Mathematically, different one dimensional real vector spaces can represent different basic dimensions. Scalar quantities of a given (derived) dimension are elements of the appropriate tensor product of these one dimensional vector spaces (and their duals). The choice of a basis in such a one dimensional space corresponds to a choice of a unit of that dimension and this is called a *gauge fixing*. The one dimensional vector spaces L^w from definition 1.1 are geometric examples of spaces carrying scalars of dimension $(\textit{length}\,)^w$.

For theoretical purposes one can reduce the number of basic dimensions by one, if one postulates that a certain proportionality factor which appears in a (fundamental) law of nature is a (fundamental) constant of nature. The first example is Newton's law of motion, which says that the force (e.g. of a spring) acting on a body is proportional to the inertial mass of the body times its acceleration: $F = \lambda m a$. The proportionality factor λ is universal. This factor is always set to unity, which turns Newton's law in a combination of a law of motion and a definition for the dimension of force (see [Par87] p. 239) $(\textit{force}\,) = (\textit{inertial mass}\,)(\textit{acceleration}\,)$.

Further examples of these laws are Einstein's relation between energy and inertial mass of massive particles $E = mc^2$, Planck's relation between energy and frequency of massless particles $E = \hbar\omega$, Coulomb's force law between two electric charges $F = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}$, Newton's force law between two gravitational masses $F = \gamma\frac{m_1 m_2}{r^2}$ and Boltzmann's law between thermical energy per degree of freedom and temperature of a molecule $E = \frac{1}{2}kT$. The dimensions of the mentioned constants of nature are as follows:

$$\begin{aligned}
\text{Velocity of light: } \dim(c) &= (\textit{length}\,)/(\textit{time}\,),\\
\text{Planck's constant: } \dim(\hbar) &= (\textit{energy}\,)(\textit{time}\,) = (\textit{force}\,)(\textit{length}\,)(\textit{time}\,),\\
\text{Coulomb's constant: } \dim(1/(4\pi\varepsilon_0)) &= (\textit{force}\,)(\textit{length}\,)^2/(\textit{electric charge}\,)^2,\\
\text{Newton's constant: } \dim(\gamma) &= (\textit{force}\,)(\textit{length}\,)^2/(\textit{gravitational mass}\,)^2,\\
\text{Boltzmann's constant: } \dim(k) &= (\textit{energy}\,)/(\textit{temperature}\,).
\end{aligned}$$

One way to deal with fundamental constants is to choose appropriate *units* in all basic dimensions such that these constants have the *value* 1. Having done this, all constants disappear in formulas. We don't want to choose units in basic dimensions here. Instead we will use fundamental constants as factors in front of physical quantities such that the product has a dimension which is simply a power of $(\textit{length}\,)$:

Example 1.5 (*Kinematic dimensions*) Using c one reduces the basic *kinematic* dimensions $(\textit{length}\,)$ and $(\textit{time}\,)$ to powers of $(\textit{length}\,)$, e.g. $(\textit{frequency}\,) = \dim(c^{-1})\,(\textit{length}\,)^{-1}$.

Example 1.6 (*Mechanical dimensions*) Using c and $\hbar$ one reduces the basic mechanical dimensions $(\textit{length}\,)$, $(\textit{time}\,)$ and $(\textit{inertial mass}\,)$ to powers of $(\textit{length}\,)$: note, $\dim\left(\frac{\hbar}{c}\right) = (\textit{length}\,)(\textit{inertial mass}\,)$ and $\dim(\hbar) = (\textit{length}\,)(\textit{linear momentum}\,)$. In classical mechanics the linear momentum is an element of the dual tangent space of the configuration space, hence has geometric dimension $(\textit{length}\,)^{-1}$ in agreement with this choice of fundamental constants.

Example 1.7 (*Electromagnetic dimensions*) In the same way one reduces the basic electromagnetic dimensions $(\textit{length}\,)$, $(\textit{time}\,)$, $(\textit{inertial mass}\,)$ and $(\textit{electric charge}\,)$ using c, $\hbar$

and $\frac{1}{4\pi\varepsilon_0}$ to powers of (*length*). Notice that $\dim(\hbar c) =$ (*force*)(*length*)^2 and hence there is a *natural unit* for electric charges since (*electric charge*)$^2 = \hbar c\, 4\pi\varepsilon_0$ and electric charge q becomes dimensionless (i.e. a real number) when divided by $\sqrt{\hbar c\, 4\pi\varepsilon_0}$.

In electrostatics there are two basic quantities. The *electric field strength* E has dimension $\dim(E) =$ (*force*)/(*electric charge*). The *dielectric displacement* D has dimension $\dim(D) =$ (*electric charge*)/(*area*). Notice,

$$\dim\left(\sqrt{\frac{4\pi\varepsilon_0}{\hbar c}}\, E\right) = \dim\left(\sqrt{\frac{1}{4\pi\varepsilon_0\, \hbar c}}\, D\right) = (\mathit{length})^{-2}.$$

In magnetostatics there are also two basic quantities. It is convenient to introduce the dimensions (*current*) := (*electric charge*)/(*time*) and (*voltage*) := (*energy*)/(*electric charge*). The *magnetic flux* B has dimension $\dim(B) =$ (*voltage*)(*time*)/(*area*) and the *magnetic loop tension* H has dimension $\dim(H) =$ (*current*)/(*length*). Notice once more,

$$\dim\left(\sqrt{\frac{4\pi\varepsilon_0\, c}{\hbar}}\, B\right) = \dim\left(\sqrt{\frac{1}{4\pi\varepsilon_0\, \hbar c^3}}\, H\right) = (\mathit{length})^{-2}.$$

In relativistic electromagnetism, the electric field E and the magnetic flux B are summarized into a 2-form F, the so called *Faraday form* F. The first Maxwell equation $dF = 0$ is expressed in terms of the natural exterior derivative see paragraphs 1.23 and 2.9. The Faraday form therefore carries the geometric dimension (*length*)$^{-2}$. Similarly, the dielectric displacement D and the magnetic loop tension H are summarized into a bivector density (denoted by G). The second Maxwell equation is expressed in terms of the natural exterior divergence see paragraphs 1.24 and 2.9. The bivector density G therefore carries the geometric dimension (*length*)$^{2-n}$, where n is the spacetime dimension. Hence with the above choice of fundamental constants and in $n = 4$ spacetime dimensions, the natural geometric dimensions and the physical dimensions are in full correspondence.

Example 1.8 (*Gravitational dimensions*) Analogously, using c, $\hbar$ and γ one reduces the basic gravitational dimensions (*length*), (*time*), (*inertial mass*) and (*gravitational mass*) to powers of (*length*). A gravitational mass becomes dimensionless, when divided by $\sqrt{\hbar c/\gamma}$.

Example 1.9 (*Thermodynamical dimensions*) Using c, $\hbar$ and k one reduces the basic thermodynamical dimensions (*length*), (*time*), (*inertial mass*) and (*temperature*) to powers of (*length*).

Fundamental constants as factors allow scalars of any physical dimension to be viewed as elements in one of the geometrically defined spaces L^w of the previous chapter. Similarly, a tensorial physical quantity is mathematically described by an element in the corresponding tensor space of the central weight according to its dimension.

Remark 1.10 The elementary electric charge e of the electron defines another natural unit of dimension (*electric charge*). Hence the quotient $e^2/(\hbar c\, 4\pi\varepsilon_0)$ determines a (dimensionless) real number, the so called *electric fine structure constant*. A gravitational analogue to an elementary electric charge has not been observed. We will not use the elementary electric charge of the electron as a unit.

Remark 1.11 The equivalence principle formulated as the proportionality between inertial and gravitational mass of a massive particle leads to another constant which one could call *Galilei's constant* g:

$$\text{Galilei's constant: } \dim(g) = (\textit{gravitational mass})/(\textit{inertial mass}).$$

For historical reasons, this constant is always set to unity (the common units for inertial mass are also used for gravitational mass). Using Galilei's constant g together with c, $\hbar$ and γ Planck found natural units for all dimensions involved, in particular $\dim(g^2\hbar\gamma c^{-3}) = (\textit{length})^2$. We will not make use of the equivalence principle in that way, hence we will not use Galilei's constant and we will not use the resulting units of Planck. (We believe that Galilei's constant is not fundamental, instead it we would like to view it as a consequence of a proper understanding of Mach's principle.)

Remark 1.12 In Einstein's theory of gravity one usually uses c, γ and Galilei's constant g from above without Planck's constant $\hbar$ to reduce the basic gravitational dimensions to powers of $(\textit{length})$ (see [MW57], p. 596). With this choice of constants we have $\dim(g\gamma c^{-2}) = (\textit{length})/(\textit{gravitational mass})$ instead of a natural unit for gravitational mass. Taking also Coulomb's electric constant $1/(4\pi\varepsilon_0)$ into account, one finds for electric charge in analogy to gravitational mass $\dim\left(gc^{-2}\sqrt{\gamma/(4\pi\varepsilon_0)}\right) = (\textit{length})/(\textit{electric charge})$. Note that in this case the electric field strength has dimension $\dim\left(gc^{-2}\sqrt{\gamma\, 4\pi\varepsilon_0}\, E\right) = (\textit{length})^{-1}$, so that we would lose the correspondence between physical dimensions and natural geometric dimensions explained in paragraph 1.7.

1.3 Density bundles and divergence operators

Applying the construction in definition 1.1 of densities of weight w pointwise to the tangent space of a n-dimensional manifold M leads to a family of natural real line bundles $L^w = L^w(TM)$ over M. Alternatively one could use the (first order) frame bundle $\mathbf{GL}(M)$ of M and the associated bundle construction for the representation $|\det|^{w/n}$ of $\mathbf{GL}(\mathbb{R}^n)$ to obtain: $L^w(TM) = \mathbf{GL}(M) \times_{|\det|^{w/n}} \mathbb{R}$. These line bundles are oriented and hence trivializable, but except for $w = 0$ do not carry a natural trivialization. The sections of L^{-n} over M play the role of natural integrands. We will follow Calderbank [Cal96] to define natural adjoints of differential operators and distributions. We apply this definition to the deRham complex to obtain the complex of exterior divergences.

Discussion 1.13 (*Integration*) We fix a notion of integration of smooth functions defined on a unit cube $[0,1]^n$ in Euclidean n-space which satisfies linearity and the transformation formula under diffeomorphisms of the cube to itself. If $F\colon [0,1]^n \to M$ is smooth and $\mu \in C^\infty(M, L^{-n})$ is a density of weight $-n$ then $x \in [0,1]^n$ and $(F^*\mu)_x := \mu_{F(x)}(\partial F_x(.) \wedge \ldots \wedge \partial F_x(.))$ define a density on $\mathbb{R}^n$. We say a real valued $\mathbb{R}$-linear functional $\int_M : C^\infty_o(M, L^{-n}) \to \mathbb{R}$ from densities of weight $-n$ with compact support *locally agrees with the Euclidean integral* if for any $F\colon [0,1]^n \to M$ and any $\mu \in C^\infty_o(M, L^{-n})$ with $\mathrm{supp}(\mu) \subset \mathrm{im}(F)$ we have:

$$\int_M \mu = \int_{[0,1]^n} F^*\mu(e_1 \wedge \ldots \wedge e_n),$$

where e_i is the standard basis in $\mathbb{R}^n$. A partition of unity argument shows existence:

Proposition 1.14 *There is a linear functional $\int_M : C^\infty_o(M, L^{-n}) \to \mathbb{R}$ which locally agrees with the Euclidean integral and which therefore is unique and invariant under diffeomorphisms of M.*

Discussion 1.15 (*Divergence of vector fields*) Let $X \in C^\infty(M, TM)$ be a vector field and $\mu \in C^\infty(M, L^{-n})$ a positive density of weight $-n$. The Lie derivative $\mathcal{L}_X\mu$ is defined by differentiating the pull back of μ by the local flow of X. The *divergence of X with respect to μ* is a function on M defined by $(div_\mu X)\mu = \mathcal{L}_X\mu$. The divergence can be interpreted as the rate of volume expansion of the flow of X with respect to μ. Notice that for any function $f \in C^\infty(M, \mathbb{R})$ we have $\mathcal{L}_{(fX)}\mu = \mathcal{L}_X(f\mu)$, such that the above formula defines a linear first order differential operator on vector field densities like $\mu \otimes X$:

$$div : C^\infty(M, L^{-n} \otimes TM) \to C^\infty(M, L^{-n}).$$

A vector field $X \in C^\infty_o(M, TM)$ with compact support on the interior of M has a complete flow. The diffeomorphism-invariance of the integral leads to the following vanishing result:

Proposition 1.16 *(Integration by parts) If $X \in C^\infty_o(M, TM)$ has compact support and $\mu \in C^\infty(M, L^{-n})$ is positive then $\int_M \mathrm{div}(\mu \otimes X) = 0$.*

Definitions 1.17 (*Jets, differential operators and pairings*) Let $EM \to M$ be a vector bundle. At a point $x \in M$ we define for any $k \in \mathbb{N}$ the vector space of sections which vanish up to order k at x to be $Z^k_x(EM) := \{e \in C^\infty(M, EM) \,|\, e(x) = 0, de(X) = 0, \ldots, d^k e(x) = 0\}$. The projection $jet^k(e)_x$ of a section $e \in C^\infty(M, EM)$ onto the quotient $Jet^k(EM)_x := C^\infty(M, EM)/Z^k_x(EM)$ is called the *k-jet of e at x*. The vector bundle $Jet^k(EM) \to M$ is called the k-jet bundle of EM. We have a bundle inclusion $\mathrm{Sym}^k T^* \otimes EM \to Jet^k(EM)$ defined at $x \in M$ by a function $f \in C^\infty(M, \mathbb{R})$ with $f(x) = 0$ as $(\partial f)^k \otimes e \mapsto jet^k(f^k e)$. This is the beginning of a short exact sequence of bundle maps:

$$0 \to \mathrm{Sym}^k T^* \otimes EM \to Jet^k(EM) \to Jet^{k-1}(EM) \to 0.$$

Let $FM \to M$ denote another vector bundle. A *linear differential operator of order k* is a $\mathbb{R}$-linear map

$$\nabla : C^\infty(M, EM) \to C^\infty(M, FM)$$

which factors through a bundle map $\pi : Jet^k(EM) \to FM$ as $\nabla = \pi \circ jet^k$ with nonzero *symbol* defined to be the composite: $\mathrm{Sym}^k T^* \otimes EM \to Jet^k(EM) \to FM$. The *universal k-jet operator* $jet^k : C^\infty(M, EM) \to C^\infty(M, Jet^k(EM))$ is a particular example. Similarly we call a $\mathbb{R}$-bilinear map

$$X : C^\infty(M, E_1M) \times C^\infty(M, E_2M) \to C^\infty(M, FM)$$

which factors through a bundle map $Q : Jet^k(E_1M) \otimes Jet^k(E_2M) \to FM$ as $X(e_1, e_2) = Q(jet^k(e_1) \otimes jet^k(e_2))$ a *bilinear differential pairing of order k*.

Discussion 1.18 (*Adjoints*) Let $EM \to M$ and $FM \to M$ be two vector bundles over M and $\nabla\colon C^\infty(M, EM) \to C^\infty(M, FM)$ be a linear differential operator. A differential operator $\nabla^*\colon C^\infty(M, L^{-n} \otimes F^*M) \to C^\infty(M, L^{-n} \otimes E^*M)$ is called *adjoint* to ∇, if for all sections with compact support in the interior of M we have $\int_M \langle \nabla e, \phi \rangle = \int_M \langle e, \nabla^* \phi \rangle$. If ∇^* exists, it is unique. One way to proof adjointness of two given operators ∇ and ∇^* is to construct a *bilinear differential pairing*

$$X_\nabla\colon C^\infty(M, EM) \times C^\infty(M, L^{-n} \otimes F^*M) \to C^\infty(M, L^{-n} \otimes TM),$$

such that a divergence formula holds:

$$div(X_\nabla(e, \phi)) = \langle \nabla e, \phi \rangle - \langle e, \nabla^* \phi \rangle.$$

The differential order of the pairing is one less then the order of the operator. In particular for first order operators the pairing is tensorial.

Definition 1.19 (*Covariant derivative*) A linear first order operator $D\colon C^\infty(M, EM) \to C^\infty(M, T^* \otimes EM)$ with symbol given by the identity on $T^* \otimes EM$ is called *covariant derivative on* EM.

Associated to a covariant derivative D is the *Riemann curvature tensor* R^D. It defines the local obstruction against the existence of parallel sections:

Proposition 1.20 *(Curvature) Let D be a covariant derivative of a vector bundle EM. The following trilinear differential pairing $R^D\colon C^\infty(M, TM) \otimes C^\infty(M, TM) \otimes C^\infty(M, EM) \to C^\infty(M, EM)$ defined by $R^D(X, Y)e := D_X(D_Y e) - D_Y(D_X e) - D_{[X,Y]}e$ (where X, Y are vector fields, e a section and the Lie bracket on functions is defined by $\partial_{[X,Y]} = \partial_X\partial_Y - \partial_Y\partial_X$) is indeed zero order and defines the* curvature *of D to be an endomorphism valued 2-form $R^D \in C^\infty(M, \Lambda^2 T^* \otimes E^* \otimes EM)$.*

Remark 1.21 A covariant derivative on EM induces a covariant derivative on the dual bundle E^*M by the product rule: $\partial_X(\langle \eta, e \rangle) = \langle D_X \eta, e \rangle + \langle \eta, D_X e \rangle$, with a vector field X and sections $e \in C^\infty(M, EM)$ and $\eta \in C^\infty(M, E^*M)$. Consider the tensorial pairing $EM \otimes (L^{-n} \otimes T \otimes E^*M) \to L^{-n} \otimes TM$ with $e \otimes Y \otimes \eta \mapsto Y \langle e, \eta \rangle$. It satisfies the divergence formula:

$$div(\langle e, (Y \otimes \eta) \rangle) = \langle D_Y e, \eta \rangle + \langle e, D_Y \eta \rangle + \langle e, \eta \rangle \, div(Y),$$

with $Y \in C^\infty(M, L^{-n} \otimes TM)$. This shows that a covariant derivative has an adjoint $D^*\colon C^\infty(M, L^{-n} \otimes T \otimes E^*M) \to C^\infty(M, L^{-n} \otimes E^*M)$ given by $D^*(Y \otimes \eta) = -D_Y \eta - div(Y) \otimes \eta$.

Discussion 1.22 (*Distributions*) Let M be a n-dimensional manifold and $EM \to M$ a vector bundle. The space of *test functions* $C^\infty_o(M, L^{-n} \otimes E^*M)$ consists of smooth sections with compact support in the interior of M. The space of sections of the k-jet bundle $C^\infty_o(M, Jet^k(L^{-n} \otimes E^*M))$ is equipped with the *compact open topology* and this pulls back under $jet^k\colon C^\infty_o(M, L^{-n} \otimes E^*M) \to C^\infty(M, Jet^k(L^{-n} \otimes E^*M))$ to induce the *C^k-topology* on $C^\infty_o(M, L^{-n} \otimes E^*M)$. A *distributional section* of EM is a linear map

$C^\infty_o(M, L^{-n} \otimes E^*M) \to \mathbb{R}$ which is continuous with respect to the C^k-topology for all $k \in \mathbb{N}$. The set of all such distributions is denoted by $\mathcal{D}(M, EM) := C^\infty_o(M, L^{-n} \otimes E^*M)^*$. Locally integrable sections of EM, i.e. elements of $\mathcal{L}^1_{\text{local}}(M, EM)$ define distributions via pointwise contraction and integration over M.

Let $\nabla : C^\infty(M, EM) \to C^\infty(M, FM)$ be a linear differential operator. Its *transpose* in terms of distributions always exists: $\nabla^{\text{trans}} : \mathcal{D}(M, L^{-n} \otimes F^*M) \to \mathcal{D}(M, L^{-n} \otimes E^*M)$. If the restriction of ∇^{trans} to smooth sections gives smooth sections, then this restriction is the adjoint $\nabla^* : C^\infty(M, L^{-n} \otimes F^*M) \to C^\infty(M, L^{-n} \otimes E^*M)$. If this adjoint exists the operator ∇ can be extended to distributional sections $\nabla : \mathcal{D}(M, EM) \to \mathcal{D}(M, FM)$ via $\langle\!\langle \nabla\, e, \phi \rangle\!\rangle = \langle\!\langle e, \nabla^*\, \phi \rangle\!\rangle$ with a distribution $e \in \mathcal{D}(M, EM)$ and a test function $\phi \in C^\infty_o(M, L^{-n} \otimes F^*M)$.

Since a covariant derivative D of EM always has an adjoint, one can say that distribution can infinitely often be differentiated.

Discussion 1.23 (*Exterior derivative*) On any manifold the exterior derivative on alternating forms

$$d : C^\infty(M, \Lambda^k T^*M) \to C^\infty(M, \Lambda^{k+1} T^*M)$$

defines a linear first order differential operator. The sequence of exterior derivatives builds the *deRham complex*: $d \circ d = 0$. The resulting cohomology is called *deRham cohomology*. The deRham complex is *locally exact* in the sense that it becomes exact when restricted to a contractible open subset of M. The wedge product is a zero order bilinear pairing:

$$\wedge : C^\infty(M, \Lambda^k T^*M) \times C^\infty(M, \Lambda^l T^*M) \to C^\infty(M, \Lambda^{k+l} T^*M),$$

subject to the following *Leibniz rule*:

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta.$$

Hence the wedge product induces a *cup product* in the deRham cohomology.

For a vector space V we have a $\mathbf{GL}\,(V)$-equivariant pairing $\langle\, , \rangle : \Lambda^k V^* \otimes \Lambda^k V \to \mathbb{R}$. We define *interior multiplication by forms* to be a linear map $\lrcorner : \Lambda^k V^* \otimes \Lambda^{k+l} V \to \Lambda^l V$ which is adjoint to the wedge product in the sense $\langle \alpha \lrcorner a, \beta \rangle = \langle a, \alpha \wedge \beta \rangle$, with $\alpha \in \Lambda^k V^*$, $a \in \Lambda^{k+l} V$ and $\beta \in \Lambda^l V$. Similarly *interior multiplication by multivectors* is a linear map $\lrcorner : \Lambda^k V \otimes \Lambda^{k+l} V^* \to \Lambda^l V^*$. On a n-dimensional manifold M the bundle of alternating forms can be tensored with the bundle of pseudoscalars $L^{-n} \otimes \Lambda^n TM$ and the above interior multiplication provides an isomorphism

$$\begin{array}{rcl} \Lambda^k T^* \otimes (L^{-n} \otimes \Lambda^n T) & \xrightarrow{\cong} & L^{-n} \otimes \Lambda^{n-k} T \\ \alpha \otimes or & \mapsto & \alpha \lrcorner or \end{array}$$

to obtain the bundle of multivector densities. Likewise we can tensor the bundle of multivector densities by the (dual) pseudoscalars $L^n \otimes \Lambda^n T^*M$ to recover the bundle of alternating forms:

$$\begin{array}{rcl} L^{-n} \otimes \Lambda^k T \otimes (L^n \otimes \Lambda^n T^*) & \xrightarrow{\cong} & \Lambda^{n-k} T^* \\ a \otimes or^* & \mapsto & a \lrcorner or^*. \end{array}$$

Discussion 1.24 (*Exterior divergence*) The bundle of dual pseudoscalar $L^n \otimes \Lambda^n T^*M$ over M has rank one, comes with a natural norm and hence with local sections or^* which are parallel. The *exterior divergence* on multivector densities

$$div : C^\infty(M, L^{-n} \otimes \Lambda^k TM) \to C^\infty(M, L^{-n} \otimes \Lambda^{k-1}TM),$$

is then defined in terms of the exterior derivative by

$$(div\, a) \lrcorner\, or^* := (-1)^{k+1} d(a \lrcorner\, or^*).$$

The sequence of divergences defines a complex $div \circ div = 0$. The resulting homology is called *deRham homology*. Interior multiplication $\lrcorner$ is a zero order pairing

$$\lrcorner : C^\infty(M, \Lambda^k T^*M) \times C^\infty(M, L^{-n} \otimes \Lambda^{k+l}TM) \to C^\infty(M, L^{-n}\Lambda^l TM).$$

For $\alpha \in C^\infty(M, \Lambda^k T^*M)$ and $a \in C^\infty(M, L^{-n} \otimes \Lambda^{k+l}TM)$ we get from the Leibniz rule the following divergence formula:

$$(-1)^k\, div(\alpha \lrcorner\, a) = d\alpha \lrcorner\, a + \alpha \lrcorner\, div\, a.$$

For that note $(\alpha \lrcorner\, a) \lrcorner\, or^* = (-1)^{k(k+l+1)} \alpha \wedge (a \lrcorner\, or^*)$. Hence the interior multiplication defines a *cap product* between deRham cohomology and homology. For $l = 1$ the above divergence formula shows that d and $-\,div$ are adjoints.

Remark 1.25 For $k = 0$ we recover the divergence on vector fields from paragraph 1.15, since for functions $f \in C^\infty(M, \mathbb{R})$ and vector fields $X \in C^\infty(M, L^{-n} \otimes TM)$ we clearly have the divergence formula $div(fX) = \langle df, X\rangle + f\, div\, X$.

Remark 1.26 Let D be a torsion-free covariant derivative of TM (see definition 1.19 and proposition 2.27). (Local coordinates induce torsion-free derivatives.) With a dual basis t_i, θ^i of TM we like the exterior derivative on a form α to be $d\alpha = \sum_i \theta^i \wedge D_{t_i}\alpha$. With the above sign convention we obtain for a multivector a applied to the divergence $div\, a = \sum_i \theta^i \lrcorner\, D_{t_i} a$.

Remark 1.27 The divergence operator can also be defined invariantly on decomposable multivector density fields using the Lie derivative: for $k = 1$ let μ denote a nonvanishing density of weight $-n$ and X, Y vector fields then $div(\mu \otimes X \wedge Y) = (\mathcal{L}_X \mu) \otimes Y - (\mathcal{L}_Y \mu) \otimes X + \mu \otimes [\,X, Y\,]$.

Application 1.28 (*Covariant exterior derivative*) Let EM be again a vector bundle over M and D a covariant derivative on EM. Since the exterior derivative on forms is first order it can be twisted by D to obtain a sequence of first order differential operators $d^D : C^\infty(M, \Lambda^k T^* \otimes EM) \to C^\infty(M, \Lambda^{k+1}T^* \otimes EM)$ between forms with values in EM. As an example, if X, Y denote vector fields and α a 1-form with values in EM then $d^D\alpha(X, Y) := D_X(\alpha(Y)) - D_Y(\alpha(X)) - \alpha([\,X, Y\,])$. The obstruction for the twisted deRham sequence to be a complex is the Riemann curvature: $R^D \in C^\infty(\Lambda^2 T^* \otimes \mathfrak{gl}(EM))$ which acts on a k-form to give a $(k+2)$-form as

$$d^D \circ d^D \alpha = \frac{1}{2} \sum_{i,j} \theta^i \wedge \theta^j \wedge R^D(t_i, t_j)\alpha =: R^D \wedge \alpha,$$

where t_i, θ^i is a dual basis of TM. Clearly D induces also a derivative on $\mathfrak{gl}(EM)$ and the Leibniz rule gives

$$\mathrm{d}^D \circ (\mathrm{d}^D \circ \mathrm{d}^D)\alpha = \mathrm{d}^D R^D \wedge \alpha + (-1)^2 R^D \wedge \mathrm{d}^D\alpha = \mathrm{d}^D R^D \wedge \alpha + (\mathrm{d}^D \circ \mathrm{d}^D) \circ \mathrm{d}^D\alpha.$$

This observation proves the following (see e.g. [BGV91]):

Proposition 1.29 *(Bianchi identity) The curvature tensor R^D of a covariant derivative D on EM satisfies a first order integrability condition: $0 = \mathrm{d}^D R^D$.*

1.4 Orientations and integration over submanifolds

We recall that the one dimensional *space of pseudoscalars* $L^{-n}(V) \otimes \Lambda^n V$ of a n-dimensional real vector space V comes with a natural norm $|\mu \otimes v| = |\mu(v)|$. The two unit elements correspond to the two orientations of V: if $v := t_1 \wedge \ldots \wedge t_n$ with $t_i \in V$ is a positive basis of V then this induces $or = \mu \otimes v$ with $0 < \mu$ determined by $\mu(v) = 1$. A unit element $or \in L^{-n}(V) \otimes \Lambda^n V$ induces a unique element $or^* \in L^n(V) \otimes \Lambda^n V^*$ in the dual space by $\langle or^*, or \rangle = 1$. Alternatively, the above norm in the one dimensional space $L^{-n}(V) \otimes \Lambda^n V$ induces an inner product, which identifies it with its dual space: $L^{-n}(V) \otimes \Lambda^n V = L^n(V) \otimes \Lambda^n V^*$

Definition 1.30 (*Orientation*) A vector bundle $EM \to M$ of rank k over a n-dimensional manifold M is called *orientable* if the bundle of pseudoscalars $L^{-k}(EM) \otimes \Lambda^k EM$ is trivializable, i.e. if there is a global section $or \in C^\infty(M, L^{-k}(EM) \otimes \Lambda^k EM)$ of norm one $|or| = 1$. A choice of such a trivialization on an orientable bundle is called an *orientation of EM* and EM with such a choice is called *oriented*.

Remark 1.31 In general, if M is connected and $x \in M$ a point then there is a group homomorphism $\pi_1(M, x) \to \{+1, -1\}$ between the fundamental group and $\mathbf{O}(\mathbb{R}) = \{+1, -1\}$ induced by parallel transport in $L^{-k}(EM) \otimes \Lambda^k EM$. This homomorphism factors through the maximal Abelian quotient $\pi_1(M, x)_{Ab}$, hence the homomorphism corresponds to a cohomology class $w_1(EM) \in H^1(M, \mathbb{Z}/2\mathbb{Z})$. The vector bundle is orientable, iff this class is trivial. In particular if M is simply connected, $\pi_1(M, x) = 1$, any vector bundle over M is orientable.

Definition 1.32 (*Orientation and coorientation*) Let $S \hookrightarrow M$ denote a k-dimensional immersed submanifold inside a n-dimensional manifold M. The submanifold $S \hookrightarrow M$ is called *orientable* iff its tangent bundle $TS \to S$ is orientable. The submanifold $S \hookrightarrow M$ is called *coorientable* iff the normal bundle (quotient bundle) $TM/TS \to S$ is orientable.

Remark 1.33 Points in M are always oriented and the manifold M itself is always cooriented. The conventional (co)orientations are often denoted by $+1$, and the opposite orientation is denoted by -1.

Application 1.34 (*Integration over oriented submanifolds*) Let $S \hookrightarrow M$ by an immersion as in definition 1.32. If S is orientable and *or* a choice of orientation then any k-form F on M can be turned into a density $\langle F, or\rangle \in C^\infty(S, L^{-k}(TS))$ over S by means of the following contraction:

$$L^{-k}(TS) \otimes \Lambda^k TS \otimes \Lambda^k T^* M \to L^{-k}(TS).$$

Hence $\langle F, or\rangle$ can be integrated naturally over S.

Proposition 1.35 *If $0 \to U \to V \to V/U \to 0$ is a short exact sequence of real vector spaces U, V and V/U, then orientations of two of them induces an orientation of the third. Indeed if $k = \dim(U)$ denote the dimension we have the following canonical isomorphisms between one dimensional spaces:*

$$\begin{gathered} L^{-k}(U) \otimes \Lambda^k U \otimes L^{-(n-k)}(V/U) \otimes \Lambda^{n-k}(V/U) \xrightarrow{\cong} L^{-n}(V) \otimes \Lambda^n V, \\ \Lambda^k U \otimes \Lambda^{n-k}(V/U) \xrightarrow{\cong} \Lambda^n V, \\ L^{-k}(U) \otimes L^{-(n-k)}(V/U) \xrightarrow{\cong} L^{-n}(V). \end{gathered}$$

Application 1.36 (*Integration over cooriented submanifolds*) Let $S \hookrightarrow M$ denote an immersion as in definition 1.32. If S is coorientable and $coor \in C^\infty(M, L^{n-k}(TM/TS) \otimes \Lambda^{n-k}(TM/TS)^*)$ a choice of coorientation then any $(n-k)$-multivector density G on M can be turned into a density $\langle G, coor\rangle \in C^\infty(S, L^{-k}(TS))$ over S, by means of the following map (we will use the notation $TS^\perp = (TM/TS)^*$):

$$\begin{aligned} L^{-n}(TM) \otimes \Lambda^{n-k} TM \otimes L^{n-k}(TM/TS) \otimes \Lambda^{n-k} TS^\perp &\to L^{-n}(TM) \otimes L^{n-k}(TM/TS) \\ &\cong L^{-k}(TS). \end{aligned}$$

Hence $\langle G, coor\rangle$ can be integrated naturally over S.

Application 1.37 (*Outer normal*) If S is a manifold with boundary ∂S then the outer normal induces an orientation of the normal bundle of the boundary $out \in L^{-1}(TS/T\partial S) \otimes \Lambda^1(TS/T\partial S)$ with $|out| = 1$ along ∂S. Applying the proposition 1.35 for $0 \to T\partial S \to TS \to TS/T\partial S \to 0$ and for $0 \to TS/T\partial S \to TM/T\partial S \to TM/TS \to 0$ we have two induced isomorphisms:

$$\begin{aligned} L^{-k}(TS) \otimes \Lambda^k(TS) &\to L^{1-k}(T\partial S) \otimes \Lambda^{k-1}(T\partial S), \\ L^{n-k}(TM/TS) \otimes \Lambda^{n-k}(TS)^\perp &\to L^{n+1-k}(TM/T\partial S) \otimes \Lambda^{n+1-k}(T\partial S)^\perp. \end{aligned}$$

Hence a (co-)orientation of S induces a (co-)orientation of ∂S.

Proposition 1.38 *(Stoke's theorem) If $F \in C^\infty(M, \Lambda^k T^* M)$ is a smooth k-form on a n-manifold M, and $\Sigma \hookrightarrow M$ an (immersed) compact oriented $(k+1)$-dimensional submanifold with boundary $\partial\Sigma$ then*

$$\int_{\partial\Sigma} \langle F, \mathrm{or}\rangle = \int_\Sigma \langle dF, \mathrm{or}\rangle.$$

Proposition 1.39 *(Divergence theorem) If $G \in C^\infty(M, L^{-n} \otimes \Lambda^{n-k}TM)$ is a smooth $(n-k)$-multivector density on an n-manifold M, and $\Sigma \hookrightarrow M$ an (immersed) compact cooriented $(k+1)$-dimensional submanifold with boundary $\partial\Sigma$ then*

$$\int_{\partial\Sigma} \langle G, coor \rangle = \int_\Sigma \langle \mathrm{div}\, G, coor \rangle.$$

Remark 1.40 In the case of M being a manifold with boundary ∂M the outer normal always induces a coorientation of ∂M and M itself is always cooriented. Hence the above divergence theorem applies to the whole of M without any orientability assumptions which is called *Gauß theorem*, whereas Stoke's theorem needs an orientable M.

1.5 Scale invariance in functional analysis

In real functional analysis the weight in definition 1.1 is related to the so called *Sobolev number*. To demonstrate this relation we will briefly recall Lebesgue and Sobolev spaces using density valued functions. The dimensional analysis then gives an explanation of the Hölder and Sobolev inequalities. Recall that the integral of a sections of the line bundle L^{-n} over a n-dimensional compact domain is invariant under diffeomorphisms of the domain.

Discussion 1.41 (*Lebesgue spaces*) A scalar function lives in the Lebesgue space $\mathcal{L}^p$ for some $p \in \mathbb{R}$ if the pth power of its modulus is integrable. Therefore such a scalar function should be considered as a section of $L^{-n/p}$, i.e. one assigns $-n/p$ as the weight of that function. Hence $\mathcal{L}^p$ can be called the *space of integrable functions of weight $w = -n/p$*. For $0 < w$ (i.e. $p < 0$) this space of functions fails to be a vector space (local problem) and for $w < -n$ (i.e. $0 \le p < 1$) it fails to satisfy the triangle inequality. Therefore a natural range for w is $-n \le w \le 0$. We may denote the norm of an integrable function u of weight w by

$$||u||_{(w)} := \int |u|^{-n/w}.$$

Hölder's inequality implies that the product of two integrable functions of weight w_1 and w_2 respectively is integrable of weight $w_1 + w_2$.

Discussion 1.42 (*Sobolev spaces*) If a function has weight w then its kth derivative will have weight $w - k$. If a function lives in the Sobolev space $\mathcal{W}^{k,p}$ then in particular its kth derivative is in $\mathcal{L}^p$. Hence it is natural to assign w with $w - k = -n/p$ as the weight of that function; w the equals the Sobolev number $w = k - n/p$. For a compact domain with smooth boundary and in the case of the first derivative $k = 1$ we reformulate Sobolev's and Hölder's inequality in terms of weights as follows: for $(1-n) \le w < 0$ we have Sobolev's inequality:

$$||u||_{(w)} \le C(n, w)||\partial u||_{(w-1)},$$

with a constant $C(n, w)$ depending on the domain and the weight w, similarly for $0 < w \le 1$ we have the Hölder estimate:

$$|u(x) - u(y)| \le C(n, w)|x - y|^w ||\partial u||_{(w-1)}.$$

Note how the dimensional analysis takes care of the correct exponents.

Example 1.43 If a differential operator ∇ of order k is selfadjoint then the pointwise inner product $\langle \nabla\, \phi, \psi \rangle$ should be considered as a density of weight $-n$ (since these are the natural integrands) leaving $w = (k-n)/2$ as natural weight for ϕ and ψ. This weight coincides with the Sobolev number of the space $\mathcal{W}^{k/2,2}$ which is the usual domain of ∇ in terms of its weak formulation.

If a partial differential equation has a geometric origin, the involved unknown function may have a natural geometric dimension $(\mathit{length}\,)^w$. Such a natural or geometric weight w usually equals the *critical Sobolev number*, i.e. natural sections just fail to be integrable. (For example the Euclidean distance $\mathrm{dist}(\Sigma, .)$ to a submanifold Σ is geometrically given by a section of L^1 which is always critical.)

Example 1.44 The Yamabe problem of finding a metric within a given conformal class which has constant scalar curvature (with respect to its own length scale) is a typical example of such a situation. The problem reduces to a nonlinear eigenvalue problem for the conformal Laplace operator (see example 4.47) $\Delta f^{(2-n)/2} = -f^{(-2-n)/2}$ with normalization $\int f^{-n} = 1$, where f is a positive section of L^1 playing the role of a length scale. The geometric context has already determined the Laplace operator to act on densities of weight $w = (2-n)/2$, which agrees with the Sobolev number in the usual analytic setting in $\mathcal{W}^{1,2}$, see [LP87].

1.6 Weyl derivatives

Since the density line bundles L^w over a manifold M have no preferred trivialization (except for $w = 0$) covariant derivatives on these line bundles are of differential geometric interest. We will indicate how densities and Weyl derivatives provide a way to geometrize electromagnetism. Any first order differential operator can be twisted by a Weyl derivative to define an operator on sections of arbitrary central weight. We apply this to the deRham complex and its adjoint.

Definition 1.45 (*Weyl derivatives*) A covariant derivative D of L^1 is called a *Weyl derivative* (see definition 1.19). It naturally induces covariant derivatives on all density bundles L^w (e.g. by means of parallel transport).

Definition 1.46 (*Faraday curvature*) The curvature (see proposition 1.20) of a Weyl derivative D in L^1 is an endomorphism valued 2-form which for a line bundle is determined by a real valued 2-form $F^D \in C^\infty(M, \Lambda^2 T^*M)$.

The differential Bianchi identity, see proposition 1.29, applied to a given Weyl derivative D means that F^D is a closed 2-form, $dF^D = 0$. Originally Weyl interpreted a Weyl derivative as electromagnetic potential and its curvature as electromagnetic field, which then automatically satisfies the first Maxwell equation. Therefore we like to call the closed two form F^D the *Faraday curvature*. The induced curvature of L^w is given by wF^D.

The Faraday curvature is the local obstruction for finding a parallel trivialization of L^1. The Weyl derivative D is called *closed* if $F^D = 0$. For a closed Weyl derivative one can find locally positive sections of L^1, which are D parallel. If there is a global positive D parallel section μ of L^1 then D is called *exact*. Conversely a positive section of L^1 determines an exact Weyl derivative D^μ via $D^\mu\mu = 0$.

The space of all Weyl derivatives forms an affine space modelled on the linear space of smooth 1-forms $C^\infty(M, T^*M)$: let D and $\widetilde{D}$ be two covariant derivatives in the line bundle L^1. They differ by an endomorphism valued 1-form $\widetilde{D} - D = \gamma$ which for a line bundle is just a real valued 1-form $\gamma \in C^\infty(M, T^*M)$. The induced derivatives in L^k differ by $\widetilde{D} - D = k\gamma$. Closed and exact Weyl derivatives are affine subspaces modelled on closed and exact 1-forms respectively.

Application 1.47 (*Twisted exterior derivative*) With the help of a Weyl derivative D the exterior derivative d can be extended to forms of arbitrary central weight (see paragraph 1.28):

$$\mathrm{d}^D : C^\infty(M, L^w \otimes \Lambda^k T^*M) \to C^\infty(M, L^w \otimes \Lambda^{k+1} T^*M).$$

The obstruction for the twisted deRham sequence to be a complex is the Faraday curvature F^D.

Application 1.48 (*Twisted exterior divergence*) Similarly a Weyl derivative D defines divergences on multivector field densities of arbitrary central weight

$$div^D : C^\infty(M, L^{w-n} \otimes \Lambda^k TM) \to C^\infty(M, L^{w-n} \otimes \Lambda^{k-1} TM).$$

For example if μ denotes a density of weight w and X a vector field of central weight $1-n$, then $div^D(\mu \otimes X) = \mathrm{tr}(D\mu \otimes X) + \mu \otimes div\, X$.

Remark 1.49 Lie derivative, Weyl derivative and twisted divergence are related as follows: for a density μ of weight w and a vector field K we have $\mathcal{L}_K\mu = D_K\mu - \frac{w}{n}\mu\, div^D K$.

The vector field K generates diffeomorphisms which leave a given Weyl derivative D invariant, iff it satisfies the following linear partial differential equation of second order: $0 = \mathcal{L}_K D$. Here $\mathcal{L}_K D$ denotes a real valued 1-form defined with the help of vector field X and a length scale $\mu \in L^1$ by

$$\begin{aligned}(\mathcal{L}_K D)_X\mu &= \mathcal{L}_K(D_X\mu) - D_{\mathcal{L}_K X}\mu - D_X(\mathcal{L}_K\mu)\\ &= F^D(K, X)\mu + \frac{1}{n}\,\mu\,\partial_X(\mathrm{div}^D K).\end{aligned}$$

Chapter 2

Elementary conformal geometry

This chapter provides the elementary background in relativity, representation theory and differential geometry which we will use in the sequel. In the first two sections we define a conformal structure on a vector space and apply this to relativistic kinematics and electromagnetism in Minkowski space. The freedom of choosing a particular inner product in the conformal class corresponds to the free choice of a unit for the dimension (*length*). Light rays, relative velocity of observers, conservation of momentum in collision experiments and the electromagnetic constitutive relation in vacuum only depend upon a conformal structure. Perihelion advance illustrates the impact of special relativity on the motion in a central force field. In section 2.3 we recall how manifolds can be equipped with a conformal structure. There is no preferred covariant derivative on a conformal manifold: instead the space of Weyl derivatives parameterizes compatible derivatives of the tangent bundle. In sections 2.4 and 2.5 we recall how vector bundles are associated to the orthogonal frame bundle and determine their curvature properties. In section 2.6 we adapt Branson's method to Weyl derivatives, which reduces the question of conformal invariance to a linear question. In the last section we recall the first few integrability conditions of the conformal curvature tensors. These Bianchi identities motivate the field equations for Einstein's and Bach's theories of gravity.

2.1 Conformal vector spaces

Conformal geometry concerns the concept of angle without that of absolute length. A conformal structure of a vector space is an equivalence class of inner products with the same notion of orthogonality. Hence two such inner products differ by a multiplicative factor in front. Since there is no preferred inner product in the class we give the following invariant definition which is taken from [Cal98b]:

Definition 2.1 (*Conformal vector space*) A *conformal structure* c of a real vector space V is a normalized inner product on the weightless space $L^{-1} \otimes V$. Normalized means that on $\Lambda^n(L^{-1} \otimes V) = L^{-n} \otimes \Lambda^n V$ the canonical inner product and the inner product induced by c coincide: $|\det \mathsf{c}| = 1$. Alternatively c can be viewed as a normalized L^2 valued inner product on V. If $\mathbf{A}$ is an affine space over the conformal vector space V then $\mathbf{A}$ carries a conformal structure.

Remark 2.2 A conformal structure can also be defined as an equivalence class of inner products: a positive element $\mu \in L^1$ is called a *length scale* and it determines by $\mu^{-2}\mathsf{c}$ a (real valued) inner product on V with μ^{-n} as volume density. On the other hand, given a (real valued) inner product g on V then g represents a conformal structure given by $\mathsf{c} := \mu(g)^2 g$, where $\mu(g)$ is the length scale of g (equivalently $\mu^{-n}(g)$ is the volume density of g).

Notation 2.3 We will use the conformal metric to identify V with its dual, precisely, we will use the isomorphisms $\sharp\colon L^k \otimes V^* \to L^{k-2} \otimes V$ and $\flat\colon L^k \otimes V \to L^{k+2} \otimes V^*$ sometimes without denoting them.

The following remark is a bit technical. We will use it to integrate sections in L^{-1} along nonlightlike curves (application 4.26) and to recover the electric field and the magnetic flux from a Faraday 2-form, see remark 2.10.

Remark 2.4 (*Pull back of densities*) If $\iota\colon U \to V$ denotes a subspace of dimension k on which the induced conformal inner product stays nondegenerated, then we have an induced map $\iota^* : L^{-k}(V) \to L^{-k}(U)$: for $\mu \in L^{-k}(V)$, $u \in \Lambda^k U$, $\operatorname{sign}(\mu) = \pm 1$ we define $\iota^*(\mu)(u) := \operatorname{sign}(\mu)\sqrt{\mathsf{c}(\mu \otimes \iota(u), \mu \otimes \iota(u))}$, where c is the induced inner product on $\Lambda^k(V^0)$. More generally for all weights w we have $\iota^*\colon L^w(V) \to L^w(U)$ with $\iota^*(\mu)(u) = \operatorname{sign}(\mu)(|(|\mu|)^{-k/w} \otimes u|_{\mathsf{c}})^{-w/k}$.

2.2 Special relativity and electromagnetism

The simplest geometrical model of physical spacetime is an affine space $\mathbf{A}$ over a real $n = 4$ dimensional vector space V. Points in $\mathbf{A}$ are called *events* and an observer living for some time traces out a *worldline* in $\mathbf{A}$. In principle an (infinitesimal) observer $v \in V$ along his worldline can measure the angle between two direction and can decide whether two rulers have equal length (at the same time). Hence (infinitesimal) spacelike measurements take place in a conformal class of (Euclidean) metrics (of the three dimensional quotient $V/\mathbb{R}v$). Note, that the focus on conformal metrics also corresponds to the following physical principle:

Discussion 2.5 (*Constant velocity of light*) Lightrays travel through spacetime in a way which is independent of the motion of the emitter. In the geometrical model these naturally given lightrays form double cones through each (emitting) event. In special relativity these light cones come from a conformal class c of inner products in V with Lorentzian signature $(3, 1)$. Note that such a lightcone $\{v \in V \mid \mathsf{c}(v, v) = 0\}$ characterizes its conformal metric.

The geometry of lightcones describes the causal relation between events:

Definition 2.6 (*Causality*) Vectors $v \in V$ inside the lightcone $\mathsf{c}(v, v) < 0$ of an event $x \in \mathbf{A}$ are called *timelike*, vectors v with $\mathsf{c}(v, v) > 0$ are called *spacelike* and vectors v with $\mathsf{c}(v, v) = 0$ are called *lightlike*.

Similarly a (piecewise) differentiable path is called timelike (respectively spacelike or lightlike), if all velocity vectors are timelike (respectively spacelike or lightlike). We say that two events x and $x+v$ with v timelike are *in causal relation*. A lightlike or timelike path joining two events x and $x+v$ with v spacelike needs to change its time direction.

Observers are smooth curves $c(t) \in \mathbf{A}$ with timelike velocity vectors $\dot{c}$. The points in the intersection of the forward lightcone of $c(t_0 - \tau)$ with the backward lightcone of $c(t_0 + \tau)$ are called *simultaneous events relative to* c. The three dimensional linear space $\dot{c}(t_0)^{\perp} := \{v \in V \mid \mathsf{c}(v, \dot{c}(t_0)) = 0\}$ perpendicular to $\dot{c}(t_0)$ describes the space of simultaneous events infinitesimally.

Remark 2.7 (*Relative velocity*) Relativistic kinematics of particles is also based upon the conformal Lorentzian metric: let $U \in V^0 := L^{-1} \otimes V$ be a unit tangent vector to a worldline of an observer, hence U is timelike and normalized $\mathsf{c}(U,U) = -1$, see also 4.10. A second observer N with $\mathsf{c}(N,N) = -1$ meeting U at an event x *points into the same time direction* as U, if $\mathsf{c}(U,N) < 0$. In that case N measures a *relative velocity* $\vec{u} \in N^{\perp} \subset V^0$ (as multiple of the velocity of light, example 1.5) in his infinitesimal space of simultaneous events $\mathsf{c}(\vec{u}, N) = 0$ given by the condition $N + \vec{u} \sim U$, hence $N + \vec{u} = -U/\,\mathsf{c}(U,N)$. Squaring this equation gives $|\vec{u}|^2 = 1 - 1/\,\mathsf{c}(U,N)^2$.

Remark 2.8 (*Collision experiments*) If two (pointlike) bodies collide at an event in space-time then their initial velocities N_1 and N_2 (weightless and normalized) and inertial masses m_1 and m_2 are related to their final velocities N_1' and N_2' (weightless and normalized) and final inertial masses m_1' and m_2' as $m_1 N_1 + m_2 N_2 = m_1' N_1' + m_2' N_2'$ (see [Par87] p. 18ff). Note that this conservation law at the (idealized) event of collision only depends upon the conformal structure. From the dimensional analysis of example 1.6 we think of inertial masses as elements in L^{-1} and $m_1 N_1 \in V^*$ is called the *linear momentum* of the body.

Electromagnetism deals with the following physical phenomenon: *motion of electric charges produces fields which travel with the speed of light and these fields influence the motion of charged particles*. The theory describing electromagnetic interactions is based upon Maxwell's equations. In their relativistic form due to Minkowski the field equations are part of the deRham sequence of the exterior derivative and its adjoint the sequence of exterior divergences:

Definition 2.9 (*Electromagnetic field theory*) A vector field j of central weight $1-n$ represents the *charge current density*. It plays the role of the *source* and has to satisfy a *conservation law*: $div\, j = 0$. We have to distinguish between two realizations of the field itself, the dynamic and the kinematic field. The *dynamic field* G is represented by a bivector density field of central weight $2-n$ and it is coupled to the source by the second Maxwell equation $j = div\, G$. The *kinematic field* F is represented by a 2-form of central weight -2 the so called *Faraday* 2-*form* and satisfies an *integrability condition* $dF = 0$ the first Maxwell equation.

Remark 2.10 (*E and B field*) Relative to an observer $\mathsf{c}(N,N) = -1$ with its space of simultaneous events $N^{\perp} \subset V^0$ the kinematic field F splits into two tensors of central weight -2 in $N^{\perp}$. We make secret use of the remark 2.4 and define a 1-form the *electric field strength*

$E(\vec{u}) := F(N, \vec{u})$ and a 2-form the *magnetic flux* $B(\vec{u}, \vec{v}) := F(\vec{u}, \vec{v})$ with $\vec{u}, \vec{v} \in N^\perp$. Similarly the dynamic field G splits into the *dielectric displacement* $D(\vec{u}) := G(N, \vec{u})$, a vector on $N^\perp$ of central weight -2 and the *magnetic loop tension* $H(\vec{u}, \vec{v}) := G(\vec{u}, \vec{v})$ a bivector on $N^\perp$ of central weight -2. For the physical dimensions see example 1.7.

Discussion 2.11 (*Constitutive relation*) Dynamic and kinematic fields are related by the so called *constitutive relation* which depends upon the matter in which the field propagates. In vacuum (and in $n = 4$ dimensions) the constitutive relation between F and G is simply given by a conformal structure c of Lorentzian signature $(n-1, 1)$, i.e. $F(X, Y) = \sum_{i,j} \mathsf{c}(X, t_i)\,\mathsf{c}(Y, t_j) G(\theta^i, \theta^j)$, where X, Y are vectors and t_i, θ^i is a dual basis of V.

Since the kinematic field F is closed there (locally) exists a 1-form A of central weight -1, called *potential*, such that $F = dA$. This potential is not uniquely determined by F, instead it is subject to (local) transformations $A \mapsto A + df$, where f is a function, called a *gauge function*.

Summary 2.12 To summarize, the electromagnetic field theory in vacuum takes place in an (affine) space modelled on a four dimensional vector space V with conformal metric of Lorentzian signature. The *kinematic sequence* is given by the beginning of the deRham sequence of the exterior derivative:

$$\begin{array}{ccccccc} \text{gauge} & & \text{potential} & & \text{kinematic field} & & \\ C^\infty(\mathbf{A}, \mathbb{R}) & \stackrel{d}{\to} & C^\infty(\mathbf{A}, V^*) & \stackrel{d}{\to} & C^\infty(\mathbf{A}, \Lambda^2 V^*) & \stackrel{d}{\to} & C^\infty(\mathbf{A}, \Lambda^3 V^*) \\ f & & A & & F & & 0 \end{array}$$

The *dynamic sequence* is given by the end of the sequence of the exterior divergence:

$$\begin{array}{ccccccc} 0 & & j & & G & & \\ C^\infty(\mathbf{A}, L^{-n}) & \stackrel{div}{\leftarrow} & C^\infty(\mathbf{A}, L^{-n} \otimes V) & \stackrel{div}{\leftarrow} & C^\infty(\mathbf{A}, L^{-n} \otimes \Lambda^2 V) & \stackrel{div}{\leftarrow} & C^\infty(\mathbf{A}, L^{-n} \otimes \Lambda^3 V) \\ & & \text{source} & & \text{dynamic field} & & \end{array}$$

The conformal metric is the constitutive relation between kinematic and dynamic field. Hence the Maxwell equations in vacuum are linear and conformally invariant given by:

$$\begin{gathered} F = G \text{ via the conformal structure } \mathsf{c}, \\ 0 = div\, j \text{ and } j = div\, G, \\ 0 = dF \text{ and } F = dA \text{ and } A \mapsto A + df. \end{gathered}$$

Remark 2.13 Note that the second Maxwell equation $div\, G = j$ between the dynamic field and the source already determines all other differential operators which are involved: the integrability condition for $div\, G = j$ gives the conservation of the source $0 = div\, div\, G = div\, j$, adjoint to this conservation law is the operator for the gauge transformation $A + df$, these transformations are in the kernel of the equation between potential and kinematic field $F = dA = d(A + df)$ (which is also adjoint to $div\, G = j$), finally the integrability of $dA = F$ gives the first Maxwell equation $0 = ddA = dF$.

Discussion 2.14 (*Lorentz force law*) Galilei's principle of inertia means that a free massive particle with initial velocity N in spacetime has the tendency to stick to that spacetime direction, i.e. in the affine model of spacetime a free massive particle moves along straight timelike lines: $D_N N = 0$, where D denotes the affine derivative. Obviously the principle of inertia isn't conformally invariant, since it involves the affine derivative. If an electromagnetic field F is present, an electrically charged particle $q \in \mathbb{R}$ is no longer free, instead it is influenced by F: a simple relativistic law of motion is then given by

$$mD_N N = q\sharp(F(N)),$$

where $m \in L^{-1}$ is the inertial mass of the charged particle and N is viewed as a test particle (q/m is small compared to F).

Example 2.15 (*Coulomb field*) Coulomb's (nonrelativistic) law determines the force between two charges q and Q which are at rest relative to the laboratory to be qQ/r^2. Viewing one charge q as test particle and the other Q as source, the Lorentz force law $mD_N N = qF(N)$ determines the corresponding Faraday 2-form F which solves Maxwell's equations - the Coulomb field: let N be the unit timelike vector of the two charges and identify the conformal affine spacetime $\mathbf{A}$ with V by taking an event $o \in \mathbf{A}$ along the straight worldline of the source as origin. If $P\colon V \to N^\perp \subset V$ denotes the projection $v \mapsto v - \langle v, N\rangle/\langle N, N\rangle N$ the radial distance $r\colon \mathbf{A} \to L^1$ to the source is given by $r(c) := |x|$, where $x = x(c) := P(c - o) \in N^\perp$ is the spacelike location of the test particle. Coulomb's force law on the one hand and Lorentz force law on the other hand gives $qQ/r^2\ \vec{r} = qF(N)$, where $\vec{r} = x/|x| = \mathit{grad}^D r$ denotes the unit radial field, which is the gradient of the distance function with respect to the (usual) affine derivative D. Therefore if $F\colon \mathbf{A} \to \Lambda^2 V^*$ is a pure electric field for N then it is given by

$$F = F_{\mathrm{Coul}} := Q/r^2\ (\langle \vec{r}, .\rangle N - \langle N, .\rangle \vec{r}).$$

Discussion 2.16 (*Motion in a central field*) Next we like to study the relativistic motion of a test particle $c\colon \mathbb{R} \to \mathbf{A}$ subject to the Lorentz force in the Coulomb field. For that we solve the ordinary differential equation

$$\ddot{c} = -a/r^2\ (\langle \vec{r}, \dot{c}\rangle N - \langle N, \dot{c}\rangle \vec{r}),$$

where $a \in \mathbb{R}$ is a number. Clearly $\langle \dot{c}, \dot{c}\rangle$ is a constant of motion. For the projection $x := P(c)$ we have

$$\ddot{x} = P(\ddot{c}) = -a/r^2(-\langle \dot{c}, N\rangle)\vec{r}.$$

The function $-\langle \dot{c}, N\rangle$ satisfies $\frac{\partial}{\partial t}(-\langle \dot{c}, N\rangle) = a\frac{\partial}{\partial t}(1/r)$, hence

$$-\langle \dot{c}, N\rangle = b + a/r,$$

where $b = -\langle \dot{c}(0), N\rangle - a/r(0)$ depends on the initial conditions only. Hence we can introduce the central potential $U\colon \mathbb{R}^+ \to \mathbb{R}$ defined by

$$U(r) := -ab/r - (\frac{1}{2}\, a^2)/r^2,$$

and (with $U(x) = U(|x|)$) the differential equation becomes

$$\begin{aligned}\ddot{x} &= -\mathit{grad}\, U_x = -U'(|x|)x/|x| \\ r(t) &= |x(t)| \\ \dot{c}(t) &= \dot{x}(t) + (b + a/r(t))N.\end{aligned}$$

We have the usual conserved quantities in a central potential: angular momentum $\vec{l} := \dot{x} \wedge x$ with length squared $l^2 = \langle \dot{x}, \dot{x} \rangle r^2 - (r\dot{r})^2$ and energy $E := \frac{1}{2}\langle \dot{x}, \dot{x} \rangle + U(|x|) = \frac{1}{2}\dot{r}^2 + \frac{1}{2} l^2/r^2 + U(r)$. The conservation of the bivector $\vec{l}$ means that the solution curve for $x(t)$ lies in a plane $x = r(\cos(\phi), \sin(\phi))$ and for the polar angle ϕ we have $\dot{\phi} = l/r^2$. Note $\langle \dot{c}, \dot{c} \rangle = 2E - b^2$.

Discussion 2.17 (*Perihelion advance*) In the case of a Coulomb field of a charge Q and an opposite test charge q with inertial mass m we have $a = -Qq/m > 0$, i.e. an attractive potential. A timelike solution curve $\langle \dot{c}, \dot{c} \rangle = -1 = 2E - b^2$ which directs into the same time $0 < -\langle \dot{c}, N \rangle$ is guaranteed if $b > 0$. If in addition $-1/2 < E < 0$ and $a^2 < l^2 < a^2/(-2E)$ we have bounded orbits $0 \leq (\dot{r})^2 = 2E - 2U(r) - l^2/r^2$ hence $r_P \leq r \leq r_A$ with $r_{A/P} = r_0 \pm q$, $(-2E)r_0 := ab$ and $(-2E)q^2 := a^2/(-2E) - l^2$. Since $(\dot{r})^2 = (-2E)(r_A - r)(r - r_P)/r^2$ the polar angle satisfies

$$\phi'(r) = \frac{\dot{\phi}}{\dot{r}} = \frac{l}{r^2\dot{r}} = \frac{l}{\sqrt{-2E}} \frac{1}{r\sqrt{(r_A - r)(r - r_P)}},$$

hence with $\phi(r_P) = 0$ we find

$$\phi(r) = \frac{l}{\sqrt{(-2E)r_A r_P}} \arccos\left(\frac{r_P r_A - r_0 r}{qr}\right).$$

Finally $\phi(r_A) = \pi l/\sqrt{(-2E)r_A r_P}$ and the total angle between two perihelion constellations (moment of nearest approach to the centre) is given by:

$$2\pi \frac{l}{\sqrt{(-2E)r_A r_P}} = 2\pi \frac{1}{\sqrt{1 - (a/l)^2}} \approx 2\pi(1 + \frac{1}{2}(a/l)^2 + \frac{3}{8}(a/l)^4 + \ldots).$$

(We could compare the above special relativistic perihelion advance with the perihelion advance in the Schwarzschild model (see e.g. [ONe83] p. 379): the qualitative behavior is similar, since the leading term is in both cases given by $(a/l)^2$, but the special relativistic factor in front is $\frac{1}{2}$ whereas in the Schwarzschild model it is 3.)

Discussion 2.18 (*Charge*) In physical situations where a smooth source j models the charge distribution with a dynamic field G subject to $\mathit{div}\, G = j$ we deduce from the divergence theorem that the integral of j over a cooriented $(n-1)$-dimensional ball $\int_B \langle j, \mathit{coor} \rangle = \int_B \langle \mathit{div}\, G, \mathit{coor} \rangle = \int_{\partial B} \langle G, \mathit{coor} \rangle$ only depends upon the boundary ∂B rather then the spanning B. This leads to the definition 2.19 of charge represented by G for the case of G being smooth in a region away from $\mathrm{supp}(j)$ even if j is distributional. We remark that the integral $\int_{\partial B} \langle G, \mathit{coor} \rangle$ only depends upon the homology class of ∂B as will be explained in proposition 2.20. In what follows we let $M \subset \mathbf{A}$ be an open region away from $\mathrm{supp}(j)$ where the field G is smooth.

Definition 2.19 (*Electric charge*) If G denotes a smooth divergence-free bivector density we define the *electric charge* (represented by G) contained in an (immersed) cooriented $(n-2)$-dimensional sphere $S^{n-2} \hookrightarrow M$ by:

$$Q_e := \int_{S^{n-2}} \langle G, coor \rangle.$$

As is well known, this quantity satisfies a conservation law, which follows from the divergence theorem:

Proposition 2.20 *If $S^{n-2} = \partial\Sigma$ is the boundary of an (immersed) compact cooriented $(n-1)$-dimensional submanifold $\Sigma \hookrightarrow M$, then $Q_e = 0$. Consequently, if S^{n-2}, coor and $S^{n-2\prime}, \mathrm{coor}'$ are two immersed spheres, such that there is an (immersed) compact cooriented $(n-1)$-dimensional submanifold Σ, coor with these two spheres as boundary: $\partial\Sigma, \mathrm{coor} = S^{n-2}, \mathrm{coor} \cup S^{n-2\prime}, -\mathrm{coor}'$, then S^{n-2} and $S^{n-2\prime}$ contain the same amount of electric charge.*

For completeness we will also define *magnetic charge* represented by a Faraday 2-form:

Definition 2.21 (*Magnetic charge*) If F denotes a smooth closed 2-form we define the *magnetic charge* (represented by F) contained in an (immersed) oriented 2-dimensional sphere $S^2 \hookrightarrow M$ by:

$$Q_m := \int_{S^2} \langle F, or \rangle.$$

This quantity satisfies a conservation law, which follows from Stoke's theorem:

Proposition 2.22 *If $S^2 = \partial\Sigma$ is the boundary of an (immersed) compact oriented 3-dimensional submanifold $\Sigma \hookrightarrow M$, then $Q_m = 0$. Consequently, if S^2, or and $S^{2\prime}, \mathrm{or}'$ are two immersed spheres, such that there is an (immersed) compact oriented 3-dimensional submanifold Σ, or with these two spheres as boundary: $\partial\Sigma, \mathrm{or} = S^2, \mathrm{or} \cup S^{2\prime}, -\mathrm{or}'$, then S^2 and $S^{2\prime}$ contain the same amount of magnetic charge.*

Remark 2.23 The differential equations of electromagnetism make sense on any conformal manifold. Source-free electromagnetic fields can then give rise to charge due to nontrivial topology of the underlying manifold. See the joint article by Hadley and the author [DH99] for a review and extension of this idea.

2.3 Conformal structure on manifolds

In this section we follow Weyl [Wey70], Bergmann and Einstein [Ber76] to define conformal structures and related geometries on arbitrary smooth manifolds. A Riemannian metric defines a conformal class of metrics, but there is no preferred metric within such a class. Consequently we define a conformal structure invariantly as a single Riemannian metric of the weightless tangent bundle. There is no preferred covariant derivative on a conformal manifold instead we deal with an affine space of derivatives modelled on the space of smooth 1-forms.

Definition 2.24 A conformal structure on a manifold M is a (smooth) normalized L^2 valued inner product c on TM (see definition 2.1). It can be viewed as a nondegenerate normalized (real valued) inner product on the weightless tangent bundle $L^{-1} \otimes TM$. This inner product between vector fields X and Y will also be denoted by $\langle X, Y \rangle = \mathsf{c}(X, Y)$.

Definition 2.25 (*Conformal geometry*) A conformal structure c defines a conformal geometry on M. Naively, a construction is called conformally invariant, if it only depends upon c, i.e. if the construction is defined using an additional choice one needs to show that the construction is indeed independent of that choice.

Definition 2.26 (*Conformal Killing fields*) A vector field K on a conformal manifold M generates diffeomorphisms which leave the conformal structure c invariant iff $0 = \mathcal{L}_K \mathsf{c}$. A solution to this linear first order partial differential equation is called a *conformal Killing field*.

Proposition 2.27 *(Torsion) Let D be a covariant derivative of the tangent bundle TM. The following bilinear differential pairing $T^D : C^\infty(M, TM) \otimes C^\infty(M, TM) \to C^\infty(M, TM)$ defined by $T^D(X, Y) := D_X Y - D_Y X - [\,X, Y\,]$ (where X, Y are vector fields and the Lie bracket on functions is defined by $\partial_{[\,X,Y\,]} = \partial_X \partial_Y - \partial_Y \partial_X$) is indeed zero order and defines the* torsion *of D to be a vector valued 2-form $T^D \in C^\infty(M, \Lambda^2 T^* \otimes TM)$.*

A covariant derivative D on TM is called *torsion-free* if $T^D = 0$. A covariant derivative D on TM induces covariant derivatives on all tensor bundles including L^w (for instance by means of parallel transport or via the induced connection on the first order frame bundle $\mathbf{GL}\,(M)$). The covariant derivative on L^1 is called a Weyl derivative, see definition 1.45. The derivative D is called *compatible with the conformal structure* if $D\mathsf{c} = 0$.

Theorem 2.28 *(Fundamental theorem of conformal geometry) On a conformal manifold M there is a one to one correspondence between covariant derivatives on L^1 and derivatives on TM, which are torsion-free and compatible with the conformal structure c. Such derivatives will be called* Weyl derivatives *and for vector fields X,Y,Z the* Koszul formula *holds:*

$$\begin{aligned} 2\,\mathsf{c}(D_X Y, Z) \;&=\; D_X(\,\mathsf{c}(Y, Z)) + D_Y(\,\mathsf{c}(Z, X)) - D_Z(\,\mathsf{c}(X, Y)) \\ &\quad + \mathsf{c}([\,X, Y\,], Z) - \mathsf{c}([\,Y, Z\,], X) + \mathsf{c}([\,Z, X\,], Y). \end{aligned}$$

Proof: The arguments needed here are the same as those used in Riemannian geometry: first of all if a covariant derivative D is torsion-free and $D\mathsf{c} = 0$ then the above Koszul formula holds. Furthermore, if $f \in C^\infty(M, \mathbb{R})$ is a function then D defined by the Koszul formula satisfies $D_{fX} Y = f D_X Y$ and $D_X(fY) = (\partial_X f)Y + f D_X Y$ since $[\,fX, Y\,] = f[\,X, Y\,] - (\partial_Y f)X$. $\square$

Definition 2.29 (*Riemannian geometry*) A length scale, i.e. a positive section μ of L^1 plus a conformal structure c defines a Riemannian geometry on M. Let D be the (exact) Weyl derivative which leaves the length scale parallel. The induced derivative on TM is then the Levi-Civita derivative of the metric $\mu^{-2}\mathsf{c}$.

A vector field K on a n-dimensional manifold M with Riemannian geometry, generates diffeomorphisms which leave the geometry invariant, iff K is a conformal Killing field which leaves the length scale invariant i.e. $0 = \mathcal{L}_K \mathsf{c}$ and $0 = \mathcal{L}_K \mu$. The second condition says that the vector field is divergence-free (with respect to the length scale). Using the Levi-Civita derivative, the two equations can be summarized by saying that DK is a skew symmetric endomorphism field.

Definition 2.30 (*Weyl geometry*) A Weyl derivative D in L^1 plus a conformal structure c defines a Weyl geometry on M.

A vector field K on a n-dimensional manifold M with Weyl geometry, generates diffeomorphisms which leave the geometry invariant, iff K is a conformal Killing field which leaves the Weyl derivative invariant i.e. $0 = \mathcal{L}_K \mathsf{c}$ and $0 = \mathcal{L}_K D = F^D(K,\ ) + \frac{1}{n} \partial \operatorname{div}^D K$. The conformal Killing equation can be rewritten in terms of the Weyl derivative D:

$$
\begin{aligned}
(\mathcal{L}_K \mathsf{c})(X,Y) &= \mathcal{L}_K(\mathsf{c}(X,Y)) - \mathsf{c}(\mathcal{L}_K X, Y) - \mathsf{c}(X, \mathcal{L}_K Y) \\
&= \mathsf{c}(D_X K, Y) + \mathsf{c}(X, D_Y K) - \frac{2}{n}(\operatorname{div}^D K)\, \mathsf{c}(X,Y).
\end{aligned}
$$

A vector field solving both equations is called a *Weyl Killing field*. Its 2-jet is already determined from its 1-jet, since for the induced derivative D on TM we clearly have $0 = \mathcal{L}_K D$ and

$$
\begin{aligned}
(\mathcal{L}_K D)_X Y &= \mathcal{L}_K(D_X Y) - D_{\mathcal{L}_K X} Y - D_X \mathcal{L}_K Y \\
&= D_K D_X Y - D_{D_X Y} K - D_{[K,X]} Y - D_X D_K Y + D_X D_Y K \\
&= R^D_{K,X} Y + D^2_{X,Y} K.
\end{aligned}
$$

Remark 2.31 (*Weyl's unified theory of electromagnetism and gravity*) Einstein's theory of gravity models the spacetime by a four dimensional manifold M with a Lorentzian metric g. Its divergence-free Ricci curvature, see remark 2.63, is coupled to the energy density of the matter in question. Electromagnetism is then modelled by an additional field F on M (which might contribute to the matter energy). In contradistinction Weyl geometry provides a geometric model of spacetime where the electromagnetic field is intrinsically represented by the Faraday curvature F^D, see [Wey70] and the next remark 2.32.

Remark 2.32 (*Twin paradox*) In case of Weyl geometry with Lorentzian signature we say that the parameterization of a timelike worldline $c\colon \mathbb{R} \to M$ is *proportional to the proper time* of the observer c if $\langle D_{\dot c} \dot c, \dot c \rangle = 0$. If two different worldlines c_1 and c_2 start at the same point $x = c_1(0) = c_2(0)$ with the same unit of proper time $\langle \dot c_1, \dot c_1 \rangle(0) = \langle \dot c_2, \dot c_2 \rangle(0)$ and both are parameterized proportional to their proper time, it is clear that if they meet again $y = c_1(t_1) = c_2(t_2)$ the two proper times t_1, t_2 will be different. This is called the twin paradox. In the context of Weyl geometry with nontrivial Faraday curvature F^D not only are the two parameters t_1, t_2 different, but also the two time units $\langle \dot c_1, \dot c_1 \rangle(t_1)$ and $\langle \dot c_2, \dot c_2 \rangle(t_2)$ are different. Similar, a parallel transported length scale along the two worldlines will be different at y even if it is the same at x. The two twins starting at x and meeting at y not only have different ages, but also different sizes, and they will age at different rates. This dependence on past history was the main reason why Weyl geometry as a unified theory of gravity and electromagnetism was rejected. This does not rule out the possibility of a very small coupling constant λ between the Faraday curvature F^D and the corresponding electromagnetic field $\lambda F = F^D$.

2.4 Elementary notions of representation theory

In this section we recall that Lie groups and their Lie algebras can act on vector spaces by linear maps. This is called a representation and we will use the basic definitions in the next section only to give a unified treatment of all relevant vector bundles on a conformal manifold.

Notation 2.33 The group of linear isomorphisms preserving the conformal structure c will be denoted by $\mathbf{CO}\,(V) := \{A \in \mathbf{GL}\,(V) \,|\, \mathsf{c}.A = \mathsf{c}\}$. It is a direct product $\mathbf{CO}\,(V) = \mathbb{R}^+ id_V \times \mathbf{O}\,(V)$ with one dimensional centre. Its Lie algebra is given by $\mathfrak{co}(V) = \mathbb{R} id_V \oplus \mathfrak{so}(V)$.

$\mathbf{CO}\,(V)$ and $\mathfrak{co}(V)$ is an example of a Lie group with its Lie algebra. In general let G be a real Lie group and $\mathfrak{g}$ be its real (or complexified) Lie algebra.

Definition 2.34 (*Group representations*) A real vector space W is called a *right representation space* for G if there is a group action from the right $\Theta\colon W \times G \to W$ by linear maps. It will also be denoted by $\Theta(w, A) := w.A$ and satisfies $w.A.B = w.(AB)$ for all $A, B \in G$ and $w \in W$. A left action of G on W would be denoted by $G \times W \to W$ satisfying $A.B.w = (AB).w$. Every left action has an associated right action defined by $w.A = A^{-1}.w$ and vice versa. If W is a right representation of G then the dual space W^* carries a left G-representation (defined without inverting group elements): $\langle A.\omega, w\rangle = \langle \omega, w.A\rangle$ for $\omega \in W^*$ and all $w \in W$.

Definition 2.35 (*Lie algebras*) Let $\mathbb{F}$ be a field (we are interested in the cases $\mathbb{F} = \mathbb{R}$ and $\mathbb{F} = \mathbb{C}$). A finite dimensional vector space $\mathfrak{g}$ over $\mathbb{F}$ with a skew symmetric bilinear multiplication $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, denoted on $X, Y \in \mathfrak{g}$ by $X \otimes Y \mapsto [\,X, Y\,]$ is called a *Lie algebra*, if it satisfies the *Jacobi identity*: for all $X, Y, Z \in \mathfrak{g}$ we have $[\,X, [\,Y, Z\,]\,] + [\,Y, [\,Z, X\,]\,] + [\,Z, [\,X, Y\,]\,] = 0$.

Remark 2.36 If G denotes a real Lie group and $\mathfrak{g}$ the space of left invariant vector fields on G, then $\mathfrak{g}$ carries a real Lie algebra structure induced by the Lie bracket of vector fields.

If $\mathfrak{p} \subset \mathfrak{g}$ is a subspace, such that $X, Y \in \mathfrak{p}$ implies $[\,X, Y\,] \in \mathfrak{p}$, then $\mathfrak{p}$ is called a *subalgebra* of $\mathfrak{g}$. A linear map $\phi\colon \mathfrak{g} \to \mathfrak{h}$ between two Lie algebras is called a *Lie algebra homomorphism*, if $\phi([\,X, Y\,]) = [\,\phi(X), \phi(Y)\,]$ for all $X, Y \in \mathfrak{g}$ (this condition is nonlinear in ϕ similar to the orthogonality condition for a linear transformation).

Example 2.37 The space of endomorphisms $\mathrm{End}\,(V) = \mathfrak{gl}(V)$ of a vector space V over the field $\mathbb{F}$ carries a natural Lie algebra structure by $[\,A, B\,] := A \circ B - B \circ A$ and is called the *general linear Lie algebra*.

Definition 2.38 (*Modules*) If $\mathfrak{g}$ denotes a Lie algebra over $\mathbb{F}$, a vector space W is said to carry a *right* $\mathfrak{g}$*-module structure* or W is called a *right* $\mathfrak{g}$*-representation space*, if there is a linear map $\theta\colon W \otimes \mathfrak{g} \to W$ denoted by $\theta(w \otimes X) =: w.X$ such that for $Y \in \mathfrak{g}$ we have $w.[\,X, Y\,] = w.X.Y - w.Y.X$.

A $\mathfrak{g}$-representation from the left $\theta\colon \mathfrak{g} \otimes W \to W$ will be denoted by $\theta(X \otimes w) = X.w$ and satisfies $[\,X, Y\,].w = X.Y.w - Y.X.w$. Every left action has an associated right action defined by $w.X = -X.w$ and vice versa. If W is a right module then W^* is a left module defined (without signs) by $\langle X.\omega, w\rangle = \langle \omega, w.X\rangle$ for $\omega \in W^*$ and all $w \in W$.

Remark 2.39 If G denotes a real Lie group with Lie algebra $\mathfrak{g}$. Let $W \times G \to W$ be a right group representation, then this induces a right representation of the Lie algebra $W \otimes \mathfrak{g} \to W$.

Examples 2.40 (*Adjoint representation*) For any Lie algebra $\mathfrak{g}$ the Jacobi identity shows, that the Lie bracket itself induces a left representation on $\mathfrak{g}$ itself, called *adjoint representation* $\mathrm{ad}\colon \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, $\mathrm{ad}(X)Y := [\,X, Y\,]$. The kernel of the adjoint representation is given by the centre $\ker \mathrm{ad} = \mathfrak{Z}(\mathfrak{g})$. The *coadjoint representation* $\mathrm{coad}\colon \mathfrak{g} \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ is given by: $\mathrm{coad} := -\mathrm{ad}^*$.

Constructions 2.41 If $W \otimes \mathfrak{g} \to W$ is a finite dimensional right representation and $X \in \mathfrak{g}$ then the action $.X$ is an endomorphism of W. Any $\mathbb{F}$ multiple of $\mathrm{tr}(\ .X)\colon \mathbb{F} \to \mathbb{F}$ defines a 1-dimensional representation. If $W_1 \otimes \mathfrak{g} \to W_1$ and $W_2 \otimes \mathfrak{g} \to W_2$ are representations, then $.X \oplus\ .X$ induces a representation on the direct sum $W_1 \oplus W_2$. Similarly $.X \otimes id_{W_2} + id_{W_1} \otimes\ .X$ induces a representation on the tensor product $W_1 \otimes W_2$.

2.5 Bundles and curvature on conformal manifolds

In this section we construct all relevant vector bundles on a conformal manifold. Riemann's curvature tensor can be defined in Weyl geometry as in Riemannian geometry and it defines an obstruction against local flatness.

Construction 2.42 (*Associated vector bundles*) Let M be a real n-dimensional conformal manifold (modelled on V). Denote by $\mathbf{CO}\,(M) \to M$ the bundle of conformal frames, i.e. the set of all orthogonal frames of equal length: for $x \in M$ we have $\mathbf{CO}\,(M)_x = \{f\colon V \to T_xM \mid f \text{ linear}, f^*\mathsf{c}_x = \mathsf{c}\}$. The group $\mathbf{CO}\,(V)$ acts naturally on $\mathbf{CO}\,(M)$ from the right. A right representation $\Theta\colon E \times \mathbf{CO}\,(V) \to E$ leads to a vector bundle over M by the *associated bundle construction*: $EM := \mathbf{CO}\,(M) \times_\Theta E := \{\{(fA, e.A) \mid A \in \mathbf{CO}\,(V)\} \mid f \in \mathbf{CO}\,(M), e \in E\}$.

If E and F are $\mathbf{CO}\,(V)$-representation with associated bundles EM and FM and $\phi\colon E \to F$ is a $\mathbf{CO}\,(V)$-equivariant map, then ϕ induces a bundle map $\phi\colon EM \to FM$. The action of the Lie algebra $E \otimes \mathfrak{co}(V) \to E$ is an example leading to the bundle map $EM \otimes \mathfrak{co}(TM) \to EM$.

Discussion 2.43 Let M be a conformal manifold with an associated vector bundle $EM \to M$ coming from a $\mathbf{CO}\,(V)$-representation space E. Any Weyl derivative D on the line bundle

L^1 induces a covariant derivative on the tangent bundle TM (which is torsion-free and compatible with the conformal structure). Hence D induces a connection on the conformal frame bundle $\mathbf{CO}(M)$. Therefore D also induces covariant derivatives on all associated bundles EM. These derivatives are also denoted by $D\colon C^\infty(M, EM) \to C^\infty(M, T^*M \otimes EM)$. The target bundle $T^*M \otimes EM$ is associated to $V^* \otimes E$.

Discussion 2.44 (*Curvature in Weyl geometry*) Let M be a conformal manifold and D a Weyl derivative. Let EM be an associated vector bundle with induced derivative D. The curvature of EM is a section $R^{E,D} \in C^\infty(M, \Lambda^2 T^* \otimes \mathfrak{gl}(EM))$ defined on vector fields X, Y and $e \in C^\infty(M, EM)$ by

$$R^{E,D}_{X,Y} e := D_X D_Y e - D_Y D_X e - D_{[X,Y]} e.$$

It is given by the Lie algebra action of the curvature $R^D := R^{T,D} \in C^\infty(M, \Lambda^2 T^* \otimes \mathfrak{co}(TM))$ of the tangent bundle as $R^{E,D}_{X,Y} e = R^D_{X,Y}.e$.

Since D is torsion-free R^D satisfies the first Bianchi identity: $0 = R^D_{X,Y} Z + R^D_{Y,Z} X + R^D_{Z,X} Y$. Two tensorial contractions of R^D are possible. The contraction of the last two indices gives back the *Faraday curvature*:

$$F^D(X,Y) = \frac{1}{n} \operatorname{tr} R^D_{X,Y}.$$

The *Ricci curvature* $\mathrm{ric} \in C^\infty(M, T^*M \otimes T^*M)$ is defined to be the (natural) trace of the curvature of the weightless tangent bundle $R^{L^{-1}\otimes T,D} = R^D - F^D \otimes id$:

$$\mathrm{ric}^D(Y,Z) := \operatorname{tr}(R^{L^{-1}\otimes T,D}_{\ ,Y} Z) = \operatorname{tr}(R^D_{\ ,Y} Z - F^D(\ ,Y)Z).$$

Since D is torsion-free, the skew part of ric^D is determined by F^D as follows:

$$\mathrm{ric}^D(Y,Z) - \mathrm{ric}^D(Z,Y) = -(n-2) F^D(Y,Z).$$

Remark 2.45 In addition to the curvature terms in Riemannian geometry, the Faraday curvature F^D occurs as a new ingredient. It is the local obstruction of finding a parallel metric in the conformal class.

In the linear algebraic context of conformal vector spaces we will define a $\mathbf{CO}(V)$-equivariant projection from $\mathfrak{gl}(V) \to \mathfrak{co}(V)$. In the rest of this chapter it will occur as a useful short hand notation. The reason why we denote this projection by a Lie bracket will become clear in section 3.2:

Definition 2.46 For a conformal vector space V we define a $\mathfrak{co}(V)$-equivariant map:

$$[\ ,\ ] \colon V^* \otimes V \to \mathfrak{co}(V); \quad [\alpha, v] := \alpha(v) id_V + \alpha \otimes v - \flat v \otimes \sharp \alpha.$$

This projection satisfies the following properties: since it has values in $\mathfrak{co}(V)$ it acts on any representation, moreover it has the following symmetries: $v, w \in V$ and $\alpha, \beta \in V^*$

$$\begin{aligned} [\alpha, v].w &= [\alpha, w].v \\ [\alpha, v].\beta &= [\beta, v].\alpha. \end{aligned}$$

Proposition 2.47 *(Splitting the curvature) There is a natural splitting of $R^{E,D}$ into two parts:*

$$R^{E,D}_{X,Y}e = (\mathrm{W}^{\mathsf{c}}{}_{X,Y} - [\,\mathrm{r}^D(X), Y\,] + [\,\mathrm{r}^D(Y), X\,]).e,$$

where

$$\mathrm{r}^D \in C^\infty(M, T^*M \otimes T^*M)$$

is the normalized Ricci bilinear form *, and*

$$\mathrm{W}^{\mathsf{c}} \in C^\infty(M, \Lambda^2 T^*M \odot \mathfrak{so}(TM))$$

is the Weyl curvature tensor *. The fact, that* W^{c} *takes values in the Cartan tensor product means, that it satisfies the first Bianchi identity (separately) and all traces applied to* W^{c} *vanish. Hence, in two and three dimensions the Weyl curvature is zero.*

Proof: The canonical Ricci curvature mostly used in Riemannian geometry was defined above by $\mathrm{ric}^D(Y,Z) := \mathrm{tr}(R^D{}_{,Y}Z - F^D(\ ,Y)Z)$. The so called *normalized Ricci curvature* r^D is much more appropriate in conformal geometry and related to the canonical Ricci as

$$\mathrm{ric}^D = (n-2)\,\mathrm{r}^D + \mathrm{s}^D\,\mathsf{c}.$$

Here we use the splitting of $V^* \otimes V^*$ into symmetric trace-free, trace and skew part:

$$\mathrm{r}^D = \mathrm{r}^D_0 + \frac{1}{n}\,\mathrm{s}^D\,\mathsf{c} - \frac{1}{2}\,F^D,$$

where

$$2(n-1)\,\mathrm{s}^D := \mathrm{tr}_{\mathsf{c}}\,\mathrm{ric}^D$$

denotes the scalar curvature and

$$(n-2)\,\mathrm{r}^D_0(Y,Z) := \frac{1}{2}\,\mathrm{ric}^D(Y,Z) + \frac{1}{2}\,\mathrm{ric}^D(Z,Y) - \frac{1}{n}\,\mathrm{tr}_{\mathsf{c}}(\mathrm{ric}^D)\langle Y,Z\rangle$$

denotes the symmetric and trace-free Ricci curvature. In $n=2$ dimensions scalar and Faraday curvature together determine the whole curvature tensor, hence in that case $\mathrm{r}^D = \frac{1}{2}\,\mathrm{s}^D\,\mathsf{c} - \frac{1}{2}\,F^D$. We refer to Calderbank [Cal98a] for more details. In $n=3$ dimensions the Ricci curvature determines the whole curvature tensor. The Ricci curvature written as curvature tensor $-[\,\mathrm{r}^D(X), Y\,].Z + [\,\mathrm{r}^D(Y), X\,].Z$ satisfies the first Bianchi identity (separately), hence so does W^{c}. $\square$

2.6 Changing the Weyl derivative

Recall that a Weyl derivative is a covariant derivative of the line bundle L^1. The set of all Weyl derivatives forms an affine space modelled on the space of 1-forms. On a conformal manifold any Weyl derivative induces a covariant derivative on the tangent bundle. Let ∇^D be a linear differential operator defined using a Weyl derivative D and the conformal structure c. The curvature tensors R^D and r^D are particular (zero order) examples of this situation.

The aim of this section is to study the behavior of such differential operators under a change of the Weyl derivative. Such operators which are independent of the choice of Weyl derivative are *conformally invariant*. We adapt Branson's method to Weyl derivatives, which reduces the question of conformal invariance to a linear question, and more generally, provides a method for computing explicitly the dependence on D of an operator which is polynomial in D and its jets. We will recall that the Weyl part of the curvature tensor only depends upon the conformal class.

Proposition 2.48 *(Linearized Koszul formula) Let D and $\widetilde{D}$ be two covariant derivatives in L^1 and $\gamma \in C^\infty(M, T^*M)$ the 1-form such that $\widetilde{D} - D = \gamma$. For $X, Y \in C^\infty(M, TM)$ the Koszul formula 2.28 determines the change of the induced covariant derivatives in the tangent bundle to be:*

$$\widetilde{D}_X Y - D_X Y = [\,\gamma, X\,].Y := \gamma(X)Y + \gamma(Y)X - \mathsf{c}(X, Y)\sharp\gamma.$$

Corollary 2.49 *Let EM be a vector bundle associated to the conformal frame bundle and $e \in C^\infty(M, EM)$ a section, then the induced change of covariant derivatives induced by $\gamma \in C^\infty(M, T^*M)$ is given by the Lie algebra action as*

$$\widetilde{D}_X e - D_X e = [\,\gamma, X\,].e = -e.[\,\gamma, X\,].$$

Discussion 2.50 (*Fundamental theorem of calculus*) Let ∇^D be a linear differential operator defined using a Weyl derivative D and the conformal structure c. If $D(s)$ is a smooth curve of Weyl derivatives parameterized by $s \in [0, 1]$ then $\nabla^{D(s)}$ defines a curve of operators. The difference $\nabla^{D(1)} - \nabla^{D(0)}$ can be calculated using the fundamental theorem of calculus

$$\nabla^{D(1)} - \nabla^{D(0)} = \int_{[0,1]} \frac{\partial}{\partial s}\, \nabla^{D(s)}\, d\sigma.$$

The velocity of $D(s)$ will be called $\gamma := \frac{\partial}{\partial s} D$. Hence ∇^D is independent of the choice of D iff the derivative $(\partial_\gamma \nabla)^D := \frac{\partial}{\partial s}\, \nabla^{D(s)}$ vanishes for all $\gamma \in C^\infty(M, T^*M)$ and all Weyl derivatives D. Hence the proof that a construction depending on a choice of Weyl derivative is indeed conformally invariant can be simplified to a linear situation.

Examples 2.51 (*First order*) The Weyl derivative itself, the exterior derivative and the exterior divergence are first order operators depending on a Weyl derivative: the induced derivative D on an associated bundle EM itself depends affinely on the Weyl derivative D by the Koszul formula, and we simply have

$$\partial_\gamma D = [\,\gamma,\ \,].$$

Since the exterior derivative, see applications 1.47, on multilinear forms of arbitrary central weight w,

$$d^D : C^\infty(M, L^{w+k} \otimes \Lambda^k T^*M) \to C^\infty(M, L^{w+k} \otimes \Lambda^{k+1} T^*M),$$

is simply the invariant d twisted by the Weyl derivative on L^{w+k}, we get

$$\partial_\gamma d^D = (w + k)\, \gamma \wedge .$$

Similarly for the exterior divergence, see applications 1.48, on multivector densities of arbitrary central weight w,

$$div^D : C^\infty(M, L^{w-k} \otimes \Lambda^k TM) \to C^\infty(M, L^{w-k} \otimes \Lambda^{k-1} TM),$$

we have

$$\partial_\gamma\, div^D = (w - k + n)\, \gamma \lrcorner\, .$$

Examples 2.52 (*Second order*) For the induced second derivative $D^2_{X,Y} e := D_X D_Y e - D_{D_X Y} e$ on an associated EM we find

$$\begin{aligned}
\partial_\gamma (D^2)_{X,Y} e &= (\partial_\gamma D)_X D_Y e + D_X((\partial_\gamma D)_Y e) - (\partial_\gamma D)_{D_X Y} e - D_{(\partial_\gamma D)_X Y} e \\
&= (\partial_\gamma D)_X D_Y e + (\partial_{D_X \gamma} D)_Y e) + (\partial_\gamma D)_Y D_X e - D_{(\partial_\gamma D)_X Y} e \\
&= [\,\gamma, X\,].D_Y e + [\,D_X \gamma, Y\,].e + [\,\gamma, Y\,].D_X e - D_{[\,\gamma, X\,].Y} e.
\end{aligned}$$

In particular the curvature tensor $R^{E,D}$ is defined to be the skew part of the second derivative which simply gives

$$(\partial_\gamma R^{E,D})_{X,Y} e = [\,D_X \gamma, Y\,].e - [\,D_Y \gamma, X\,].e.$$

Application 2.53 Notice that the curvature tensor only changes in its Ricci part, which already shows that the Weyl curvature is conformally invariant, i.e. W^{c} only depends upon the conformal structure.

Example 2.54 (*Linear change of Ricci curvature*) More explicitly we can calculate the linear change of the Faraday and Ricci curvature terms:

$$\begin{aligned}
(\partial_\gamma F^D)_{X,Y} &= \mathrm{tr}([\,D_X \gamma, Y\,] - [\,D_Y \gamma, X\,]) \\
&= d\gamma(X, Y), \\
(\partial_\gamma\, \mathrm{ric}^D)_{Y,Z} &= \mathrm{tr}(\partial_\gamma R^D_{\ ,Y} Z - \partial_\gamma F^D(\ , Y) Z) \\
&= \mathrm{tr}([\,D\gamma, Y\,].Z - [\,D_Y \gamma,\ \,].Z) - d\gamma(Z, Y) \\
&= (2 - n) D_Y \gamma(Z) - \mathsf{c}(Y, Z)\, div^D \gamma, \\
(\partial_\gamma\, \mathrm{r}^D)_{Y,Z} &= -D_Y \gamma(Z).
\end{aligned}$$

I like to thank D. Calderbank for pointing out to me the following observation:

Discussion 2.55 (*Taylor's theorem of calculus*) Let ∇^D be again a linear differential operator defined using a Weyl derivative D and the conformal structure c. For a smooth curve $D(s)$ of Weyl derivatives parameterized by $s \in [0, 1]$ we defined the first derivative at $D = D(0)$ by $(\partial_\gamma\, \nabla)^D := \frac{\partial}{\partial s} \nabla^{D(s)}$ with $\gamma = D'(0)$. Similarly we can define higher derivatives as

$$(\partial^2_{\gamma,\gamma}\, \nabla)^D = \frac{\partial}{\partial s} (\partial_\gamma\, \nabla)^{D(s)}, \text{ etc.}$$

Since ∇^D is assumed to be of finite order, all higher derivatives will vanish eventually. Then Taylor's theorem applies to give

$$\nabla^{D+\gamma} = \nabla^D + (\partial_\gamma\, \nabla)^D + \frac{1}{2} (\partial^2_{\gamma,\gamma}\, \nabla)^D + \frac{1}{3!} (\partial^3_{\gamma,\gamma,\gamma}\, \nabla)^D + \dots\ .$$

Example 2.56 (*Ricci curvature*) We can calculate the change of the Ricci curvature as quadratic polynomial in γ: we only need to differentiate $(\partial_\gamma \mathrm{r}^D)_{Y,Z} = -D_Y\gamma(Z)$ once more to obtain $(\partial^2_{\gamma,\gamma} \mathrm{r}^D)_{Y,Z} = -[\,\gamma, Y\,].\gamma(Z)$ and finally

$$\mathrm{r}^{D+\gamma}(Y,Z) \;=\; \mathrm{r}^D(Y,Z) - D_Y\gamma(Z) + \frac{1}{2}\,\gamma([\,\gamma, Y\,]Z).$$

2.7 Differential Bianchi identities

The Riemann curvature tensor of a covariant derivative D of a vector bundle EM satisfies a first order integrability condition, proposition 1.29. Since there is no prefered derivative on a conformal manifold we will construct conformally invariant operators annihilating the Weyl curvature tensor. In $n = 3$ dimensions the Weyl curvature vanishes and the Cotton York curvature defined below is a first invariant obstruction against local flatness in three dimensions. In this section we will follow Gauduchon's notes [Gau90], personal communications with Calderbank and his paper [Cal98b].

Definition 2.57 (*Cotton York tensor*) For any Weyl derivative D on a conformal manifold M define the *Cotton York tensor* C^D to be the skew derivative of the normalized Ricci curvature: $\mathrm{C}^D := d^D\,\mathrm{r}^D$, where r^D is viewed as a covector valued 1-form and C^D is a covector valued 2-form. On vector fields X, Y, Z we have

$$\mathrm{C}^D_{X,Y}(Z) := D_X\,\mathrm{r}^D(Y,Z) - D_Y\,\mathrm{r}^D(X,Z).$$

Proposition 2.58 *The linear change of the Cotton York tensor under a change of Weyl derivative is given by the Weyl curvature tensor as:*

$$(\partial_\gamma \mathrm{C})^D = \gamma(\mathrm{W}^{\mathsf{c}}).$$

Since the linear change is independent of the Weyl derivative D we can even conclude $\mathrm{C}^{D+\gamma} - \mathrm{C}^D = \gamma(\mathrm{W}^{\mathsf{c}})$.

Proof: The calculation is a straight forward application of the method developed in the previous chapter:

$$\begin{aligned}(\partial_\gamma \mathrm{C})^D_{X,Y} &= [\,\gamma, X\,].\,\mathrm{r}^D(Y) - [\,\gamma, Y\,].\,\mathrm{r}^D(X) - D^2_{X,Y}\gamma + D^2_{Y,X}\gamma\\ &= [\,\mathrm{r}^D(Y), X\,].\gamma - [\,\mathrm{r}^D(X), Y\,].\gamma - R^D_{X,Y}.\gamma\\ &= -\,{\mathrm{W}^{\mathsf{c}}}_{X,Y}\,.\gamma.\end{aligned}$$

The final remark is elementary calculus e.g. paragraph 2.50 or 2.55. □

Corollary 2.59 *The Cotton York tensor is conformally invariant in $n = 3$ dimensions since then the Weyl curvature vanishes. Hence in three dimensions the Cotton York tensor is an invariant obstruction against conformal flatness.*

Discussion 2.60 (*Second Bianchi identity*) The differential Bianchi identity of the Riemann curvature tensor R^D of the tangent bundle of a Weyl manifold reads:

$$0 = d^D R^D_{X,Y,Z} := D_X R^D_{Y,Z} + D_Y R^D_{Z,X} + D_Z R^D_{X,Y}.$$

With the splitting $R^D_{X,Y} = \mathrm{W^c}_{X,Y} - [\, \mathrm{r}^D(X), Y\,] + [\, \mathrm{r}^D(Y), X\,]$ in proposition 2.47 of R^D into Weyl and Ricci part we can rewrite the Bianchi identity using the Cotton York tensor:

$$d^D R^D_{X,Y,Z} = d^D \,\mathrm{W^c}_{X,Y,Z} - [\, \mathrm{C}^D_{X,Y}, Z\,] - [\, \mathrm{C}^D_{Y,Z}, X\,] - [\, \mathrm{C}^D_{Z,X}, Y\,].$$

A first contraction of this expression gives a covector valued 2-form (note $\mathrm{tr}\,\mathrm{W^c} = 0$ and $dF^D = 0$):

$$\begin{aligned} 0 &= \mathrm{tr}\, d^D R^D_{Y,Z} \\ &= \mathrm{tr}\, D\,\mathrm{W^c}_{Y,Z} - (n-3)\mathrm{C}_{Y,Z} \\ &\quad + \flat Z \otimes \mathrm{tr_c}\, D\,\mathrm{r}^D(Y) - \flat Y \otimes \mathrm{tr_c}\, D\,\mathrm{r}^D(Z) - \flat Z \otimes D_Y\, \mathrm{s}^D + \flat Y \otimes D_Z\, \mathrm{s}^D\,. \end{aligned}$$

One further contraction using c shows (note $\mathrm{tr_c}\,\mathrm{W^c} = 0$):

$$\begin{aligned} 0 &= -(n-3)\,\mathrm{tr_c}\, D\,\mathrm{r}^D(Z) + (n-3) D_Z\, \mathrm{s}^D \\ &\quad + \mathrm{tr_c}\, D\,\mathrm{r}^D(Z) - n\,\mathrm{tr_c}\, D\,\mathrm{r}^D(Z) - D_Z\, \mathrm{s}^D + n D_Z\, \mathrm{s}^D \\ &= -(2n-4)\,\mathrm{tr_c}\, D\,\mathrm{r}^D(Z) + (2n-4) D_Z\, \mathrm{s}^D\,. \end{aligned}$$

We summarize these calculations in the following:

Proposition 2.61 *(Contracted Bianchi identity) The derivative of the normalized scalar curvature is a divergence of the normalized Ricci curvature:*

$$\mathrm{tr_c}\, D\,\mathrm{r}^D(Z) = D_Z\, \mathrm{s}^D,$$

and a multiple of the Cotton York tensor is a divergence of the Weyl curvature:

$$div^D\, \mathrm{W^c}_{Y,Z} := \mathrm{tr}\, D\,\mathrm{W^c}_{Y,Z} = (n-3)\mathrm{C}_{Y,Z}.$$

Remark 2.62 The first equation means that the Cotton York tensor is trace-free and by $dF^D = 0$ it is also alternating-free, hence we have identified C^D as a section of $\mathrm{C}^D \in C^\infty(M, \Lambda^2 T^* \odot T^* M)$.

Remark 2.63 (*Einstein equation*) In closed Weyl geometry, i.e. $F^D = 0$, the Ricci curvature is symmetric and $\mathrm{r}^D - \mathrm{s}^D\,\mathsf{c}$ is a symmetric bilinear form, which is divergence-free by the above contracted Bianchi identity. General relativity models spacetime by a $n = 4$ dimensional conformal manifold of Lorentzian signature together with a closed (usually exact) Weyl derivative D. The geometry is coupled to the matter in question by the *Einstein equation* $\mathrm{r}^D - \mathrm{s}^D\,\mathsf{c} = T$ where T is the central weight -2 symmetric and divergence-free energy momentum stress density tensor of the matter. The dimensional analysis uses Galilei's constant as was explained in paragraph 1.12.

By the second part of the above proposition 2.61 together with paragraph 2.60, it is clear that the Weyl curvature tensor satisfies its own first order Bianchi identity in $n \geq 5$ dimensions: $0 = \mathit{Bianchi}(\mathrm{W}^{\mathfrak{c}})$ with the operator *Bianchi* given by the following:

Proposition 2.64 *(First order Bianchi identity) The following linear first order differential operator*

$$\mathit{Bianchi}\colon C^\infty(M, \Lambda^2 T^* \odot \mathfrak{so}(TM)) \to C^\infty(\Lambda^3 T^* \odot \mathfrak{so}(TM)),$$

defined on $W \in C^\infty(M, \Lambda^2 T^* \odot \mathfrak{so}(TM))$ by

$$\mathit{Bianchi}(W)_{X,Y,Z} := d^D W_{X,Y,Z} - \frac{1}{n-3}\left([\,\mathit{div}^D\, W_{X,Y}, Z\,] + [\,\mathit{div}^D\, W_{Y,Z}, X\,] + [\,\mathit{div}^D\, W_{Z,X}, Y\,]\right),$$

is nonzero and conformally invariant in $n \geq 5$ dimensions. It provides an integrability condition for the Weyl curvature tensor in $n \geq 5$ dimensions: $0 = \mathit{Bianchi}(\mathrm{W}^{\mathfrak{c}})$.

Proof: That this definition is independent of the choice of the Weyl derivative D is a lengthy but elementary calculation. It is also a special case of Fegan's result, theorem 4.36. In $n = 4$ dimensions this operator is zero. $\square$

The next aim will be to find a conformally invariant second order Bianchi identity in $n = 4$ dimensions.

Proposition 2.65 *(Third Bianchi identity) The Cotton York tensor viewed as a covector valued 2-form satisfies on vector fields X, Y, Z and U the following equation:*

$$d^D \mathrm{C}^D_{X,Y,Z}(U) + \mathrm{r}^D(X, {\mathrm{W}^{\mathfrak{c}}}_{Y,Z}\, U) + \mathrm{r}^D(Y, {\mathrm{W}^{\mathfrak{c}}}_{Z,X}\, U) + \mathrm{r}^D(Z, {\mathrm{W}^{\mathfrak{c}}}_{X,Y}\, U) = 0.$$

Proof: Applying d^D to $d^D\, \mathrm{r}^D$ gives $R^D . \wedge \mathrm{r}^D$, i.e. cyclic permutations like:

$$\begin{aligned} d^D \mathrm{C}^D_{X,Y,Z} &= (R^D_{X,Y} . \,\mathrm{r}^D)(Z) + \mathrm{cyc}\,(X, Y, Z) \\ &= R^D_{X,Y} .(\mathrm{r}^D(Z)) + \mathrm{cyc}\,(X, Y, Z), \end{aligned}$$

by the first Bianchi identity for R^D. The splitting of R^D gives

$$R^D_{X,Y} .(\mathrm{r}^D(Z)) = {\mathrm{W}^{\mathfrak{c}}}_{X,Y} .(\mathrm{r}^D(Z)) - [\,\mathrm{r}^D(X), Y\,].(\mathrm{r}^D(Z)) + [\,\mathrm{r}^D(Y), X\,].(\mathrm{r}^D(Z)).$$

Summing the cyclic permutations of the last two terms gives zero because $[\,\alpha, v\,].\beta$ is symmetric in α and β. $\square$

Remark 2.66 In three dimensions the Weyl curvature vanishes and the Cotton York tensor is conformally invariant. Hence the above leads to a first order integrability condition $d\mathrm{C} = 0$ for the Cotton York tensor, where the operator $d\colon C^\infty(M, \Lambda^2 T^* \odot T^*) \to C^\infty(M, \Lambda^3 T^* \odot T^*)$ is also conformally invariant (either by an elementary calculation or by theorem 4.36).

The contracted Bianchi identity $(n-3)\mathrm{C}^D_{X,Y} U = \mathit{div}^D\, {\mathrm{W}^{\mathfrak{c}}}_{X,Y}\, U$ together with the third Bianchi identity gives:

Corollary 2.67 *The Weyl curvature tensor is in the kernel of a second order differential operator:*

$$(\mathrm{d}^D\,\mathrm{div}^D\,\mathrm{W}^{\mathsf{c}})_{X,Y,Z}U + (n-3)(\mathrm{r}^D(X, \mathrm{W}^{\mathsf{c}}{}_{Y,Z}\,U) + \mathrm{r}^D(Z, \mathrm{W}^{\mathsf{c}}{}_{X,Y}\,U) + \mathrm{r}^D(Y, \mathrm{W}^{\mathsf{c}}{}_{Z,X}\,U)) = 0.$$

In $n = 4$ this is a conformally invariant Bianchi identity since the above operator is conformally invariant which we will prove in proposition 4.51, see also [Bra98]:

Proposition 2.68 *(Second Order Bianchi identity) For $n = 4$ dimensions there is a second order conformally invariant differential operators: Bianchi* $: C^\infty(M, \Lambda^2 T^* \odot \mathfrak{so}(TM)) \to C^\infty(M, \Lambda^3 T^* \odot T^*M)$ *given on an endomorphism valued 2-form W by*

$$\mathit{Bianchi}(W)_{X,Y,Z} := (\mathrm{d}^D\,\mathrm{div}^D\,W)_{X,Y,Z} + \mathrm{r}^D(X, W_{Y,Z}) + \mathrm{r}^D(Y, W_{Z,X}) + \mathrm{r}^D(Z, W_{X,Y}).$$

It provides an integrability condition for the Weyl curvature tensor in $n = 4$ dimensions Bianchi$(\mathrm{W}^{\mathsf{c}}) = 0$.

Remark 2.69 We have presented a number of integrability conditions on Weyl manifolds and conformal manifolds. As was remarked before the Bianchi identity provides a motivation for the Einstein equation in the Riemannian case. It is not clear to us how to couple matter to Weyl's geometric theory of gravity and electromagnetism (briefly described in remark 2.31), i.e. it is not clear how to relate the curvature invariants in Weyl geometry to the matter energy density.

We will finish this section by introducing the *Bach tensor* B^{c} in $n = 4$ dimension which is a curvature invariant of quartic order. In fact B^{c} is obtained by applying a conformally invariant second order operator to the Weyl curvature tensor. The invariance of the following operator will be proved in proposition 4.51, see also [Bra98]:

Proposition 2.70 *In arbitrary dimensions $n \geq 3$ there is a second order conformally invariant differential operator: Bach* $: C^\infty(M, L^{4-n} \otimes \Lambda^2 T^* \odot \mathfrak{so}(TM)) \to C^\infty(M, L^{2-n} \otimes T^* \odot T^*M)$ *given on an endomorphism valued 2-form W by*

$$\mathit{Bach}(W)_{Y,U} := \sum_{i,j} (D^2_{t_i,t_j} + \mathrm{r}^D(t_i, t_j))\,\mathsf{c}(W(\theta^i, Y)U, \theta^j) + \mathit{sym}\,(Y, U),$$

where t_i, θ^i is a dual basis of TM.

Definition 2.71 (*Bach tensor*) On a four dimensional conformal manifold we define the *Bach tensor* to be $\mathrm{B}^{\mathsf{c}} := \mathit{Bach}(\mathrm{W}^{\mathsf{c}})$, which is a trace-free symmetric tensor of central weight -4.

Proposition 2.72 *The Bach tensor on a conformal four manifold is divergence-free: $0 = \mathrm{div}\,\mathrm{B}^{\mathsf{c}}$, where div $: C^\infty(M, L^{2-n} \otimes T^* \odot T^*M) \to C^\infty(M, L^{-n} \otimes T^*M)$ is invariant, see theorem 4.36.*

Proof: This is an application of the Bianchi identities. For simplicity choose a flat Weyl derivative $F^D = 0$, then r^D is symmetric. Weyl curvature and Cotton York tensor are trace-free and alternating-free. For $n = 4$ we have from proposition 2.61 that $div^D\,\mathrm{W}^{\mathsf{c}} = \mathrm{C}^D$ and from proposition 2.65 we know $\sum_i D_{t_i}\mathrm{C}^D_{Y,Z}\sharp\theta^i = 0$. A lengthy but elementary calculation shows the result. $\square$

Remark 2.73 (*Bach's theory of gravity*) A couple of years after Weyl published his unified theory of electromagnetism and gravity, Bach obtained in [Bac21] his conformal curvature invariant B^{c} together with its conservation law $div\,\mathrm{B}^{\mathsf{c}} = 0$ on a $n = 4$ dimensional conformal manifold. Using fundamental constants as in paragraph 1.8 the Bach tensor B^{c} has the dimension $(\mathit{energy})/(\mathit{volume})$. The field equation in Bach's conformal theory of gravity is $\mathrm{B}^{\mathsf{c}} = T$ where T is the conserved energy momentum stress tensor of the matter in question. This theory is nonlinear in the conformal structure c since $\mathsf{c} \mapsto \mathrm{W}^{\mathsf{c}}$ is nonlinear. Bach's field equation is fourth order in the conformal structure. This was one reason for most physicists to reject this theory. The induced conformal structure of the Schwarzschild geometry solves the source free Bach equation. On the other hand, the cosmological models due to Friedmann Robertson Walker in Einstein's theory are conformally flat and hence only provide trivial or flat solutions of Bach's equation.

Chapter 3

Algebra of conformal geometry

This chapter uses the Taylor expansion of smooth functions to determine in section 3.1 and 3.2 all vector fields on an affine conformal space $\mathbf{A}$ over V which leave the (flat) conformal metric c invariant. The space of these fields forms the conformal Lie algebra $\mathfrak{g}$, which is also called the *Möbius Lie algebra*. The algebra structure provides a link between V^*, the infinitesimal translations at infinity of $\mathbf{A}$, and the linear change of a Weyl derivative on a curved conformal manifolds, see remark 3.4. The Möbius algebra acts on smooth polynomial functions through the Lie derivative, see paragraph 3.7. The above mentioned link to linear changes of Weyl derivatives guides the search for higher jet operators on conformal manifolds, see proposition 4.45.

In section 3.3 we identify the Möbius Lie algebra as a linear Lie algebra $\mathfrak{g} = \mathfrak{so}(\hat{V})$, where $\hat{V}$ is a natural vector space associated to a conformal affine space $\mathbf{A}$. This allows to use tensor products of $\hat{V}$ as representations W for $\mathfrak{g}$. An element of such a representation space is called a *twistor* and it induces a *twistor field* on $\mathbf{A}$, i.e. a polynomial with values in the vector space W_{V^*} of *coinvariants* of W. Twistors and twistor fields have a vivid history and play the key role in this dissertation. They occur here in their most elementary context. The inclusion $W \to C^\infty(\mathbf{A}, W_{V^*})$ is the beginning of the conformal Bernstein Gelfand Gelfand complex, see theorem 5.12. In the last section 3.4 we will identify the conformal sphere as a homogeneous space.

3.1 Lie derivatives on affine space

Let $\mathbf{A}$ be an affine space modelled on a vector space V. Let F be a finite dimensional right representation space of $\mathfrak{gl}(V)$ (convention: for $A, B \in \mathfrak{gl}(V)$ we have $[\,A, B\,] = A \circ B - B \circ A$). Denote by $f \in C^\infty(\mathbf{A}, F)$ a function on $\mathbf{A}$ with values in F (i.e. a section of F). If $F \otimes \mathfrak{gl}(V) \to F$ comes from a group representation $F \times \mathbf{GL}\,(V) \to F$ and if $\phi \colon A \to A$ is a (local) diffeomorphism, then the *pull back* of f with respect to ϕ is defined to be $\phi^* f(x) := f(\phi(x)).(\partial\phi|_x)$. For any vector field $X \in C^\infty(\mathbf{A}, V)$ let ϕ_t be its local flow, i.e. $\frac{\partial}{\partial t}\phi_t(x) = X|_{\phi_t(x)}$. We define the *Lie derivative of f in the direction of X* by (convention):

$$\mathcal{L}_X f := \frac{\partial}{\partial t}|_{t=0} {\phi_t}^* f = \partial_X f + f.\partial X.$$

Here ∂ denotes the affine derivative on $\mathbf{A}$ in V and F. In particular, if $Y \in C^\infty(\mathbf{A}, V)$ is another vector field, then $\mathcal{L}_X Y = \partial_X Y - \partial_Y X$. The space of smooth vector fields forms a Lie algebra under $\mathcal{L}$. This Lie algebra structure turns the space of sections with the above Lie derivative $\mathcal{L}$ into a left representation of $C^\infty(\mathbf{A}, V)$, since

$$\mathcal{L}_X \mathcal{L}_Y f - \mathcal{L}_Y \mathcal{L}_X f = \mathcal{L}_{\mathcal{L}_X Y} f.$$

In what follows we like $\mathcal{L}$ to be a right representation of $C^\infty(\mathbf{A}, V)$ hence we will use in this section the opposite Lie bracket between smooth vector fields: $[\,X, Y\,] := -\mathcal{L}_X Y = \partial_Y X - \partial_X Y$. With these convention every right $\mathfrak{gl}(V)$ representation F induces a right $C^\infty(\mathbf{A}, V)$-representation:

$$\mathcal{L} \colon C^\infty(\mathbf{A}, F) \otimes C^\infty(\mathbf{A}, V) \to C^\infty(\mathbf{A}, F).$$

Discussion 3.1 (*Taylor expansion*) We fix a point $x \in \mathbf{A}$. Any smooth section $f \in C^\infty(\mathbf{A}, F)$ has a Taylor expansion at x: if $v \in V$ denotes a direction then $(f + \partial_v f + \ldots + \frac{1}{k!}\,\partial^k_{v^k} f)\big|_x$ is a polynomial in v of degree k. We assume, that F has a certain central weight $w \in \mathbb{R}$. We will label tensors in $\mathrm{Sym}^{\,k}(V^*) \otimes F$ by their central weight $w - k$. An element

$$f = (f_w, f_{w-1}, \ldots, f_{w-k}) \in \mathrm{Sym}\,(V^*) \otimes F := (\mathbb{R} \oplus V^* \oplus \mathrm{Sym}^{\,2} V^* \oplus \ldots) \otimes F$$

can be viewed as an F-valued polynomial on V (of finite degree k) given by ($e^v = 1 + v + \frac{1}{2}\,v^2 + \ldots$):

$$v \mapsto f(e^v) := f_w + f_{w-1}(v) + \ldots + \frac{1}{k!}\,f_{w-k}(v^k).$$

The Lie derivative of smooth sections in the direction of smooth vector fields $\mathcal{L}_X f = \partial_X f + f.\partial X$ induces a natural right action of polynomial vector fields $X = (X_1, X_0, \ldots) \in \mathrm{Sym}\,(V^*) \otimes V$ on polynomial sections:

$$(\mathrm{Sym}\,(V^*) \otimes F) \;\otimes\; (\mathrm{Sym}\,(V^*) \otimes V) \to \mathrm{Sym}\,(V^*) \otimes F,$$

given by $f \otimes X \mapsto \mathcal{L}_X f$. We use the symmetric algebra $\mathrm{Sym}\,(V)$ where the product of a and b is denoted by ab and e^v is viewed as a formal element $e^v = 1 + v + \frac{1}{2}\,v^2 + \ldots$. Then we find:

$$\mathcal{L}_X f(e^v) = f(X(e^v)e^v) + f(e^v).(X(e^v)).$$

The Lie derivative on smooth vector fields $C^\infty(V, V)$ induces a Lie algebra structure on $\mathrm{Sym}\,(V^*) \otimes V$. Note, that the Lie algebra $\mathrm{Sym}\,(V^*) \otimes V$ comes with a natural grading, such that elements in $\mathrm{Sym}^{\,k}(V^*) \otimes V$ are labled by $1 - k$. We will specialize the above formula to the Lie bracket $[\,X, Y\,] = -\mathcal{L}_X Y$ (note that the right action of $\mathfrak{gl}(V)$ on V is given by $v.A = -A(v)$):

$$\begin{array}{rcl}
[\,X_1, Y_1\,] &=& 0, \\
[\,X_0, Y_1\,] &=& X_0(Y_1), \\
[\,X_{-1}, Y_1\,](v) &=& X_{-1}(Y_1, v), \\
[\,X_0, Y_0\,](v) &=& X_0(Y_0(v)) - Y_0(X_0(v)), \\
[\,X_{-1}, Y_0\,](v, v) &=& 2X_{-1}(Y_0(v), v) - Y_0(X_{-1}(v, v)), \\
[\,X_{-1}, Y_{-1}\,](v, v, v) &=& 3X_{-1}(Y_{-1}(v, v), v) - 3Y_{-1}(X_{-1}(v, v), v).
\end{array}$$

In general we have

$$[X, Y](e^v) = X(Y(e^v)e^v) - Y(X(e^v)e^v).$$

3.2 Conformal Killing fields

Let $\mathbf{A}$ be an affine space modelled over the vector space V which is equipped with a conformal inner product c (see definition 2.1). In this chapter we characterize polynomial vector fields K on $\mathbf{A}$ which have local diffeomorphisms leaving the conformal structure of the affine space invariant: $\mathcal{L}_K\mathsf{c} = 0$. Such vector fields are called *conformal Killing fields* and represent the infinitesimal symmetries of conformal geometry. In $n \geq 3$ dimensions the space of these fields forms a finite dimensional Lie algebra.

Proposition 3.2 *The following sub space* $\mathfrak{g}$ *of quadratic vector fields on a conformal vector space* V, c

$$\mathfrak{g} := V \oplus \mathfrak{co}(V) \oplus V^* \quad \subset \quad V \,\oplus\, V^* \otimes V \,\oplus\, \mathrm{Sym}^2 V^* \otimes V,$$

with inclusion $V^* \hookrightarrow \mathrm{Sym}^2 V^* \otimes V$ *defined by*

$$\alpha \mapsto ((v, w) \mapsto \alpha(v)w + \alpha(w)v - \mathsf{c}(v, w)\sharp\alpha = [[\alpha, v], w]),$$

defines a Lie algebra of polynomial vector fields K*, which leave the conformal structure invariant:* $\mathcal{L}_K\mathsf{c} = 0$.

Proof: If (a, A, α) and (b, B, β) denote elements in $\mathfrak{g} = V \oplus \mathfrak{co}(V) \oplus V^*$ then the Lie bracket in $\mathrm{Sym}(V^*) \otimes V$ under the above inclusion is given by:

$$\begin{aligned} [a, b] &= 0, \\ [A, b] &= Ab, \\ [\alpha, b] &= \alpha(b)id + \alpha \otimes b - \flat b \otimes \sharp\alpha, \\ [A, B] &= A \circ B - B \circ A, \\ [\alpha, B] &= \alpha \circ B, \\ [\alpha, \beta] &= 0. \end{aligned}$$

This shows that $\mathfrak{g}$ is indeed a subalgebra. Since the conformal structure on affine space is parallel we know $\mathcal{L}_a\mathsf{c} = 0$. Also $\mathcal{L}_A\mathsf{c} = \mathsf{c}.A = 0$ by definition of $\mathfrak{co}(V)$. Finally $\mathcal{L}_\alpha\mathsf{c}(v) = \mathsf{c}.[\alpha, v] = 0$. $\square$

Note, that the Lie algebra $\mathfrak{g}$ inherits the grading from $\mathrm{Sym}(V^*) \otimes V$ namely $\mathfrak{g}_1 := V$, $\mathfrak{g}_0 := \mathfrak{co}(V)$ and $\mathfrak{g}_{-1} := V^*$. The subalgebra of vector fields which vanish at the origin will be called $\mathfrak{p} := \mathfrak{co}(V) \oplus V^*$.

Remark 3.3 (*Translations at infinity*) The Abelian subalgebra V^* generates local diffeomorphisms which can be called *translations at infinity*: if $\mu \in L^1$ is a length scale and $g = \mu^{-2}\mathsf{c}$ a Euclidean metric then the *inversion at the unit sphere* I is defined to be a map $V \setminus \{x \mid \mathsf{c}(x,x) \neq 0\} \to V$ with $I(x) := \frac{x}{g(x,x)}$. It leaves the conformal structure invariant $I^*\mathsf{c} = \mathsf{c}$ and is idempotent $I^2 = id$. Finally, for $\alpha \in V^*$ define $x \mapsto I(I(x) + \alpha)$, which is independent of μ and defines a group homomorphism between $V^*, +$ and conformal diffeomorphisms of V.

Remark 3.4 (*Change of Weyl derivative*) The Lie bracket $[\gamma, v]$ between a linear form $\gamma \in V^*$ and a vector $v \in V$ is an element in $\mathfrak{co}(V)$ and hence acts on any representation of $\mathbf{CO}(V)$. The corresponding bundle map on a conformal manifold coincides with the linearized Koszul formula 2.48, where γ represents a (linear) change of Weyl derivative and v a direction in which to differentiate. Hence the translations at infinity on the affine space can be interpreted as linear changes of Weyl derivatives on a curved conformal manifold.

Proposition 3.5 *For $n \geq 3$ the Lie subalgebra of all polynomial vector fields K with $\mathcal{L}_K\mathsf{c} = 0$ is finite dimensional and given by the above $\mathfrak{g} := V \oplus \mathfrak{co}(V) \oplus V^*$.*

Proof: If $K \in \mathrm{Sym}^k V^* \otimes V$ with $k \geq 1$ represents a polynomial vector field, then $0 = \mathcal{L}_K\mathsf{c}$ forces K to be in the kernel of the map

$$\mathrm{Sym}^k V^* \otimes V \to \mathrm{Sym}^{k-1} V^* \otimes (V^* \odot V),$$

where $V^* \odot V$ denote the endomorphisms of V which are symmetric and trace-free, i.e. $V^* \otimes V = (V^* \odot V) \oplus \mathfrak{co}(V)$. For $n \geq 3$ and $k \geq 3$ the above map is injective. For $k = 2$ we have a short exact sequence

$$0 \to V^* \to \mathrm{Sym}^2 V^* \otimes V \to V^* \otimes (V^* \odot V) \to 0. \quad \square$$

Remark 3.6 Conformal geometry in two dimensions $n = 2$ is different reflected by the richness of holomorphic functions (which are conformal diffeomorphisms at points where the first derivative is nonzero). Indeed for $n = 2$ and all $k \geq 2$ we have a short exact sequence

$$0 \to \mathrm{Sym}_0^{k-1} V^* \to \mathrm{Sym}^k V^* \otimes V \to \mathrm{Sym}^{k-1} V^* \otimes (V^* \odot V) \to 0,$$

where $\mathrm{Sym}_0^{k-1} V^*$ is the two dimensional space of trace-free symmetric forms. Consequently the Lie algebra of polynomial vector fields is infinite dimensional and given by

$$V \oplus \mathfrak{co}(V) \oplus V^* \oplus \mathrm{Sym}_0^2 V^* \oplus \mathrm{Sym}_0^3 V^* \oplus \ldots \quad .$$

This dissertation only covers the $n \geq 3$ dimensional case. Conformal vector fields on a two dimensional conformal space which also leave the additional choice of a *Möbius structure* invariant (see [KP88] for the flat case, and [Cal98a] for the curved case) are then again only given by the above quadratic polynomials.

Discussion 3.7 ($\mathfrak{g}$-*action on functions*) Let $F \otimes \mathfrak{co}(V) \to F$ be a right representation. As before we assume, that F has a certain central weight $w \in \mathbb{R}$. Polynomials with values in F are elements $f = (f_w, f_{w-1}, \ldots, f_{w-k}) \in \mathrm{Sym}(V^*) \otimes F$ (of finite degree k) given by $v \mapsto f(e^v) := f_w + f_{w-1}(v) + \ldots + \frac{1}{k!} f_{w-k}(v^k)$. The Lie derivative $\mathcal{L}$ induces a right action of $(a, A, \alpha) \in \mathfrak{g} = V \oplus \mathfrak{co}(V) \oplus V^*$ on $f \in \mathrm{Sym}(V^*) \otimes F$:

$$\mathcal{L} \colon \mathrm{Sym}(V^*) \otimes F \;\otimes\; (V \oplus \mathfrak{co}(V) \oplus V^*) \to \mathrm{Sym}(V^*) \otimes F,$$

given by

$$\begin{aligned}
\mathcal{L}_a f(e^v) &= f(ae^v),\\
\mathcal{L}_A f(e^v) &= f((Av)e^v) + f(e^v).A,\\
\mathcal{L}_\alpha f(e^v) &= f(\frac{1}{2}([\alpha, v]v)e^v) + f(e^v).[\alpha, v].
\end{aligned}$$

3.3 Tensorial twistors in conformal geometry

Penrose discovered the physical significance of elements in a representation space of the Lie algebra of conformal Killing fields $\mathfrak{g}$. He called these elements *twistors* (see [PR84]) and in the present context of affine conformal geometry any such element induces a special section of a certain bundle, a *twistor field*. Constant functions and conformal Killing fields are particular examples. Twistors play a central role in this dissertation and occur here as polynomials. Later on they will be realized as solutions of overdetermined partial differential equations and will play the role of generalized charges (such as gravitational masses) in linear field theories.

Definition 3.8 (*Conformal twistors*) Let W be a right representation space of the conformal Lie algebra $\mathfrak{g} = V \oplus \mathfrak{co}(V) \oplus V^*$. An element of W is called a *conformal twistor*.

Our first aim is to construct such representations W by identifying $\mathfrak{g}$ as a linear Lie algebra: if V is a vector space then we denote by $V^0 := L^{-1} \otimes V$ the weightless vector space.

Definition 3.9 If V carries a conformal structure c with signature $\mathrm{sig}(\mathsf{c}) = (p, n-p)$ then the $(n+2)$-dimensional space

$$\hat{V} := L^1 \oplus V^0 \oplus L^{-1}$$

carries a natural (real valued) inner product of signature $\mathrm{sig}(\mathsf{c}) + (1,1) = (p+1, n-p+1)$. It is given on two vectors $\hat{v} = (v_1, v_0, v_{-1})$ and $\hat{w} = (w_1, w_0, w_{-1})$ in $\hat{V}$ by

$$\langle \hat{v}, \hat{w} \rangle := v_1 w_{-1} + \mathsf{c}(v_0, w_0) + v_{-1} w_1.$$

The space Skew $(\hat{V}) = \mathfrak{so}(\hat{V})$ of skew symmetric endomorphisms of $\hat{V}$ forms a Lie algebra under $[\hat{A}, \hat{B}] = \hat{A} \circ \hat{B} - \hat{B} \circ \hat{A}$.

Proposition 3.10 *The following identification:*

$$V \oplus \mathfrak{co}(V) \oplus V^* \xrightarrow{\simeq} \mathfrak{so}(\hat{V}) \;;\; (a, A, \alpha) \mapsto \hat{A}$$

defined on $\hat{v} = (v_1, v_0, v_{-1}) \in \hat{V}$ *by*

$$\hat{A}\hat{v} := \left(\frac{1}{n}(\mathrm{tr}\, A) v_1 + \mathsf{c}(a, v_0)\,,\, A_0 v_0 - a \otimes v_{-1} - \alpha \otimes v_1\,,\, \alpha(v_0) - \frac{1}{n}(\mathrm{tr}\, A) v_{-1} \right),$$

is a Lie algebra isomorphism, where $A = A_0 + \frac{1}{n}\,\mathrm{tr}\, A id$.

Tensor products of $\hat{V} = L^1 \oplus V^0 \oplus L^{-1}$ are therefore examples of representations for $\mathfrak{g} = V \oplus \mathfrak{co}(V) \oplus V^*$. Let $\mathbf{A}$ be an affine space over V and $x \in \mathbf{A}$ a point. Each element of $\mathfrak{so}(\hat{V})$ (the right adjoint representation) induces a polynomial vector field on $\mathbf{A}$, a so called *conformal Killing field* via $\mathfrak{so}(\hat{V}) = V \oplus \mathfrak{co}(V) \oplus V^* \to \mathrm{Sym}\,(V^*) \otimes V \to C^\infty(\mathbf{A}, V)$. In general if W is a right representation of $\mathfrak{g} = V \oplus \mathfrak{co}(V) \oplus V^*$ then $W \otimes \mathfrak{g} \to W$ induces a map $W \otimes V^* \to W$ with image denoted by $W.V^* \subset W$.

Definition 3.11 (*Coinvariants*) Elements of the quotient space $W_{V^*} := W/W.V^*$ are called *coinvariants* of W with respect to V^*.

Since the map $W \otimes V^* \to W$ is $\mathfrak{p} = \mathfrak{co}(V) \oplus V^*$ equivariant the space of coinvariants W_{V^*} is indeed a $\mathfrak{p}$-representation. The nilpotent part V^* of $\mathfrak{p}$ acts trivially on W_{V^*}.

Examples 3.12 If W is an irreducible subspace of a tensor product of $\hat{V} = L^1 \oplus V^0 \oplus L^{-1}$ with itself, then W splits under the action of the centre of $\mathfrak{co}(V)$ into a direct sum of subspaces with elements of the same central weight $W = W_w \oplus W_{w-1} \oplus \ldots \oplus W_{-w}$. The linear action of elements in V^* lowers the central weight by -1. From this it is clear that the space of coinvariants W_{V^*} is isomorphic to W_w as $\mathfrak{co}(V)$-representation (we will use the following notation: if E is a $\mathfrak{co}(V)$-representation of central weight w_E, then we denote by $E^w := L^{w-w_E} \otimes E$ the same $\mathfrak{so}(V)$-representation but of central weight w):

$$
\begin{aligned}
W &= \hat{V} = L^1 \oplus V^0 \oplus L^{-1}, \text{ hence } W_{V^*} = L^1, \\
W &= \mathfrak{so}(\hat{V}) = V \oplus \mathfrak{co}(V) \oplus V^*, \text{ hence } W_{V^*} = V, \\
W &= \Lambda^{k+1}(\hat{V}) = (\Lambda^k V)^1 \oplus \Lambda^{k-1} V^0 \oplus \Lambda^{k+1} V^0 \oplus (\Lambda^k V)^{-1}, \text{ hence } W_{V^*} = L^{k+1} \otimes \Lambda^k V^*, \\
W &= \mathrm{Sym}\,_0^2(\hat{V}) = L^2 \oplus V \oplus \mathrm{Sym}\,^2 V^0 \oplus V^* \oplus L^{-2}, \text{ hence } W_{V^*} = L^2.
\end{aligned}
$$

Remark 3.13 Suppose that a finite dimensional irreducible representation W of $\mathfrak{so}(\hat{V})$ is characterized by a Young diagram (see [FH91]) for $\hat{V}$. In that case, the coinvariants W_{V^*} as $\mathfrak{so}(V)$-representation can be characterized by the Young diagram for V which arises after deleting the first row. The central weight of W_{V^*} is given by the number of entries of the first row, which determines W_{V^*} as $\mathfrak{co}(V)$-representation. The nilpotent part of $\mathfrak{p} = \mathfrak{co}(V) \oplus V^*$ will act trivially on W_{V^*}.

Each element of W induces (in a linear way) a section with values in W_{V^*}. More precisely:

Proposition 3.14 *(Conformal twistor fields) The following linear inclusion is equivariant under the right $\mathfrak{g}$-actions:*

$$\iota\colon W \to \mathrm{Sym}\,(V^*) \otimes W_{V^*}; \quad \iota(w)(v^k) = [w(.v)^k]_{V^*}.$$

The resulting sections of $C^\infty(\mathbf{A}, W_{V^*})$ are called conformal twistor fields *induced by W.*

Proof: Let $(a, A, \alpha) \in \mathfrak{g}$ be a conformal Killing field then

$$\mathcal{L}_a(\iota(w))(e^v) = \iota(w)(ae^v) = [w.a.e^v]_{V^*} = \iota(w.a)(e^v).$$

As elements acting from the right on W (alternatively in the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$) we have $.A.e^v = [\,A, v\,].e^v + .e^v.A$ from which we get

$$\begin{aligned}\mathcal{L}_A(\iota(w))(e^v) &= \iota(w)((Av)e^v) + \iota(w)(e^v).A \\ &= [w.[\,A, v\,].e^v + w.e^v.A]_{V^*} \\ &= [w.A.e^v]_{V^*} = \iota(w.A)(e^v).\end{aligned}$$

Finally for the action of α note $.\alpha.e^v = .(\frac{1}{2}\,[\,\alpha, v\,]v).e^v + .e^v.[\,\alpha, v\,] + .e^v.\alpha$ hence

$$\begin{aligned}\mathcal{L}_\alpha(\iota(w))(e^v) &= \frac{1}{2}\,\iota(w)(([\,\alpha, v\,]v)e^v) + \iota(w)(e^v).[\,\alpha, v\,] \\ &= [\frac{1}{2}\,w.([\,\alpha, v\,]v).e^v + w.e^v.[\,\alpha, v\,]]_{V^*} \\ &= [w.\alpha.e^v - w.e^v.\alpha]_{V^*} \\ &= [w.\alpha.e^v]_{V^*} = \iota(w.\alpha)(e^v). \quad \square\end{aligned}$$

Example 3.15 The conformal twistor fields induced by $W = \Lambda^{k+1}\hat{V}$ with $k = 0 \ldots n$ are quadratic polynomials with values in $L^{k+1} \otimes \Lambda^k V^*$. In the case $k = 0$ the scalar valued polynomial coming from $\hat{a} = (a_1, a_0, a_{-1}) \in L^1 \oplus V^0 \oplus L^{-1}$ is a called a *Fierz twistor (for gravity)* and is given by

$$\begin{aligned}\iota(\hat{a})(e^v) &= [\hat{a} + \hat{a}.v + \frac{1}{2}\,\hat{a}.v.v + \ldots]_{V^*} \\ &= a_1 - \,\mathsf{c}(a_0, v) - \frac{1}{2}\,a_{-1}\,\mathsf{c}(v, v).\end{aligned}$$

In the case $k = 1$ we reproduce the *conformal Killing fields*: the element $\hat{A} = (a, A, \alpha) \in V \oplus \mathfrak{co}(V) \oplus V^*$ induces

$$\begin{aligned}\iota(\hat{A})(e^v) &= [\hat{A} + \hat{A}.v + \frac{1}{2}\,\hat{A}.v.v + \ldots]_{V^*} \\ &= [\hat{A} + [\,\hat{A}, v\,] + \frac{1}{2}\,[\,[\,\hat{A}, v\,], v\,]]_{V^*} \\ &= a + Av + \frac{1}{2}\,[\,\alpha, v\,]v.\end{aligned}$$

In the case $k \geq 2$ the twistor fields induced by $\hat{A} = (A_1, A_0^-, A_0^+, A_{-1}) \in (\Lambda^k V)^1 \oplus (\Lambda^{k-1}V)^0 \oplus (\Lambda^{k+1}V)^0 \oplus (\Lambda^k)^{-1}$ are given by

$$\begin{aligned}\iota(\hat{A})(e^v) &= [\hat{A} + \hat{A}.v + \frac{1}{2}\,\hat{A}.v.v + \ldots]_{V^*} \\ &= A_1 + v \wedge A_0^- + v \lrcorner A_0^+ + \frac{1}{2}\,\mathsf{c}(v, v)A_{-1}.\end{aligned}$$

3.4 Projective light cone

A projective model of conformal geometry can be constructed as follows: let $\hat{V}$ be a $(n+2)$-dimensional vector space with inner product of signature $(p+1, n-p+1)$. A nonzero vector $\hat{l} \in \hat{V}$ is called *lightlike*, *null* or *isotropic*, if $\langle \hat{l}, \hat{l} \rangle = 0$. The set of light like vectors forms the light cone $\mathrm{Cone} := \{\hat{l} \in \hat{V} - \{0\} \mid \langle \hat{l}, \hat{l} \rangle = 0\}$.

Definition 3.16 (*Conformal sphere*) The set of null lines $S := \mathrm{Cone}/\mathbb{R}^\times = \{\mathbb{R}\hat{l} \mid \hat{l} \in \mathrm{Cone}\}$ defines a compact real smooth manifold and will be called the *real conformal sphere of signature* $(p, n-p)$.

Remark 3.17 The real conformal sphere is diffeomorphic to a product of spheres $S^p \times S^{n-p}/\sim$ where points like $(x,y) \sim (-x,-y)$ are identified. It plays the role of the *flat model* of conformal geometry. The case $p = n-1$ and $n = 4$ refers to the *real conformal Minkowski space* of special relativity.

From this definition there are various tautological vector bundles over S: the trivial bundle $\hat{V} \times S$ has two natural sub bundles: the canonical real line bundle denoted by $\mathrm{Can} \to S$ with fibre $\mathbb{R}\hat{l}$ over $\mathbb{R}\hat{l} \in S$ and the bundle of tangent spaces to the light cone with fibre at $\mathbb{R}\hat{l} \in S$ given by $T_{\hat{l}}\mathrm{Cone} = \{\hat{j} \in \hat{V} \mid \langle \hat{j}, \hat{l} \rangle = 0\}$. Since the canonical line sits inside the tangent cone $\mathrm{Can} \subset T\mathrm{Cone}$ the quotient bundle $T\mathrm{Cone}/\mathrm{Can}$ is also natural. Indeed all intrinsic tensor bundles of S can naturally be rediscovered: for the tangent, cotangent and density bundles we have $TS = \mathrm{Can}^* \otimes T\mathrm{Cone}/\mathrm{Can}$, $T^*S = \mathrm{Can} \otimes (T\mathrm{Cone}/\mathrm{Can})^*$ and $L^w = L^w(TS) = \mathrm{Can}^{-w}$.

Notice that the weightless tangent bundle $L^{-1} \otimes TS = T\mathrm{Cone}/\mathrm{Can}$ inherits a Riemannian metric of signature $(p, n-p)$ from $\hat{V}$, and this defines a conformal inner product on each tangent space of S, i.e. a *conformal structure* on S.

Theorem 3.18 *(Möbius group) The orthogonal group* $\mathbf{O}(\hat{V})$ *acts linearly upon* $\hat{V}$ *and leaves the light cone invariant, which induces a smooth transitive action on the conformal sphere* $\mathbf{O}(\hat{V}) \times S \to S$ *by conformal diffeomorphisms, the so called* Möbius transformations*. For* $n \geq 3$ *all conformal diffeomorphisms of* S *come from* $\mathbf{O}(\hat{V})$*.*

For a proof of the last part of this theorem we refer to [KP88].

Remark 3.19 Let $G := \mathbf{O}(\hat{V})$ be a short hand for the Möbius transformations and $\mathbb{R}\hat{l} \in S$ be a point, then $P := \{g \in G \mid \mathbb{R}\hat{g}l = \mathbb{R}\hat{l}\}$ denotes the stabilizer sub group. By use of the orbit stabilizer theorem S can be identified as homogeneous space $S = G/P$. In the case $\hat{V} = L^1 \oplus V^0 \oplus L^{-1}$ the sub line L^{-1} is null and the stabilizer group of $\mathbb{R}\hat{l} = L^{-1}$ can be identified with $P = \mathbf{CO}(V) \ltimes V^*$.

Chapter 4

Differential conformal invariants

The aim of this chapter is to study local differential invariants on conformal manifolds. The conformal metric itself and the Weyl curvature are simple examples. We are concerned here with first and second order invariants in $n \geq 3$ dimensions. On conformal manifolds of indefinite signature lightlike geodesics are well known to be invariant and because of their physical significance we investigate them in the first section. Conformal geodesics generalize circles to curved conformal manifolds. We derive new conserved properties along conformal geodesics using twistors. In the last two sections we discuss differential invariants for sections of associated bundles. We recall Fegan's classification of linear conformally invariant differential operators of first order and apply it to give new bilinear invariants. These bilinear invariants are crucial when dealing with linear field theories as we will explain in the next chapter. The key tool to study second order invariants is the definition 4.44 of a 2-jet operator in terms of a Weyl derivative, which behaves well under a change of Weyl derivative. With this operator in hand all second order invariants of the affine space have analogues on curved conformal manifolds.

4.1 Conformal invariants along lightlike curves

A curve $\alpha\colon \mathbb{R} \to M$ in a conformal manifold of indefinite signature is called *lightlike* if $\langle \dot{\alpha}, \dot{\alpha} \rangle = 0$. Lightlike curves play the role of light rays in relativistic geometric optics. If the acceleration $D_{\dot{\alpha}}\dot{\alpha}$ is proportional to the tangent vector $D_{\dot{\alpha}}\dot{\alpha} \sim \dot{\alpha}$ the curve is called a *lightlike geodesic*. The above condition is independent of the parameterization and of the choice of the Weyl derivative D. Hence lightlike geodesics are conformally invariant. Together with the conformal invariance of electromagnetism we suggest a simple law how an electromagnetic field can influence a light ray. Unfortunately it is very difficult to verify such a law in a laboratory, since the interaction is expected to be very weak. Motivated by astronomical distance measurements we will define a new invariant distance on a conformal manifold along the backward lightcone of an observer.

One way to see the conformal invariance of lightlike acceleration is to attach a never vanishing vector field $L_\alpha \in L^{w-1} \otimes T_\alpha M$ of central weight w along α which is proportional to the tangent vector $L_\alpha \sim \dot{\alpha}$. For the acceleration in terms of L we find for a 1-form γ

representing a linear change of Weyl derivative, see example 2.51:

$$\partial_\gamma D_L L = w\gamma(L)L + \gamma(L)L - \langle L, L\rangle \sharp\gamma,$$

hence for a lightlike L of central weight $w = -1$ the acceleration $D_L L$ is conformally invariant.

Definition 4.1 (*Lightlike geodesic*) A lightlike curve $\alpha : \mathbb{R} \to M$ with an attached cotangent vector $0 \neq L_\alpha \in T^*_\alpha M$ with $L_\alpha \sim \dot{\alpha}$ is called a *lightlike geodesic* if $D_L L = 0$.

Remark 4.2 (*Frequency*) Lightlike acceleration being conformally invariant for vectors L of central weight -1 corresponds to the fact that in physics one usually attaches the dimension (*frequency*) to the tangent vector L: with the velocity of light as fundamental constant this has geometric dimension (*length*)$^{-1}$, see paragraph 1.5. Indeed the *frequency of L measured by an observer N* (with $\langle N, N\rangle = -1$) is defined to be $-\langle L, N\rangle$. This is a way to normalize lightlike vectors in a conformally invariant and physically meaningful way.

Proposition 4.3 *(Conservation law along lightlike geodesics) If K denotes a conformal Killing field on M and L is the attached covector of a lightlike geodesic, then $\langle L, K\rangle$ is a weightless conformally invariant constant of motion.*

Proof: This is a simple application of the product rule:

$$\begin{aligned}\partial_L(\langle L, K\rangle) &= \langle D_L L, K\rangle + \langle L, D_L K\rangle \\ &= \langle D_L L, K\rangle + \langle L, \mathit{Kill}(K)(L)\rangle + \frac{1}{n}\langle L, L\rangle \operatorname{div}^D K,\end{aligned}$$

and all three summands are zero by assumption. □

The conformal invariance of the lightlike acceleration $D_L L$ can be used to suggest an interaction between L and a Faraday 2-form F as follows:

Remark 4.4 (*Interaction with 2-forms*) Let $F \in C^\infty(M, \Lambda^2 T^*M)$ denote a closed 2-form. A conformally invariant interaction between a lightlike curve L and F is given by $D_L L = \lambda\sharp(F(L))$, where $\lambda \in \mathbb{R}$ is a coupling constant. If F is a kinematic electromagnetic field, then the above is a simple description of a light-light interaction in terms of classical geometric optics. The number λ needs to be determined by experiment: according to the above law a monochromatic laser light ray of (low) frequency ν should bend in a (small) circle of radius R when moving through a (strong) homogeneous magnetic field B such that:

$$R = \frac{1}{\lambda}\sqrt{\frac{\hbar}{4\pi\varepsilon_0 c^3}}\,\frac{\nu}{B} \quad .$$

Remark 4.5 If $K \in C^\infty(M, TM)$ denotes a vector field and $A \in C^\infty(M, T^*M)$ a (local) potential of $F = dA$ then $q := \langle L, K\rangle - \lambda A(K)$ is a weightless function along the light ray of L, which satisfies

$$\begin{aligned}\partial_L q &= \langle D_L L, K\rangle + \langle L, D_L K\rangle - \lambda\partial_L(K \lrcorner A) \\ &= -\lambda(K \lrcorner dA)(L) + \langle L, \mathit{Kill}(K)(L)\rangle - \lambda d(K \lrcorner A)(L) \\ &= -\lambda\mathcal{L}_K A(L) + \langle L, \mathit{Kill}(K)(L)\rangle.\end{aligned}$$

Hence if K is a conformal Killing field representing a symmetry of A, i.e. $\mathcal{L}_K(A) = 0$, then q is a conformally invariant conserved quantity along L.

Remark 4.6 (*Redshift and bending of light*) We will apply the study of motion in a Coulomb field, see paragraph 2.15, also to the case of a lightlike solution curve $L = \dot{c}$ and $\langle \dot{c}, \dot{c} \rangle = 0$ subject to the equation $D_L L = \lambda F(L)$. The variables a, b, E, l are defined in paragraph 2.16 and we will use their properties here. The law of motion fixes the parameter a to be $a = -\lambda Q$. The frequency of L measured by the static observer N is $-\langle \dot{c}, N \rangle = b + a/r$ hence $b > 0$ has the interpretation of the frequency at infinity. In the case of an attractive potential $a > 0$ the light has a higher frequency (it gains energy) when it comes nearer to the centre (when it looses potential energy). This is a redshift phenomenon. From $0 = \langle \dot{c}, \dot{c} \rangle$ we know $2E = b^2$ and the extremal distances r_0 with $\dot{r} = 0$ satisfy $E = \frac{1}{2} l^2/r_0^2 + U(r_0)$, i.e. $r_0 = (-a \pm l)/b$ which has exactly one positive solution if $l > |a| > 0$ given by $r_0 = (l - a)/b$. In the case $l > |a|$ the light ray passes the centre and comes nearest at $t = 0$ with distance r_0. The total polar angle is given by

$$\begin{aligned}
\phi(\infty) - \phi(-\infty) &= 2(\phi(\infty) - \phi(0)) \\
&= 2\int_{r_0}^{\infty} \frac{\dot{\phi}(t)}{\dot{r}}\, dr = 2\int_{r_0}^{\infty} \frac{l}{r^2 \dot{r}}\, dr = 2\int_0^{\infty} \frac{1}{\cosh(x) - a/l}\, dx \\
&= \frac{4}{\sqrt{1 - (a/l)^2}} \arctan\Big(\frac{e^x - a/l}{\sqrt{1 - (a/l)^2}}\Big)\Big|_0^{\infty} \\
&\approx \pi + 2(a/l) + \frac{\pi}{2}(a/l)^2 + \dots .
\end{aligned}$$

The qualitative behavior of this bending depends on the sign of $a = -\lambda Q$. Moreover from $l = a + br_0$ we find $\phi(\infty) - \phi(-\infty) \approx \pi + 2a/(br_0)$ which shows that the bending also depends upon the frequency b of the light. The bending of light in the Schwarzschild model (see e.g. [ONe83] p. 384) does not depend upon the frequency (in consistency with the equivalence of energy and gravitational mass) and is given by $\phi(\infty) - \phi(-\infty) \approx \pi + 4(a/r_0)$.

Remark 4.7 (*Luminosity distance*) In special relativity the intensity of electromagnetic radiation (also called luminosity) falls off like $1/r^2$ where r is the (retarded) spacelike distance to the source (the intensity is an energy density quadratic in the field). This fall off behavior allows one to measure astronomical distances of shining stars by comparing their radiation intensity to that of comparable stars with known distance. In special relativity this so called *luminosity distance* satisfies a differential equation (see paragraph 5.10) which generalizes to the following equation on a (curved) conformal manifold: along a lightlike geodesic $\alpha : \mathbb{R} \to M$ with $D_L L = 0$ we can define a section $r_\alpha \in L^1_\alpha$ subject to $D_L D_L r + \mathrm{r}^D(L, L) r = 0$, where the initial conditions $r(0) = 0$ and $D_L r(0) = -\langle L, N \rangle$ depend upon the choice of an observer $\langle N, N \rangle = -1$ at $\alpha(0)$ and r^D is the normalized Ricci curvature of D, see proposition 2.47. The resulting length scale along α is called the *luminosity distance* with respect to the observer N, the latter can be thought of as the radiation source or as the measuring astronomer.

Definition 4.8 A vector field L of central weight -1 on a conformal manifold of indefinite signature is called *lightlike* if $\langle L, L \rangle = 0$. It is called *geodetic* if $D_L L = 0$ (which is also a conformally invariant condition). The *frequency* measured by an observer N with $\langle N, N\rangle = -1$ is a density of weight -1 defined to be $-\langle L, N\rangle$.

Example 4.9 (*Light from a point source*) Let $c(t)$ be a parameterized timelike curve representing an emitter. Let M be a neighborhood of the worldline c such that the spacetime double lightcone of each point $x \in M$ intersects the worldline c twice: once with the forward lightcone at $c(\phi^+(x))$ and once with the backward lightcone at $c(\phi^-(x))$. This defines two function $\phi^\pm : M \to \mathbb{R}$ (*advanced* and *retarded* parameter). The two gradients $L^\pm := -\,grad\,\phi^\pm$ are lightlike geodetic vector fields. Along the worldline c we have $t = \phi^\pm(c(t))$ which gives $1 = \partial\phi^\pm(\dot c) = -\langle L^\pm, \dot c\rangle$, hence $L^\pm$ are fields pointing in the same time direction as $\dot c$ and the emitted frequency (relative to c) is incorporated into the parameterization of c.

4.2 Conformal invariants along nonlightlike curves

In this section we focus on invariants along timelike or spacelike curves. The Riemannian geodesic equation is not conformally invariant, but there is a conformally invariant third derivative of a curve. Curves which are uniformly accelerated are called *conformal geodesics*. The Fermi transport along arbitrary curves is a simpler example of an invariant, which we will need in the next section to construct conserved properties along conformal geodesics. None of the results of this section are new, although we simplified the conformal geodesic equation to a single third order equation, which will turn out to be useful in the next section.

Discussion 4.10 (*Geometry of curves*) Let $c : \mathbb{R} \to M$ with $t \mapsto c(t)$ be a parameterized nonlightlike curve $\langle \dot c, \dot c\rangle \neq 0$ on a conformal manifold M. The parameterization induces a positive section ν of L^{-1} along c defined by $\pm 1 = \nu^2\langle \dot c, \dot c\rangle$. The normalized weightless tangent vector $N := \nu\dot c$ is independent of the parameterization. Given a Weyl derivative D we call $D_N N$ the *acceleration* of c. The skew symmetric endomorphism $K^D \in L_c^{-1} \otimes \mathfrak{so}(T_cM)$ defined by $\langle N, N\rangle K^D := N \vartriangle D_N N = \langle\ , N\rangle D_N N - \langle\ , D_N N\rangle N$ is called the *curvature* of c with respect to D.

Definition 4.11 (*Fermi derivative along curves*) Let X be a vector field along a nonlightlike curve c, then N splits X into tangential and normal parts $X = X^\top + X^\perp$, where $\langle N, N\rangle X^\top := \langle X, N\rangle N$. The Fermi derivative with respect to a Weyl derivative D is defined by

$$\begin{aligned} \mathit{Fermi}^D_N X \;&:=\; (D_N X^\top)^\top + (D_N X^\perp)^\perp \\ &=\; D_N X - K^D(X). \end{aligned}$$

This definition on vector fields extends to arbitrary sections associated to a $\mathfrak{co}(V)$-representation by means of the left K^D action. The Fermi derivative leaves the conformal structure and the projections on tangential respectively normal parts parallel. A vector field X with $0 = \mathit{Fermi}^D_N X$ is called *minimally rotating* or *Fermi parallel*.

Proposition 4.12 *The Fermi derivative is conformally invariant for weightless tensors and therefore weightless tensors can be transported in a conformally invariant way along any curve.*

Proof: Let Z a vector field of central weight w along c. Under a linear change of Weyl derivative γ, the Fermi derivative of Z in the direction N changes as follows:

$$\begin{aligned}
\partial_\gamma \, \mathit{Fermi}_N^D Z &= \partial_\gamma D_N Z - \partial_\gamma K^D(Z) \\
&= w\gamma(N)Z. \quad \square
\end{aligned}$$

On the conformal sphere S (see definition 3.16) circles are preferred curves, since they are mapped onto circles under all Möbius transformations. In Euclidean geometry, circles have constant geodesic curvature, i.e. their third derivative is proportional to their velocity. On an arbitrary conformal manifold M, c with $n \geq 3$ dimensions, there is a conformally invariant third derivative for non lightlike curves $c\colon \mathbb{R} \to M$: denote by N_c the normalized tangent vector and choose a Weyl derivative D. From $\langle N, N\rangle = \pm 1$ we have $\langle D_N N, N\rangle = 0$ and for the tangential component of $D_N D_N N$ we find $\langle D_N D_N N, N\rangle = -\langle D_N N, D_N N\rangle$.

Proposition 4.13 *(Conformal geodesics) In $n \geq 3$ dimensions the following parameter independent third derivative of c*

$$\mathit{Geod}\, N := (D_N D_N N - \langle N, N\rangle \sharp\, \mathrm{r}^D(N))^\perp$$

is conformally invariant. A nonlightlike curve c with $\mathit{Geod}\, N = 0$ is called a conformal geodesic *.*

Proof: Under a linear change γ of the Weyl derivative D the normal component of the third derivative changes like the normalized Ricci curvature, see example 2.54:

$$\begin{aligned}
\partial_\gamma (D_N D_N N)^\perp &= ([\gamma, N].D_N N)^\perp + (D_N([\gamma, N].N))^\perp \\
&= -\gamma(N) D_N N + (D_N(\gamma(N)N - \langle N, N\rangle \sharp\gamma))^\perp \\
&= \langle N, N\rangle (-D_N \sharp\gamma)^\perp \\
&= \langle N, N\rangle (\sharp\partial_\gamma \, \mathrm{r}^D(N))^\perp. \quad \square
\end{aligned}$$

Remark 4.14 The above third derivative occurs as *Abraham vector* in the Lorentz Dirac equation for the classical motion of a charged particle influenced by its own electromagnetic field in four dimensional Minkowski space (see Rohrlich [Roh65]).

Remark 4.15 In a conformal affine space $\mathbf{A}$ with base point x_0 we could use the translations at infinity $\phi\colon \mathbf{A} \to \mathbf{A}$ with $\phi(x) := x_0 + I(I(x - x_0) + \alpha)$ with $\alpha \in V^*$ and the inversion I (see paragraph 3.3) to produce general conformal geodesics $t \mapsto \phi(x_0 + tN)$ i.e. (hyperbolic) circles out of straight lines $t \mapsto x_0 + tN$. The conformal invariance of electromagnetism allows to pull back the Coulomb field F_{Coul}, see paragraph 2.15, associated to the straight worldline $x_0 + tN$ to produce a solution of Maxwell's equations $\phi^* F_{\mathrm{Coul}}$ with source supported along the conformal geodesic $\phi(x_0 + tN)$. These solutions are called *radiation fields from uniformly accelerated charges* see Rohrlich [Roh65]. They are special cases of the Lienard Wiechert fields discussed in paragraph 5.10.

4.3 Weight one twistors and conservation laws

This section contains new conserved conformal invariants along conformal geodesics. In Riemannian geometry it is well known that a Killing field $\mathcal{L}_K g = 0$ (leaving the Riemannian metric invariant) can be contracted with the velocity of a curve $g(\dot{c}, K_c)$ to produce a number along c which is constant if $\ddot{c} = D_{\dot{c}}\dot{c} = 0$, i.e. if c is a Riemannian geodesic (see also paragraph 4.3). We will present such pairings in conformal geometry where we need to consider bilinear differential pairings rather then just tensorial contractions as in $g(\dot{c}, K)$. Moreover we will use a class of twistor fields rather than only Killing fields to construct these conserved invariants.

In the case of an affine conformal space $\mathbf{A}$ over V we already constructed conformal twistor fields in proposition 3.14 which were induced by elements in a representation space W for the Lie algebra of conformal Killing fields $\mathfrak{g} = \mathfrak{so}(\hat{V})$, with $\hat{V} = L^1 \oplus V^0 \oplus L^{-1}$. Differential operators annihilating such twistor fields are also called *twistor operators*. In this section we define twistor operators covering the cases $W = \Lambda^{k+1}\hat{V}$ for $k = 0 \ldots n$. The twistor fields take values in the coinvariants $W_{V^*} = L^{k+1} \otimes \Lambda^k V^*$, see the examples following definition 3.11. These twistor operators annihilating the twistor fields on affine space make sense on any (curved) conformal manifold. Hence we will define twistor fields on manifolds to be sections of the bundle $L^{k+1} \otimes \Lambda^k T^*M$ which lie in the kernel of the twistor operator. The existence of a nontrivial twistor places constraints on the curvature.

The Fermi derivative is defined on weightless tensors and the above mentioned twistor fields f (including Killing fields) have central weight $+1$. Along conformal geodesics we will construct weightless Fermi parallel tensors out of twistor fields. From the dimensional analysis we expect first order pairings between f and N the normalized weightless tangent vector.

We begin with the defining $\mathfrak{g}$-representation $W = \hat{V} = L^1 \oplus V^0 \oplus L^{-1}$. In the conformal affine space the resulting twistor fields are scalar valued quadratic polynomials with values in L^1, see paragraph 3.15. There is no restriction on the linear part of these polynomials, but the quadratic part has to be tracelike. Therefore the annihilating operator must be second order:

Discussion 4.16 (*Second derivative on scalars*) Let D be a Weyl derivative on a conformal manifold M and γ a 1-form. The linear change $\partial_\gamma D^2$ of the second covariant derivative D^2 applied to scalar densities f of weight w is given in terms of vector fields X, Y by example 2.52:

$$\begin{aligned}
\partial_\gamma {D^2}_{X,Y} f &= [\,\gamma, X\,].D_Y f + [\,D_X\gamma, Y\,].f + [\,\gamma, Y\,].D_X f - D_{[\,\gamma,X\,].Y} f \\
&= w\gamma(X) D_Y f + w D_X\gamma(Y) f + w\gamma(Y) D_X f - D_{\gamma(X)Y + \gamma(Y)X - \langle X,Y\rangle \sharp\gamma} f \\
&= (w-1)\gamma(X) D_Y f + w D_X\gamma(Y) f + (w-1)\gamma(Y) D_X f + \langle X, Y\rangle D_{\sharp\gamma} f.
\end{aligned}$$

To find conformally invariant parts of the second derivative we like to add on the right amount of Ricci curvature to kill the $D\gamma$ term in the above expression. This is possible in $n \geq 3$ dimensions:

$$\partial_\gamma (D^2 + w\,\mathrm{r}^D)_{X,Y} f = (w-1)\gamma(X) D_Y f + (w-1)\gamma(Y) D_X f + \langle X, Y\rangle D_{\sharp\gamma} f.$$

This proves the following:

Proposition 4.17 *(Conformal Hessian) For scalar densities of weight $w = +1$ in $n \geq 3$ dimensions the trace-free part of the Ricci corrected second covariant derivative is a conformally invariant differential operator called* conformal Hessian *:*

$$\begin{aligned} Hesse &: C^\infty(M, L^1) \to C^\infty(M, L^1 \otimes \mathrm{Sym}_0^2 T^*M), \\ Hesse\, f\,(X,Y) &:= \mathrm{Sym}_0^2 (D^2 + \mathrm{r}^D) f\,(X,Y) \\ &= \frac{1}{2}\,(D^2_{X,Y} f + D^2_{Y,X} f) - \frac{1}{n}\,\mathrm{tr}_{\mathsf{c}}\, D^2 f\, \mathsf{c}(X,Y) + \mathrm{r}_0^D(X,Y) f. \end{aligned}$$

In the case of a conformal affine space this operator annihilates the twistor fields coming from $W = \hat{V}$, hence Hesse is a twistor operator.

Remark 4.18 (*Einstein metrics from the conformal class*) Sections f of L^1 which are in the kernel of *Hesse* are also called *Fierz twistors* (for gravity): *Hesse*$(f) = 0$. In a region where a Fierz twistors is positive $f > 0$, it defines a length scale and induces a metric $g = f^{-2}\mathsf{c}$ from the conformal structure which is *Einstein*: for the induced exact Weyl derivative D we have $Df = 0$, hence $0 = Hesse(f) = \mathrm{Sym}_0(D^2 f + \mathrm{r}^D f) = \frac{1}{n-2} f\, \mathrm{ric}_0^D$. Hence the Ricci curvature of g is a constant multiple of g.

Proposition 4.19 *(Conservation law along conformal geodesics) Let $f \in C^\infty(M, L^1)$ be a density of weight $+1$ and c a nonlightlike curve with normalized weightless tangent vector N, then*

$$f.N := \langle N, N\rangle(\mathrm{grad}^D f)^\perp + f D_N N$$

is independent of the Weyl derivative D and hence defines a conformally invariant weightless vector field along c perpendicular to N. For the Fermi derivative we have

$$Fermi_N\, f.N = f\, Geod(N) + \langle N, N\rangle(\sharp\, Hesse\, f(N))^\perp.$$

Hence if f is a twistor field Hesse $f = 0$ and c is a conformal geodesic, then the induced $f.N$ is Fermi parallel.

Proof: The term $f.N$ changes under a change of Weyl derivative by the 1-form γ as

$$\partial_\gamma (f.N)^D = \langle N, N\rangle \sharp\gamma \otimes f - \gamma(N) f N + f\gamma(N) N - f\langle N, N\rangle\sharp\gamma = 0.$$

It is straight forward to calculate the Fermi derivative of $f.N$:

$$\begin{aligned} Fermi_N(f.N) &= (D_N(f.N))^\perp \\ &= \langle N, N\rangle(D_N\, \mathrm{grad}^D f)^\perp - (D_N f) D_N N + D_N f D_N N + f(D_N D_N N)^\perp \\ &= \langle N, N\rangle\sharp(D_N^2 f + \mathrm{r}^D(N))^\perp + f(D_N D_N N - \langle N, N\rangle\, \mathrm{r}^D(N))^\perp \\ &= \langle N, N\rangle\sharp(Hesse\, f(N))^\perp + f\, Geod(N). \quad \square \end{aligned}$$

Application 4.20 In the case of an affine conformal space $\mathbf{A}$ we have an injection $\iota \colon \hat{V} \to C^\infty(\mathbf{A}, L^1)$ form the $(n+2)$ dimensional vector space $\hat{V} = L^1 \oplus V^0 \oplus L^{-1}$ into the kernel of the conformal Hessian. If c is a conformal geodesic and J a Fermi parallel vector field along c, then $\langle J, \iota(\hat{v}).N\rangle$ is a constant real number linear in $\hat{v} \in \hat{V}$. Hence a given conformal geodesic with attached Fermi parallel vector field induces a linear from on $\hat{V}$. Since $\hat{V}$ is equipped with an inner product the conformal geodesic with attached Fermi parallel vector field provides us with a unique twistor.

Definition 4.21 *Conformal Killing fields* $K \in C^\infty(M, TM)$ are in the kernel of the *conformal Killing operator* $\mathit{Kill}\colon C^\infty(M,TM) \to C^\infty(M, L^2 \otimes \operatorname{Sym}^2_0(T^*M))$ which in terms of a Weyl derivative D is given by

$$\begin{aligned}\mathit{Kill}(K)(X,Y) &:= \frac{1}{2}\,(\mathcal{L}_K\mathsf{c})(X,Y)\\ &= \frac{1}{2}\,\langle D_X K, Y\rangle + \frac{1}{2}\,\langle X, D_Y K\rangle - \frac{1}{n}\,(\operatorname{div}^D K)\langle X,Y\rangle\\ &= \langle D_X K, Y\rangle - \frac{1}{2}\,d^D(\flat K)(X,Y) - \frac{1}{n}\,(\operatorname{div}^D K)\langle X,Y\rangle,\end{aligned}$$

where X,Y are vector fields.

Proposition 4.22 *Let $K \in C^\infty(M,TM)$ be a vector field (of central weight $+1$) and c a nonlightlike curve with normalized weightless tangent vector N. Let t_i, θ^i be a dual basis of T_cM, then*

$$K.N := K^\perp \wedge D_N N + \frac{1}{2}\,\langle N,N\rangle \sum_i (D_{t_i}K)^\perp \wedge (\sharp\theta^i)^\perp$$

is independent of the Weyl derivative D and hence defines a conformally invariant weightless bivector field along c perpendicular to N. For the Fermi derivative of $K.N$ we have:

$$\mathit{Fermi}_N(K.N) = K^\perp \wedge \mathit{Geod}(N) + \mathit{Kill}(K).N + \langle N,N\rangle \sum_i (\mathrm{W^c}_{N,t_i}\, K)^\perp \wedge (\sharp\theta^i)^\perp,$$

where $\mathit{Kill}(K).N$ is another conformally invariant bilinear combination along c: if $h \in C^\infty(M, L^2 \otimes \operatorname{Sym}^2_0(T^*))$ is a weightless bilinear form then define:

$$h.N := (\sharp h(N))^\perp \wedge D_N N + \langle N,N\rangle \sum_i (\sharp D_{t_i} h(N))^\perp \wedge (\sharp\theta^i)^\perp.$$

Hence if K is a conformal Killing field $\mathit{Kill}\,K = 0$ in $n = 3$ dimensions (where $\mathrm{W^c} = 0$) and c is a conformal geodesic, then the induced $K.N$ is Fermi parallel.

Proof: The term $K.N$ changes under a change of Weyl derivative by the 1-form γ as

$$\begin{aligned}\partial_\gamma(K.N)^D &= -K^\perp \wedge \langle N,N\rangle(\gamma)^\perp + \frac{1}{2}\,\langle N,N\rangle \sum_i ([\,\gamma, t_i\,]K)^\perp \wedge (\theta^i)^\perp\\ &= -\langle N,N\rangle K^\perp \wedge (\gamma)^\perp + \langle N,N\rangle K^\perp \wedge (\gamma)^\perp + 0 = 0.\end{aligned}$$

Let U be a weightless vector and t a vector of central weight $+1$, then

$$\begin{aligned}[\,\gamma, t\,].h(N,U) &= -h([\,\gamma,t\,].N, U) - h(N, [\,\gamma,t\,].U)\\ &= -\gamma(N)h(t,U) + \langle t,N\rangle h(\gamma,U) - \gamma(U)h(N,t) + \langle t,U\rangle h(N,\gamma).\end{aligned}$$

Hence we find

$$\begin{aligned}\partial_\gamma(h.N)^D &= -\langle N,N\rangle(h(N))^\perp\wedge\gamma^\perp + \langle N,N\rangle\sum_i([\gamma,t_i].h(N))^\perp\wedge(\sharp\theta^i)^\perp\\ &= -\langle N,N\rangle(h(N))^\perp\wedge\gamma^\perp - \langle N,N\rangle\gamma^\perp\wedge(h(N))^\perp = 0.\end{aligned}$$

Let t_i and θ^i be Fermi parallel with respect to D, hence $0 = D_N t_i - K^D(t_i)$. The claimed identification of the Fermi derivative of $K.N$ is an elementary application of the product rule using the above Fermi parallel basis. $\square$

On a conformal vector space V we define the *Cartan product* of a 1-form $\alpha\in V^*$ with a k-form $\beta\in\Lambda^kV^*$ to be an element $\alpha\odot\beta\in V^*\otimes\Lambda^kV^*$ which is alternating-free and trace-free: evaluation on $v\in V$ gives

$$(\alpha\odot\beta)(v) := \alpha(v)\beta - \frac{1}{k+1}\,v\lrcorner\alpha\wedge\beta - \frac{1}{n-(k-1)}\sum_i \mathsf{c}(v,t_i)\theta^i\wedge(\sharp\alpha\lrcorner\beta).$$

The subspace spanned by these elements is denoted by $V^*\odot\Lambda^kV^*$.

Definition 4.23 (*First order operator on forms*) In terms of a Weyl derivative D we define a linear first order operator acting on alternating k-forms (with $k\geq 1$) of central weight w as follows:

$$\begin{aligned}\mathit{Twist}^D &: C^\infty(M,L^{w+k}\otimes\Lambda^kT^*)\to C^\infty(M,L^{w+k}\otimes T^*\odot\Lambda^kT^*);\\ \mathit{Twist}^D(f) &:= \sum_i\theta^i\odot D_{t_i}f.\end{aligned}$$

Proposition 4.24 *(Conformal twistor operator for weight one) For $w=1$ the above operator is conformally invariant. In case of conformal affine space it annihilates the twistor fields coming from $W=\Lambda^{k+1}\hat{V}$, hence Twist is a twistor operator.*

Proof: The linear change under a change of Weyl derivative is given by:

$$\begin{aligned}\partial_\gamma\,\mathit{Twist}^D(f)(X) &= [\gamma,X].f - \frac{w+k}{k+1}X\lrcorner(\gamma\wedge f) - \frac{w-k+n}{n-(k-1)}\langle X,\ \rangle\wedge(\gamma\lrcorner f)\\ &= w\gamma(X)f + \langle X,\ \rangle\wedge(\gamma\lrcorner f) - \gamma\wedge(X\lrcorner f)\\ &\quad -\frac{w+k}{k+1}X\lrcorner(\gamma\wedge f) - \frac{w-k+n}{n-(k-1)}\langle X,\ \rangle\wedge(\gamma\lrcorner f)\\ &= (w-1)(\gamma(X)f - \frac{1}{k+1}\,X\lrcorner\gamma\wedge f - \frac{1}{n-(k-1)}\,\langle X,\ \rangle\wedge(\gamma\lrcorner f))\\ &= (w-1)(\gamma\odot f)(X).\quad\square\end{aligned}$$

Proposition 4.25 *Let $f\in C^\infty(M,L^3\otimes\Lambda^2T^*M)$ be a 2-form of central weight $+1$ and c a nonlightlike curve with normalized weightless tangent vector N, then*

$$f.N := f(D_NN,N) + \frac{1}{n-1}\,\langle N,N\rangle\,\mathrm{div}^D f(N)$$

is independent of the Weyl derivative D and hence defines a conformally invariant real number along c. For the derivative of $f.N$ we have

$$\partial_N\ f.N = f(Geod(N), N) + (Twist\, f).N,$$

where $(Twist\, f).N$ is another invariant pairing: for a section $h \in C^\infty(M, L^3 \otimes T^* \odot \Lambda^2 T^*)$ the following scalar of weight -1 is invariant:

$$h.N := h(N, D_N N, N) + \frac{1}{n-2} \langle N, N\rangle \operatorname{tr} Dh(N, N),$$

with $\operatorname{tr} Dh(N,N) := \sum_i D_{t_i} h(N, \sharp\theta^i, N)$. Hence if f is a twistor field $Twist\, f = 0$ and c is a conformal geodesic, then the induced $f.N$ is constant.

Proof: The term $f.N$ changes under a change of Weyl derivative by the 1-form γ as

$$\partial_\gamma (f.N)^D = -\langle N, N\rangle f(\gamma, N) + \frac{1-(2-n)}{n-1} \langle N, N\rangle f(\gamma, N) = 0.$$

To calculate the derivative of $f.N$ notice

$$Twist\, f(N, D_N N, N) = D_N f(D_N N, N) + \frac{1}{n-1} \langle N, N\rangle\, div^D f(D_N N),$$

and

$$\operatorname{tr} D\, Twist\, f(N, N) \quad = \quad (n-2)(f(\mathrm{r}^D(N), N) + \frac{1}{n-1} D_N\, div^D f(N)).$$

Finally we obtain:

$$\begin{aligned}
&\partial_N\ f.N \\
&= D_N f(D_N N, N) + f(D_N D_N N, N) \\
&\quad + \frac{1}{n-1} \langle N, N\rangle D_N\, div^D f(N) + \frac{1}{n-1} \langle N, N\rangle\, div^D f(D_N N) \\
&= Twist\, f(N, D_N N, N) + f(D_N D_N N, N) + \frac{1}{n-1} \langle N, N\rangle D_N\, div^D f(N) \\
&= Twist\, f(N, D_N N, N) + f(Geod(N), N) + \frac{1}{n-2} \langle N, N\rangle \operatorname{tr} D\, Twist\, f(N, N). \quad \square
\end{aligned}$$

Application 4.26 Any nonlightlike curve c induces a distributional section J of $L^{-n-3} \otimes TM \odot \Lambda^2 TM$ as follows: the evaluation on a test section $h \in C_0^\infty(M, L^3 \otimes T^*M \odot \Lambda^2 T^*M)$ is defined by $\langle\!\langle J, h\rangle\!\rangle := \int_c h.N$. (Note that $h.N$ is a section in $L^{-1}(TM)$ which can be integrated over c using the conformal structure to identify $L^{-1}(Tc) \cong L^{-1}(TM)$, see remark 2.4). The above theorem shows that for any $f \in C_0^\infty(M, L^3 \otimes \Lambda^2 T^*M)$ we have

$$\langle\!\langle J, Twist\, f\rangle\!\rangle = \int_c (Twist\, f).N = -\int_c f(Geod(N), N).$$

Hence the distributional J is divergence-free $Twist^*\, J = 0$ iff c is a conformal geodesic.

Remark 4.27 (*Pointlike sources in Einstein's theory*) The above application is a conformal analogue of a well known observation in general relativity: a curve gives rise to a distributional symmetric tensor T (of central weight $1-n$) defined by $\langle\!\langle T, \alpha \rangle\!\rangle := \int_c \alpha(N,N)$ and T is divergence-free iff c is a geodesic $D_N N = 0$. If m denotes a mass then mT has the interpretation being an energy momentum tensor of the particle. Viewed as a source for gravity, the conservation law $0 = div^D(mT)$ forces the particle to move along a geodesic.

Remark 4.28 For twistors induced by $W = \Lambda^{k+1}\hat{V}$ with $k \geq 2$ (i.e. elements in the kernel of the above defined operator *Twist* on k-forms) it is possible to construct an invariant weightless $(k-2)$-form and a weightless $(k+1)$-form along a curve. In the flat case they are Fermi parallel along conformal geodesics, but on a curved conformal manifold the Weyl curvature prevents a true conservation law. An example of this phenomenon was given in proposition 4.22 with Killing fields in $n > 3$ dimensions.

4.4 First order conformal invariants

The previous three sections dealt with conformal invariants along curves. We will now focus on conformal differential invariants for sections of associated bundles. We begin with the first order linear invariants of Fegan [Feg76], who classified all conformally invariant differential operators of first order. For the proof we will follow Gauduchon [Gau91]. We will apply this result to give new bilinear invariants: a favourite example of such a pairing is the Lie bracket between vector fields:

$$[\,,\,] \colon C^\infty(M, TM) \times C^\infty(M, TM) \to C^\infty(M, TM).$$

Another examples is a pairing between two scalars to give a 1-form of weight +1:

$$\vee \colon C^\infty(M, L^1) \times C^\infty(M, L^1) \to C^\infty(M, L^2 \otimes T^*M),$$

which is defined on scalar densities f_1 and f_2 by

$$f_1 \vee f_2 := Df_1 \otimes f_2 - f_1 \otimes Df_2.$$

Here D is a chosen Weyl derivative, but $\vee$ is indeed independent of that choice: if $\widetilde{D} = D + \gamma$ is another Weyl derivative, then $\widetilde{D}f_1 - Df_1 = \gamma \otimes f_1$ and $\widetilde{D}f_2 - Df_2 = \gamma \otimes f_2$. These two examples of pairings in fact only need the smooth structure, whereas we are interested in invariants involving the conformal structure as well: note that the Lie bracket of two conformal Killing fields is a conformal Killing field, since the following product rule holds:

$$\mathcal{L}_{[X,Y]}\mathsf{c} = \mathcal{L}_X\mathcal{L}_Y\mathsf{c} - \mathcal{L}_Y\mathcal{L}_X\mathsf{c}.$$

Likewise the above pairing $\vee$ between scalar densities gives a vector field which can be applied to the Killing operator $\mathit{Kill}(X) = \frac{1}{2}\mathcal{L}_X\mathsf{c} = \mathrm{Sym}^2_0 DX$ (see definition 4.21) to give:

$$\begin{aligned} \mathit{Kill}(f_1 \vee f_2) &= \mathit{Kill}(Df_1 \otimes f_2 - f_1 \otimes Df_2) \\ &= (\mathrm{Sym}^2_0 D^2 f_1) \otimes f_2 + Df_1 \odot Df_2 - Df_1 \odot Df_2 - f_1 \otimes \mathrm{Sym}^2_0 D^2 f_2 \\ &= (\mathit{Hesse}\, f_1) \otimes f_2 - f_1 \otimes (\mathit{Hesse}\, f_2). \end{aligned}$$

(The two individual Ricci corrections for *Hesse*, see proposition 4.17, cancel in the difference). In particular two Fierz twistors f_1, f_2 (in the kernel of the conformal Hessian) induce a conformal Killing field. In the case of an affine conformal space $\mathbf{A}$ over V we already constructed conformal twistor fields in proposition 3.14 which were induced by elements in a representation space W for the Lie algebra of conformal Killing fields $\mathfrak{g} = \mathfrak{so}(\hat{V})$. Fierz twistors are induced by elements in the defining representation $\hat{V}$ (see remark 4.18) and Killing fields are induced by elements in the adjoint representation $\mathfrak{so}(\hat{V})$. On the algebraic level we have a skew symmetric $\mathfrak{g}$-equivariant map

$$\hat{V} \otimes \hat{V} \to \mathfrak{so}(\hat{V}),$$

and the above pairing $\vee$ is an extension of that map to curved conformal manifolds. Likewise the Lie bracket between smooth vector fields is an extension of the algebraic Lie bracket

$$\mathfrak{so}(\hat{V}) \otimes \mathfrak{so}(\hat{V}) \to \mathfrak{so}(\hat{V}).$$

From these two examples where algebraic pairings correspond to differential pairings we also expect a differential pairing corresponding to $\mathfrak{so}(\hat{V}) \otimes \hat{V} \to \hat{V}$. This is given by the Lie derivative of a scalar density in the direction of a vector field:

$$\mathcal{L}\colon C^\infty(M, TM) \times C^\infty(M, L^1) \to C^\infty(M, L^1).$$

The induced product rule $\mathit{Hesse}(\mathcal{L}_X f) = \dots$ becomes more involved, since not only a second order pairing will be needed on the right hand side (between $\mathit{Kill}\, X$ and f) but there will be also an additional curvature term. This and the next section cover first and second order operators and pairings. More examples of product rules can be found in chapter 5 on linear field theories. We obtain a general result on the conformal affine space, see theorems 5.12 and 5.13.

We turn now to a systematic study of first order invariants: in terms of a Weyl derivative D we will define a first order differential operator, encoding, in a trivial fashion, all the information of the 1-jet of a section: $\mathit{jet}^1(e) \in C^\infty(M, \mathit{Jet}^1(EM))$. It will turn out that the operator behaves well if we would change the Weyl derivative by a 1-form $\gamma \in C^\infty(M, T^*M)$ in the sense that the first order information transforms as expected from the algebraic action of $V^* \subset \mathfrak{g}$ on affine functions (polynomials of degree 1) see paragraph 3.7. This allows to classify algebraically all first order invariants discussed here.

Definition 4.29 A Weyl derivative D induces a first order jet operator defined by

$$\mathit{jet}^{1,D}\colon C^\infty(M, EM) \to C^\infty(M, E \oplus T^* \otimes EM); \quad \mathit{jet}^{1,D} e := e \oplus De.$$

Any linear first order differential operator $\nabla\colon C^\infty(M, EM) \to C^\infty(M, FM)$ between associated bundles in Weyl geometry is given by a $\mathbf{CO}\,(V)$-equivariant linear map $E \oplus V^* \otimes E \to F$, which is nontrivial on $V^* \otimes E$. This in terms gives a bundle map $\pi\colon EM \oplus T^*M \otimes EM \to FM$ and finally the differential operator $\nabla\, e = \pi \circ \mathit{jet}^{1,D}$. If E and F are irreducible then E and F have central weights w_E and w_F and $V^* \otimes E \to F$ being nontrivial implies $w_F = w_E - 1$ and $E \to F$ is trivial. Hence in Weyl geometry the study of first order operators between irreducible representations reduces to the classification of irreducible sub representations of $V^* \otimes E$: an application of Weyl's character formula in representation theory (see [Feg76] or [FH91]) gives the following

Proposition 4.30 *If E is a finite dimensional irreducible representation of $\mathfrak{co}(V)$, then $V^*\otimes E$ splits under the action of $\mathfrak{co}(V)$ into a direct sum of pairwise nonisomorphic irreducible representations.*

Example 4.31 Let E be a $\mathfrak{co}(V)$-representation of central weight w_E, then we denote by $E^w := L^{w-w_E}\otimes E$ the same $\mathfrak{so}(V)$-representation but of central weight w. With this notation we give the following examples to the above proposition (to ensure irreducibility of all the below representations we may assume $n > 5$ and $n > 2(k+1)$):

$$\begin{aligned} V^*\otimes L^w &= V^{w-1}, \\ V^*\otimes V^w &= L^{w-1}\oplus(\Lambda^2V^*)^{w-1}\oplus(\mathrm{Sym}\,^2_0(V^*))^{w-1}, \\ V^*\otimes(\Lambda^kV^*)^w &= (\Lambda^{k-1}V^*)^{w-1}\oplus(\Lambda^{k+1}V^*)^{w-1}\oplus(V^*\odot\Lambda^kV^*)^{w-1}. \end{aligned}$$

Definition 4.32 (*Stein Weiss operators*) Let E and F be a finite dimensional irreducible $\mathbf{CO}\,(V)$-representation and denote by $\mathrm{Hom}_{\,\mathbf{CO}\,(V)}(V^*\otimes E, F)$ the space of $\mathbf{CO}\,(V)$-equivariant linear maps $\pi\colon V^*\otimes E\to F$. Associated to a nontrivial π is a linear first order differential operator on a conformal manifold M with Weyl derivative D:

$$\nabla^D : C^\infty(M,EM)\to C^\infty(M,FM);\quad \nabla^D(e) := \pi(De).$$

Remark 4.33 A Stein Weiss operator as above on a Weyl manifold always has an adjoint which is also Stein Weiss: the zero order pairing $X: E\otimes L^{-n}\otimes F^*\to L^{-n}\otimes V$ is given by the symbol $X(e,\phi) := \sum_i t_i\otimes\langle\pi(\theta^i\otimes e),\phi\rangle$ with a dual basis t_i and θ^i. The divergence formula follows

$$\begin{aligned} div(X(e,\phi)) &= \langle\pi(De),\phi\rangle + \sum_i\langle\pi(\theta^i\otimes e), D_{t_i}\phi\rangle \\ &= \langle\nabla^D e,\phi\rangle - \langle e,\nabla^{*D}\phi\rangle. \end{aligned}$$

The next aim is to study the behavior of a Stein Weiss operator ∇^D under a change of Weyl derivative. In example 2.51 we already observed that on a conformal manifold M the covariant derivative of an associated bundle EM changes under a linear change of Weyl derivative $\gamma\in C^\infty(M,T^*M)$ as:

$$(\partial_\gamma D)_X e = [\,\gamma, X\,].e,$$

where $e\in C^\infty(M,EM)$ is a section and $X\in C^\infty(M,TM)$ is a vector field. As was explained in paragraph 3.7 for a conformal vector space V any $\mathfrak{co}(V)$-representation E induces a $\mathfrak{g} = V\oplus\mathfrak{co}(V)\oplus V^*$ representation on $\mathrm{Sym}\,V^*\otimes E$ via the Lie derivative. In what follows we like to view E as a left $\mathfrak{co}(V)$-representation and we will use the induced left action of $\mathfrak{g}$ on $\mathrm{Sym}\,V^*\otimes E$. The restriction of this action to the Abelian subalgebra $V^*\subset\mathfrak{g}$ is then given by

$$\mathcal{L}\colon V^*\otimes\mathrm{Sym}\,^{k-1}V^*\otimes E\to\mathrm{Sym}\,^kV^*\otimes E,$$

with

$$\frac{1}{k!}\,\mathcal{L}_\gamma f(v^k) := \frac{1}{(k-1)!}\,[\,\gamma, v\,].(f(v^{k-1})) - \frac{1}{2(k-2)!}\,f(([\,\gamma, v\,]v)v^{k-2}).$$

Special cases are as follows:

$$
\begin{aligned}
k=0 \quad &: \quad \mathcal{L}_\gamma f = \mathcal{L}(\gamma\otimes f) = 0,\\
k=1 \quad &: \quad \mathcal{L}_\gamma f(v) = \mathcal{L}(\gamma\otimes f)(v) = [\,\gamma, v\,].f.
\end{aligned}
$$

The cases $k=0$ and $k=1$ show the following:

Proposition 4.34 *Under a change of Weyl derivative the linear change of the above first jet operator is given by the Lie derivative applied to the zero jet of the section (with a dual basis t_i, θ^i of TM):*

$$(\partial_\gamma jet^{1,D})(e) = \mathcal{L}_\gamma e = 0 \oplus \sum_i \theta^i \otimes [\,\gamma, t_i\,].e.$$

For a Stein Weiss operator ∇^D we get $(\partial_\gamma \nabla^D)(e) = \pi\circ(\partial_\gamma D)(e) = \pi\circ\mathcal{L}(\gamma\otimes e)$. Those operators which do not change $\partial_\gamma\nabla^D = 0$ for all γ and D are conformally invariant. Algebraically this question reduces to the study of the composition:

$$V^*\otimes E \xrightarrow{\mathcal{L}} V^*\otimes E \xrightarrow{\pi} F:$$

Proposition 4.35 *The equivariant linear maps $\pi \in \mathrm{Hom}_{\mathbf{CO}(V)}(V^*\otimes E, F)$ with $\pi\circ\mathcal{L}=0$ classify all conformally invariant linear first order operators between bundles associated to irreducible $\mathbf{CO}(V)$-representations E and F.*

Following Gauduchon [Gau91] we will split $[\,\gamma, X\,] \in \mathfrak{co}(V)$ into its trace and skew part: if w_E denotes the central weight of E (under the action of the centre of $\mathfrak{co}(V)$) then define B to be the induced $\mathfrak{so}(V)$-action

$$B: V^*\otimes E \to V^*\otimes E; \quad \mathcal{L}(\gamma\otimes e) = \sum_i \theta^i\otimes[\,\gamma, t_i\,].e = w_E\,\gamma\otimes e + B(\gamma\otimes e),$$

where t_i, θ^i is a dual basis of V. Explicitly $B(\gamma\otimes e) = \sum_i \theta^i\otimes(\gamma\otimes t_i - \flat t_i\otimes\sharp\gamma).e$.

According to proposition 4.30 the tensor product $V^*\otimes E$ splits into a direct sum of pairwise nonisomorphic $\mathfrak{co}(V)$-sub representations $F^k\subset V^*\otimes E$ with $k=1\ldots N$. We will use the inclusions $\iota_k: F^k\to V^*\otimes E$ and projections $\pi_k: V^*\otimes E\to F^k$ with normalization $\pi_k\circ\iota_k = id_{F^k}$. For any $\mathfrak{co}(V)$-equivariant map $B\in\mathfrak{gl}(V^*\otimes E)$ we know that $\pi_l\circ B\circ\iota_k$ is trivial for $l\neq k$, since the F^k are irreducible and pairwise nonisomorphic. In case $l=k$ we apply Schur's lemma (see e.g. [FH91]) to get multiples of the identity: $\pi_k\circ B\circ\iota_k = b_k id_{F^k}$. Therefore any such $\mathfrak{co}(V)$-equivariant map B is determined by its eigenvalues b_k:

$$B = \sum_{k=1}^N b_k\pi_k\circ\iota_k.$$

Theorem 4.36 *(Fegan's first order conformal invariants) Let E and F be irreducible representations of $\mathbf{CO}(V)$ and $\pi\in\mathrm{Hom}_{\mathbf{CO}(V)}(V^*\otimes E,F)$ a nontrivial map. The associated Stein Weiss operator $\nabla^D = \pi\circ D$ changes under a change of Weyl derivative D by γ as*

$$\nabla^{D+\gamma}e - \nabla^D e = \partial_\gamma\nabla^D(e) = \pi\circ\mathcal{L}(\gamma\otimes e) \stackrel{!}{=} (w_E+b)\pi(\gamma\otimes e),$$

where e is a section of EM, w_E is the central weight of E and b is a number directly associated to the two $\mathfrak{so}(V)$-representations E and F. Conformally invariant linear first order operators are characterized by $w_E = -b$.

Proof: After writing B in terms of its eigenvalues $B = \sum_k^N b_k \pi_k \circ \iota_k$ on the invariant subspaces $F^k \subset V^* \otimes E$ we find for the difference (with $F = F^l$):

$$\begin{aligned} \pi \circ \mathcal{L}(\gamma \otimes e) &= \pi \circ \sum_i \theta^i \otimes [\,\gamma, t_i\,].e \\ &= \pi \circ (w_E\, \gamma \otimes e + B(\gamma \otimes e)) \\ &= (w_E + b_l)\pi \circ (\gamma \otimes e). \quad \square \end{aligned}$$

Remark 4.37 The *Killing form* on $\mathfrak{so}(V)$ is proportional to the inner product induced from the defining representation V which is given by $\langle A, B\rangle := -\frac{1}{2}\,\mathrm{tr}(A \circ B)$ with $A, B \in \mathfrak{so}(V)$. An orthonormal basis in $\mathfrak{so}(V)$ is given by $E_{ij} := \langle e_i, \ \rangle e_j - \langle e_j, \ \rangle e_i \in \mathfrak{so}(V)$ with $1 \le i < j \le n$ and $e_i \in V$ an orthonormal basis of V. With this convention the *Casimir element* $\mathrm{Cas} \in \mathfrak{Z}(\mathfrak{U}(\mathfrak{so}(V)))$ is $\mathrm{Cas} = \sum_{i<j} E_{ij}E_{ij}$. Denote by $\mathrm{Cas}(W) \in \mathrm{End}\,(W)$ the *induced endomorphism* on the $\mathfrak{so}(V)$-representation W. Applied to the defining representation we find $\mathrm{Cas}(V) = (1-n)id_V$ and $\mathrm{Cas}(\Lambda^k V) = k(k-n)id$. In general, if W is irreducible then by Schur's lemma $\mathrm{Cas}(W)$ is a multiple of the identity and that multiple is called the *Casimir value* $\mathrm{Cas}(W) = \mathrm{cas}(W)id_W$. Fegan observed, that the endomorphism B can be written in terms of various images of the Casimir element:

$$B = \frac{1}{2}\,(\mathrm{Cas}(V^* \otimes E) - id_{V^*} \otimes \mathrm{Cas}(E) - \mathrm{Cas}(V^*) \otimes id_E).$$

This allows to calculate eigenvalues of B once the Casimir values of the irreducible sub representations $F^k \subset V^* \otimes E$ are known:

$$b_k = \frac{1}{2}\,(\mathrm{cas}(F^k) - \mathrm{cas}(E) - (1-n)).$$

Discussion 4.38 (*First order pairings*) The next aim is to study first order bilinear pairings in general. In the introduction of this chapter we mentioned two simple examples. In general suppose E_1, E_2 and F are $\mathbf{CO}\,(V)$-representations with a $\mathbf{CO}\,(V)$-equivariant linear map

$$Q\colon (E_1 \oplus V^* \otimes E_1) \otimes (E_2 \oplus V^* \otimes E_2) \to F,$$

then $\vee^D\colon C^\infty(M, E_1M) \times C^\infty(M, E_2M) \to C^\infty(M, FM)$ defined by $e_1 \vee^D e_2 := Q(jet^{1,D}(e_1) \otimes jet^{1,D}(e_2))$ is a bilinear differential pairing of first order in Weyl geometry. We will consider those pairings which are truly first order in both e_1 and e_2, hence we assume such a Q to be nontrivial when restricted to $(V^* \otimes E_1) \otimes (E_2 \oplus V^* \otimes E_2)$ and $(E_1 \oplus V^* \otimes E_1) \otimes (V^* \otimes E_2)$. If E_1, E_2 and F are irreducible the action of the centre of $\mathbf{CO}\,(V)$ determines such a Q to be nontrivial only on the following subspace

$$Q\colon (V^* \otimes E_1 \otimes E_2) \oplus (V^* \otimes E_1 \otimes E_2) \to F.$$

The corresponding pairing is then clearly given by

$$e_1 \vee^D e_2 := Q(\sum_i \theta^i \otimes D_{t_i} e_1 \otimes e_2 \oplus \sum_i \theta^i \otimes e_1 \otimes D_{t_i} e_2).$$

This dependence of $\vee^D$ on the Weyl derivative leads to study the following composition of maps:

$$V^* \otimes E_1 \otimes E_2 \overset{\mathcal{L}\otimes id_{E_2} \oplus id_{E_1} \otimes \mathcal{L}}{\longrightarrow} (V^* \otimes E_1 \otimes E_2) \oplus (V^* \otimes E_1 \otimes E_2) \overset{Q}{\longrightarrow} F,$$

where the first map is given by $\gamma \otimes e_1 \otimes e_2 \mapsto \sum_i \theta^i \otimes [\gamma, t_i].e_1 \otimes e_2 \oplus \sum_i \theta^i \otimes e_1 \otimes [\gamma, t_i].e_2$.

Proposition 4.39 *Conformally invariant bilinear first order differential pairings*

$$\vee : C^\infty(M, E_1 M) \times C^\infty(M, E_2 M) \to C^\infty(M, FM),$$

between associated bundles from irreducible $\mathbf{CO}\,(V)$*-representations* E_1*,* E_2 *and* F *are classified by equivariant linear maps*

$$Q \in \mathrm{Hom}_{\mathbf{CO}\,(V)}(V^* \otimes E_1 \otimes E_2 \oplus V^* \otimes E_1 \otimes E_2, F),$$

such that $Q \circ (\mathcal{L} \otimes id_{E_2} \oplus id_{E_1} \otimes \mathcal{L}) = 0$*.*

A simple subclass of examples is given in the following case:

Definition 4.40 (*Simple projections*) Suppose that E_1, E_2 and $F \subset V^* \otimes E_1 \otimes E_2$ are irreducible $\mathbf{CO}\,(V)$-representations with irreducible $F_1 \subset V^* \otimes E_1$ and $F_2 \subset V^* \otimes E_2$, such that the projection

$$Q : V^* \otimes E_1 \otimes E_2 \to F$$

factors through $F_1 \otimes E_2 \to F$ and $E_1 \otimes F_2 \to F$. Such a map Q will be called *simple*.

Remark 4.41 If F has multiplicity one inside $V^* \otimes E_1 \otimes E_2$, i.e. $1 = \dim(\mathrm{Hom}_{\mathbf{CO}\,(V)}(V^* \otimes E_1 \otimes E_2, F))$ then there are unique spaces F_1, F_2 as required above and $Q : V^* \otimes E_1 \otimes E_2 \to F$ is simple.

Proposition 4.42 *(Simple conformally invariant first order pairings) Suppose that* $Q : V^* \otimes E_1 \otimes E_2 \to F$ *is simple. Then there is a conformally invariant first order differential pairing*

$$\vee : C^\infty(M, E_1 M) \times C^\infty(M, E_2 M) \to C^\infty(M, FM),$$

which in terms of a Weyl derivative D *applied to sections* e_1 *and* e_2 *is given by*

$$e_1 \vee e_2 := (w_2 + b_2) Q(\sum_i \theta^i \otimes D_{t_i} e_1 \otimes e_2) - (w_1 + b_1) Q(\sum_i \theta^i \otimes e_1 \otimes D_{t_i} e_2),$$

where w_1*,* w_2 *are the central weights of* E_1*,* E_2 *(the central weight of* F *is clearly* $w_1 + w_2 - 1$*) and* $-b_1$*,* $-b_2$ *are the weights for which the Stein Weiss operators associated to* E_1*,* F_1 *and* E_2*,* F_2 *are conformally invariant.*

Proof: Applying Fegan's result, theorem 4.36, the following composition is zero:

$$\begin{aligned} &((w_2 + b_2) Q \oplus (-w_1 - b_1) Q) \circ (\mathcal{L} \otimes id_{E_2} \oplus id_{E_1} \otimes \mathcal{L})(\gamma \otimes e_1 \otimes e_2) \\ = \;& (w_2 + b_2)(w_1 + b_1) Q(\gamma \otimes e_1 \otimes e_2) - (w_1 + b_1)(w_2 + b_2) Q(\gamma \otimes e_1 \otimes e_2) = 0. \quad \square \end{aligned}$$

4.5 Second order conformal invariants

In terms of a Weyl derivative D we will define a second order differential operator, encoding all the information of the 2-jet of a section: $jet^2(e) \in C^\infty(M, Jet^2(EM))$. We will use the Ricci curvature to correct the second covariant derivative such that this operator behaves well if we change the Weyl derivative by a 1-form $\gamma \in C^\infty(M, T^*M)$. This allows to classify algebraically all second order operators and pairings. We will give examples which will become relevant in the next chapter on linear field theories for gravity.

In example 2.52 we already observed that on a conformal manifold M the second covariant derivative of an associated bundle EM changes under a change of Weyl derivative as:

$$(\partial_\gamma D^2)_{X,Y} e = [\,\gamma, X\,].D_Y e + [\,D_X\gamma, Y\,].e + [\,\gamma, Y\,].D_X e - D_{[\,\gamma,X\,].Y} e,$$

where $e \in C^\infty(M, EM)$ is a section, $X, Y \in C^\infty(M, TM)$ are vector fields and $\gamma \in C^\infty(M, T^*M)$ is the change of the Weyl derivative D. The $D\gamma$ term can be expressed as the linear change of the normalized Ricci curvature:

$$(\partial_\gamma \, \mathrm{r}^D)_{X,Y} = -D_X\gamma(Y).$$

Definition 4.43 . For a Weyl derivative D we call

$$D^2_{X,Y} e + [\,\mathrm{r}^D(X), Y\,].e$$

the *Ricci corrected second Weyl derivative* of the bundle EM.

The skew part of the above Ricci corrected second Weyl derivative is given by the Weyl curvature tensor, hence

$$D^2_{X,Y} e + [\,\mathrm{r}^D(X), Y\,].e - \frac{1}{2}\,\mathrm{W^c}_{X,Y}\,.e$$

is symmetric in X and Y. Note that this is not simply the symmetric part of the second derivative. Instead it is characterized by the fact that under a linear change of the Weyl derivative only first order terms algebraic in γ and no $D\gamma$ terms occur. In terms of a Weyl derivative D we will next define a linear second order operator containing all the information of the 2-jet of a section $jet^2(e) \in C^\infty(M, Jet^2(EM))$ (we will use the following notation: $\mathrm{Sym}^{\leq k} V^* := \mathbb{R} \oplus V^* \oplus \ldots \oplus \mathrm{Sym}^k V^*$):

Definition 4.44 (*Second jet operator*) In Weyl geometry we define the 2*-jet operator* to be the following linear differential operators of second order:

$$jet^{2,D} : C^\infty(M, EM) \to C^\infty(M, \mathrm{Sym}^{\leq 2} T^* \otimes EM),$$

with

$$jet^{2,D} e := e \oplus De \oplus D^2 e + [\,\mathrm{r}^D, \;\,].e - \frac{1}{2}\,\mathrm{W^c}\,.e \quad .$$

Any linear second order differential operator $\nabla\colon C^\infty(M,EM)\to C^\infty(M,FM)$ between associated bundles in Weyl geometry is given by a linear map $(\mathrm{Sym}^{\leq 2}V^*)\otimes E\to F$, which is $\mathbf{CO}(V)$-equivariant and nontrivial on $\mathrm{Sym}^2V^*\otimes E$. This in terms gives a bundle map and the differential operator $\nabla^D=\pi\circ jet^{2,D}$.

If E and F are irreducible then E and F have central weights w_E and w_F and $\mathrm{Sym}^2V^*\otimes E\to F$ being nontrivial implies $w_F=w_E-2$ and $E\to F$ as well as $V^*\otimes E\to F$ are trivial. Hence in Weyl geometry the study of second order operators between irreducible representations reduces to the classification of irreducible sub representations of $\mathrm{Sym}^2V^*\otimes E$. Hence given a nonzero $\mathbf{CO}(V)$-equivariant linear map $\pi\colon\mathrm{Sym}^2V^*\otimes E\to F$ with E and F irreducible we get a differential operator $\nabla^D:=\pi\circ(D^2+[\,\mathrm{r}^D,\ \,].-\frac{1}{2}\,\mathrm{W}^{\mathsf{c}}.)$.

As for the 1-jet in Weyl geometry, the linear change of the curvature corrected second derivative under a change of Weyl derivative is given by the algebraic Lie derivative:

$$\mathcal{L}\colon V^*\otimes\mathrm{Sym}^{k-1}V^*\otimes E\to\mathrm{Sym}^kV^*\otimes E,$$

with

$$k=2\ :\ \mathcal{L}_\gamma f(v^2)=\mathcal{L}(\gamma\otimes f)(v^2)=2[\gamma,v].(f(v))-f([\gamma,v]v).$$

Proposition 4.45 *(Linear change of the second jet) Under a change of Weyl derivative the linear change of the above jet operator is given by the Lie derivative applied to the lower jet:*

$$(\partial_\gamma jet^{2,D})(e)\ =\ \mathcal{L}_\gamma(jet^{1,D}e)=0\oplus[\,\gamma,\ \,].e\oplus([\,\gamma,\ \,].De+[\,\gamma,\ \,].De-D_{[\gamma,\ ]}\,e).$$

The linear change of a second order operator ∇^D between irreducible E, F is then $\partial_\gamma(\nabla^D)(e)=\pi\circ\mathcal{L}_\gamma(De)$. Hence ∇^D is conformally invariant if the following composition of maps is zero:

$$V^*\otimes(V^*\otimes E)\xrightarrow{\mathcal{L}}\mathrm{Sym}^2V^*\otimes E\xrightarrow{\pi}F\ :$$

Proposition 4.46 *The linear equivariant maps $\pi\in\mathrm{Hom}_{\mathbf{CO}(V)}(\mathrm{Sym}^2V^*\otimes E,F)$ with $\pi\circ\mathcal{L}=0$ classify all conformally invariant linear second order operators between bundles associated to irreducible $\mathbf{CO}(V)$-representations E and F.*

Such a classification has been given by Branson in [Bra96] and [Bra98]. Instead of presenting the full classification, we will only construct examples directly or with the help of Fegan's result.

Example 4.47 (*Conformal Laplacian*) For $E=L^w$ and $F=L^{w-2}$ the only map $\mathrm{Sym}^2V^*\otimes E\to F$ is given by (multiples of) the conformal structure: $\pi(\alpha^2\otimes\mu)=\mathsf{c}(\alpha,\alpha)\mu$. In terms of a Weyl derivative D this defines a Laplace operator

$$\Delta^D\colon C^\infty(M,L^w)\to C^\infty(M,L^{w-2}),$$

given by

$$\begin{aligned}\Delta^D(e)\ &=\ \mathrm{tr}_{\mathsf{c}}\,jet^{2,D}(e)=\mathrm{tr}_{\mathsf{c}}\,D^2e+\mathrm{tr}_{\mathsf{c}}[\,\mathrm{r}^D,\ \,].e=\mathrm{tr}_{\mathsf{c}}\,D^2e+w\,\mathrm{tr}_{\mathsf{c}}\,\mathrm{r}^D\otimes e\\ &=\ \mathrm{tr}_{\mathsf{c}}\,D^2e+w\,\mathrm{s}^D\,e,\end{aligned}$$

with the normalized scalar curvature $\mathrm{s}^D := \mathrm{tr}_{\mathsf{c}}\, \mathrm{r}^D$. For the linear change we directly find:

$$\begin{aligned}\partial_\gamma \Delta^D(e) &= \sum_{i,j} \mathsf{c}(\theta^i, \theta^j)([\,\gamma, t_i\,].D_{t_j} e + [\,\gamma, t_j\,].D_{t_i} e - D_{[\,\gamma, t_i\,] t_j} e) \\ &= 2(w - (2-n)/2) D_{\sharp\gamma} e.\end{aligned}$$

Hence for $w = (2-n)/2$ this composite is zero and we get a conformally invariant Laplace operator

$$\Delta \colon C^\infty(M, L^{(2-n)/2}) \to C^\infty(M, L^{(-2-n)/2}).$$

Example 4.48 (*Branson's operator on forms*) We will define Branson's operator acting between k-forms for $k \geq 1$ which is part of his classification [Bra96]. There are three Stein Weiss operators defined on k-forms according to the decomposition $V^* \otimes \Lambda^k V^* = \Lambda^{k+1} V^* \oplus L^{-2} \otimes \Lambda^{k-1} V^* \oplus V^* \odot \Lambda^k V^*$, which are the exterior derivative d^D, exterior divergence div^D and the twistor operator $Twist^D$ see definition 4.23. The second order operator we are looking for is based upon a linear combination of $d^D \circ div^D$ and $div^D \circ d^D$ acting on k-forms $f \in C^\infty(M, L^{w+k} \otimes \Lambda^k T^* M)$ of central weight w. Hence the symbol is a linear combination of $\wedge(id_{V^*} \otimes \lrcorner)$ and $\lrcorner(id_{V^*} \otimes \wedge)$ which we will apply to the Ricci corrected second derivative. For the individual linear changes we find using $\mathfrak{co}(V)$-equivariance and the examples in 2.51:

$$\begin{aligned}&\partial_\gamma \wedge (id_{V^*} \otimes \lrcorner)(D^2 + [\,\mathrm{r}^D,\ \,])f \\ &\quad = \ (w+k-2)\,\gamma \wedge \mathrm{div}^D f - (w+n-k)\,\gamma \lrcorner\, \mathrm{d}^D f + (w+n-k) D_{\sharp\gamma} f, \\ &\partial_\gamma \lrcorner(id_{V^*} \otimes \wedge)(D^2 + [\,\mathrm{r}^D,\ \,])f \\ &\quad = \ -(w+k)\,\gamma \wedge \mathrm{div}^D f + (w+n-k-2)\,\gamma \lrcorner\, \mathrm{d}^D f + (w+k) D_{\sharp\gamma} f.\end{aligned}$$

The linear combination killing the $D_{\sharp\gamma} f$ terms is called *Branson's operator*:

$$\begin{aligned}Bran^D \ &: \ C^\infty(M, L^{w+k} \otimes \Lambda^k T^* M) \to C^\infty(M, L^{w-2+k} \otimes \Lambda^k T^* M); \\ Bran^D \ &:= \ ((w+k) \wedge (id_{V^*} \otimes \lrcorner) - (w+n-k) \lrcorner(id_{V^*} \otimes \wedge))(D^2 + [\,\mathrm{r}^D,\ \,]).\end{aligned}$$

The linear change under changing the Weyl derivative is therefore:

$$\begin{aligned}\partial_\gamma\, Bran^D &= ((w+k)(w+k-2) + (w+n-k)(w+k))\gamma \wedge \mathrm{div}^D f \\ &\quad -((w+k)(w+n-k) + (w+n-k)(w+n-k-2))\gamma \lrcorner\, \mathrm{d}^D f \\ &= (2w+n-2)\Big((w+k)\gamma \wedge \mathrm{div}^D f - (w+n-k)\gamma \lrcorner\, \mathrm{d}^D f\Big).\end{aligned}$$

Hence for $w = (2-n)/2$ we obtain a conformally invariant operator

$$Bran \colon C^\infty(M, L^{(2-n)/2+k} \otimes \Lambda^k T^* M) \to C^\infty(M, L^{(-2-n)/2+k} \otimes \Lambda^k T^* M).$$

The last two examples where selfadjoint operators. The conformal Hessian from proposition 4.17, the Bianchi and Bach operator in $n = 4$ dimensions, see propositions 2.68 and 2.70, are examples of the following definition:

Definition 4.49 (*Simple projection*) Suppose that E, F and G are irreducible $\mathbf{CO}(V)$-representations with irreducible $F \subset V^* \otimes E$ and $G \subset \mathrm{Sym}^2_0 V^* \otimes E$, such that the projection

$$\pi_G \colon V^* \otimes V^* \otimes E \to G$$

factors through the projection $\pi_F \colon V^* \otimes E \to F$. Such a map π_G will be called *simple*.

Associated to a simple $\pi_G \colon V^* \otimes V^* \otimes E \to G$ is a second order operator on a Weyl manifold defined by the projected Ricci corrected second derivative on EM:

$$\nabla^D := \pi_G(D^2 + [\,\mathrm{r}^D,\ \,]) \colon C^\infty(M, EM) \to C^\infty(M, GM).$$

Such operators will be called *simple second order operators*.

Remark 4.50 Using Fegan's result, theorem 4.36 the Ricci curvature correction simplifies in the simple second order case to

$$\begin{aligned} \nabla^D e &= \pi_G(X, Y \mapsto D^2_{X,Y} e + [\,\mathrm{r}^D(X), Y\,].e) \\ &= \pi_G(X, Y \mapsto D^2_{X,Y} e + (w_E + b)\,\mathrm{r}^D(X, Y) \otimes e) \\ &= \pi_G(D^2 e) + (w_E + b)\pi_G(\mathrm{r}^D \otimes e), \end{aligned}$$

where w_E is the central weight of E and $-b$ is the weight on which the Stein Weiss operator associated to E and F is conformally invariant. The significance of the Ricci correction for a composite of two Stein Weiss operators as above was pointed out to me by D. Calderbank. This has been generalized here to a Ricci correction of the full second derivative, definition 4.43.

Proposition 4.51 *(Simple second order differential operators) Let ∇^D be a simple second order operator associated to E, F, G and denote by w_E the central weight of E. Let $-b$ be the weight on which the Stein Weiss operator associated to E and F is conformally invariant (see theorem 4.36). The linear change of ∇^D under changing the Weyl derivative is then given by*

$$(\partial_\gamma \nabla^D)e = \partial_\gamma \pi_G(D^2 e + [\,\mathrm{r}^D,\ \,].e) \overset{!}{=} 2(w_E + b - 1)\pi_G(\gamma \otimes De).$$

Hence, conformally invariant simple second order operators are characterized by $w_E = 1 - b$ and the Ricci correction simplifies to:

$$\nabla e = \pi_G(D^2 e + \mathrm{r}^D \otimes e).$$

Proof: This is a direct computation using proposition 4.45 and theorem 4.36:

$$\begin{aligned} &\partial_\gamma \pi_G(X, Y \mapsto D^2_{X,Y} e + [\,\mathrm{r}^D(X), Y\,].e) \\ &\quad = \pi_G(X, Y \mapsto [\,\gamma, X\,].D_Y e + [\,\gamma, Y\,].D_X e - D_{[\,\gamma, X\,]Y} e) \\ &\quad = \pi_G(X, Y \mapsto 2[\,\gamma, X\,].D_Y e - D_{2\gamma(X)Y} e) \\ &\quad = \pi_G(X, Y \mapsto 2(w_E + b - 1)\gamma(X) D_Y e). \quad \square \end{aligned}$$

We now turn to the study of second order bilinear differential pairings. The general theory is slightly more involved than in the first order case. For this reason we start with three examples of pairings:

Example 4.52 (*Laplace pairing*) There is a conformally invariant bilinear differential pairing between scalar densities

$$\vee : C^\infty(M, L^{w_1}) \times C^\infty(M, L^{w_2}) \to C^\infty(M, L^{w_1+w_2-2}).$$

It is based upon the Laplace operator example 4.47 and it is defined in terms of a Weyl derivative by

$$e_1 \vee e_2 := w_2 q_2\, (\Delta^D e_1) \otimes e_2 - 2q_1 q_2\, \mathsf{c}(De_1, De_2) + w_1 q_1\, e_1 \otimes (\Delta^D e_2),$$

where $q_1 := (w_1 - (2-n)/2)$ and $q_2 := (w_2 - (2-n)/2)$. The linear change of the Weyl derivative is zero, since

$$\begin{aligned} &\partial_\gamma (e_1 \vee^D e_2) \\ &= w_2 q_2 (2q_1)\, D_\gamma e_1 \otimes e_2 - 2q_1 q_2 w_1 e_1 \otimes D_\gamma e_2 - 2q_1 q_2 w_2 D_\gamma e_1 \otimes e_2 + w_1 q_1 (2q_2)\, e_1 \otimes D_\gamma e_2 \\ &= (2w_2 q_2 q_1 - 2q_1 q_2 w_2) D_\gamma e_1 \otimes e_2 - (2q_1 q_2 w_1 - 2w_1 q_1 q_2) e_1 \otimes D_\gamma e_2. \end{aligned}$$

Example 4.53 (*Branson pairing*) There is also a scalar valued pairing between k-forms based upon Branson's operator, example 4.48:

$$\vee : C^\infty(M, L^{w_1+k} \otimes \Lambda^k T^*M) \times C^\infty(M, L^{w_2+k} \otimes \Lambda^k T^*M) \to C^\infty(M, L^{w_1+w_2-2}).$$

In terms of a Weyl derivative it is given by

$$\begin{aligned} f_1 \vee f_2 \;:=\; & p_2 q_2 r_2\, \mathsf{c}(Bran^D f_1, f_2) + p_1 q_1 r_1\, \mathsf{c}(f_1, Bran^D f_2) \\ & -2p_1 p_2 r_1 r_2\, \mathsf{c}(div^D f_1, div^D f_2) + 2q_1 q_2 r_1 r_2\, \mathsf{c}(d^D f_1, d^D f_2), \end{aligned}$$

with $p_1 := w_1 - (-k)$, $q_1 := w_1 - (k-n)$, $r_1 := w_1 - (2-n)/2$ and similar for the index 2. It is straight forward to check that the linear change by changing the Weyl derivative is zero using the results in examples 4.48 and 2.51.

Example 4.54 (*Hesse pairing*) There is a conformally invariant bilinear differential pairing between scalar densities to give trace-free symmetric bilinear forms:

$$\vee : C^\infty(M, L^{w_1}) \times C^\infty(M, L^{w_2}) \to C^\infty(M, L^{w_1+w_2} \otimes \mathrm{Sym}\,^2_0 T^*M).$$

It is based upon the Hessian operator, proposition 4.17, and it is defined in terms of a Weyl derivative by

$$e_1 \vee e_2 := w_2 q_2\, (Hesse^D\, e_1) \otimes e_2 - 2q_1 q_2\, De_1 \odot De_2 + w_1 q_1\, e_1 \otimes (Hesse^D\, e_2),$$

where $q_1 := w_1 - 1$ and $q_2 := w_2 - 1$.

The last example is a special a case of a class of pairings associated to simple second order operators:

Definition 4.55 (*Simple projection*) Let E_1, E_2 be irreducible $\mathbf{CO}\,(V)$-representation and $G \subset \mathrm{Sym}\,^2_0 V^* \otimes E_1 \otimes E_2$ an irreducible subrepresentation. If there are irreducible subrepresentations $F_1 \subset V^* \otimes E_1$ and $F_2 \subset V^* \otimes E_2$ such that $Q : V^* \otimes V^* \otimes E_1 \otimes E_2 \to G$ factors through $V^* \otimes F_1 \otimes E_2$ and through $V^* \otimes E_1 \otimes F_2$ then Q is called *simple*.

Proposition 4.56 *(Simple conformally invariant second order pairings) Let Q be a simple projection as above and denote by w_1 and w_2 the central weights of E_1 and E_2. Let $-b_1$ respectively $-b_2$ be the weights for which the Stein Weiss operators associated to E_1, F_1 respectively E_2, F_2 are conformally invariant. Then the following differential pairing is conformally invariant:*

$$\vee \colon C^\infty(M, E_1M) \times C^\infty(M, E_2M) \to C^\infty(M, GM),$$

which in terms of a Weyl derivative D applied to sections e_1 and e_2 is given by

$$\begin{aligned} e_1 \vee e_2 \;&:=\; (w_2 + b_2)(w_2 + b_2 - 1)Q((D^2 + [\,\mathrm{r}^D,\;\,])e_1 \otimes e_2) \\ &\quad -2(w_1 + b_1 - 1)(w_2 + b_2 - 1)Q(De_1 \otimes De_2) \\ &\quad +(w_1 + b_1)(w_1 + b_1 - 1)Q(e_1 \otimes (D^2 + [\,\mathrm{r}^D,\;\,])e_2). \end{aligned}$$

Proof: From the propositions 4.51 and 4.36 we immediately have:

$$\begin{aligned} &Q\;\;(\partial_\gamma((D^2 + [\,\mathrm{r}^D,\;\,])e_1 \otimes e_2)) = 2(w_1 + b_1 - 1)Q(\gamma \otimes De_1 \otimes e_2), \\ &Q\;\;(\partial_\gamma(e_1 \otimes (D^2 + [\,\mathrm{r}^D,\;\,])e_2)) = 2(w_2 + b_2 - 1)Q(e_1 \otimes \gamma \otimes De_2), \\ &Q\;\;(\partial_\gamma(De_1 \otimes De_2)) = (w_1 + b_1)Q(\gamma \otimes e_1 \otimes De_2) + (w_2 + b_2)Q(De_1 \otimes \gamma \otimes e_2). \quad \square \end{aligned}$$

Discussion 4.57 (*Second order pairings*) We now turn to the study of second order bilinear differential pairings in general. Suppose E_1, E_2 and F are $\mathbf{CO}\,(V)$-representations and

$$Q \colon (\mathrm{Sym}^{\leq 2} V^* \otimes E_1) \otimes (\mathrm{Sym}^{\leq 2} V^* \otimes E_2) \to G$$

is $\mathbf{CO}\,(V)$-equivariant, then $\vee^D \colon C^\infty(M, E_1M) \times C^\infty(M, E_2M) \to C^\infty(M, GM)$ defined by $e_1 \vee^D e_2 := Q(jet^{2,D}(e_1) \otimes jet^{2,D}(e_2))$ is a bilinear differential pairing of second order in Weyl geometry. As in the first order case we will consider those pairings which are truly second order in both e_1 and e_2. If E_1, E_2 and G are irreducible the action of the centre of $\mathbf{CO}\,(V)$ determines such a Q to be nontrivial only on the following subspace

$$Q \colon (\mathrm{Sym}^2 V^* \otimes E_1 \otimes E_2) \oplus (V^* \otimes V^* \otimes E_1 \otimes E_2) \oplus (\mathrm{Sym}^2 V^* \otimes E_1 \otimes E_2) \to G.$$

The corresponding pairing is then clearly given by

$$e_1 \vee^D e_2 := Q(((D^2 + [\,\mathrm{r}^D,\;\,])e_1 \otimes e_2) \oplus (De_1 \otimes De_2) \oplus (e_1 \otimes (D^2 + [\,\mathrm{r}^D,\;\,])e_2)).$$

The dependence of $\vee^D$ on the Weyl derivative leads to study the following map:

$$\begin{aligned} &\mathcal{L}^{\mathrm{bi}} \colon V^* \otimes (V^* \otimes E_1 \otimes E_2 \oplus V^* \otimes E_1 \otimes E_2) \\ &\quad \to \;\; (\mathrm{Sym}^2 V^* \otimes E_1 \otimes E_2) \oplus (V^* \otimes V^* \otimes E_1 \otimes E_2) \oplus (\mathrm{Sym}^2 V^* \otimes E_1 \otimes E_2) \end{aligned}$$

which is given by

$$\begin{aligned} &\mathcal{L}^{\mathrm{bi}}(\gamma \otimes ((\alpha_1 \otimes e_1 \otimes e_2) \oplus (e_1 \otimes \alpha_2 \otimes e_2))) \\ &\;:=\;\; (\mathcal{L}(\gamma \otimes \alpha_1 \otimes e_1) \otimes e_2) \\ &\qquad \oplus(\mathcal{L}(\gamma \otimes e_1) \otimes \alpha_2 \otimes e_2 + \alpha_1 \otimes e_1 \otimes \mathcal{L}(\gamma \otimes e_2)) \\ &\qquad \oplus(e_1 \otimes \mathcal{L}(\gamma \otimes \alpha_2 \otimes e_2)). \end{aligned}$$

We summarize the above observations in the following:

Proposition 4.58 *Conformally invariant bilinear second order differential pairings*

$$\vee\colon C^\infty(M, E_1M) \times C^\infty(M, E_2M) \to C^\infty(M, GM),$$

between associated bundles from irreducible $\mathbf{CO}\,(V)$*-representations* E_1*,* E_2 *and* G *are classified by* $\mathbf{CO}\,(V)$*-equivariant linear maps*

$$Q\colon (\mathrm{Sym}\,^2 V^* \otimes E_1 \otimes E_2) \oplus (V^* \otimes V^* \otimes E_1 \otimes E_2) \oplus (\mathrm{Sym}\,^2 V^* \otimes E_1 \otimes E_2) \to G,$$

such that $Q \circ \mathcal{L}^{\mathrm{bi}} = 0$*.*

Chapter 5

Conformal linear field theories

Soon after Weyl developed differential conformal geometry and his unified theory of electromagnetism and gravity (see [Wey70], or remark 2.31) Bach came up with a nonlinear conformally invariant geometric theory of gravity (see [Bac21], or remark 2.73). We will present its linearized version here. Fierz in [Fie39] invented a class of theories which shares analogous properties to electromagnetism, except these theories fail to be conformally invariant. We will give a variation of Fierz's theory for gravity which is conformally invariant. The so called *zero rest mass field equations* due to Penrose (see [PR84]) are conformally invariant and provide another large class of field equations analogous to Maxwell's equations. We will discuss Penrose's linear theory of gravity here. I like to thank D. Calderbank for drawing my attention to the fact that these conformal field theories are Bernstein Gelfand Gelfand complexes—the latter were studied in Minkowski space by Eastwood and Rice in [ER87].

The aim of this chapter is to analyse the structure of Maxwell's equations and the Lorentz force law to extract general principles a linear field theory should follow. This happens in the first three sections. The key ingredient which extends the basic field equations are bilinear differential pairings imitating the exterior and interior multiplication of forms and multivectors known from electromagnetism. These pairings allow us to suggest the beginning of a theory of motion: charged particles induce fields which travel with the speed of light and these fields influence the motion of charged particles. This is summarized in section 5.4. In the last three sections we investigate the three field theories in connection with linear gravity. We explicitly define the field equation and bilinear pairings needed and hence prove special cases of theorem 5.13, on an elementary level. These explicit results and calculations arose out of joint efforts with D. Calderbank to understand conformal linear field theories.

5.1 Ingredients of a general field theory

We would like a general linear field theory to share the following ingredients and properties with electromagnetism: a tensor field represents the *source* and satisfies a *conservation law*. A charged particles along an arbitrary worldline naturally gives rise to a conserved source. We distinguish between two realizations of the field itself, the *dynamic and the kinematic field*. The dynamic field is coupled to the source by the *dynamic field equation*. The kinematic field satisfies an *integrability condition*. Kinematic and dynamic fields are related by a *constitutive relation*, which in vacuum only involves the conformal structure. It is the kinematic field which determines the *motion of charged test particles* by a general force law. The integrability condition of the kinematic field means that it is (locally) in the image of a linear differential operator applied to a *potential*. This potential is not uniquely determined, instead it is subject to (local) *gauge transformations*. The operator of the dynamic field equation is adjoint to the potential operator, similarly the operator for the conservation law of the source is adjoint to the operator between gauges and potentials.

To summarize, a relativistic linear field theory in vacuum takes place on a $n = 4$ dimensional conformal manifold M of Lorentzian signature. The beginning of a complex of conformally invariant linear differential operators defines the *kinematic sequence* of the theory:

$$\begin{array}{ccccccc} \text{gauge} & & \text{potential} & & \text{kinematic field} & & \\ C^\infty(M, H_0) & \overset{d_H}{\to} & C^\infty(M, H_1) & \overset{d_H}{\to} & C^\infty(M, H_2) & \overset{d_H}{\to} & C^\infty(M, H_3) \end{array}$$

Here H_k denote associated vector bundles in which the fields (gauges, potentials, kinematic fields) take their values. Adjoint to this is the end of a complex defining the *dynamic sequence* of the theory. Sources and dynamic fields take values in $L^{-n} \otimes H^k$ with $H^k = {H_k}^*$:

$$\begin{array}{ccccccc} C^\infty(M, L^{-n} \otimes H^0) & \overset{div_H}{\leftarrow} & C^\infty(M, L^{-n} \otimes H^1) & \overset{div_H}{\leftarrow} & C^\infty(M, L^{-n} \otimes H^2) & \overset{div_H}{\leftarrow} \\ & & \text{source} & & \text{dynamic field} & \end{array}$$

To be consistent with electromagnetism we like $-\,div_H$ to be adjoint to d_H, i.e. according to paragraph 1.18 we postulate the existence of bilinear differential pairings

$$\lrcorner_H : C^\infty(M, H_k) \times C^\infty(M, L^{-n} \otimes H^{k+1}) \to C^\infty(M, L^{-n} \otimes TM),$$

such that the following divergence formula holds:

$$div_H(\alpha \lrcorner_H a) = \langle d_H\, \alpha, a\rangle + \langle \alpha, div_H\, a\rangle.$$

5.2 Charges in field theories

We will follow Penrose who interpreted the kernel of the gauge operator

$$W := \ker\, d_H : C^\infty(M, H^0) \to C^\infty(M, H^1)$$

as the *space of charges* which characterize the field theory in question. In case $M = \mathbf{A}$ being a conformal affine space W is a representation space for the Möbius Lie algebra $\mathfrak{so}(\hat{V})$ of conformal Killing fields in M, since d_H is conformally invariant. Penrose calls d_H the *twistor operator* of the theory and the solutions W are called *twistors*. In electromagnetism electric charges are real numbers, and we have $\mathbb{R} = \ker d : C^\infty(M, \mathbb{R}) \to C^\infty(M, T^*M)$. For the linearized conformal geometry, remark 2.73, the charges are given by the space of conformal Killing fields in affine background given by $\mathfrak{so}(\hat{V}) = \ker\, Kill$. In case of a linear theory for gravity W has the interpretation as *space of gravitational masses*. This interpretation is not entirely obvious and this section is only a first step towards this task: we will show how a dynamic field represents an element in W by integrating it over a cooriented sphere, how a worldline with associated element in W naturally gives rise to a conserved distributional source and how a kinematic field induces an acceleration on a test particle with associated element in W. To make these three constructions work in a general field theory we need bilinear pairings imitating the interior and exterior multiplication known from the deRham complex and its adjoint (electromagnetism).

Remark 5.1 As will be explained in section 5.4, twistors in W behaves more like magnetic charges whereas *dual twistors*, i.e. elements in W^*, behave more like electric charges, see definitions 2.19 and 2.21. Since W is a $\mathfrak{so}(\hat{V})$ representation it carries an inner product which allows to identify W with its dual W^*. Hence we will not distinguish twistors from dual twistors until section 5.4.

Discussion 5.2 (*Charge represented by dynamic fields*) We recall that in electromagnetism the electric charge generating a field G can be rediscovered from G by integration: Maxwell's second equation $div\, G = j$ allows to define the charge contained inside a cooriented $(n-2)$-dimensional sphere S^{n-2} (away from the source $\mathrm{supp}(j)$) to be $q = \int_{S^{n-2}} \langle G, coor \rangle$. The divergence theorem provides a conservation law for this integral. To make a similar construction work for a general field theory we need the following *bilinear differential pairings*:

$$\begin{aligned}
\lrcorner_H : C^\infty(M, H_0) \times C^\infty(M, L^{-n} \otimes H^2) &\rightarrow C^\infty(M, L^{-n} \otimes \Lambda^2 TM), \\
\lrcorner_H : C^\infty(M, H_1) \times C^\infty(M, L^{-n} \otimes H^2) &\rightarrow C^\infty(M, L^{-n} \otimes TM), \\
\lrcorner_H : C^\infty(M, H_0) \times C^\infty(M, L^{-n} \otimes H^1) &\rightarrow C^\infty(M, L^{-n} \otimes TM),
\end{aligned}$$

such that for sections $f \in C^\infty(M, H_0)$ and $g \in C^\infty(M, H^2)$ the following *codimension one divergence formula* holds:

$$div(f \lrcorner_H g) = (d_H f) \lrcorner_H g + f \lrcorner_H div_H\, g.$$

In that case we say that the gauge operator $d_H : C^\infty(M, H_0) \to C^\infty(M, H_1)$ and the operator of the dynamic field equation $div_H : C^\infty(M, L^{-n} \otimes H^2) \to C^\infty(M, L^{-n} \otimes H^1)$ are *codimension one adjoints* (see the general definition below). From this its clear that for a charge $q \in W = \ker\, d_H$ and in a regions where the dynamic field G satisfies $div_H\, G = 0$ we can produce a divergence-free bivector field $q \lrcorner_H G$. Hence if S^{n-2} is a cooriented sphere (away from $\mathrm{supp}(j)$), then $\int_{S^{n-2}} \langle q \lrcorner_H G, coor \rangle$ is a real number linear in q. This determines an element in the dual space $\int_{S^{n-2}} \langle . \lrcorner_H G, coor \rangle \in W^*$. As was already mentioned in the affine case $M = \mathbf{A}$ the space W is a representation of the Möbius Lie algebra, hence it can carry an invariant inner product to identify W with its dual W^*.

Definition 5.3 (*Codimension one adjoints*) Let EM, FM and GM be vector bundles over M and $\nabla_1 : C^\infty(M, EM) \to C^\infty(M, FM)$ and $\nabla_2 : C^\infty(M, FM) \to C^\infty(M, GM)$ be two differential operators of order d_1 and d_2 respectively. Suppose that both operators have adjoints $\nabla_1^* : C^\infty(M, L^{-n} \otimes F^*M) \to C^\infty(M, L^{-n} \otimes E^*M)$ and $\nabla_2^* : C^\infty(M, L^{-n} \otimes G^*M) \to C^\infty(M, L^{-n} \otimes F^*M)$ coming from $L^{-n} \otimes TM$ valued bilinear differential pairings X_1 and X_2, such that

$$\begin{aligned}
div(X_1(e, \phi)) &= \langle \nabla_1 e, \phi \rangle - \langle e, \nabla_1^* \phi \rangle, \quad \text{and} \\
div(X_2(f, \psi)) &= \langle \nabla_2 f, \psi \rangle - \langle f, \nabla_2^* \psi \rangle.
\end{aligned}$$

Then ∇_1 and ∇_2^* are called *codimension one adjoints* if there is a $L^{-n} \otimes \Lambda^2 TM$ valued bilinear differential pairing Y of order $d_1 + d_2 - 2$ between sections of EM and sections of $L^{-n} \otimes G^*M$, such that

$$div(Y(e, \psi)) = X_2(\nabla_1 e, \psi) + X_1(e, \nabla_2^* \psi).$$

Remark 5.4 Applying the operator *div* once more to the codimension one divergence formula we obtain a necessary condition to hold for codimension one adjoints. This is satisfied if ∇^1, ∇^2 from a complex, i.e. $\nabla_2 \circ \nabla_1 = 0$ and hence also $\nabla_1^* \circ \nabla_2^* = 0$.

Remark 5.5 A complex of first order Stein Weiss operators always has codimension one adjoints: We assume that the operators are given in terms of a Weyl derivative D and symbols $\pi_1 : T^* \otimes E \to F$ and $\pi_2 : T^* \otimes F \to G$ as $\nabla_1 = \pi_1 \circ D$ and $\nabla_2 = \pi_2 \circ D$. With a dual basis t_i, θ^i of TM the zero order pairings are $X_1(e, \phi) = \sum_i t_i \otimes \langle \pi_1(\theta^i \otimes e), \phi\rangle$ and similar X_2. For $\alpha, \beta \in T^*M$ we note that the symbol of $\nabla_2 \circ \nabla_1$ is $\pi_{21}(\alpha \otimes \alpha \otimes e) = \pi_2(\alpha \otimes \pi_1(\alpha \otimes e))$. Since $0 = \nabla_2 \circ \nabla_1$ we know that $\pi_{21}(\alpha \otimes \beta \otimes e) = \pi_2(\alpha \otimes \pi_1(\beta \otimes e))$ needs to be skew in α and β, which defines $Y(e, \psi) := \frac{1}{2} \sum_{i,j} t_i \wedge t_j \langle \pi_{21}(\theta^j \otimes \theta^i \otimes e), \psi\rangle$.

Discussion 5.6 (*Charged particles as sources*) An oriented worldline $c \colon \mathbb{R} \to M$, $t \mapsto c(t)$ of an electrically charged particle $q \in \mathbb{R}$ gives rise to a distributional source $j \in \mathcal{D}(M, L^{-n} \otimes TM)$ in electromagnetism: $\langle\!\langle j, \alpha\rangle\!\rangle := \int_{\mathbb{R}} q\alpha(\dot{c})$ with $\alpha \in C_o^\infty(M, T^*M)$. This source is divergence-free, since for a real valued test function $f \in C_o^\infty(M, \mathbb{R})$ we have $\langle\!\langle j, df\rangle\!\rangle := \int_{\mathbb{R}} q df(\dot{c}) = \int_{\mathbb{R}} \frac{\partial}{\partial t}(qf(c(t))) = 0$. A similar construction works for a general linear field theory if it is equipped with two further bilinear differential pairings

$$\begin{aligned} \wedge_H : C^\infty(M, H_0) \times C^\infty(M, H_0) &\to C^\infty(M, \mathbb{R}), \\ \wedge_H : C^\infty(M, H_1) \times C^\infty(M, H_0) &\to C^\infty(M, T^*M), \end{aligned}$$

such that the following product rule holds for any $f, g \in C^\infty(M, H_0)$:

$$d(f \wedge_H g) = (d_H f) \wedge_H g + f \wedge_H (d_H g).$$

If $c \colon \mathbb{R} \to M$ is an oriented worldline and $q \in W$ an associated charge then this gives rise to a conserved distributional source $j \in \mathcal{D}(M, L^{-n} \otimes H^1)$ of the general linear field theory: $\langle\!\langle j, \alpha\rangle\!\rangle := \int_{\mathbb{R}}(q \wedge_H \alpha)(\dot{c})$, where $\alpha \in C_o^\infty(M, H_1)$ is a test section. This source is conserved, since for a test gauge section $f \in C_o^\infty(M, H_0)$ we find

$$\begin{aligned} \langle\!\langle j, d_H f\rangle\!\rangle &= \int_{\mathbb{R}} (q \wedge_H d_H f)(\dot{c}) \\ &= \int_{\mathbb{R}} d(q \wedge_H f)(\dot{c}) - ((d_H q) \wedge_H f)(\dot{c}) \\ &= 0. \end{aligned}$$

Discussion 5.7 (*Lorentz force law in general field theories*) A worldline c with normalized timelike tangent vector N, associated electric charge $q \in \mathbb{R}$ and inertial mass $m \in L^{-1}$ can represent a test particle moving in an electromagnetic field F. The Lorentz force law determines the acceleration on c to be $mD_N N = q\sharp(F(N))$ (in terms of a Weyl derivative D with $Dm = 0$). To establish a simple force law in a general linear field theory we need three further bilinear differential pairings:

$$\begin{array}{rcl} \wedge_H : C^\infty(M, H_0) \times C^\infty(M, H_2) & \to & C^\infty(M, \Lambda^2 T^* M), \\ \wedge_H : C^\infty(M, H_1) \times C^\infty(M, H_2) & \to & C^\infty(M, \Lambda^3 T^* M), \\ \wedge_H : C^\infty(M, H_0) \times C^\infty(M, H_3) & \to & C^\infty(M, \Lambda^3 T^* M), \end{array}$$

such that for sections $f \in C^\infty(M, H_0)$ and $F \in C^\infty(M, H_2)$ the following product rule holds:

$$d(f \wedge_H F) = (d_H f) \wedge_H F + f \wedge_H (d_H F).$$

From this its clear that for a charge $q \in W = \ker d_H$ and a kinematic field F subject to the integrability condition $d_H F = 0$ we can produce a closed 2-form $q \wedge_H F$. Hence a test particle with inertial mass m and charge $q \in W$ is accelerated due to the kinematic field F as

$$mD_N N = \sharp((q \wedge_H F)(N)),$$

where D denotes a Weyl derivative of M with $Dm = 0$.

5.3 Lienard Wiechert fields

The long history of electromagnetism provides us with a long list of solutions of the field equations which have elaborated physical interpretations. In this chapter we will describe how bilinear differential pairings imitating the wedge product of the deRham complex allow to generate solutions in general field theories from solutions of Maxwell's equations. In particular a single electrically charged particle along a worldline produces an electromagnetic field. The corresponding solution of Maxwell's equation is due to Lienard and Wiechert. We will describe how analogous solutions can be obtained in any general linear field theory.

Discussion 5.8 (*Kinematic helicity raising*) We begin with a general field theory based upon the following kinematic sequence:

$$W \to C^\infty(M, H_0) \to C^\infty(M, H_1) \to C^\infty(M, H_2) \to \ldots$$

Suppose there are bilinear differential pairings like

$$\begin{array}{rcl} \wedge_H & : & C^\infty(M, H_0) \times C^\infty(M, T^* M) \to C^\infty(M, H_1), \\ \wedge_H & : & C^\infty(M, H_0) \times C^\infty(M, \Lambda^2 T^* M) \to C^\infty(M, H_2), \\ \wedge_H & : & C^\infty(M, H_1) \times C^\infty(M, T^* M) \to C^\infty(M, H_2), \end{array}$$

so that for sections $f \in C^\infty(M, H_0)$ and $A \in C^\infty(M, T^* M)$ the following product rule holds:

$$d_H(f \wedge_H A) = (d_H f) \wedge_H A + f \wedge_H (dA).$$

The first two pairings allow to produce a general potential $f \wedge_H A$ respectively a general kinematic field $f \wedge_H F$ from an electromagnetic potential A respectively from an electromagnetic field F with the help of a general gauge f. If the gauge is indeed a general charge $d_H f = 0$ then the product rule ensures that the general kinematic field comes from the general potential: $f \wedge_H F = d_H(f \wedge_H A)$ if $F = dA$. Such a translation of solutions from electromagnetism to a general field theory is called *helicity raising* (where the *helicity* of a field theory is a number reflecting a particular *polarization property* of plane wave solutions of that theory - for example electromagnetic waves have helicity one and gravitational waves are expected to have helicity two).

Discussion 5.9 (*Dynamic helicity raising*) For the dynamic sequence of a general field theory

$$C^\infty(M, L^{-n} \otimes H^0) \leftarrow C^\infty(M, L^{-n} \otimes H^1) \leftarrow C^\infty(M, L^{-n} \otimes H^2) \leftarrow \dots$$

suppose there are bilinear differential pairings like

$$\begin{aligned} \lrcorner_H &: C^\infty(M, H_0) \times C^\infty(M, L^{-n} \otimes \Lambda^2 TM) \to C^\infty(M, L^{-n} \otimes H^2), \\ \lrcorner_H &: C^\infty(M, H_0) \times C^\infty(M, L^{-n} \otimes TM) \to C^\infty(M, L^{-n} \otimes H^1), \\ \lrcorner_H &: C^\infty(M, H_1) \times C^\infty(M, L^{-n} \otimes \Lambda^2 TM) \to C^\infty(M, L^{-n} \otimes H^1), \end{aligned}$$

so that for sections $f \in C^\infty(M, H_0)$ and $G \in C^\infty(M, L^{-n} \otimes \Lambda^2 TM)$ the following product rule holds:

$$div_H(f \lrcorner_H G) = (d_H f) \lrcorner_H G + f \lrcorner_H (div\, G).$$

The first two pairings allow to produce a general dynamic field $f \wedge_H G$ respectively a general source $f \wedge_H j$ from an electromagnetic field G respectively from an electromagnetic source j with the help of a general gauge f. If the gauge is indeed a general charge $d_H f = 0$ then the product rule ensures that the general source comes from the general dynamic field: $f \wedge_H j = div_H(f \wedge_H G)$ if $j = div\, G$.

Discussion 5.10 (*Electromagnetic Lienard Wiechert fields*) In affine conformal Minkowski space $\mathbf{A}$ modelled over the vector space V let $c\colon \mathbb{R} \to \mathbf{A}$; $t \mapsto c(t)$ be a timelike curve, which represents an emitter of light rays. The backward lightcone of each point $x \in \mathbf{A}$ intersects the worldline at $c(\phi(x))$, which defines a function $\phi\colon \mathbf{A} \to \mathbb{R}$ the retarded parameter. Its gradient $L := -\,grad\,\phi$ is a lightlike geodetic vector field in $\mathbf{A}$ (compare example 4.9). This function ϕ satisfies $0 = \langle x - c(\phi(x)), x - c(\phi(x))\rangle$ and differentiating along a curve $x\colon \mathbb{R} \to \mathbf{A}$; $s \mapsto x(s)$ gives $0 = \langle x' - \dot{c}(\phi(x))\partial_{x'}\phi|_x, x - c(\phi(x))\rangle$, hence the lightlike vector field is explicit:

$$L_x = \frac{x - c(\phi(x))}{-\langle \dot{c}(\phi(x)), x - c(\phi(x))\rangle}.$$

Let $N = \nu\dot{c} \in V^0$ with $\nu \in L^{-1}$ along c denote the normalized tangent vector $\langle N, N\rangle = -1$, whereas ν depends upon the parameterization of c. It can be viewed as being the emitted *frequency* of L with respect to N, since $\nu = -\langle L, N\rangle$. The *luminosity distance* to the worldline $r\colon \mathbf{A} \to L^1$ (compare remark 4.7) is defined by

$$r(x) := -\langle N_{c(\phi(x))}, x - c(\phi(x))\rangle.$$

Along the light rays r has initial conditions $r(c(t)) = 0$, $D_L r(c(t)) = -\langle L, N\rangle = \nu$ and satisfies $D_L D_L r = 0$ with the affine derivative D.

With this in hand we can define the electromagnetic *Lienard Wiechert potential* $A\colon \mathbf{A} \to V^*$ in $n = 4$ dimensions to be

$$A(x) := q\frac{N}{r},$$

where $q \in \mathbb{R}$ is a constant number. From the potential we have the kinematic Lienard Wiechert field $F = dA$ and its dynamic field $G = F$ (identified using the conformal structure). The source of the dynamic field is a distribution with support on the worldline given by (see [Thi90]):

$$\langle\!\langle div\, G, \alpha\rangle\!\rangle = 4\pi \int_{\mathbb{R}} q\alpha(\dot{c}).$$

Hence the Lienard Wiechert field solves Maxwell's equation with a single charged particle along a worldline as source. The *Coulomb field* (compare paragraph 2.15) is the case when the worldline follows a straight line.

Discussion 5.11 (*Distributional sources and general Lienard Wiechert fields*) In paragraph 5.9 we explained what pairings were needed to raise a smooth electromagnetic field G with sources $j = div\, G$ to a general field $q \lrcorner_H G$ with source $q \lrcorner_H j = div_H(q \lrcorner_H G)$. We used the following pattern of pairings:

$$\lrcorner_H\, : C^\infty(M, H_k) \times C^\infty(M, L^{-n} \otimes \Lambda^{k+l}TM) \to C^\infty(M, L^{-n} \otimes H^l),$$

with $k = 0, 1$ and $k + l = 1, 2$. In paragraph 5.6 we considered pairings like

$$\wedge_H\colon C^\infty(M, H_k) \times C^\infty(M, H_l) \quad\to\quad C^\infty(M, \Lambda^{k+l}T^*M),$$

(with $k = 0, 1$ and $l = 0$) to construct conserved distributional sources from worldlines with an attached charge $q \in W$. More generally if $j \in \mathcal{D}(M, L^{-n} \otimes TM)$ is a distributional source in electromagnetism, then $J \in \mathcal{D}(M, L^{-n} \otimes H^1)$ defined by $\langle\!\langle J, \alpha\rangle\!\rangle = \langle\!\langle j, q \wedge_H \alpha\rangle\!\rangle$ with $\alpha \in C_o^\infty(M, H_1)$ is conserved if j is conserved. To make the smooth and distributional construction of sources consistent we need the operators $q\wedge_H$ and $q \lrcorner_H$ to be adjoint in the sense that there is a trilinear differential pairing:

$$X\colon C^\infty(M, H_0) \times C^\infty(M, L^{-n} \otimes \Lambda^l TM) \times C^\infty(M, H_l) \to C^\infty(M, L^{-n} \otimes TM),$$

such that

$$div(X(q, a, \alpha)) = \langle q \lrcorner_H a, \alpha\rangle - \langle a, q \wedge_H \alpha\rangle,$$

for $l = 1$. Moreover for $l = 2$ such a pairing guarantees $\langle\!\langle div_H(q \lrcorner_H G), \alpha\rangle\!\rangle = \langle\!\langle div\, G, q \wedge_H \alpha\rangle\!\rangle$ since then the following calculation holds

$$\begin{aligned}
\langle\!\langle div_H(q \lrcorner_H G), \alpha\rangle\!\rangle &= -\langle\!\langle q \lrcorner_H G, d_H\, \alpha\rangle\!\rangle \\
&= -\langle\!\langle G, q \wedge_H d_H\, \alpha\rangle\!\rangle \\
&= -\langle\!\langle G, d(q \wedge_H \alpha) - (d_H\, q) \wedge_H \alpha\rangle\!\rangle \\
&= \langle\!\langle div\, G, (q \wedge_H \alpha)\rangle\!\rangle.
\end{aligned}$$

With such pairings in hands the electromagnetic Lienard Wiechert fields of the last paragraph 5.10 can be raised into any general field theory, and the sources of these general Lienard Wiechert fields are the expected ones.

5.4 General field theory

This section begins with quoting a special case of a general result which we will prove at the end of this dissertation. The first of these theorems is due to Eastwood and Rice [ER87] whereas the second statement concerning differential pairings is new: let $\mathbf{A}$ be an affine space modelled on a $n = 4$ dimensional vector space V with conformal metric c. We already identified the Möbius Lie algebra (of vector fields on $\mathbf{A}$ leaving the affine conformal structure invariant) with the graded Lie algebra $\mathfrak{g} = V \oplus \mathfrak{co}(V) \oplus V^*$ and also with the orthogonal Lie algebra $\mathfrak{g} = \mathfrak{so}(\hat{V})$ where $\hat{V} = L^1 \oplus V^0 \oplus L^{-1}$.

Theorem 5.12 *(Conformal Bernstein Gelfand Gelfand complex) Associated to any finite dimensional representation W of the Lie algebra $\mathfrak{g}$ is a sequence of finite dimensional $\mathfrak{co}(V)$-representations denoted by $H_k(W)$ with $k = 0 \ldots n$, beginning with the coinvariants $H_0(W) = W_{V^*}$ of W with respect to V^*. The natural inclusion $\iota_H : W \to C^\infty(\mathbf{A}, W_{V^*})$ is the beginning of a locally exact complex of $\mathfrak{g}$-equivariant linear differential operators:*

$$W \xrightarrow{\iota_H} C^\infty(\mathbf{A}, H_0(W)) \xrightarrow{d_H} C^\infty(\mathbf{A}, H_1(W)) \xrightarrow{d_H} C^\infty(\mathbf{A}, H_2(W)) \xrightarrow{d_H} \ldots \quad .$$

This theorem provides us with a kinematic sequence of a general linear field theory characterized by W. For the trivial representation $W = \mathbb{R}$ we recover the deRham complex $H_k(\mathbb{R}) = \Lambda^k V^*$. The next theorem guarantees the existence of bilinear pairings imitating the wedge product on forms in electromagnetism:

Theorem 5.13 *(Conformal exterior pairings) If W_1, W_2 and W_3 are three finite dimensional $\mathfrak{g}$-representations and $F : W_1 \otimes W_2 \to W_3$ is a linear $\mathfrak{g}$-equivariant map, then there is a $\mathfrak{g}$-equivariant bilinear differential* exterior pairing

$$\wedge_H : C^\infty(\mathbf{A}, H_k(W_1)) \times C^\infty(\mathbf{A}, H_l(W_2)) \to C^\infty(\mathbf{A}, H_{k+l}(W_3)).$$

For $k = l = 0$ this pairing is an extension of F since it satisfies $\wedge_H \circ (\iota_H \otimes \iota_H) = \iota_H \circ F$. More generally for $s \in C^\infty(\mathbf{A}, H_k(W_1))$ and $t \in C^\infty(\mathbf{A}, H_l(W_2))$ the following Leibniz rule holds:

$$d_H(s \wedge_H t) = (d_H\, s) \wedge_H t + (-1)^k s \wedge_H (d_H\, t).$$

To construct an adjoint dynamic sequence we quote the relation between $H_k(W)$ and its dual space $H^k(W^*) := (H_k(W))^*$ which is given by the following observation which will also be proved later:

Proposition 5.14 *(Duality) For the sequence of $\mathfrak{co}(V)$-representation $H_k(W^*)$ we have natural isomorphisms*

$$\lrcorner : H^k(W^*) \otimes \Lambda^n V^* \xrightarrow{\cong} H_{n-k}(W^*).$$

Remark 5.15 For the natural pairing $F : W \otimes W^* \to \mathbb{R}$ the bilinear differential pairing $\wedge_H$ in degrees with $k + l = n$ is indeed zero order and induced by the above duality rewritten as: $H_k(W) \otimes H_{n-k}(W^*) = \Lambda^n V^*$.

The duality allows to twist the kinematic sequence d_H induced by W^* with the bundle of pseudoscalar $L^{-n} \otimes \Lambda^n V$ as in paragraph 1.24:

Definition 5.16 In analogy to the exterior divergence we define the *divergence operators associated to* W: $div_H : C^\infty(M, L^{-n} \otimes H^k(W^*)) \to C^\infty(M, L^{-n} \otimes H^{k-1}(W^*))$, by

$$(div_H\, a) \lrcorner \operatorname{or}^* := (-1)^{k+1}\, d_H(a \lrcorner \operatorname{or}^*),$$

where $or^* \in L^n \otimes \Lambda^n V^*$ is a parallel element of the (dual) space of pseudoscalars and the operator $d_H : C^\infty(M, H_{n-k}(W^*)) \to C^\infty(M, H_{n-k+1}(W^*))$ is induced by W^*, see theorem 5.12. Moreover, if W_1, W_2 and W_3 are three finite dimensional $\mathfrak{g}$-representations and $F : W_1 \otimes W_2 \to W_3$ is a linear $\mathfrak{g}$-equivariant map, then we define bilinear differential *interior pairings*

$$\lrcorner_H\, : C^\infty(\mathbf{A}, H_k(W_1)) \times C^\infty(\mathbf{A}, L^{-n} \otimes H^{k+l}(W_2)) \to C^\infty(\mathbf{A}, L^{-n} \otimes H^l(W_3)),$$

by

$$(\alpha \lrcorner_H a) \lrcorner \operatorname{or}^* := (-1)^{k(k+l+1)} \alpha \wedge_H (a \lrcorner \operatorname{or}^*).$$

Corollary 5.17 *The resulting sequence of differential operators* div_H

$$C^\infty(\mathbf{A}, L^{-n} \otimes H^0(W^*)) \overset{\mathrm{div}_H}{\longleftarrow} C^\infty(\mathbf{A}, L^{-n} \otimes H^1(W^*)) \overset{\mathrm{div}_H}{\longleftarrow} C^\infty(\mathbf{A}, L^{-n} \otimes H^2(W^*)) \overset{\mathrm{div}_H}{\longleftarrow} \ldots$$

defines a locally exact complex which for $W = \mathbb{R}$ *reduces to the exterior divergence. For the interior pairing* $\lrcorner_H$ *induced by* $F : W_1 \otimes W_2 \to W_3$ *we obtain from the Leibniz rule the following formula:*

$$(-1)^k\, div_H(\alpha \lrcorner_H a) = (d_H\, \alpha) \lrcorner_H a + \alpha \lrcorner_H (div_H\, a),$$

with $\alpha \in C^\infty(M, H_k(W_1))$ *and* $a \in C^\infty(M, L^{-n} \otimes H^{k+l}(W_2))$. *In particular applied to the natural pairing* $W \otimes W^* \to \mathbb{R}$ *with* $l = 1$ *the above formula shows that* d_H *of* W *and the above* $-\,\mathrm{div}_H$ *are adjoints.*

Proof: Unraveling the definitions and using the above two theorems gives

$$\begin{aligned} div_H(div_H(a)) \lrcorner_H \operatorname{or}^* &= -(-1)^{k-1}\, d_H(div(a) \lrcorner_H \operatorname{or}^*) \\ &= -\, d_H(d_H(a \lrcorner_H \operatorname{or}^*)) = 0. \end{aligned}$$

Similarly for the divergence formula note

$$\begin{aligned} &(-1)^k (div_H(\alpha \lrcorner_H a)) \lrcorner_H \operatorname{or}^* \\ &= -(-1)^{k+l}\, d_H((\alpha \lrcorner_H a) \lrcorner_H \operatorname{or}^*) \\ &= (-1)^{(k+1)(k+l+1)}\, d_H(\alpha \wedge_H (a \lrcorner_H \operatorname{or}^*)) \\ &= (-1)^{(k+1)(k+l+1)} (d_H\, \alpha) \wedge_H (a \lrcorner_H \operatorname{or}^*) + (-1)^{(k+1)(k+l+1)+k} \alpha \wedge_H d_H(a \lrcorner_H \operatorname{or}^*) \\ &= ((d_H\, \alpha) \lrcorner_H a) \lrcorner_H \operatorname{or}^* + (-1)^{k(k+l)} \alpha \wedge_H ((div_H\, a) \lrcorner_H \operatorname{or}^*) \\ &= ((d_H\, \alpha) \lrcorner_H a) \lrcorner_H \operatorname{or}^* + (\alpha \lrcorner_H (div_H\, a)) \lrcorner_H \operatorname{or}^*. \end{aligned}$$

For $F\colon W\otimes W^* \to \mathbb{R}$ the interior pairing $\lrcorner_H$ between $\beta \in C^\infty(\mathbf{A}, H_k(W))$ and $b \in C^\infty(\mathbf{A}, L^{-n}\otimes H^k(W^*))$ takes values in $C^\infty(\mathbf{A}, L^{-n})$ and is zero order and induced by the natural pairing $H_k(W)\otimes H^k(W^*) \to \mathbb{R}$. Hence the pairings $\lrcorner_H$ on the right hand side of the divergence formula for $l=1$ are simple contractions. $\square$

We will now apply the above results to put the ideas of the previous two sections 5.2 and 5.3 into proper definitions: *Dynamic helicity lowering* is based upon the contraction $W\otimes W^* \to \mathbb{R}$ inducing the following interior pairing:

$$\lrcorner_H : C^\infty(\mathbf{A}, H_k(W)) \times C^\infty(\mathbf{A}, L^{-n}\otimes H^{k+l}(W^*)) \to C^\infty(\mathbf{A}, L^{-n}\otimes \Lambda^l V).$$

Definition 5.18 (*General electric charge*) (Compare paragraphs 2.19 and 5.2) For a dynamic field $G \in C(\mathbf{A}, L^{-n}\otimes H^2(W^*))$ of the theory associated to W we define the *general electric charge* (represented by G) contained in an cooriented $(n-2)$-dimensional sphere $S^{n-2}\subset \mathbf{A}$ to be the following element in W^*:

$$W \to \mathbb{R}; \quad q \mapsto \int_{S^{n-2}} \langle q \lrcorner_H G, coor\rangle.$$

In regions where $div_H\, G = 0$ this charge satisfies the conservation law from the divergence theorem, see proposition 2.20. *Kinematic helicity lowering* is based upon the contraction $W^*\otimes W \to \mathbb{R}$ inducing the exterior pairing:

$$\wedge_H : C^\infty(\mathbf{A}, H_k(W^*)) \times C^\infty(\mathbf{A}, H_l(W)) \to C^\infty(\mathbf{A}, \Lambda^{k+l}V^*).$$

Definition 5.19 (*General magnetic charge*) (Compare paragraph 2.21) For a kinematic field $F \in C(\mathbf{A}, H_2(W))$ of the theory associated to W we define the *general magnetic charge* (represented by F) contained in an oriented 2-dimensional sphere $S^2\subset\mathbf{A}$ to be the following element in W:

$$W^* \to \mathbb{R}; \quad q \mapsto \int_{S^2} \langle q \wedge_H F, or\rangle.$$

In regions where $d_H\, F = 0$ this charge satisfies the conservation law from Stoke's theorem, see proposition 2.22.

Definition 5.20 (*General Lorentz force law*) (Compare paragraphs 2.14 and 5.7) The force of a kinematic field F felt by a test particle N with an attached dual twistor $q \in W^*$ (interpreted as general electric test charge) is given by $(q\wedge_H F)(N)$.

Definition 5.21 (*General particles as sources*) (Compare paragraph 5.6) A dual twistor $q \in W^*$ (a general electric charge) attached to an oriented worldline $t \mapsto c(t)$ induces a distributional source $j \in \mathcal{D}(\mathbf{A}, L^{-n}\otimes H^1(W^*)) = (C^\infty_o(\mathbf{A}, H_1(W)))^*$ by $\langle\!\langle j, \alpha\rangle\!\rangle := \int_{\mathbb{R}} (q \wedge_H \alpha)(\dot c)$.

The induced exterior pairing from scalar multiplication $W\otimes\mathbb{R}\to W$ allows *kinematic helicity raising*, as in paragraph 5.8, from electromagnetism to the general field theory associated to W:

$$\wedge_H : C^\infty(\mathbf{A}, H_k(W)) \otimes C^\infty(\mathbf{A}, \Lambda^l V^*) \to C^\infty(\mathbf{A}, H_{k+l}(W)).$$

This establishes Lienard Wiechert fields in general field theories. The interior pairing from scalar multiplication $W \otimes \mathbb{R} \to W$ allows *dynamic helicity raising*, as in paragraph 5.9, from electromagnetism to the general field theory associated to W:

$$\lrcorner_H : C^\infty(\mathbf{A}, H_k(W)) \otimes C^\infty(\mathbf{A}, L^{-n} \otimes \Lambda^{k+l} V) \to C^\infty(\mathbf{A}, L^{-n} \otimes H^l(W^*)).$$

5.5 Fierz's theory of gravity

The next three sections study in some detail the three linear field theories which are candidates for a conformally invariant linear field theory for gravity. To relate physical dimensions and central weights of tensors we will use fundamental constants as in example 1.8. The three theories are based upon the representations $W = \Lambda^k \hat{V}$ with $k = 1, 2, 3$. Hence we will prove special cases of the claims about sequences of differential operators and pairings quoted in the last section. The conformal differential geometry developed in chapters 2 and 4 is sufficient to develop these theories on an arbitrary conformal manifold rather than on the conformal affine space. The presence of curvature introduces some phenomena that do not arise in the flat case.

Twistor fields induced by elements in the defining representation $\hat{V}$ of the Möbius Lie algebra are the gravitational masses of Fierz's theory of gravity. Hence Fierz gravitational masses occur as sections f of the density bundle L^1 which lie in the kernel of the conformal Hessian $\mathit{Hesse}\, f = 0$, see proposition 4.17. The theory is based upon the following kinematic sequence of differential operators: (we will use the following short hand notations $T^0 := L^1 \otimes T^*M$ and $C(L^1) = C^\infty(M, L^1)$ etc.)

$$\begin{array}{ccccccccc} \text{charge} & & \text{gauge} & & \text{potential} & & \text{kinematic field} & & \\ \hat{V} & \to & C(L^1) & \overset{\mathit{Hesse}}{\to} & C(T^* \odot T^0) & \overset{\mathit{Coda}}{\to} & C(\Lambda^2 T^* \odot T^0) & \overset{\mathit{Fierz}}{\to} & C(\Lambda^3 T^* \odot T^0) \\ & & f & & A & & F & & \end{array}$$

The gauges are scalar densities of weight $+1$, the potentials are symmetric trace-free bilinear forms of weight -1, the kinematic fields are covector valued 2-forms of weight -2 which are alternating-free and trace-free. To summarize: $H_0(\hat{V}) = L^1$, $H_1(\hat{V}) = V^* \odot V$ and $H_2(\hat{V}) = \Lambda^2 V^* \odot V^0$. The gauge operator is second order and given by the conformal Hessian. Potential and kinematic field operator are first order Stein Weiss operators 4.32 and hence their conformal invariance is given by Fegan's result, theorem 4.36. We will define them explicitly in terms of a Weyl derivative D with symmetric and trace-free normalized Ricci curvature r_0^D: let X, Y, Z and U be vector fields, then

$$\begin{aligned} \mathit{Hesse}\, f(X,U) &:= \frac{1}{2}(D^2_{X,U} f + D^2_{U,X} f) - \frac{1}{n} \sum_i (D^2_{t_i,\theta^i} f)\, \mathsf{c}(X,U) + f\, \mathrm{r}_0^D(X,U), \\ \mathit{Coda}\, A(X,Y,U) &:= D_X A(Y,U) + \frac{1}{n-1} \sum_i D_{t_i} A(\theta^i, X)\, \mathsf{c}(Y,U) - \mathrm{sym}\,(X,Y), \\ \mathit{Fierz}\, F(X,Y,Z,U) &:= D_X F(Y,Z,U) - \frac{1}{n-2} \sum_i D_{t_i} F(Y,Z,\theta^i)\, \mathsf{c}(X,U) + \mathrm{cyc}\,(X,Y,Z). \end{aligned}$$

It is elementary to check that this defines a complex in a conformally flat $\mathrm{W}^{\mathsf{c}} = 0$ spacetime. However, in a curved background this is just a sequence of operators and the Weyl curvature is the obstruction for being a complex. The dynamic sequence is given by:

$$\begin{array}{ccccccc} 0 & & j & & G & & \\ C(L^{-n-1}) & \overset{divdiv}{\longleftarrow} & C(L^{-n}\otimes T\odot T^0) & \overset{Symdiv}{\longleftarrow} & C(L^{-n}\otimes\Lambda^2T\odot T^0) & \overset{Fierzdiv}{\longleftarrow} & C(L^{-n}\otimes\Lambda^3T\odot T^0) \\ & & \text{source} & & \text{dynamic field} & & \end{array}$$

The operators are explicitly defined by:

$$\begin{aligned} \mathit{Fierzdiv}\, Q(X,Y,U) &:= \sum_i D_{t_i} Q(\theta^i, X, Y, U) - \frac{1}{3}\sum_i D_{t_i} Q(X,Y,U,\theta^i), \\ \mathit{Symdiv}\, G(X,U) &:= \frac{1}{2}(\sum_i D_{t_i} G(\theta^i, X, U) + \sum_i D_{t_i} G(\theta^i, U, X)), \\ \mathit{divdiv}\, j &:= \sum_{i,j} (D^2_{t_i,t_j} j(\theta^i,\theta^j) + \mathrm{r}_0^D(t_i,t_j)\otimes j(\theta^i,\theta^j)). \end{aligned}$$

The constitutive relation in vacuum and in $n = 4$ dimensions is given by the conformal structure, which identifies kinematic and dynamic fields.

Remark 5.22 Note that the conservation law for the source is second order. This is a consequence of conformal invariance of the theory. In the original noninvariant version Fierz in [Fie39] used a first order divergence as a candidate for a conservation law. His sources given by $\mathit{Symdiv}\, G$, where $G = F = \mathit{Coda}\, A$ are only conserved $\mathit{div}^D\, \mathit{Symdiv}\, G = 0$ if the potential A satisfies an additional strong *gauge condition* $\mathit{div}^D\, A = 0$ (which is also not conformally invariant).

Discussion 5.23 (*Product rules for adjoints*) To see that those two sequences are adjoint to each other we observe the following pairings and divergence formulas. Note, that the central weights of the bundles determine the order of operators and pairings. The operators *Hesse* and *divdiv* are second order and the required pairing is a simple first order pairing, see proposition 4.42:

$$\begin{aligned} \lrcorner_H &: C^\infty(M, L^1)\times C^\infty(M, L^{-n}\otimes T\odot T^0) \to C^\infty(M, L^{-n}\otimes T), \\ f \lrcorner_H j &:= \sum_i D_{t_i} f\otimes j(\theta^i,\ ) - \sum_i f\otimes D_{t_i} j(\theta^i,\ ), \\ \mathit{div}(f\lrcorner_H j) &= \langle \mathit{Hesse}\, f, j\rangle + \langle f, -\,\mathit{divdiv}\, j\rangle. \end{aligned}$$

The pairing between potentials and dynamic fields is purely tensorial:

$$\begin{aligned} \lrcorner_H &: (T^*\odot T^0)\otimes(L^{-n}\otimes\Lambda^2T\odot T^0)\to L^{-n}\otimes T, \\ -\,\mathit{div}(A\lrcorner_H G) &= \langle \mathit{Coda}\, A, G\rangle + \langle A, \mathit{Symdiv}\, G\rangle. \end{aligned}$$

The pairing for the kinematic integrability operator is also tensorial:

$$\begin{aligned} \lrcorner_H &: (\Lambda^2T^*\odot T^0)\otimes(L^{-n}\otimes\Lambda^3T\odot T^0)\to L^{-n}\otimes T, \\ (-1)^2\,\mathit{div}(F\lrcorner_H J) &= \langle \mathit{Fierz}\, F, J\rangle + \langle F, \mathit{Fierzdiv}\, J\rangle. \end{aligned}$$

Discussion 5.24 (*Mass represented by dynamic Fierz fields*) To rediscover the gravitational mass, i.e. a Fierz twistor, represented by a dynamic Fierz field we need the following pairings and product rules: the pairing between gauges and dynamic Fierz fields is simple first order, see proposition 4.42:

$$\begin{aligned}
\lrcorner_H &: C^\infty(M, L^1) \times C^\infty(M, L^{-n} \otimes \Lambda^2 T \odot T^0) \to C^\infty(M, L^{-n} \otimes \Lambda^2 T), \\
f \lrcorner_H G &:= \sum_i (D_{t_i} f) G(\ ,\ , \theta^i) - \frac{1}{2} f \sum_i D_{t_i} G(\ ,\ , \theta^i), \\
\lrcorner_H &: (T^* \odot T^0) \otimes (L^{-n} \otimes \Lambda^2 T \odot T^0) \to (L^{-n} \otimes T), \\
\lrcorner_H &: C^\infty(M, L^1) \times C^\infty(M, L^{-n} \otimes T \odot T^0) \to C^\infty(M, L^{-n} \otimes T), \\
\mathit{div}(f \lrcorner_H G) &= \mathit{Hesse}\, f \lrcorner_H G + f \lrcorner_H \mathit{Symdiv}\, G - f \frac{1}{2} \sum_{i,j,k} G(\theta^i, \theta^j, \theta^k)\, \mathrm{W}^{\mathsf{c}}{}_{t_i, t_j}\, t_k.
\end{aligned}$$

Note that the third pairing needed is the first order pairing from the above paragraph 5.23 and that the appearance of the Weyl curvature tensor prevents a true codimension one divergence formula in the curved case. As was explained in paragraph 5.2 the first pairing together with a cooriented $(n-2)$-sphere S and a dynamic Fierz field G induces a linear form on the kernel of the conformal Hessian $\ker(\mathit{Hesse}) \ni f \mapsto \int_S \langle f \lrcorner_H G, \mathit{coor} \rangle$. In the affine case this linear form represents a Fierz twistor using the inner product from definition 3.9 on $\hat{V}$.

The next operators in the kinematic and dynamic sequences are true codimension one adjoints, see definition 5.3 even in the curved case, since the pairing is purely tensorial and the operators are first order:

$$\begin{aligned}
\lrcorner_H &: (T^* \odot T^0) \otimes (L^{-n} \otimes \Lambda^3 T \odot T^0) \to L^{-n} \otimes \Lambda^2 T, \\
-\mathit{div}(A \lrcorner_H J) &= \mathit{Coda}\, A \lrcorner_H J + A \lrcorner_H \mathit{Fierzdiv}\, J.
\end{aligned}$$

Discussion 5.25 (*Fierz particles as sources*) To view a Fierz twistor along a worldline as a pointlike source for gravity, as in paragraph 5.6, we need a pairing between Fierz potentials and Fierz gauges to give 1-forms, which is simple first order:

$$\begin{aligned}
\wedge_H &: C^\infty(M, L^1) \times C^\infty(M, T^* \odot T^0) \to C^\infty(M, T^*), \\
f \wedge_H A &:= f \otimes \mathit{div}^D A - (n-1)\, \mathit{grad}^D f \lrcorner A.
\end{aligned}$$

Given $f \in \ker(\mathit{Hesse})$ along an oriented worldline c we define a distributional source $j \in \mathcal{D}(M, L^{-n} \otimes T \odot T^0) = (C^\infty_o(M, T^* \odot T^0))^*$ by evaluating it on $\alpha \in C^\infty_o(M, T^* \odot T^0)$ as $\langle\!\langle j, \alpha \rangle\!\rangle := \int_{\mathbb{R}} (f \wedge_H \alpha)(\dot{c})\, dt$. To check that this source is conserved we need a pairing between Fierz gauges and Fierz gauges to give functions, which is a Laplace pairing, example 4.52:

$$\begin{aligned}
\wedge_H &: C^\infty(M, L^1) \times C^\infty(M, L^1) \to C^\infty(M, \mathbb{R}), \\
f \wedge_H g &:= \frac{n-1}{n} \left((\Delta^D f) \otimes g - n\, \mathsf{c}(Df, Dg) + f \otimes (\Delta^D g) \right).
\end{aligned}$$

To verify the product rule notice $(D^2 + \mathrm{r}^D) f = \mathit{Hesse}\, f + \frac{1}{n} (\Delta^D f) \mathsf{c}$ and recall the contracted second Bianchi identity, proposition 2.61: $D_X\, \mathrm{s}^D = \mathrm{tr}_{\mathsf{c}}\, D\, \mathrm{r}^D(X)$. Together this gives

$$D_X \Delta^D f = (2-n)\, \mathrm{r}^D(X, \mathit{grad}^D f) + \mathit{div}^D\, \mathit{Hesse}\, f(X) + \frac{1}{n} D_X \Delta^D f,$$

and finally

$$d(f \wedge_H g) = (\mathit{Hesse}\, f) \wedge_H g + f \wedge_H (\mathit{Hesse}\, g).$$

Discussion 5.26 (*Kinematic Fierz fields as accelerations*) The Lorentz force law in Fierz theory is based upon a pairing between Fierz gauges (gravitational test masses) and kinematic Fierz fields to give a 2-form. This pairing is simple first order:

$$\begin{aligned} \wedge_H \quad &: \quad C^\infty(M, L^1) \times C^\infty(M, \Lambda^2 T^* \odot T^0) \to C^\infty(M, \Lambda^2 T^*), \\ (f \wedge_H F)(X, Y) \quad &:= \quad (n-2) F(X, Y, \mathit{grad}^D f) - f \otimes \sum_i D_{t_i} F(X, Y, \theta^i). \end{aligned}$$

Discussion 5.27 (*Lienard Wiechert fields in Fierz's theory*) Given a solution of the Maxwell equation $F_{\mathrm{em}} = dA_{\mathrm{em}}$, like the Lienard Wiechert field, the following pairing between a Fierz twistor and a 1-form respectively a 2-form induces a Fierz potential respectively a Fierz field: the pairings are simple first order:

$$\begin{aligned} \wedge_H \quad &: \quad C^\infty(M, L^1) \times C^\infty(M, T^*) \to C^\infty(M, T^* \odot T^0), \\ (f \wedge_H A_{\mathrm{em}}) \quad &:= \quad (-2) A_{\mathrm{em}} \odot Df - f \otimes \mathit{Kill}^D A_{\mathrm{em}}, \\ \wedge_H \quad &: \quad C^\infty(M, L^1) \times C^\infty(M, \Lambda^2 T^*) \to C^\infty(M, \Lambda^2 T^* \odot T^0), \\ (f \wedge_H F_{\mathrm{em}}) \quad &:= \quad (-3) F_{\mathrm{em}} \odot Df - f \otimes \mathit{Twist}^D F_{\mathrm{em}}, \end{aligned}$$

where $\odot$ and Twist^D are defined in definition 4.23.

5.6 Bach's theory of gravity

Linearized conformal geometry and *Linearized Bach's theory of gravity* are the same in $n = 4$ dimensions. This theory is a linearization of Bach's conformal theory of gravity which we briefly recalled in remark 2.73. It is based upon the adjoint representation $W = \mathfrak{g}$, hence elements in W have the geometric interpretation as being conformal Killing fields and play the role of gravitational masses in Bach's theory of gravity. The first operator in the kinematic sequence is the conformal Killing operator:

$$\mathit{Kill} = \frac{1}{2}\, \mathcal{L}\mathsf{c} \colon C^\infty(M, TM) \to C^\infty(M, L^2 \otimes \mathrm{Sym}\,^2_0 T^*M),$$

with $H_0(\mathfrak{g}) := TM$ and $H_1(\mathfrak{g}) := L^2 \otimes \mathrm{Sym}\,^2_0 T^*M$. *Vector fields* play the role of gauges and the potentials $h \in C^\infty(M, H_1(\mathfrak{g}))$ are weightless symmetric bilinear forms and have the interpretation of *linearized conformal metrics*: if c_t denotes a family of conformal structures on M with $\mathsf{c}_0 := \mathsf{c}$ being the background conformal structure, then $\dot{\mathsf{c}} := \frac{\partial}{\partial t} \mathsf{c}_t|_{t=0}$ is trace-free with respect to c_0, since all c_t are normalized $|\det \mathsf{c}_t| = 1$. To understand the next operator in the kinematic sequence ($n \geq 4$) note that each conformal metric c_t has an associated Weyl curvature tensor $\mathrm{W}^{\mathsf{c}_t}$ and the linearization of this process $\mathsf{c}_t \mapsto \mathrm{W}^{\mathsf{c}_t}$ defines a differential operator $\mathit{LinWeyl}(\dot{c}) := \frac{\partial}{\partial t} \mathrm{W}^{\mathsf{c}_t}$ with

$$\mathit{LinWeyl} \colon C^\infty(M, L^2 \otimes \mathrm{Sym}\,^2_0 T^*M) \to C^\infty(M, \Lambda^2 T^* \odot \mathfrak{so}(TM)),$$

where $H_2(\mathfrak{g}) := \Lambda^2 T^*M \odot \mathfrak{so}(TM)$ is the bundle of Weyl tensors of central weight -2, where the kinematic fields take their values. This is the beginning of a complex only if the background metric is conformally flat, otherwise the Weyl curvature is an obstruction: if X is an arbitrary vector field with (local) flow Φ_t, then

$$\begin{aligned} \mathit{LinWeyl} \circ \mathit{Kill}(X) &= \frac{1}{2}\, \mathit{LinWeyl}(\frac{\partial}{\partial t}\Phi_t^* \mathsf{c}) = \frac{1}{2}\frac{\partial}{\partial t} \mathrm{W}^{\Phi_t^* \mathsf{c}} = \frac{1}{2}\frac{\partial}{\partial t}\Phi_t^*(\mathrm{W}^{\mathsf{c}_0}) \\ &= \frac{1}{2}\, \mathcal{L}_X \mathrm{W}^{\mathsf{c}_0} \end{aligned}$$

If the background conformal structure $\mathsf{c}_0 = \mathsf{c}$ has zero Weyl curvature $\mathrm{W}^{\mathsf{c}} = 0$ then this Lie derivative vanishes. The Weyl curvature tensor $\mathrm{W}^{\mathsf{c}_t}$ on a conformal manifold M, c_t satisfies a differential Bianchi identity $\mathit{Bianchi}_{\mathsf{c}_t}(\mathrm{W}^{\mathsf{c}_t}) = 0$, see propositions 2.64, 2.68. Here $\mathit{Bianchi}_{\mathsf{c}_t}$ is a linear conformally invariant operators which (as any invariant operator) depends upon the conformal structure (in a nonlinear way). Hence the kinematic integrability condition on Weyl tensors is given by the differential Bianchi identity: for $n \geq 5$ dimensions this is

$$\mathit{Bianchi}\colon C^\infty(M, \Lambda^2 T^* \odot \mathfrak{so}(TM)) \to C^\infty(M, \Lambda^3 T^* \odot \mathfrak{so}(TM)),$$

and in $n = 4$ dimension this is a second order operator:

$$\mathit{Bianchi}\colon C^\infty(M, \Lambda^2 T^* \odot \mathfrak{so}(TM)) \to C^\infty(M, \Lambda^3 T^* \odot T^*M).$$

For the composite $\mathit{Bianchi} \circ \mathit{LinWeyl}$ note that for all t we know $\mathit{Bianchi}_{\mathsf{c}_t} \mathrm{W}^{\mathsf{c}_t} = 0$ and differentiating gives:

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}(\mathit{Bianchi}_{\mathsf{c}_t} \mathrm{W}^{\mathsf{c}_t}) \\ &= \frac{\partial}{\partial t}\mathit{Bianchi}_{\mathsf{c}_t} \mathrm{W}^{\mathsf{c}_0} + \frac{\partial}{\partial t}\mathit{Bianchi}_{\mathsf{c}_0} \mathrm{W}^{\mathsf{c}_t} \\ &= (\frac{\partial}{\partial t}\mathit{Bianchi}_{\mathsf{c}_t})\mathrm{W}^{\mathsf{c}_0} + \mathit{Bianchi}_{\mathsf{c}_0} \circ \mathit{LinWeyl}(\dot{\mathsf{c}}). \end{aligned}$$

Hence we have $\mathit{Bianchi} \circ \mathit{LinWeyl}(\dot{\mathsf{c}}) = -(\frac{\partial}{\partial t}\mathit{Bianchi}_{\mathsf{c}_t})\, \mathrm{W}^{\mathsf{c}}$, which is zero in the flat case. The beginning of the kinematic sequence:

$$\begin{array}{ccccccccc} \text{Killing fields} & & \text{vector fields} & & \text{lin. conf. metrics} & & \text{Weyl fields} & & \\ \mathfrak{so}(\hat{V}) & \to & C(T) & \xrightarrow{\mathit{Kill}} & C(L^2 \otimes \mathrm{Sym}\,^2_0 T^*) & \xrightarrow{\mathit{LinWeyl}} & C(\Lambda^2 T^* \odot \mathfrak{so}(T)) & \xrightarrow{\mathit{Bianchi}} & , \end{array}$$

has and adjoint providing the end of the dynamic sequence:

$$\begin{array}{ccccc} C(L^{-n-2} \otimes T) & \xleftarrow{\mathit{div}} & C(L^{-n-2} \otimes \mathrm{Sym}\,^2_0 T) & \xleftarrow{\mathit{Bach}} & C(L^{-n} \otimes \Lambda^2 T \odot \mathfrak{so}(T)), \\ & & \text{source} & & \text{dynamic field} \end{array}$$

where the adjoint of *Kill* is a first order divergence operator *div* acting on symmetric bilinear forms of central weight $-n$ and the operator adjoint to *LinWeyl* is second order and called the *Bach operator*, see 2.70, acting on Weyl tensors of central weight $2 - n$. In $n = 4$

dimensions the constitutive relation in vacuum between kinematic and dynamic fields is simply the conformal structure, since both fields are Weyl tensors of central weight -2.

For convenience we will write down the linearized Weyl operator which is simple second order, proposition 4.51, and associated to $E := V^* \odot V$, $F := \Lambda^2 V^* \odot V$, $G := \Lambda^2 V^* \odot \mathfrak{so}(V)$: the aim is to make the projection $\pi_W : V^* \otimes V^* \otimes E \to G$ explicit. For this we write π_W as a composition $\pi_W := \pi_L \circ (id_{V^*} \otimes \pi_C)$ with $\pi_C : V^* \otimes E \to F$ and $\pi_L : V^* \otimes F \to G$. The Stein Weiss operator of these projections are called *Codazzi operator* and *Lanczos operator* and are given by (X, Y, U, V are vector fields):

$$\begin{aligned}
\mathit{Coda}^D\, h(X,Y,U) &:= \pi_C \circ Dh(X,Y,U) \\
&= D_X h(Y,U) + \frac{1}{n-1} \sum_i D_{t_i} h(\theta^i, X)\, \mathsf{c}(Y,U) - \mathrm{sym}\,(X,Y), \\
\mathit{Lanc}^D\, A(X,Y,U,V) &:= \pi_L \circ DA(X,Y,U,V) \\
&= D_X A(U,V,Y) - D_Y A(U,V,X) + D_U A(X,Y,V) - D_V A(X,Y,U) \\
&\quad - \frac{2}{n-2}\Big(\mathit{Symdiv}^D\, A(Y,V)\, \mathsf{c}(X,U) - \mathit{Symdiv}^D\, A(X,V)\, \mathsf{c}(Y,U) \\
&\qquad - \mathit{Symdiv}^D\, A(Y,U)\, \mathsf{c}(X,V) + \mathit{Symdiv}^D\, A(X,U)\, \mathsf{c}(Y,V)\Big), \\
\mathit{Symdiv}^D\, A(X,U) &:= \frac{1}{2}(\mathrm{tr}\, DL(\ ,X,U) + \mathrm{tr}\, DL(\ ,U,X)), \\
\mathit{LinWeyl}\, h &:= \pi_L (id \otimes \pi_C)(D^2 h + \mathrm{r}^D \otimes h).
\end{aligned}$$

The Bach operator is adjoint to this and therefore given by:

$$\mathit{Bach}^D\, W(Y,U) := \sum_{i,j} (D^2_{t_i,t_j} + \mathrm{r}^D(t_i,t_j)) W(\theta^i, Y, U, \theta^j) - \mathrm{sym}\,(Y,U).$$

Discussion 5.28 (*Product rules for adjoints*) To see that those two sequences are adjoint to each other we remark that first order Stein Weiss operators like *Kill* always have an obvious divergence formula, see remark 4.33. For convenience we will write out the tensorial pairing between vector fields and Bach sources:

$$\begin{aligned}
\lrcorner_H &: T \otimes (L^{-n-2} \otimes T \odot T) \to L^{-n} \otimes T, \\
f \lrcorner_H j &:= \sum_i \mathsf{c}(f, t_i) j(\theta^i,\ ), \\
\mathit{div}(f \lrcorner_H j) &= \langle \mathit{Kill}\, f, j \rangle + \langle f, \mathit{div}\, j \rangle.
\end{aligned}$$

The second order operators *LinWeyl* and *Bach* need the following simple first order pairing:

$$\begin{aligned}
\lrcorner_H &: C(M, L^2 \otimes T^* \odot T^*) \times C(M, L^{-n} \otimes \Lambda^2 T \odot \mathfrak{so}(T)) \to C(M, L^{-n} \otimes T), \\
h \lrcorner_H W &:= \frac{1}{2} \sum_{i,j,k} h(t_j,t_k) D_{t_i} W(\theta^j, \theta^k)\theta^i - D_{t_i} h(t_j,t_k) W(\theta^j,\theta^k)\theta^i, \\
-\,\mathit{div}(h \lrcorner_H W) &= \langle \mathit{LinWeyl}\, h, W \rangle + \langle h, -\,\mathit{Bach}\, W \rangle.
\end{aligned}$$

Discussion 5.29 (*Mass represented by dynamic Bach fields*) To rediscover the gravitational mass, a Killing field, represented by a dynamic Bach field, a Weyl tensor, we need the following pairings and product rules: the pairing between gauges and dynamic Bach fields is simple first order:

$$\begin{aligned}
\lrcorner_H \quad &: \quad C^\infty(M,T) \times C^\infty(M, L^{-n} \otimes \Lambda^2 T \odot \mathfrak{so}(T)) \to C^\infty(M, L^{-n} \otimes \Lambda^2 T),\\
f \lrcorner_H W \quad &:= \quad \sum_{i,j} D_{t_i} f(\theta^j)\langle W(\ ,\ ,t_j), \theta^i\rangle - 2f(\theta^j)\langle D_{t_i} W(\ ,\ ,t_j), \theta^i\rangle,\\
\lrcorner_H \quad &: \quad C^\infty(M, L^2 \otimes T^* \odot T^*) \times C^\infty(M, \Lambda^2 T^* \odot \mathfrak{so}(T)) \to C^\infty(M, L^{-n} \otimes T),\\
\lrcorner_H \quad &: \quad T \otimes (\Lambda^2 T^* \odot \mathfrak{so}(T)) \to (L^{-n} \otimes T),\\
div(f \lrcorner_H W) \quad &= \quad Kill\, f \lrcorner_H W + f \lrcorner_H (-\,Bach\, W) + \langle W \otimes K, \mathrm{W}^{\mathfrak{c}}\rangle.
\end{aligned}$$

Note that the second pairing is the first order pairing from the above paragraph 5.28. The appearance of the Weyl curvature tensor prevents a true codimension one divergence formula in the curved case. As was explained in paragraph 5.2 the first pairing together with a cooriented $(n-2)$-sphere S and a dynamic Bach Weyl field W induces a linear form on the kernel of the conformal Killing equation $\ker(Kill) \ni f \mapsto \int_S \langle f \lrcorner_H W, coor\rangle$. In the affine case this linear form represents a Killing field using the Killing form on $\mathfrak{so}(\hat{V})$.

Discussion 5.30 (*Bach particles as sources*) To view a Killing field along a worldline as a pointlike source for gravity we need a pairing between vector fields and linearized metrics to give 1-forms:

$$\wedge_H : C^\infty(M,T) \times C^\infty(M, L^2 \otimes T^* \odot T^*) \to C^\infty(M, T^*).$$

This pairing is second order and not simple. Its existence follows from the general theory, theorem 5.13, applied to $\mathfrak{so}(\hat{V}) \otimes \mathfrak{so}(\hat{V}) \to \mathbb{R}$. To check that this source is conserved we need the Branson pairing, example 4.53, with $k = 1$.

Discussion 5.31 (*Kinematic Bach fields as accelerations*) The Lorentz force law in Bach's theory is based upon a pairing between Killing fields and kinematic Bach fields to give a 2-form. This pairing is simple first order:

$$\begin{aligned}
\wedge_H \quad &: \quad C^\infty(M,T) \times C^\infty(M, \Lambda^2 T^* \odot \mathfrak{so}(T)) \to C^\infty(M, \Lambda^2 T^*),\\
(f \wedge_H W)(X,Y) \quad &:= \quad (n-2)F(X,Y,\mathrm{grad}^D f) - f \otimes \sum_i D_{t_i} F(X,Y,\theta^i).
\end{aligned}$$

Discussion 5.32 (*Lienard Wiechert fields in Bach's theory*) Given a solution of the Maxwell equation $F = dA$, like the Lienard Wiechert field, a tensorial pairing between a Killing field and a 1-form induces a Bach potential:

$$\begin{aligned}
\wedge_H \quad &: \quad T \otimes T^* \to L^2 \otimes T^* \odot T^*,\\
(f \wedge_H A) \quad &:= \quad f \odot A.
\end{aligned}$$

5.7 Penrose's theory of gravity

Twistor fields induced by elements in the representation on trivectors $\Lambda^3\hat{V}$ of the Möbius Lie algebra $\mathfrak{so}(\hat{V})$ are the gravitational masses of Penrose's theory of gravity. Hence Penrose gravitational masses are 2-forms f of central weight $+1$ in the kernel of the Twistor operator $\mathit{Twist}\, f = 0$, see definition 4.23. The theory is based upon the following kinematic sequence:

$$\begin{array}{ccccccc} \text{charge} & & \text{gauge} & & \text{potential} & & \text{kinematic field} \\ \Lambda^3\hat{V} & \to & C(\Lambda^2 T^0 \otimes L^1) & \stackrel{\mathit{Twist}}{\to} & C(T^* \odot \Lambda^2 T^0 \otimes L^1) & \stackrel{\mathit{Lanc}}{\to} & C(\Lambda^2 T^* \odot \Lambda^2 T^0 \otimes L^1) \;\to \\ & & f & & A & & F \end{array}$$

The kinematic fields are Weyl tensors of central weight -1. The two operators above are first order Stein Weiss operators, see definition 4.32, and were already defined in the last section 5.6. It is a beginning of a complex in the flat case. They have natural adjoints:

$$\begin{array}{ccccc} & & j & & G \\ C(L^{-n-1} \otimes \Lambda^2 T^0) & \stackrel{\mathit{Skewdiv}}{\leftarrow} & C(L^{-n-1} \otimes T \odot \Lambda^2 T^0) & \stackrel{\mathit{div}}{\leftarrow} & C(L^{-n-1} \otimes \Lambda^2 T \odot \Lambda^2 T^0) \;\leftarrow \\ & & \text{source} & & \text{dynamic field} \end{array}$$

Note the the dynamic field is a Weyl tensor of central weight -3, whereas the kinematic field is a Weyl tensor of central weight -1. Hence the constitutive relation in vacuum is more involved, since we can't use the conformal metric to identify kinematic and dynamic fields as in electromagnetism or Fierz's and Bach's theory. Theorem 5.12 yields in $n = 4$ dimensions an additional third order operator from potential to fields, since $H_2(\Lambda^3\hat{V})$ splits into a direct sum, see example 6.30:

$$H_2(\Lambda^3\hat{V}) = \Lambda^2 V^* \odot \Lambda^2 V^0 \otimes (L^1 \oplus L^{-1}).$$

Hence in $n = 4$ dimensions there is an invariant third order differential operator

$$\mathit{UltraLanc}\colon C^\infty(M, T^* \odot \Lambda^2 T^0 \otimes L^1) \to C^\infty(M, \Lambda^2 T^* \odot \Lambda^2 T^0 \otimes L^{-1}),$$

which annihilates the image of *Twist* in the flat case, i.e. $\mathit{UltraLanc} \circ \mathit{Twist} = 0$. This allows to suggest that kinematic and dynamic Weyl field F and G come from the same kinematic potential $A \in C^\infty(M, T^* \odot \Lambda^2 T^0 \otimes L^1)$ using

$$\begin{array}{rcl} F & := & \mathit{Lanc}\, A, \\ G & := & \mathit{UltraLanc}\, A, \end{array}$$

which can be interpreted as a conformally invariant constitutive relation in $n = 4$ dimensions, since it does not depend upon the choice of the potential. This third order operator is related to the kinematic integrability condition for F, which is also third order, since by example 6.30: $H_3(\Lambda^3\hat{V}) = \Lambda^3 V^* \odot \Lambda^2 V^0 \otimes L^{-1}$, which has central weight -4.

Discussion 5.33 (*Mass represented by dynamic Penrose Weyl fields*) To rediscover the gravitational mass, i.e. a Penrose twistor, represented by a dynamic Penrose Weyl field we just need a zero order pairing:

$$\begin{aligned}
\lrcorner_H \quad &: \quad (\Lambda^2 T^0 \otimes L^1) \otimes L^{-1-n} \otimes \Lambda^2 T \odot \Lambda^2 T^0) \to L^{-n} \otimes \Lambda^2 T, \\
f \lrcorner_H G \quad &:= \quad \frac{1}{2} \sum_{i,j} f(t_i, t_j) G(\ ,\ , \theta^i, \theta^j), \\
div(f \lrcorner_H G) \quad &= \quad Twist\, f \lrcorner_H G + f \lrcorner_H\, div\, G.
\end{aligned}$$

This divergence formula and related product rules based upon zero order pairings were first observed by Penrose [PR84]. As was explained in paragraph 5.2 this pairing together with a cooriented $(n-2)$-sphere S and a dynamic Penrose Weyl field G induces a linear form on the kernel of the twistor operator $\ker(Twist) \ni f \mapsto \int_S \langle f \lrcorner_H G, coor\rangle$. In the affine case this linear form represents a twistor using the inner product on $\Lambda^3 \hat{V}$.

Discussion 5.34 (*Penrose particles as sources*) To view a Penrose twistor along a worldline as a pointlike source for gravity, as in paragraph 5.6, we need a pairing between Penrose gauges and Penrose potentials to give 1-forms, which is second order, but not simple. We refer to the general theory, theorem 5.13, applied to $\Lambda^3 \hat{V} \otimes \Lambda^3 \hat{V} \to \mathbb{R}$:

$$\wedge_H \quad : \quad C^\infty(M, \Lambda^2 T^0 \otimes L^1) \times C^\infty(M, T^* \odot \Lambda^2 T^0 \otimes L^1) \to C^\infty(M, T^*).$$

Given $f \in \ker(Twist)$ along an oriented worldline c we define a distributional source $j \in \mathcal{D}(M, L^{-n-1} \otimes T \odot \Lambda^2 T^0) = (C_o^\infty(M, T^* \odot \Lambda^2 T^0 \otimes L^1)^*$ by evaluating it on α as $\langle\!\langle j, \alpha \rangle\!\rangle := \int_{\mathbb{R}} (f \wedge_H \alpha)(\dot{c})\, dt$. The product rule which ensures conservation of this source involves a pairing between Penrose gauges and Penrose gauges to give functions, which is a Branson pairing, example 4.53.

Discussion 5.35 (*Kinematic Penrose Weyl fields as accelerations*) The Lorentz force law in Penrose's theory is based upon a pairing between Penrose gauges (gravitational test masses) and kinematic Penrose Weyl fields to give a 2-form. This pairing is second order and not simple. We refer to the general theory, theorem 5.13, applied to $\Lambda^3 \hat{V} \otimes \Lambda^3 \hat{V} \to \mathbb{R}$:

$$\wedge_H \quad : \quad C^\infty(M, \Lambda^2 T^0 \otimes L^1) \times C^\infty(M, \Lambda^2 T^* \odot \Lambda^2 T^0 \otimes L^1) \to C^\infty(M, \Lambda^2 T^*).$$

Discussion 5.36 (*Lienard Wiechert fields in Penrose's theory*) Given a solution of the Maxwell equation $F = dA$, like the Lienard Wiechert field, a tensorial pairing between a Penrose twistor and a 1-form respectively a 2-form induces a Penrose potential respectively a kinematic Penrose field:

$$\begin{aligned}
\wedge_H \quad &: \quad (\Lambda^2 T^0 \otimes L^1) \otimes T^* \to (T^* \odot \Lambda^2 T^0 \otimes L^1), \\
\wedge_H \quad &: \quad (\Lambda^2 T^0 \otimes L^1) \otimes \Lambda^2 T^* \to (\Lambda^2 T^* \otimes \Lambda^2 T^0 \otimes L^1).
\end{aligned}$$

Remark 5.37 (*Schwarzschild Weyl field*) Penrose's linear theory of gravity provides a direct link to Newton's theory of gravity via the Weyl tensor of the Schwarzschild solution. The Schwarzschild geometry in Einstein's theory is static and spherically symmetric. It's Riemann curvature tensor only has a Weyl curvature part. Linearizing the Schwarzschild solution (by the mass parameter) with Minkowski background we obtain a tensor with the symmetries of a Weyl tensor in Minkowski space. It is divergence-free by the contracted second Bianchi identity 2.61. On the other hand, let $c(t) := c_0 + tN$ be the straight worldline in the affine space representing the pointlike gravitational source. The Schwarzschild Weyl field is (up to scale) uniquely determined to be the dynamic Penrose Weyl field G_{Schw} which is static and spherically symmetric and satisfies $div_H(G_{\text{Schw}}) = 0$ away from the worldline c:

$$G_{\text{Schw}} = \frac{M}{r^3}\,(\vec{r} \wedge N) \odot (\vec{r} \wedge N),$$

where $M \in \mathbb{R}$ is a parameter interpreted as gravitational mass of the source, r is the luminosity distance to the source, $\vec{r} := grad^D\, r$ is the spacelike radial vector field and D is the affine derivative. The Schwarzschild Weyl field falls off like $1/r^3$ in agreement with the central weight of G_{Schw} being -3. It has a potential A_{Schw} and a kinematic field F_{Schw} given by

$$\begin{aligned}
A_{\text{Schw}} &= M(\vec{r} \wedge N) \odot N, \\
F_{\text{Schw}} &= Lanc\, A_{\text{Schw}} = \frac{M}{r}\,(\vec{r} \wedge N) \odot (\vec{r} \wedge N), \\
G_{\text{Schw}} &= UltraLanc\, A_{\text{Schw}}.
\end{aligned}$$

Note that c is in particular a conformal geodesic and along a conformal geodesic any Penrose twistor in $\Lambda^3 \hat{V}$ induces a constant real number along c according to proposition 4.25. The inner product on $\Lambda^3 \hat{V}$ turns this linear form on $\Lambda^3 \hat{V}$ into a Penrose twistor f_{Schw} induced by c. This twistor is up to scale the unique static and spherically bivector field in the kernel of the twistor operator *Twist* given by the linear polynomial

$$f_{\text{Schw}} := r\,(\vec{r} \wedge N).$$

Obviously it is this twistor which can be used to raise the Coulomb potential and field to the Schwarzschild potential and kinematic field: $A_{\text{Schw}} = A_{\text{Coul}} \odot (M f_{\text{Schw}})$ with $A_{\text{Coul}} = \frac{1}{r}\,N$ and $F_{\text{Schw}} = F_{\text{Coul}} \odot (M f_{\text{Schw}})$ with $F_{\text{Coul}} = \frac{1}{r^2}\,\vec{r} \wedge N$. To calculate the predicted force of F_{Schw} on a test particle we assume a multiple of f_{Schw} to be the gravitational mass of the test particle, like $m f_{\text{Schw}}$, with $m \in \mathbb{R}$. Although we haven't calculated the relevant second order pairing $(m f_{\text{Schw}}) \wedge_H F_{\text{Schw}}$ we know that the resulting closed 2-form is static and spherically symmetric, hence a multiple of F_{Coul}:

$$(m f_{\text{Schw}}) \wedge_H F_{\text{Schw}} = \frac{mM}{r^2}\,\vec{r} \wedge N.$$

(The second order pairing is in this case only first order, since f_{Schw} is linear, which also explains why the force falls off like $1/r^2$.) Hence the general Lorentz force law from 5.35 specializes to the Newtonian force law in the spherically symmetric case. Paragraph 2.17 applies also to this gravitational force law and predicts a perihelion advance in this linear theory of gravity.

Chapter 6

Bernstein Gelfand Gelfand theory

In this chapter we develop the Bernstein Gelfand Gelfand resolution together with its coproduct. This chapter is algebraic in flavour and the link to differential geometry is given in the first section 6.1. For a Lie group G with closed subgroup $P \subset G$, we define an invariant jet operator, see 6.9, acting on sections of a homogeneous bundle over the homogeneous space G/P. This jet operator encodes the information of all derivatives of a section at a point into an algebraic object, a linear form on the *induced module*. This jet operator allows to translate differential operators into homomorphisms and differential pairings into coproducts between induced modules. From section 6.2 on we deal with a semisimple G and a parabolic P. Parabolic subgroups contain a further subgroup $L \subset P \subset G$ which can be thought of as the structure group of the reduced frame bundle of G/P. For instance for the conformal sphere G/P we have $L := \mathbf{CO}(V)$ and $P := \mathbf{CO}(V) \ltimes V^*$, see section 3.4. Note that in general a P-representation leads to vector bundle over G/P which is not associated to the L-reduction of the linear frame bundle of G/P. Only those P-representations on which V^* acts trivially lead to tensor bundles. The most substantial piece of representation theory that we will need is relative Lie algebra homology. The elementary definitions and properties are presented in section 6.3 with detailed proofs. If $\mathfrak{p} \subset \mathfrak{g}$ denote the Lie algebras with $V := \mathfrak{g}/\mathfrak{p}$, then the chains of the homology theory are $\Lambda^k V \otimes W$, where W is a G-representation. The homologies $H_k(W)$ are examples of P-representations on which V^* acts trivially. In section 6.4 we twist the exterior deRham complex by a representation W of G. The resulting complex is too big to be geometrically relevant, since V^* acts nontrivially on $\Lambda^k V^* \otimes W$. In section 6.5 we construct a projection S onto the homologies and the BGG complex between sections in the homology occurs as homotopy equivalent to the twisted deRham complex between sections of the chains. Adjoint operators can also be studied purely algebraically, which we will do in section 6.7.

6.1 Verma modules and the holonomic jet operator

Let G denote a (real) Lie group with closed subgroup $P \subset G$ and Lie algebras $\mathfrak{g} \subset \mathfrak{p}$. We are interested in invariant differential operators and pairings on the following manifold:

Definition 6.1 (*Homogeneous model*) The quotient G/P where P acts as subgroup on G

from the right is called a *Kleinian* or *homogeneous model*.

The differential operators and pairings act between *homogeneous vector bundles*: any P-representation E from the right leads to a bundle $G \times_P E := G \times E/\sim$ where pairs $(g, e) \sim (gp, e.p)$ in the orbit of $p \in P$ are identified. Sections of that bundle can be viewed as P-equivariant functions on G with values in E: $C^\infty(G, E)^P := \{e \in C^\infty(G, E) \,|\, e(gp) = e(g).p\}$. On these bundles we have various group action which we need to distinguish:

Definition 6.2 (*Regular right G-actions*) The map $R_h : C^\infty(G, E) \to C^\infty(G, E)$ for $h \in G$ defined on functions e by right multiplication $R_h e(g) := e(gh)$ is called the *regular right action*. This is a G-action from the left:

$$R : G \times C^\infty(G, E) \to C^\infty(G, E).$$

Definition 6.3 (*Induced left P-action*) A function e is P-equivariant, if $R_p e = e.p$ for all $p \in P$. Hence e is P-equivariant, if it is invariant under the following *induced left P-action*:

$$P \times C^\infty(G, E) \to C^\infty(G, E),$$

defined by $p.e := R_p e.p^{-1}$, i.e. $(p.e)(g) := e(gp).p^{-1}$.

Definition 6.4 (*Regular left G-action*) The map $L_h : C^\infty(G, E) \to C^\infty(G, E)$ for $h \in G$ defined on functions e by left multiplication $L_h e(g) := e(hg)$ is called the *regular left action*. This is a G-action from the right:

$$L : C^\infty(G, E) \times G \to C^\infty(G, E).$$

Remark 6.5 Regular right and left action commute, hence if E is a right P-representation, and a section e is P-equivariant, then so is $L_h(e)$. I.e. we have a right G-action on sections of the homogeneous bundle:

$$L : C^\infty(G, E)^P \times G \to C^\infty(G, E)^P.$$

Associated to a Lie algebra $\mathfrak{g}$ is the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$, which is associative and has a unit (see Appendix). The algebra $\mathfrak{U}(\mathfrak{g})$ can be interpreted as the algebra of scalar differential operators on G which are equivariant under the regular left G-action. This is how we will use it in proposition 6.9, when we define a universal jet operator on G/P, which is the aim of the next paragraphs.

The Lie algebra $\mathfrak{g}$ acts upon $\mathfrak{U}(\mathfrak{g})$ by multiplication from the left, from the right and by conjugation:

$$\begin{aligned} l : \mathfrak{g} \otimes \mathfrak{U}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g}) \quad &; \quad l_X(U) = XU, \\ r : \mathfrak{g} \otimes \mathfrak{U}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g}) \quad &; \quad r_X(U) = -UX, \\ \mathrm{ad} : \mathfrak{g} \otimes \mathfrak{U}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g}) \quad &; \quad \mathrm{ad}_X(U) = (l_X - r_X)(U) = XU - UX. \end{aligned}$$

In what follows it is the left action which is *the* action of $\mathfrak{g}$ on $\mathfrak{U}(\mathfrak{g})$ and other actions will be mentioned explicitly.

Definition 6.6 (*Induced modules or Verma modules*) If $\mathfrak{p}$ is a subalgebra of $\mathfrak{g}$ and E^* is a left $\mathfrak{p}$-module, then $\mathfrak{U}(\mathfrak{g}) \otimes E^*$ contains an *induced left* $\mathfrak{U}(\mathfrak{g})$ *ideal* generated by

$$I(\mathfrak{g}, \mathfrak{p}, E^*) := < X \otimes \eta - 1 \otimes X.\eta \,|\, X \in \mathfrak{p}, \eta \in E^* >.$$

The quotient

$$\mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, E^*) := \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{p})} E^* := \mathfrak{U}(\mathfrak{g}) \otimes E^* / I(\mathfrak{g}, \mathfrak{p}, E^*),$$

is called the *induced left* $\mathfrak{g}$*-module* by the left $\mathfrak{p}$-module E^*. The action of $\mathfrak{g}$ on $\mathfrak{U}(\mathfrak{g})$ by multiplication from the left clearly leaves the ideal invariant and descends to a left $\mathfrak{g}$-action on $\mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, E^*)$.

Definition 6.7 (*P-actions on Verma modules*) Let $P \subset G$ be Lie groups with Lie algebras $\mathfrak{p} \subset \mathfrak{g}$. The group G acts upon $\mathfrak{U}(\mathfrak{g})$ by the adjoint action, which corresponds to the action of $\mathfrak{g}$ by conjugation on $\mathfrak{U}(\mathfrak{g})$. If E^* is a left P-representation, then P as subgroup acts upon $\mathfrak{U}(\mathfrak{g})$ by the adjoint action and hence on the tensor product:

$$\begin{gathered} \mathrm{Ind}\colon P \times \mathfrak{U}(\mathfrak{g}) \otimes E^* \to \mathfrak{U}(\mathfrak{g}) \otimes E^*, \\ \mathrm{Ind}_p(U \otimes \eta) := \mathrm{Ad}_p(U) \otimes p.\eta. \end{gathered}$$

This action leaves the induced ideal $I(\mathfrak{g}, \mathfrak{p}, E^*)$ invariant, since it maps generators to generators: $\mathrm{Ind}_p(X \otimes \eta - 1 \otimes X.\eta) = (\mathrm{Ad}_p(X) \otimes p.\eta - 1 \otimes p.X.\eta) = (\mathrm{Ad}_p(X) \otimes p.\eta - 1 \otimes \mathrm{Ad}_p(X).p.\eta)$. Hence the action descends to the quotient

$$P \times \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, E^*) \to \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, E^*).$$

Remark 6.8 Infinitesimally we get a left $\mathfrak{p}$-action

$$\begin{gathered} \mathrm{ind}\colon \mathfrak{p} \otimes \mathfrak{U}(\mathfrak{g}) \otimes E^* \to \mathfrak{U}(\mathfrak{g}) \otimes E^*, \\ \mathrm{ind}_X(U \otimes \eta) := \mathrm{ad}_X(U) \otimes \eta + U \otimes X.\eta, \end{gathered}$$

leaving the ideal $I(\mathfrak{g}, \mathfrak{p}, E^*)$ invariant: $\mathrm{ind}_X(Y \otimes \eta - 1 \otimes Y.\eta) = (\mathrm{ad}_X(Y) \otimes \eta + Y \otimes X.\eta - 1 \otimes X.Y.\eta) = ([\,X, Y\,] \otimes \eta - 1 \otimes [\,X, Y\,].\eta) + (Y \otimes X.\eta - 1 \otimes Y.X.\eta)$. The corresponding $\mathfrak{p}$-action on $\mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, E^*)$ is indeed the restriction of the $\mathfrak{g}$-action by left multiplication, since

$$\begin{aligned} \mathrm{ind}_X(U \otimes_{\mathfrak{U}(\mathfrak{p})} \eta) &= (XU - UX) \otimes_{\mathfrak{U}(\mathfrak{p})} \eta + U \otimes_{\mathfrak{U}(\mathfrak{p})} X.\eta \\ &= XU \otimes_{\mathfrak{U}(\mathfrak{p})} \eta. \end{aligned}$$

Following Weingart [Wei99] we will use the induced module $\mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, E^*)$ by a right P-representation E to define a *jet operator* for sections of the homogeneous bundle $G \times_P E$ over G/P: the left action on functions $R\colon G \times C^\infty(G, E) \to C^\infty(G, E)$; $R_h e(g) := e(gh)$, induces a left representation of the Lie algebra $\mathfrak{g}$ and hence a left action R_X by elements of the universal enveloping algebra $X \in \mathfrak{U}(\mathfrak{g})$:

$$R\colon \mathfrak{U}(\mathfrak{g}) \otimes C^\infty(G, E) \to C^\infty(G, E).$$

Proposition 6.9 *(Jet operator) The linear differential operator:*

$$jet^\infty : C^\infty(G,E) \to C^\infty(G,(\mathfrak{U}(\mathfrak{g})\otimes E^*)^*)$$

given by

$$\langle jet^\infty(e)(g), U\otimes\eta\rangle := \langle (R_U e)(g),\eta\rangle = (R_U\langle e,\eta\rangle)(g)$$

is G-equivariant under the regular action and P-equivariant under the induced left action. Its restriction to P-invariant sections vanishes on the induced ideal $I(\mathfrak{g},\mathfrak{p},E^*)$:

$$jet^\infty : C^\infty(G,E)^P \to C^\infty(G,\mathrm{Verma}\,(\mathfrak{g},\mathfrak{p},E^*)^*)^P.$$

Proof: The operator jet^∞ is G-equivariant since regular left and right multiplication commute: $L_h\circ R_U = R_U\circ L_h$ for $h\in G$. To show P-equivariance note for $p\in P$:

$$\begin{aligned}
\langle p.(jet^\infty(e))(g), U\otimes\eta\rangle &= \langle (jet^\infty(e))(gp).p^{-1}, U\otimes\eta\rangle\\
&= \langle (jet^\infty(e))(gp), \mathrm{Ad}_{p^{-1}}U\otimes p^{-1}.\eta\rangle\\
&= \langle (R_{\mathrm{Ad}_{p^{-1}}U}e)(gp), p^{-1}.\eta\rangle\\
&= \langle R_U(R_p e)(g), p^{-1}.\eta\rangle\\
&= \langle R_U((R_p e).p^{-1})(g), \eta\rangle\\
&= \langle R_U(p.e)(g), \eta\rangle\\
&= \langle jet^\infty(p.e)(g), U\otimes\eta\rangle.
\end{aligned}$$

A function $e\in C^\infty(G,E)$ is P-equivariant, if $R_p e = e.p$ for all $p\in P$, which infinitesimally means $R_X e = e.X$ for $X\in\mathfrak{p}$. That shows $\langle jet^\infty(e)(g), I(\mathfrak{g},\mathfrak{p},E^*)\rangle = 0$. □

Using this operator, we will present next the result, independently due to Baston Eastwood Rice [BE89], [ER87], Collingwood Shelton [CS90] and Soergel [Soe90], that G-equivariant linear differential operators are in one to one correspondence with $\mathfrak{g}$-equivariant homomorphisms between the induced Verma modules:

Discussion 6.10 (*Correspondence between operators and homomorphisms*) Let E and F be two right P-representations. The above jet operator translates $\mathfrak{g}$-equivariant homomorphisms between the induced Verma modules

$$\nabla : \mathrm{Verma}\,(\mathfrak{g},\mathfrak{p},F^*) \to \mathrm{Verma}\,(\mathfrak{g},\mathfrak{p},E^*)$$

into a G-equivariant differential operator

$$\nabla : C^\infty(G,E)^P \to C^\infty(G,F)^P$$

as follows: if $e\in C^\infty(G,E)$ is a P-equivariant section, $\phi\in F^*$ an element of the dual space and $g\in G$ a point, then

$$\langle (\nabla\, e)(g),\phi\rangle = \langle jet^\infty(e)(g), \nabla(1\otimes_{\mathfrak{U}(\mathfrak{p})}\phi)\rangle.$$

This is a one to one correspondence.

Hence the geometric question of finding G-equivariant differential operators between sections of homogeneous bundles has been translated into the algebraic question of finding homomorphisms between Verma modules. Similarly, if E_1 and E_2 are two right P representations, then a G-equivariant differential pairing

$$\vee\colon C^\infty(G,E_1)^P \times C^\infty(G,E_2)^P \to C^\infty(G,F)^P,$$

corresponds to a $\mathfrak{g}$-equivariant coproduct:

$$\vee\colon \mathrm{Verma}\,(\mathfrak{g},\mathfrak{p},F^*) \to \mathrm{Verma}\,(\mathfrak{g},\mathfrak{p},E_1^*) \otimes \mathrm{Verma}\,(\mathfrak{g},\mathfrak{p},E_2^*).$$

We will finish this section with some general remarks about $\mathfrak{g}$-homomorphisms between Verma modules:

Remark 6.11 Let W^* be some (finite or infinite dimensional) left $\mathfrak{g}$-representation and E^* a left $\mathfrak{p}$-representation. Clearly, any $\mathfrak{g}$-homomorphism $W^* \to \mathfrak{U}(\mathfrak{g}) \otimes E^*$ induces a $\mathfrak{g}$-homomorphism $W^* \to \mathrm{Verma}\,(\mathfrak{g},\mathfrak{p},E^*)$ by projection into the quotient.

Proposition 6.12 *(Frobenius reciprocity) Let E^* be a left $\mathfrak{p}$-representation and W^* a left $\mathfrak{g}$-representation. Any $\mathfrak{p}$-equivariant linear map $\phi_0\colon E^* \to W^*$ induces a $\mathfrak{g}$-equivariant homomorphism $\phi\colon \mathrm{Verma}\,(\mathfrak{g},\mathfrak{p},E^*) \to W^*$ via $\phi(U \otimes_{\mathfrak{U}(\mathfrak{p})} \eta) := U.(\phi_0(\eta))$, with $U \in \mathfrak{U}(\mathfrak{g})$ and $\eta \in E^*$. This defines an isomorphism:*

$$\mathrm{Hom}_{\mathfrak{p}}(E^*,W^*) \xrightarrow{\cong} \mathrm{Hom}_{\mathfrak{g}}(\mathrm{Verma}\,(\mathfrak{g},\mathfrak{p},E^*),W^*); \quad \phi_0 \mapsto \phi.$$

Proof: Any linear map $\phi_0\colon E^* \to W^*$ induces a $\mathfrak{g}$-homomorphism $\phi\colon \mathfrak{U}(\mathfrak{g}) \otimes E^* \to W^*$ via $\phi(U \otimes \eta) := U.(\phi_0(\eta))$. If ϕ_0 is $\mathfrak{p}$-equivariant then $\phi_0(X.\eta) = X.(\phi_0(\eta))$ for all $X \in \mathfrak{p}$. In that case ϕ vanishes on the generators of the induced ideal $I(\mathfrak{g},\mathfrak{p},E^*)$ since $\phi(X \otimes \eta - 1 \otimes X.\eta) = X.(\phi(1 \otimes \eta)) - \phi(1 \otimes X.\eta) = X.(\phi_0(\eta)) - \phi_0(X.\eta)$. Hence ϕ descends to the quotient $\phi\colon \mathrm{Verma}\,(\mathfrak{g},\mathfrak{p},E^*) \to W^*$. On the other hand, given a $\mathfrak{g}$-homomorphism $\phi\colon \mathrm{Verma}\,(\mathfrak{g},\mathfrak{p},E^*) \to W^*$, there is an induced $\phi_0\colon E^* \to W^*$ defined by $\phi_0(\eta) = \phi(1 \otimes_{\mathfrak{U}(\mathfrak{p})} \eta)$ and ϕ_0 satisfies $\phi_0(X.\eta) = \phi(1 \otimes_{\mathfrak{U}(\mathfrak{p})} X.\eta) = \phi(X \otimes_{\mathfrak{U}(\mathfrak{p})} \eta) = X.(\phi(1 \otimes_{\mathfrak{U}(\mathfrak{p})} \eta)) = X.(\phi_0(\eta))$. $\square$

Let E^* and F^* be two left $\mathfrak{p}$-representations. In the sequel we will construct $\mathfrak{g}$-homomorphisms between Verma modules like $\mathrm{Verma}\,(\mathfrak{g},\mathfrak{p},F^*) \to \mathrm{Verma}\,(\mathfrak{g},\mathfrak{p},E^*)$ which lift to $\mathfrak{g}$-homomorphism between $\mathfrak{U}(\mathfrak{g}) \otimes F^* \to \mathfrak{U}(\mathfrak{g}) \otimes E^*$. In general a $\mathfrak{g}$-homomorphism $\phi\colon \mathfrak{U}(\mathfrak{g}) \otimes F^* \to \mathfrak{U}(\mathfrak{g}) \otimes E^*$ induces a homomorphism between the corresponding Verma modules, iff ϕ maps the induced ideal into the induced ideal: $\phi(I(\mathfrak{g},\mathfrak{p},F^*)) \subset I(\mathfrak{g},\mathfrak{p},E^*)$. To evaluate ϕ on the ideal we summaries the relevant left $\mathfrak{p}$-actions on $\mathfrak{U}(\mathfrak{g}) \otimes E^*$ as follows: for $X \in \mathfrak{p}$, $U \in \mathfrak{U}(\mathfrak{g})$ and $\eta \in E^*$ we have

$$\begin{aligned}
l_X(U \otimes \eta) &:= XU \otimes \eta, \\
r_X(U \otimes \eta) &:= -UX \otimes \eta, \\
\mathrm{ad}_X(U \otimes \eta) &:= XU \otimes \eta - UX \otimes \eta, \\
\mathrm{ind}_X(U \otimes \eta) &:= XU \otimes \eta - UX \otimes \eta + U \otimes X.\eta, \\
(\mathrm{ind} - l)_X(U \otimes \eta) &:= -UX \otimes \eta + U \otimes X.\eta.
\end{aligned}$$

Notice that the last action maps onto the ideal $I(\mathfrak{g}, \mathfrak{p}, E^*)$. We will often use the following criterion:

Proposition 6.13 *A $\mathfrak{g}$-homomorphism $\phi\colon \mathfrak{U}(\mathfrak{g}) \otimes F^* \to \mathfrak{U}(\mathfrak{g}) \otimes E^*$ descends to a homomorphism between the corresponding Verma modules iff its commutator with the induced $\mathfrak{p}$-actions maps into the induced ideal:* $\mathrm{im}\,[\,\phi, \mathrm{ind}_X\,] \subset I(\mathfrak{g}, \mathfrak{p}, E^*)$ *with* $X \in \mathfrak{p}$.

Proof: The action of $(\mathrm{ind}-l)_X$ maps onto the induced ideal, hence ϕ descends iff $\phi\circ(\mathrm{ind}-l)_X$ takes values in the ideal. Since ϕ commutes with the left l_X action we have:

$$\begin{aligned} \phi \circ (\mathrm{ind} - l)_X &= [\,\phi, (\mathrm{ind} - l)_X\,] + (\mathrm{ind} - l)_X \circ \phi \\ &= [\,\phi, \mathrm{ind}_X\,] + (\mathrm{ind} - l)_X \circ \phi, \end{aligned}$$

where the second summand already is in the ideal. □

6.2 Homogeneous parabolic geometry

We will now focus on the class of parabolic homogeneous spaces G/P which will be characterized algebraically on the level of the Lie algebras $\mathfrak{p} \subset \mathfrak{g}$. We will use the following definition which emphasizes exactly the properties which we will need in the sequel:

Definition 6.14 (*Parabolic geometry*) Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra, which splits as vector space into a direct sum $\mathfrak{g} = V \oplus \mathfrak{l} \oplus V^*$, where V, $\mathfrak{l}$, V^* are subalgebras, V^* and V are dual to each other (via the Killing form of $\mathfrak{g}$), and we have the following properties of the Lie bracket: $[\,V, \mathfrak{l}\,] \subset V$, $[\,V^*, \mathfrak{l}\,] \subset V^*$. Denote by $\mathfrak{p} := \mathfrak{l} \oplus V^*$ the complementary subalgebra to V. Such a Lie algebra $\mathfrak{g} = V \oplus \mathfrak{l} \oplus V^*$ characterizes a so called *parabolic geometry*. If V is Abelian, then this is called the *Abelian* or *almost Hermitian* case. If G denotes a Lie group with Lie algebra $\mathfrak{g}$ (e.g. $G = \mathbf{Aut}\,(\mathfrak{g})$) then $P := \{g \in G \,|\, \mathrm{Ad}_g(\mathfrak{p}) = \mathfrak{p}\}$ is a closed subgroup with Lie algebra $\mathfrak{p}$ and we refer to G/P as the *homogeneous model of parabolic geometry*.

Remark 6.15 In the above situation $\mathfrak{l}$ is reductive, V and V^* are nilpotent and $\mathfrak{p}$ is a parabolic subalgebra. Alternatively, if $\mathfrak{g}$ is semisimple and $\mathfrak{p} \subset \mathfrak{g}$ a parabolic subalgebra, then $\mathfrak{g}$ splits into a direct sum as explained above. Hence a parabolic geometry is characterized by a semisimple $\mathfrak{g}$ and a choice of a parabolic $\mathfrak{p} \subset \mathfrak{g}$. In the Abelian case $\mathfrak{Z}(\mathfrak{l})$, the centre of $\mathfrak{l}$, is one dimensional. For completeness we mention the dual parabolic subalgebra, defined by $\mathfrak{p}^* := V \oplus \mathfrak{l}$.

Example 6.16 (*Conformal geometry*) Let V be an n-dimensional conformal vector space. The Möbius Lie algebra $\mathfrak{g} = V \oplus \mathfrak{co}(V) \oplus V^*$ with $\mathfrak{p} := \mathfrak{co}(V) \oplus V^*$ is a special case of a parabolic geometry. In view of the isomorphism $\mathfrak{g} = \mathfrak{so}(\hat{V})$ with $\hat{V} := L^1 \oplus V^0 \oplus L^{-1}$ we can take as Möbius group $G := \mathbf{O}\,(\hat{V})$ and as subgroup $P = \mathbf{CO}\,(V) \ltimes V^*$ (see also section 3.4). For positive definite signature we obtain the sphere $S^n = G/P$ as homogenous model of parabolic geometry.

Example 6.17 (*Projective geometry*) Let V be an n-dimensional vector space. The Lie algebra of projective geometry is $\mathfrak{g} = V \oplus \mathfrak{gl}(V) \oplus V^*$ with $\mathfrak{p} = \mathfrak{gl}(V) \oplus V^*$, where the Lie bracket on $(a, A, \alpha), (b, B, \beta) \in \mathfrak{g}$ is given by

$$\begin{aligned} [\,a, b\,] &= 0, \\ [\,A, b\,] &= Ab, \\ [\,\alpha, b\,] &= -\alpha(b)id - \alpha \otimes b, \\ [\,A, B\,] &= A \circ B - B \circ A, \\ [\,\alpha, B\,] &= \alpha \circ B, \\ [\,\alpha, \beta\,] &= 0. \end{aligned}$$

We have a Lie algebra isomorphism $\psi\colon \mathfrak{g} \to \mathfrak{sl}(\hat{V})$ where $\hat{V} := L^{-n/(n+1)} \otimes V \oplus L^{-n/(n+1)}$ is $(n+1)$-dimensional and

$$\psi(a, A, \alpha)(v, t) := (at - \frac{1}{n+1}\,\mathrm{tr}(A)v + Av, -\frac{1}{n+1}\,\mathrm{tr}(A)t + \alpha(v)),$$

with $v \in L^{-n/(n+1)} \otimes V$ and $t \in L^{-n/(n+1)}$. The central element $id_V \in \mathfrak{gl}(V)$ acts as $\psi(id_V)(v,t) = (1/(n{+}1)v, -n/(n{+}1)t)$ which explains the choice of weights in the definition. $\hat{V}$ is equipped with a canonical density. As Lie group we can take the Lie group $G := \mathbf{SL}\,(\hat{V})$ leaving this density invariant and $P := \mathbf{GL}\,(V) \ltimes V^*$. As homogeneous model we obtain n-dimensional projective space $G/P = \mathbb{R}P^n$.

Construction 6.18 (*Penrose's twistor fields*) Let W be a right G-representation. This clearly induces a right $\mathfrak{g}$-action on W. Elements of W are called *parabolic twistors*. The space of coinvariants $W_{V^*} := W/WV^*$ obviously defines a right P-representation. Each twistor $w \in W$ induces (in a linear way) a section of the homogeneous bundle $G \times_P W_{V^*}$, a so called *twistor field* by mapping $g \in G$ to $[wg]_{V^*} \in W_{V^*}$. This gives a G-equivariant linear inclusion:

$$\iota_H : W \to C^\infty(G, W_{V^*})^P,$$

i.e. $\iota_H(wh) = L_h(\iota_H(w))$ for all $h \in G$.

The aim of the next sections is to identify the twistor fields as solution of a G-equivariant linear differential operator. For this we assume W to be finite dimensional. In fact the above inclusion ι_H is the beginning of a G-equivariant differential complex:

Theorem 6.19 *(Parabolic Bernstein Gelfand Gelfand complex) If we denote by $H_k(V^*, W)$ the relative Lie algebra homology spaces defined in the next section in paragraph 6.23, then $H_k(V^*, W)$ are right P-representations and there is a locally exact complex*

$$0 \to W \xrightarrow{\iota_H} C^\infty(G, H_0(V^*, W))^P \xrightarrow{d_H} C^\infty(G, H_1(V^*, W))^P \xrightarrow{d_H} C^\infty(G, H_2(V^*, W))^P \xrightarrow{d_H} \dots \quad ,$$

of G-equivariant linear differential operators.

In particular we have $H_0(V^*, W) = W_{V^*}$. Such a complex can be viewed as a variation of the deRham complex of alternating multilinear forms. So we denoted the resulting differential operators by d_H where the subscript H stands for homology. The first differential operator $C^\infty(G, H_0(V^*, W))^P \to C^\infty(G, H_1(V^*, W))^P$ will be called the *twistor operator*. On the deRham complex we have the wedge product on forms and the exterior derivative satisfies the Leibniz rule. Similarly we have the following:

Theorem 6.20 *If W_1, W_2 and W_3 are three finite dimensional G-representations from the right and $F\colon W_1 \otimes W_2 \to W_3$, is a linear (nontrivial) G-equivariant map, then there is a G-equivariant bilinear differential pairing*

$$\wedge_H : C^\infty(G, H_k(V^*, W_1))^P \otimes C^\infty(G, H_l(V^*, W_2))^P \to C^\infty(G, H_{k+l}(V^*, W_3))^P.$$

For $k = l = 0$ this pairing is an extension of F since it satisfies

$$\wedge_H \circ (\iota_H \otimes \iota_H) = \iota_H \circ F.$$

More generally for $s \in C^\infty(G, H_k(V^*, W_1))^P$ and $t \in C^\infty(G, H_l(V^*, W_2))^P$ the following Leibniz rule holds:

$$d_H(s \wedge_H t) = (d_H\, s) \wedge_H t + (-1)^k s \wedge_H (d_H\, t).$$

Examples 6.21 Such pairings between G-representations are as follows: the G-equivariant contraction $W \otimes W^* \to \mathbb{R}$, the trivial multiplication $W \otimes \mathbb{R} \to W$, the Lie algebra action $\mathfrak{g} \otimes W \to W$ and its dual $W \otimes W^* \to \mathfrak{g}$.

We will prove the above results not on the level of differential operators acting on sections (differential geometric picture). Instead we will work in the dual (algebraic) picture with homomorphisms between induced modules, see definition 6.6. In the parabolic context, when $\mathfrak{g}$ is semisimple and $\mathfrak{p}$ is parabolic, the induced module $\mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, E^*)$ is also called a *parabolic Verma module*.

Remark 6.22 (*Universal lowest weight modules*) The case when $\mathfrak{b}$ is a *Borel subalgebra* of a semisimple $\mathfrak{g}$ containing a *Cartan subalgebra* $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$ is a special (non Abelian) parabolic geometry $\mathfrak{g} = V \oplus \mathfrak{l} \oplus V^*$ with $\mathfrak{l} = \mathfrak{h}$, $\mathfrak{p} = \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^-$ and $\mathfrak{n}^- = V^*$. Let W^* be a left $\mathfrak{g}$-representation. An element $\lambda \in \mathfrak{h}^*$ is called a *weight of W^** if the corresponding *weight space* $W^*_\lambda = \{\omega \in W^* \mid H.\omega = \lambda(H)\omega, \forall H \in \mathfrak{h}\}$ is nontrivial. Nonzero elements of nontrivial weight spaces are called *weight vectors*. A weight vector $\omega \in W^*_\lambda$ is called *lowest weight vector* if $\mathfrak{n}^- \subset \mathfrak{g}$ acts trivially on it: $\mathfrak{n}^-.\omega = 0$. A representation W^* is called *lowest weight module* if it is generated by a lowest weight vector ω, i.e. $W^* = \mathfrak{U}(\mathfrak{g})\omega$. In this context Verma defined in [Ver68] to each *weight* $\lambda \in \mathfrak{h}^*$ a *universal lowest weight module* as quotient $\mathfrak{U}(\mathfrak{g})/I(\mathfrak{g}, \mathfrak{b}, \lambda)$, where $I(\mathfrak{g}, \mathfrak{b}, \lambda)$ is the left $\mathfrak{U}(\mathfrak{g})$ ideal generated by $\mathfrak{n}^-$ and $<X - \lambda(X) \mid X \in \mathfrak{h}>$. This defines a *lowest weight module* which is universal in the sense that it maps onto all other lowest weight modules of weight λ. Indeed all finite dimensional irreducible $\mathfrak{g}$-representations occur as quotients. Lepowsky in [Lep77] generalized this notion to the above parabolic Verma modules.

6.3 Eilenberg's relative Lie algebra homology

In case of a parabolic geometry $\mathfrak{p} \subset \mathfrak{g}$ see definition 6.14, we will consider a $\mathfrak{p}$-equivariant relative Lie algebra homology theory with values in a right $\mathfrak{g}$-representation W: the boundary operator is a map between

$$\delta_{V^*} : \Lambda^k V^* \otimes W \to \Lambda^{k-1} V^* \otimes W,$$

where the subscript V^* denotes the Lie algebra which effectively acts. The homology spaces $H_k(V^*, W)$ will define the right $\mathfrak{p}$-representations relevant in the sequence of differential operators (geometric picture). Completely dual to the above homology theory is a cohomology theory with values in W^*:

$$d_{V^*} = \delta^*_{V^*} : \Lambda^k V \otimes W^* \to \Lambda^{k+1} V \otimes W^*.$$

This coboundary operator will define left $\mathfrak{p}$-representations $H^k(V^*, W^*) = H_k(V^*, W)^*$ which occur when the differential operators are expressed in terms of Verma module homomorphisms (algebraic picture).

In addition we like to mention another boundary operator

$$\delta_V : \Lambda^k V \otimes W^* \to \Lambda^{k-1} V \otimes W^*$$

which will appear in the next section as zero order part of the twisted deRham homomorphism. Therefore we define it here, although its $\mathfrak{p}^*$-equivariant homology theory is not relevant in our context. For completeness we also like to mention its dual coboundary operator, which is the zero order part of the twisted deRham operator (geometric picture):

$$d_V = \delta^*_V : \Lambda^k V^* \otimes W \to \Lambda^{k+1} V^* \otimes W.$$

Definition 6.23 (V^* *homology with values in* W) Let W be a right representation of $\mathfrak{g}$. We denote the right action of $X \in \mathfrak{g}$ on $w \in W$ by $w.X \in W$. The *k-chains* are defined by

$$C_k(V^*, W) := \Lambda^k V^* \otimes W.$$

Note that $\mathfrak{p}$ acts upon V^* by the adjoint representation (from the right) and therefore $\mathfrak{p}$ acts on $\Lambda^k V^*$ by the product rule: let t_i, θ^i be a dual basis of V, then we have for $X \in \mathfrak{p}$ and $\alpha \in \Lambda^k V^*$:

$$\alpha.X := \sum_i [\,\theta^i, X\,] \wedge t_i \lrcorner \alpha.$$

In the Abelian case the action of $V^* \subset \mathfrak{p}$ on $\Lambda^k V^*$ is trivial. Clearly $\mathfrak{p}$ as subalgebra of $\mathfrak{g}$ acts also on W from the right and hence $C_k(V^*, W)$ is a right $\mathfrak{p}$-representation $C_k(V^*, W) \otimes \mathfrak{p} \to C_k(V^*, W)$: for $w \in W$ we have: $(\alpha \otimes w).X := (\alpha.X) \otimes w + \alpha \otimes (w.X)$. Next we define the boundary operator δ_{V^*} where the subscript denotes the Lie algebra which effectively acts upon W in the following definition: $\delta_{V^*} : C_k(V^*, W) \to C_{k-1}(V^*, W)$;

$$\delta_{V^*}(\alpha \otimes w) := \frac{1}{2} \sum_i (t_i \lrcorner \alpha).\theta^i \otimes w + \sum_i (t_i \lrcorner \alpha) \otimes w.\theta^i.$$

For $k = 0, 1$ this definition means $\delta_{V^*}(w) = 0$ and $\delta_{V^*}(\alpha \otimes w) = w.\alpha$.

Proposition 6.24 *This defines a complex $\delta_{V^*} \circ \delta_{V^*} = 0$, which is equivariant under the right action of $X \in \mathfrak{p}$, i.e. $[\,.X, \delta_{V^*}\,] := .X \circ \delta_{V^*} - \delta_{V^*} \circ .X = 0$. Hence the kernel, image and homology of δ_{V^*} give right $\mathfrak{p}$-representations:*

$$\begin{aligned}
Z_k(V^*, W) &:= \ker \delta_{V^*} : C_k(V^*, W) \to C_{k-1}(V^*, W), \\
B_k(V^*, W) &:= \operatorname{im} \delta_{V^*} : C_{k+1}(V^*, W) \to C_k(V^*, W), \\
H_k(V^*, W) &:= Z_k(V^*, W)/B_k(V^*, W).
\end{aligned}$$

For $\gamma \in V^*$ and $c \in C_k(V^*, W)$ we have Cartan's identity:

$$\delta_{V^*}(\gamma \wedge c) + \gamma \wedge (\delta_{V^*} c) = c.\gamma.$$

Hence, V^* acts trivially on $H_k(V^*, W)$, such that $H_k(V^*, W)$ can equally well be viewed as a right $\mathfrak{l}$-representation extending trivially to a $\mathfrak{p}$-representation.

Proof: To do these calculation we remark the following formulas: the right adjoint action of V^* on V^* is clearly given by $\theta.\gamma = [\,\theta, \gamma\,]$. Hence for the right coadjoint action of V^* on V we have $v.\gamma = \sum_k t_k \langle v.\gamma, \theta^k \rangle = \sum_k t_k \langle v, \gamma.\theta^k \rangle = \sum_k t_k \langle v, [\,\gamma, \theta^k\,] \rangle$, with $v \in V$. The identity on V is invariant under this action $0 = id_V.\gamma = \sum_i (\theta^i \otimes t_i).\gamma$. To show $\delta_{V^*}^2 = 0$ note that the square of the first summand in the definition of δ_{V^*} has to be zero separately, since this is the only summand in case of a trivial $\mathfrak{g}$-representation W. This follows from the Jacobi identity in V^*:

$$\begin{aligned}
\sum_{i,j} (t_j \lrcorner (t_i \lrcorner \alpha).\theta^i).\theta^j &= \sum_{i,j} (t_j \lrcorner t_i \lrcorner \alpha).\theta^i.\theta^j - (t_j.\theta^i \lrcorner t_i \lrcorner \alpha).\theta^j \\
&= \frac{1}{2} \sum_{i,j} (t_j \lrcorner t_i \lrcorner \alpha).[\,\theta^i, \theta^j\,] - \sum_{i,k} (t_k \lrcorner t_i \lrcorner \alpha).[\,\theta^i, \theta^k\,] \\
&= \frac{1}{2} \sum_{i,j} (t_i \lrcorner t_j \lrcorner \alpha).[\,\theta^i, \theta^j\,] \\
&= \frac{1}{2} \sum_{i,j,k} [\,\theta^k, [\,\theta^i, \theta^j\,]\,] \wedge (t_k \lrcorner t_i \lrcorner t_j \lrcorner \alpha) = 0.
\end{aligned}$$

In general we have

$$\begin{aligned}
\delta_{V^*}(\delta_{V^*}(\alpha \otimes w)) &= \frac{1}{4} \sum_{i,j} (t_j \lrcorner (t_i \lrcorner \alpha).\theta^i).\theta^j \otimes w + \frac{1}{2} \sum_{i,j} t_j \lrcorner (t_i \lrcorner \alpha).\theta^i \otimes w.\theta^j \\
&\quad + \frac{1}{2} \sum_{i,j} (t_j \lrcorner t_i \lrcorner \alpha).\theta^j \otimes w.\theta^i + \sum_{i,j} (t_j \lrcorner t_i \lrcorner \alpha) \otimes w.\theta^i.\theta^j \\
&= 0 + \frac{1}{2} \sum_{i,j} (t_j \lrcorner t_i \lrcorner \alpha).\theta^i \otimes w.\theta^j - \frac{1}{2} \sum_{i,j} (t_j.\theta^i \lrcorner t_i \lrcorner \alpha) \otimes w.\theta^j \\
&\quad + \frac{1}{2} \sum_{i,j} (t_j \lrcorner t_i \lrcorner \alpha).\theta^j \otimes w.\theta^i + \frac{1}{2} \sum_{i,j} (t_j \lrcorner t_i \lrcorner \alpha) \otimes w.[\,\theta^i, \theta^j\,] \\
&= -\frac{1}{2} \sum_{i,k} (t_k \lrcorner t_i \lrcorner \alpha) \otimes w.[\,\theta^i, \theta^k\,] + \frac{1}{2} \sum_{i,j} (t_j \lrcorner t_i \lrcorner \alpha) \otimes w.[\,\theta^i, \theta^j\,] = 0.
\end{aligned}$$

For the equivariance under $X \in \mathfrak{p}$ note once more $v.X = \sum_k t_k \langle v, [\,X, \theta^k\,] \rangle$. The following calculation shows $\mathfrak{p}$-equivariance in case of a trivial $\mathfrak{g}$-representation W:

$$\begin{aligned}
(\sum_i (t_i \lrcorner \alpha).\theta^i).X - \sum_i (t_i \lrcorner (\alpha.X)).\theta^i &= \sum_i (t_i \lrcorner \alpha).\theta^i.X - (t_i \lrcorner \alpha).X.\theta^i + (t_i.X \lrcorner \alpha).\theta^i \\
&= \sum_i (t_i \lrcorner \alpha).[\,\theta^i, X\,] + \sum_k (t_k \lrcorner \alpha).[\,X, \theta^k\,] = 0.
\end{aligned}$$

In general we have

$$\begin{aligned}
&[\,.X, \delta_{V^*}\,](\alpha \otimes w) \\
&= (\delta_{V^*}(\alpha \otimes w)).X - \delta_{V^*}(\alpha.X \otimes w) - \delta_{V^*}(\alpha \otimes w.X) \\
&= \frac{1}{2} \sum_i (t_i \lrcorner \alpha).\theta^i.X \otimes w + \frac{1}{2} \sum_i (t_i \lrcorner \alpha).\theta^i \otimes w.X + \sum_i (t_i \lrcorner \alpha).X \otimes w.\theta^i \\
&\quad + \sum_i (t_i \lrcorner \alpha) \otimes w.\theta^i.X - \frac{1}{2} \sum_i (t_i \lrcorner \alpha.X).\theta^i \otimes w - \frac{1}{2} \sum_i (t_i \lrcorner \alpha).\theta^i \otimes w.X \\
&\quad - \sum_i (t_i \lrcorner \alpha.X) \otimes w.\theta^i - \sum_i (t_i \lrcorner \alpha) \otimes w.X.\theta^i \\
&= 0 + \sum_i (t_i.X \lrcorner \alpha) \otimes w.\theta^i + \sum_i (t_i \lrcorner \alpha) \otimes w.\theta^i.X - \sum_i (t_i \lrcorner \alpha) \otimes w.X.\theta^i \\
&= \sum_k (t_k \lrcorner \alpha) \otimes w.[\,X, \theta^k\,] + \sum_i (t_i \lrcorner \alpha) \otimes w.[\,\theta^i, X\,] = 0.
\end{aligned}$$

Cartan's identity is a straight forward calculation: $c = \alpha \otimes w$,

$$\begin{aligned}
&\delta_{V^*}(\gamma \wedge c) + \gamma \wedge (\delta_{V^*} c) \\
&= \frac{1}{2} \sum_i (t_i \lrcorner (\gamma \wedge \alpha)).\theta^i \otimes w + \sum_i t_i \lrcorner (\gamma \wedge \alpha) \otimes w.\theta^i \\
&\quad + \frac{1}{2} \sum_i \gamma \wedge (t_i \lrcorner \alpha).\theta^i \otimes w + \sum_i \gamma \wedge (t_i \lrcorner \alpha) \otimes w.\theta^i \\
&= \frac{1}{2} \alpha.\gamma \otimes w - \frac{1}{2} \sum_i (\gamma \wedge t_i \lrcorner \alpha).\theta^i \otimes w + \alpha \otimes w.\gamma - \sum_i \gamma \wedge t_i \lrcorner \alpha \otimes w.\theta^i \\
&\quad + \frac{1}{2} \sum_i \gamma \wedge (t_i \lrcorner \alpha).\theta^i \otimes w + \sum_i \gamma \wedge (t_i \lrcorner \alpha) \otimes w.\theta^i \\
&= \frac{1}{2} \alpha.\gamma \otimes w - \frac{1}{2} \sum_i \gamma.\theta^i \wedge t_i \lrcorner \alpha \otimes w + \alpha \otimes w.\gamma \\
&= \alpha.\gamma \otimes w + \alpha \otimes w.\gamma. \quad \square
\end{aligned}$$

Remark 6.25 If G is a (real) Lie group with (complexified) Lie algebra $\mathfrak{g}$ and if W is indeed a right G-representation, then all the above constructed $\mathfrak{p}$-representation are also P-representation with $P = \{g \in G \,|\, \mathrm{Ad}_g(\mathfrak{p}) = \mathfrak{p}\}$.

Definition 6.26 The zero homology $H_0(V^*, W)$ is also called the *space of coinvariants* of W with respect to V^*, since in that case $Z_0(V^*, W) = W$ and $B_0(V^*, W) = \mathrm{im}\,(W \otimes V^* \to W) = W.V^*$, i.e. $H_0(V^*, W) = W/W.V^* =: W_{V^*}$.

Example 6.27 (*Maxwell's homology*) The homology for the trivial representation $W = \mathbb{R}$ in the Abelian case gives back the usual multilinear forms:

$$H_k(V^*, \mathbb{R}) = \Lambda^k V^*.$$

Example 6.28 (*Fierz's homology*) In case of V being a conformal vector space, the conformal Lie algebra $\mathfrak{g}$ is given by $\mathfrak{g} = V \oplus \mathfrak{co}(V) \oplus V^*$, which is semisimple, since we have an isomorphism $\mathfrak{g} = \mathfrak{so}(\hat{V})$ with $\hat{V} := L^1 \oplus V^0 \oplus L^{-1}$. In particular $\hat{V}$ itself is a $\mathfrak{g}$-representation:

$$\begin{aligned} H_0(V^*, \hat{V}) &= L^1, \\ H_k(V^*, \hat{V}) &= \Lambda^k V^* \odot V^0, \\ H_n(V^*, \hat{V}) &= \Lambda^n V^* \otimes L^{-1}. \end{aligned}$$

Here elements in the Cartan product $\Lambda^k V^* \odot V^*$ are tensors in $\Lambda^k V^* \otimes V^*$ which are alternating-free and trace-free.

Example 6.29 (*Linearized conformal geometry, Bach's homology*) The adjoint representation $W = \mathfrak{g} = V \oplus \mathfrak{co}(V) \oplus V^*$ in the conformal case gives:

$$\begin{aligned} H_0(V^*, \mathfrak{g}) &= V, \\ H_1(V^*, \mathfrak{g}) &= V^* \odot V, \\ H_k(V^*, \mathfrak{g}) &= \Lambda^k V^* \odot \mathfrak{so}(V), \\ H_{n-1}(V^*, \mathfrak{g}) &= \Lambda^{n-1} V^* \odot V^*, \\ H_n(V^*, \mathfrak{g}) &= \Lambda^n V^* \otimes V^*. \end{aligned}$$

Elements in the homology for $k = 0, 1, 2$ have immediate geometric interpretations as vectors, linearized conformal metrics and conformal Weyl curvature tensors.

Example 6.30 (*Penrose's homology*) The representation on trivectors $W = \Lambda^3 \hat{V}$ in the conformal case gives for $n > 4$:

$$\begin{aligned} H_0(V^*, W) &= \Lambda^2 V^0 \otimes L^1, \\ H_1(V^*, W) &= V^* \odot \Lambda^2 V^0 \otimes L^1, \\ H_2(V^*, W) &= \Lambda^2 V^* \odot \Lambda^2 V^0 \otimes L^1, \\ H_k(V^*, W) &= \Lambda^k V^* \odot \Lambda^3 V^0, \\ H_{n-2}(V^*, W) &= \Lambda^{n-2} V^* \odot \Lambda^2 V^0 \otimes L^{-1}, \\ H_{n-1}(V^*, W) &= \Lambda^{n-1} V^* \odot \Lambda^2 V^0 \otimes L^{-1}, \\ H_n(V^*, \mathfrak{g}) &= \Lambda^n V^* \otimes \Lambda^2 V^0 \otimes L^{-1}. \end{aligned}$$

For $n = 4$ the middle dimensional homology becomes:

$$H_2(V^*, W) = \Lambda^2 V^* \odot \Lambda^2 V^0 \otimes (L^1 \oplus L^{-1}).$$

Example 6.31 (*Spinor homology*) In the even dimensional conformal case $n = 2l$ let S^+ and S^- be the two (weightless) (fundamental) spinor representations for $\mathfrak{co}(V)$ (on which the centre acts trivially). The spinor representation of central weight w will be denoted by $S^{+,w}$ and $S^{-,w}$. Then (by periodicity) $\hat{S}^+ := S^{+,\frac{1}{2}} \oplus S^{-,-\frac{1}{2}}$ and $\hat{S}^- := S^{-,\frac{1}{2}} \oplus S^{+,-\frac{1}{2}}$ are the two spinor representations for $\mathfrak{so}(\hat{V})$. For the homology we find:

$$H_k(V^*, \hat{S}^+) = \Lambda^k V^* \odot S^{+,\frac{1}{2}}.$$

The elements in the Cartan product $\Lambda^k V^* \odot S$ are elements in the tensors product which are in the kernel of the Clifford multiplication.

Example 6.32 (*Linearized projective geometry*) The adjoint representation $W = \mathfrak{g} = V \oplus \mathfrak{gl}(V) \oplus V^*$ in the projective case gives:

$$\begin{aligned} H_0(V^*, \mathfrak{g}) &= V, \\ H_1(V^*, \mathfrak{g}) &= \mathrm{Sym}^2 V^* \odot V, \\ H_k(V^*, \mathfrak{g}) &= \Lambda^k V^* \odot \mathfrak{sl}(V), \\ H_{n-1}(V^*, \mathfrak{g}) &= \Lambda^{n-1} V^* \odot V^* \odot V, \\ H_n(V^*, \mathfrak{g}) &= \Lambda^n V^* \otimes V^*. \end{aligned}$$

Elements in the Cartan product are in the kernel of the $\mathfrak{gl}(V)$ equivariant alternation or contraction. Elements in the homology for $k = 0, 1, 2$ have immediate geometric interpretations as vectors, linearized torsion-free derivatives (up to projective equivalence) and projective Weyl curvature tensors.

Discussion 6.33 (V^* *cohomology with values in* W^*) Secondly we describe the V^* cohomology with values in W^* and here we like to view W^* as a left $\mathfrak{g}$-representation. It is then completely dual to the above homology theory: the *k-cochains* are defined by

$$C^k(V^*, W^*) := \Lambda^k V \otimes W^* = (C_k(V^*, W))^*.$$

The left action of $X \in \mathfrak{p}$ on $v \in V$ is given by projecting the adjoint action onto V: $[X, v]_V = X.v = \sum_k \langle \theta^k, X.v \rangle t_k = \sum_k \langle \theta^k.X, v \rangle t_k = \sum_k \langle [\theta^k, X], v \rangle t_k$. This extends as derivation on $\Lambda^k V$, hence the left action of $X \in \mathfrak{p}$ on $a \otimes \omega \in C^k(V^*, W^*)$ is therefore given by:

$$\begin{aligned} X.(a \otimes \omega) &:= \sum_i ([X, t_i]_V \wedge \theta^i \lrcorner a) \otimes \omega + a \otimes X.\omega \\ &= \sum_j t_j \wedge ([\theta^j, X] \lrcorner a) \otimes \omega + a \otimes X.\omega. \end{aligned}$$

We define the coboundary operator $d_{V^*} \colon C^k(V^*, W^*) \to C^{k+1}(V^*, W^*)$ to be dual to the boundary operator $d_{V^*} = \delta^*_{V^*}$:

$$d_{V^*}(a \otimes \omega) := \frac{1}{2} \sum_i (t_i \wedge \theta^i.a) \otimes \omega + \sum_i (t_i \wedge a) \otimes \theta^i.\omega.$$

Therefore it satisfies $d_{V^*} \circ d_{V^*} = 0$ and it is equivariant under the action of $X \in \mathfrak{p}$, i.e. $[\,X.,d_{V^*}\,] = 0$. Hence the kernel, image and cohomology of d_{V^*} give left $\mathfrak{p}$-representations:

$$\begin{aligned}
Z^k(V^*,W^*) &:= \ker d_{V^*} : C^k(V^*,W^*) \to C^{k+1}(V^*,W^*),\\
B^k(V^*,W^*) &:= \operatorname{im} d_{V^*} : C^{k-1}(V^*,W^*) \to C^k(V^*,W^*),\\
H^k(V^*,W^*) &:= Z^k(V^*,W^*)/B^k(V^*,W^*).
\end{aligned}$$

The zero cohomology $H^0(V^*,W^*)$ is also called the *space of invariants* of W^* with respect to V^*, since in that case $B^0(V^*,W^*) = 0$ and $H^0(V^*,W^*) = Z^0(V^*,W^*) = W^{*V^*}$ with $W^{*V^*} = \{\omega \in W^* \,|\, \gamma.\omega = 0\ \forall \gamma \in V^*\}$. Again V^* acts trivially on $H^k(V^*,W^*)$ since Cartan's identity holds for $\gamma \in V^*$ and $c \in C^k(V^*,W^*)$:

$$d_{V^*}(\gamma \lrcorner c) + \gamma \lrcorner (d_{V^*}c) = \gamma.c.$$

Proposition 6.34 *(Poincaré duality) For* $\dim V = n$ *there is a canonical isomorphism* $\lrcorner : \Lambda^k V^* \otimes \Lambda^n V \to \Lambda^{n-k} V$ *which induces isomorphisms* $C_k(V^*,W) \otimes \Lambda^n V \cong C^{n-k}(V^*,W)$ *and*

$$\begin{aligned}
H_k(V^*,W) \otimes \Lambda^n V &\cong H^{n-k}(V^*,W)\\
&= H_{n-k}(V^*,W^*)^*.
\end{aligned}$$

Proof: Let $v \in \Lambda^n V$ be nontrivial and $\theta \in V^*$. For the action of θ on the volume element v note $v.\theta = \operatorname{tr}_{V^*}(\alpha \mapsto \alpha.\theta)v = \sum_k \langle \theta^k.\theta, t_k \rangle v = \langle \theta, \sum_k t_k.\theta^k \rangle v$. Then we find that $\lrcorner\, v$ intertwines d_{V^*} and δ_{V^*}:

$$\begin{aligned}
&(\delta_{V^*}(\alpha \otimes w)) \lrcorner v\\
&= \frac{1}{2} \sum_i ((t_i \lrcorner \alpha).\theta^i \otimes w + 2 t_i \lrcorner \alpha \otimes w.\theta^i) \lrcorner v\\
&= \frac{1}{2} \sum_i (t_i.\theta^i \lrcorner \alpha \otimes w + t_i \lrcorner \alpha.\theta^i \otimes w + 2 t_i \lrcorner \alpha \otimes w.\theta^i) \lrcorner v\\
&= \frac{1}{2} (-1)^k \sum_i (t_i.\theta^i \wedge (\alpha \lrcorner v) \otimes w + t_i \wedge (\alpha.\theta^i \lrcorner v) \otimes w + 2 t_i \wedge (\alpha \lrcorner v) \otimes w.\theta^i)\\
&= \frac{1}{2} (-1)^k \sum_i (t_i \wedge (\alpha \lrcorner v).\theta^i \otimes w + 2 t_i \wedge (\alpha \lrcorner v) \otimes w.\theta^i)\\
&= \frac{1}{2} (-1)^{k+1} \sum_i (t_i \wedge \theta^i.(\alpha \lrcorner v) \otimes w + 2 t_i \wedge (\alpha \lrcorner v) \otimes \theta^i.w)\\
&= (-1)^{k+1} d_{V^*}((\alpha \otimes w) \lrcorner v). \quad \square
\end{aligned}$$

Discussion 6.35 (V *homology with values in* W^*) Thirdly we discuss the V homology with values in W^* viewed as a left $\mathfrak{g}$-representation: the *k-chains* are defined by

$$C_k(V,W^*) := \Lambda^k V \otimes W^* = C^k(V^*,W^*)$$

Note, that $\mathfrak{p}^*$ acts upon V by the adjoint representation and therefore $\mathfrak{p}^*$ acts on $C_k(V, W^*)$: for $Y \in \mathfrak{p}^*$ and $a \in \Lambda^k V$, $\omega \in W^*$:

$$Y.(a \otimes \omega) := \sum_i [\, Y, t_i \,] \wedge \theta^i \lrcorner\, a \otimes \omega + a \otimes Y.\omega.$$

The boundary operator $\delta_V : C_k(V, W^*) \to C_{k-1}(V, W^*)$ is defined to be

$$\delta_V(a \otimes \omega) := -\frac{1}{2} \sum_i t_i.(\theta^i \lrcorner\, a) \otimes \omega - \sum_i (\theta^i \lrcorner\, a) \otimes t_i.\omega.$$

It satisfies $\delta_V \circ \delta_V = 0$ and is equivariant under the action of $Y \in \mathfrak{p}^*$, i.e. $[\, Y., \delta_V \,] = 0$.

Remark 6.36 The commutator on the spaces $C^k(V^*, W^*) = C_k(V, W^*) = \Lambda^k V \otimes W^*$ of δ_V with the left $X \in \mathfrak{p}$ action is given by

$$[\, X., \delta_V \,] = -\sum_i [\, X, t_i \,]_{\mathfrak{p}}.(\theta^i \lrcorner \otimes id_{W^*}).$$

To check this we recall that the left action of $X \in \mathfrak{p}$ on $v \in V$ is given by projecting the adjoint action onto V, see paragraph 6.33. The extention as derivation on $a \in \Lambda^k V$ is then given by $X.a = \sum_i [\, X, t_i \,]_V \wedge \theta^i \lrcorner\, a = \sum_k t_k \wedge [\, \theta^k, X \,] \lrcorner\, a$. Notice the following commutator rules on $\Lambda^k V$ for $t \in V$ and $\theta \in V^*$:

$$\begin{aligned} [\, X., t \wedge \,] &= [\, X, t \,]_V \wedge \,, \\ [\, X., \theta \lrcorner \,] &= [\, X, \theta \,] \lrcorner \,. \end{aligned}$$

Then we get:

$$\begin{aligned}
[\, X., \delta_V \,] &= [\, X., -\frac{1}{2} \sum_{i,j} [\, t_i, t_j \,] \wedge \theta^j \lrcorner\, \theta^i \lrcorner \otimes id_{W^*} - \sum_i \theta^i \lrcorner \otimes t_i. \,] \\
&= -\frac{1}{2} \sum_{i,j} [\, X, [\, t_i, t_j \,] \,]_V \wedge \theta^j \lrcorner\, \theta^i \lrcorner \otimes id_{W^*} - \frac{1}{2} \sum_{i,j} [\, t_i, t_j \,] \wedge [\, X, \theta^j \,] \lrcorner\, \theta^i \lrcorner \otimes id_{W^*} \\
&\quad -\frac{1}{2} \sum_{i,j} [\, t_i, t_j \,] \wedge \theta^j \lrcorner [\, X, \theta^i \,] \lrcorner \otimes id_{W^*} - \sum_i [\, X, \theta^i \,] \lrcorner \otimes t_i. - \sum_i \theta^i \lrcorner \otimes [\, X, t_i \,]. \\
&= -\sum_{i,j} [\, [\, X, t_i \,], t_j \,]_V \wedge \theta^j \lrcorner\, \theta^i \lrcorner \otimes id_{W^*} - \sum_{i,j} [\, t_i, t_j \,] \wedge [\, X, \theta^j \,] \lrcorner\, \theta^i \lrcorner \otimes id_{W^*} \\
&\quad - \sum_i [\, X, \theta^i \,] \lrcorner \otimes t_i. - \sum_i \theta^i \lrcorner \otimes [\, X, t_i \,]. \\
&= -\sum_{i,j} [\, [\, X, t_i \,]_{\mathfrak{p}}, t_j \,]_V \wedge \theta^j \lrcorner\, \theta^i \lrcorner \otimes id_{W^*} - \sum_i [\, X, \theta^i \,] \lrcorner \otimes t_i. - \sum_i \theta^i \lrcorner \otimes [\, X, t_i \,]. \\
&= -\sum_i [\, X, t_i \,]_{\mathfrak{p}}.(\theta^i \lrcorner) \otimes id_{W^*} - \sum_i [\, X, \theta^i \,] \lrcorner \otimes t_i. - \sum_i \theta^i \lrcorner \otimes [\, X, t_i \,]. \\
&= -\sum_i [\, X, t_i \,]_{\mathfrak{p}}.(\theta^i \lrcorner \otimes id_{W^*}).
\end{aligned}$$

6.4 Twisted deRham resolution

Let $\mathfrak{p} \subset \mathfrak{g}$ be again a Cartan geometry with $\mathfrak{g} = V \oplus \mathfrak{l} \oplus V^*$ and $\mathfrak{p} = \mathfrak{l} \oplus V^*$. Let t_i, θ^i be a dual basis of V. The exterior deRham operator on differential forms is given in terms of a $\mathfrak{g}$-equivariant homomorphism between Verma modules by the following:

Definition 6.37 We define a $\mathfrak{g}$-homomorphism $\delta^{dR} \colon \mathfrak{U}(\mathfrak{g}) \otimes \Lambda^k V \to \mathfrak{U}(\mathfrak{g}) \otimes \Lambda^{k-1} V$ generated on $a \in \Lambda^k V$ by

$$\delta^{dR}(1 \otimes a) \;\; := \;\; \sum_i t_i \otimes \theta^i \lrcorner\, a + \frac{1}{2} \sum_{i,j} 1 \otimes [\, t_i, t_j\,] \wedge \theta^i \lrcorner\, \theta^j \lrcorner\, a.$$

For low degrees $k = 1, 2$ the above definition of δ^{dR} means

$$\begin{aligned} \delta^{dR}(a) &= a \otimes 1, \\ \delta^{dR}(a \wedge b) &= a \otimes b - b \otimes a - 1 \otimes [\, a, b\,]. \end{aligned}$$

In general we notice that the deRham homomorphism and the boundary operator δ_V of the V homology with values in the trivial $\mathbb{R}$, from paragraph6.35, are related as:

$$\begin{aligned} \delta^{dR}(1 \otimes a) &= \sum_i t_i \otimes \theta^i \lrcorner\, a - \frac{1}{2} \sum_{i,j} 1 \otimes t_i.(\theta^i \lrcorner\, a) \\ &= \sum_i t_i \otimes \theta^i \lrcorner\, a + \delta_V a. \end{aligned}$$

Proposition 6.38 *(deRham resolution) The homomorphism δ^{dR} maps the induced ideal $I(\mathfrak{g}, \mathfrak{p}, \Lambda^k V)$ to $I(\mathfrak{g}, \mathfrak{p}, \Lambda^{k-1} V)$ and thus induces a $\mathfrak{g}$-homomorphism*

$$\delta^{dR} \colon \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, \Lambda^k V) \to \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, \Lambda^{k-1} V).$$

This leads to the deRham complex $\delta^{dR} \circ \delta^{dR} = 0$. In zero degree $k = 0$ we can replace the trivial $\delta^{dR} = 0$ by $\mathrm{ev} \colon \mathfrak{U}(\mathfrak{g}) \otimes \mathbb{R} \to \mathbb{R}$ (with $\mathbb{R}$ as trivial $\mathfrak{g}$-representation) given by $1 \otimes 1 \mapsto 1$ and still obtain a complex:

$$0 \leftarrow \mathbb{R} \stackrel{ev}{\leftarrow} \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, \mathbb{R}) \stackrel{\delta^{dR}}{\leftarrow} \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, V) \stackrel{\delta^{dR}}{\leftarrow} \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, \Lambda^2 V) \leftarrow \ldots \quad .$$

Proof: The homomorphism δ^{dR} maps the induced ideal $I(\mathfrak{g}, \mathfrak{p}, \Lambda^k V)$ into $I(\mathfrak{g}, \mathfrak{p}, \Lambda^{k-1} V)$ since for the commutator with the induced action of $X \in \mathfrak{p}$ we find:

$$\begin{aligned} [\, \delta^{dR}, \mathrm{ind}_X\,] &= [\sum_i r_{t_i} \otimes \theta^i \lrcorner + 1 \otimes \delta_V, -r_X \otimes id_{C^k} + 1 \otimes X.\,] \\ &= [\sum_i r_{t_i} \otimes \theta^i \lrcorner, -r_X \otimes id_{C^k}\,] + [\sum_i r_{t_i} \otimes \theta^i \lrcorner, 1 \otimes X.\,] + [\, 1 \otimes \delta_V, 1 \otimes X.\,] \\ &= \sum_i r_{[\, t_i, X\,]} \otimes \theta^i \lrcorner + \sum_i r_{t_i} \otimes [\, \theta^i, X\,] \lrcorner + 1 \otimes [\, \delta_V, X.\,] \\ &= \sum_i -r_{[\, X, t_i\,]_{\mathfrak{p}}} \otimes \theta^i \lrcorner + 1 \otimes [\, X, t_i\,]_{\mathfrak{p}}.(\theta^i \lrcorner). \end{aligned}$$

Hence $\delta^{dR}\colon \mathrm{Verma}\,(\mathfrak{g},\mathfrak{p},\Lambda^k V) \to \mathrm{Verma}\,(\mathfrak{g},\mathfrak{p},\Lambda^{k-1}V)$ is well defined. For the composite we have:

$$
\begin{aligned}
\delta^{dR}\circ\delta^{dR} &= \delta^{dR}\circ\sum_i t_i\otimes\theta^i\lrcorner \;+\frac{1}{2}\,\delta^{dR}\circ\sum_{i,j}1\otimes[\,t_i,t_j\,]\wedge\theta^i\lrcorner\,\theta^j\lrcorner \\
&= \sum_{i,j}t_it_j\otimes\theta^j\lrcorner\,\theta^i\lrcorner \;+\frac{1}{2}\sum_{i,j,k}t_i\otimes[\,t_j,t_k\,]\wedge\theta^j\lrcorner\,\theta^k\lrcorner\,\theta^i\lrcorner \\
&\quad +\frac{1}{2}\sum_{i,j,k}t_k\otimes\theta^k\lrcorner\,([\,t_i,t_j\,]\wedge\theta^i\lrcorner\,\theta^j\lrcorner\,) \\
&\quad +\frac{1}{4}\sum_{i,j,k,l}1\otimes[\,t_k,t_l\,]\wedge\theta^k\lrcorner\,\theta^l\lrcorner\,([\,t_i,t_j\,]\wedge\theta^i\lrcorner\,\theta^j\lrcorner\,) \\
&= \frac{1}{2}\sum_{i,j,k}t_i\otimes[\,t_j,t_k\,]\wedge\theta^j\lrcorner\,\theta^k\lrcorner\,\theta^i\lrcorner \;-\frac{1}{2}\sum_{i,j,k}t_k\otimes[\,t_i,t_j\,]\wedge\theta^k\lrcorner\,\theta^i\lrcorner\,\theta^j\lrcorner \\
&\quad +\frac{1}{4}\sum_{i,j,k,l}1\otimes[\,t_k,[\,t_i,t_j\,]\,]\wedge\theta^k\lrcorner\,\theta^i\lrcorner\,\theta^j\lrcorner \\
&\quad -\frac{1}{4}\sum_{i,j,k,l}1\otimes[\,t_k,t_l\,]\wedge\theta^k\lrcorner\,([\,t_i,t_j\,]\wedge\theta^l\lrcorner\,\theta^i\lrcorner\,\theta^j\lrcorner\,) \\
&= -\frac{1}{4}\sum_{i,j,k,l}1\otimes[\,t_k,t_l\,]\wedge[\,t_i,t_j\,]\wedge\theta^k\lrcorner\,\theta^l\lrcorner\,\theta^i\lrcorner\,\theta^j\lrcorner = 0.
\end{aligned}
$$

For the last statement note $ev(\delta^{dR}(1\otimes a)) = ev(a\otimes 1) = a.1 = 0$. $\square$

Remark 6.39 If G is a Lie group with Lie algebra $\mathfrak{g}$ and $P := \{g\in G \,|\, \mathrm{Ad}_g(\mathfrak{p}) = \mathfrak{p}\}$ the (closed) sub group with Lie algebra $\mathfrak{p}$, then the above homomorphism δ^{dR} induces the exterior derivative as differential operator on G/P: indeed the Verma module homomorphism δ^{dR} corresponds to a differential operator like

$$d\colon C^\infty(G,\Lambda^{k-1}V^*)^P \to C^\infty(G,\Lambda^k V^*)^P,$$

which on a $(k-1)$-form $F\in C^\infty(G,\Lambda^{k-1}V^*)^P$ can be evaluated on a k-multivector $a\in\Lambda^k V$ at $g\in G$ to give

$$
\begin{aligned}
\langle (dF)(g),a\rangle &= \langle jet^\infty(F)(g),\delta^{dR}(1\otimes_{\mathfrak{U}(\mathfrak{p})}a)\rangle \\
&= \langle jet^\infty(F)(g),\sum_i t_i\otimes_{\mathfrak{U}(\mathfrak{p})}\theta^i\lrcorner\, a+\frac{1}{2}\sum_{i,j}1\otimes_{\mathfrak{U}(\mathfrak{p})}[\,t_i,t_j\,]\wedge\theta^i\lrcorner\,\theta^j\lrcorner\, a\rangle \\
&= \sum_i R_{t_i}\langle\theta^i\wedge F,a\rangle(g)+\frac{1}{2}\sum_{i,j}\langle\theta^j\wedge\theta^i\wedge[\,t_i,t_j\,]\lrcorner\, F,a\rangle(g).
\end{aligned}
$$

Remark 6.40 A complex of homomorphisms is called a *resolution*, if it is exact. From local exactness of the deRham complex, the Poincaré lemma, we deduce with the above remark that the deRham homomorphisms of proposition 6.38 form a resolution of $\mathbb{R}$.

Discussion 6.41 (*Hopf map*) The universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is an associative algebra, hence $\mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g})$ carries an obvious algebra structure. Moreover there is a coproduct, the *Hopf map*:

$$\triangle : \mathfrak{U}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g}),$$

defined to be an algebra homomorphism induced by

$$1 \mapsto 1 \otimes 1 \text{ and } X \mapsto X \otimes 1 + 1 \otimes X,$$

for $X \in \mathfrak{g}$. Similarly if W_1^* and W_2^* are left $\mathfrak{g}$-representations then their tensor product $W_1^* \otimes W_2^*$ is a left $\mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g})$ module, hence $\mathfrak{U}(\mathfrak{g})$ acts upon $W_1^* \otimes W_2^*$ through $\triangle$.

Proposition 6.42 *(Co-Leibniz rule) The wedge coproduct $\triangle^{kl} : \Lambda^{k+l}V \to \Lambda^k V \otimes \Lambda^l V$ defined on $c \in \Lambda^{k+l}V$ with the help of $\alpha \in \Lambda^k V^*$ and $\beta \in \Lambda^l V^*$ by:*

$$\langle \triangle^{kl}\, c, \alpha \otimes \beta \rangle := \langle c, \alpha \wedge \beta \rangle,$$

is $\mathfrak{p}$-equivariant. This induces $\mathfrak{g}$-equivariant pairings

$$\triangle^{kl} : \mathfrak{U}(\mathfrak{g}) \otimes \Lambda^{k+l}V \to (\mathfrak{U}(\mathfrak{g}) \otimes \Lambda^k V) \otimes (\mathfrak{U}(\mathfrak{g}) \otimes \Lambda^l V), \quad \triangle^{kl}(U \otimes c) := \triangle U \otimes \triangle^{kl} c,$$

and

$$\triangle^{kl} : \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, \Lambda^{k+l}V) \to \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, \Lambda^k V) \otimes \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, \Lambda^l V).$$

This pairing is compatible with the deRham homomorphisms in the sense that the following co-Leibniz rule holds:

$$\triangle^{kl} \circ \delta^{dR} = (\delta^{dR} \otimes id) \circ \triangle^{k+1,l} + (-1)^k (id \otimes \delta^{dR}) \circ \triangle^{k,l+1}.$$

Proof: The $\mathfrak{p}$-equivariance of $\triangle^{kl}$ is clear. To verify the Co-Leibniz rule note that the associativity of the wedge product with $\theta \in V^*$ means $\triangle^{kl} \circ \theta \lrcorner = (\theta \lrcorner \otimes id) \circ \triangle^{k+1,l} = (-1)^k (id \otimes \theta \lrcorner) \circ \triangle^{k,l+1}$. Similar the interior multiplication by $t \in V$ is a derivation hence $\triangle^{kl} \circ t \wedge = (t \wedge \otimes id) \circ \triangle^{k-1,l} + (-1)^k (id \otimes t \wedge) \circ \triangle^{k,l-1}$. With this in hands we find:

$$\begin{aligned}
\triangle^{kl} \circ \delta^{dR} &= \sum_i (t_i \otimes 1 + 1 \otimes t_i) \otimes \triangle^{kl} \circ \theta^i \lrcorner + \frac{1}{2} \sum_{i,j} (1 \otimes 1) \otimes \triangle^{kl} \circ [\, t_i, t_j \,] \wedge \theta^i \lrcorner \theta^j \lrcorner \\
&= \sum_i (t_i \otimes 1) \otimes (\theta^i \lrcorner \otimes id) \circ \triangle^{k+1,l} + (-1)^k \sum_i (1 \otimes t_i) \otimes (id \otimes \theta^i \lrcorner) \circ \triangle^{k,l+1} \\
&\quad + \frac{1}{2} \sum_{i,j} (1 \otimes 1) \otimes ([\, t_i, t_j \,] \wedge \theta^i \lrcorner \theta^j \lrcorner \otimes id) \circ \triangle^{k+1,l} \\
&\quad + \frac{1}{2} (-1)^k \sum_{i,j} (1 \otimes 1) \otimes (id \otimes [\, t_i, t_j \,] \wedge \theta^i \lrcorner \theta^j \lrcorner) \circ \triangle^{k,l+1} \\
&= (\delta^{dR} \otimes id) \circ \triangle^{k+1,l} + (-1)^k (id \otimes \delta^{dR}) \circ \triangle^{k,l+1}. \quad \square
\end{aligned}$$

In what follows we will twist the deRham homomorphism and the wedge coproduct by a left $\mathfrak{g}$-representation W^*. This is easily done after appreciating the following proposition: if W^* is a left $\mathfrak{g}$-representation, then $\mathfrak{g}$ acts on $(\mathfrak{U}(\mathfrak{g}) \otimes E^*) \otimes W^*$ by the product rule. Also W^* is a left $\mathfrak{p}$-representation by restriction and therefore $(E^* \otimes W^*)$ is a left $\mathfrak{p}$-representation.

Proposition 6.43 *There is an isomorphism between $\mathfrak{g}$-representations:*

$$\Psi : \mathfrak{U}(\mathfrak{g}) \otimes (E^* \otimes W^*) \xrightarrow{\cong} (\mathfrak{U}(\mathfrak{g}) \otimes E^*) \otimes W^*,$$

induced by $1 \otimes (\eta \otimes \omega) \mapsto (1 \otimes \eta) \otimes \omega$. For $X \in \mathfrak{g}$ this means $\Psi(X \otimes (\eta \otimes \omega)) = (X \otimes \eta) \otimes \omega + (1 \otimes \eta) \otimes X.\omega$ and hence $\Psi^{-1}((X \otimes \eta) \otimes \omega) = X \otimes (\eta \otimes \omega) - 1 \otimes (\eta \otimes X.\omega)$. This isomorphism maps $I(\mathfrak{g}, \mathfrak{p}, (E^* \otimes W^*))$ onto $I(\mathfrak{g}, \mathfrak{p}, E^*) \otimes W^*$ and hence descends to an isomorphism

$$\Psi : \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, E^* \otimes W^*) \xrightarrow{\cong} \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, E^*) \otimes W^*.$$

Definition 6.44 (*Twisted deRham homomorphism*) If W^* is a left $\mathfrak{g}$-representation, then $\delta^{dR} \otimes id_{W^*}$ leads with the above isomorphism to the so called *twisted deRham homomorphism* between the Verma modules induced by $C^k(V^*, W^*) := \Lambda^k V \otimes W^*$:

$$\delta := \Psi^{-1} \circ (\delta^{dR} \otimes id_{W^*}) \circ \Psi \;:\; \mathfrak{U}(\mathfrak{g}) \otimes (\Lambda^k V \otimes W^*) \to \mathfrak{U}(\mathfrak{g}) \otimes (\Lambda^{k-1} V \otimes W^*).$$

Proposition 6.45 *The twisted deRham homomorphism is induced by the $\mathfrak{g}$-homomorphism defined on $a \otimes \omega \in \Lambda^k V \otimes W^*$ by*

$$\delta(a \otimes \omega) = \sum_i t_i \otimes \theta^i \lrcorner\, a \otimes \omega + 1 \otimes \delta_V(a \otimes \omega),$$

where δ_V is the boundary operator of paragraph 6.35. The ideal $I(\mathfrak{g}, \mathfrak{p}, (\Lambda^k V \otimes W^*))$ is mapped into $I(\mathfrak{g}, \mathfrak{p}, (\Lambda^{k-1} V \otimes W^*))$ hence we have and induced map

$$\delta : \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, C^k(V^*, W^*)) \to \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, C^{k-1}(V^*, W^*)).$$

Proof: We simply have to unravel the definitions:

$$\begin{aligned}
\delta(a \otimes \omega) &= \Psi^{-1} \circ (\delta^{dR} \otimes id_{W^*}) \circ \Psi(a \otimes \omega) \\
&= \Psi^{-1} \circ (\delta^{dR} \otimes id_{W^*})(a \otimes \omega) \\
&= \Psi^{-1}(\delta^{dR} a \otimes \omega) \\
&= \Psi^{-1}(\sum_i t_i \otimes \theta^i \lrcorner\, a \otimes \omega + \frac{1}{2} \sum_{i,j} 1 \otimes [\, t_i, t_j\,] \wedge \theta^i \lrcorner\, \theta^j \lrcorner\, a \otimes \omega) \\
&= \sum_i (t_i \otimes \theta^i \lrcorner\, a \otimes \omega - 1 \otimes \theta^i \lrcorner\, a \otimes t_i.\omega) + \frac{1}{2} \sum_{i,j} 1 \otimes [\, t_i, t_j\,] \wedge \theta^i \lrcorner\, \theta^j \lrcorner\, a \otimes \omega \\
&= \sum_i t_i \otimes \theta^i \lrcorner\, a \otimes \omega + 1 \otimes \delta_V(a \otimes \omega).
\end{aligned}$$

For the commutator with the induced action of $X \in \mathfrak{p}$ we have:

$$[\, \delta, \mathrm{ind}_X\,] = \sum_i -r_{[\,X, t_i\,]_\mathfrak{p}} \otimes \theta^i \lrcorner + 1 \otimes [\, X, t_i\,]_\mathfrak{p}.(\theta^i \lrcorner). \quad \Box$$

Remark 6.46 Note that $C^k(V^*, W^*) = C_k(V, W^*)$ and that in this formula the $\mathfrak{p}^*$-equivariant boundary operator δ_V of the V homology with values in W^* occurs as zero order part: the twisted deRham homomorphism can therefore be viewed as a $\mathfrak{p}$-equivariant extension of $\delta_V : C^k(V^*, W^*) \to C^{k-1}(V^*, W^*)$ to $\delta : C^k(V^*, W^*) \to \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, C^{k-1}(V^*, W^*))$

Proposition 6.47 *(Twisted deRham resolution) The twisted deRham homomorphisms define a complex $\delta \circ \delta = 0$. In degree zero $k = 0$ we can replace the trivial homomorphism $\delta = 0$ by the $\mathfrak{g}$-equivariant evaluation $\mathrm{ev} : \mathrm{Verma}\,(W^*) \to W^*$ generated by $1 \otimes_{\mathfrak{U}(\mathfrak{p})} \omega \mapsto \omega$ and still obtain a complex:*

$$0 \leftarrow W^* \xleftarrow{\mathrm{ev}} \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, W^*) \xleftarrow{\delta} \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, V \otimes W^*) \xleftarrow{\delta} \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, \Lambda^2 V \otimes W^*) \leftarrow \dots \quad .$$

Proposition 6.48 *(Twisted Co-Leibniz rule) Let W_1^*, W_2^* and W_3^* be three $\mathfrak{g}$-representations and $F^* : W_3^* \to W_1^* \otimes W_2^*$ be a (non trivial) $\mathfrak{g}$-equivariant homomorphism. Then F^* together with the wedge coproduct $\triangle^{kl} : \Lambda^{k+l} V \to \Lambda^k V \otimes \Lambda^l V$ defines a $\mathfrak{p}$-equivariant coproduct on the level of the cochains $\triangle^{kl} \otimes F^* : C^{k+l}(V^*, W_3^*) \to C^k(V^*, W_1^*) \otimes C^l(V^*, W_2^*)$ and thus induces a $\mathfrak{g}$-equivariant pairing:*

$$\triangle^{kl} : \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, C^{k+l}(V^*, W_3^*)) \to \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, C^k(V^*, W_1^*)) \otimes \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, C^l(V^*, W_2^*)).$$

For $k = l = 0$ this coproduct satisfies

$$F^* \circ \mathrm{ev} = (ev \otimes ev) \circ \triangle^{00},$$

and for general k, l the following Co-Leibniz rule holds for the twisted deRham operators:

$$(\triangle^{kl} \otimes F^*) \circ \delta = (\delta \otimes id) \circ (\triangle^{k+1,l} \otimes F^*) + (-1)^k (id \otimes \delta) \circ (\triangle^{k,l+1} \otimes F^*).$$

6.5 Projection on Verma cochains

From now on we assume that the left $\mathfrak{g}$-representation W^* is a finite dimensional representation. In that case Kostant's Hodge theory provides a splitting of $C^k(V^*, W^*) := \Lambda^k V \otimes W^*$ (the chains of the V^* cohomology with values in W^*) into a direct sum, where one summand is isomorphic to $H^k(V^*, W^*)$. This splitting is not $\mathfrak{p}$-invariant, but only $\mathfrak{l}$-invariant. The basic ingredient is the so called quabla operator defined by the following anti commutator:

$$\Box_{\mathfrak{l}} := d_{V^*} \circ \delta_V + \delta_V \circ d_{V^*} \quad : \quad C^k(V^*, W^*) \to C^k(V^*, W^*).$$

Since d_{V^*} is $\mathfrak{p}$-equivariant and δ_V is $\mathfrak{p}^*$ equivariant, the quabla operator is only $\mathfrak{l} = \mathfrak{p} \cap \mathfrak{p}^*$ equivariant. Clearly quabla commutes with d_{V^*}, in particular quabla maps the image $B^k(V^*, W^*) := \mathrm{im}\,(d_{V^*})$ to itself.

Theorem 6.49 *(Kostant's inversion) For a finite dimensional $\mathfrak{g}$-module W^*, Kostant's quabla operator $\Box_{\mathfrak{l}} = d_{V^*}\delta_V + \delta_V d_{V^*}$ is invertible on the image of the coboundary operator $\mathrm{im}\,(d_{V^*}) = B^k(V^*, W^*)$.*

Proof: In [Kos61] Kostant demonstrates that $\Box_{\mathfrak{l}}$ is diagonalizable. He calculates the eigenvalues of $\Box_{\mathfrak{l}}$ which are nonzero on the image of d_{V^*}. $\Box$

Corollary 6.50 *The above proposition allows to define a homomorphism like*

$$Q_{\mathfrak{l}} := \Box_{\mathfrak{l}}^{-1} \circ d_{V^*} \ : \ C^{k-1}(V^*, W^*) \to C^k(V^*, W^*),$$

which maps into $B^k(V^*, W^*)$. Furthermore we define a map

$$S_{\mathfrak{l}} = id - \delta_V \circ Q_{\mathfrak{l}} - Q_{\mathfrak{l}} \circ \delta_V \ : \ C^k(V^*, W^*) \to C^k(V^*, W^*),$$

which satisfies the following properties:

$$\begin{aligned} d_{V^*} \circ S_{\mathfrak{l}} &= 0, \\ S_{\mathfrak{l}} \circ d_{V^*} &= 0, \\ \mathrm{proj} \circ S \circ \mathrm{repr} &= id : H^k(V^*, W^*) \to H^k(V^*, W^*), \end{aligned}$$

where $\mathrm{proj} : Z^k(V^*, W^*) \to H^k(V^*, W^*)$ *denotes the $\mathfrak{p}$-equivariant projection from the kernel of d_{V^*} to cohomology and* repr *means the choice of a representative of the cohomology class. Furthermore we have*

$$\begin{aligned} S_{\mathfrak{l}} \circ S_{\mathfrak{l}} &= S_{\mathfrak{l}}, \\ \delta_V \circ S_{\mathfrak{l}} &= S_{\mathfrak{l}} \circ \delta_V. \end{aligned}$$

Proof: The properties have simple verifications:

$$\begin{aligned} d_{V^*} S_{\mathfrak{l}} &= d_{V^*} - d_{V^*} \delta_V \Box_{\mathfrak{l}}^{-1} d_{V^*} = d_{V^*} - \Box_{\mathfrak{l}} \Box_{\mathfrak{l}}^{-1} d_{V^*} = 0, \\ S_{\mathfrak{l}} d_{V^*} &= d_{V^*} - \Box_{\mathfrak{l}}^{-1} d_{V^*} \delta_V d_{V^*} = d_{V^*} - \Box_{\mathfrak{l}}^{-1} \Box_{\mathfrak{l}} d_{V^*} = 0. \end{aligned}$$

For $z \in Z^k(V^*, W^*)$ (i.e. $d_{V^*} z = 0$) we have $\mathrm{proj}\,(S_{\mathfrak{l}}(z)) = \mathrm{proj}\,(z - \delta_V Q_{\mathfrak{l}}(z) - Q_{\mathfrak{l}} \delta_V(z)) = \mathrm{proj}\,(z)$, since $Q_{\mathfrak{l}}(z) = \Box_{\mathfrak{l}}^{-1} d_{V^*} z = 0$ and $\mathrm{im}\,(Q_{\mathfrak{l}}) \subset \mathrm{im}\,(d_{V^*})$. For the projection property note:

$$Q_{\mathfrak{l}} \delta_V Q_l \ = \ \Box_{\mathfrak{l}}^{-1} d_{V^*} \delta_V \Box_{\mathfrak{l}}^{-1} d_{V^*} = \Box_{\mathfrak{l}}^{-1} \Box_{\mathfrak{l}} \Box_{\mathfrak{l}}^{-1} d_{V^*} = Q_{\mathfrak{l}},$$

which gives:

$$\begin{aligned} S_{\mathfrak{l}} S_{\mathfrak{l}} &= (id - \delta_V Q_{\mathfrak{l}} - Q_{\mathfrak{l}} \delta_V)^2 \\ &= id - \delta_V Q_{\mathfrak{l}} - Q_{\mathfrak{l}} \delta_V - \delta_V Q_{\mathfrak{l}} + (\delta_V Q_{\mathfrak{l}})^2 - \delta_V Q_{\mathfrak{l}} Q_{\mathfrak{l}} \delta_V - Q_{\mathfrak{l}} \delta_V + Q_{\mathfrak{l}} \delta_V \delta_V Q_{\mathfrak{l}} + (Q_{\mathfrak{l}} \delta_V)^2 \\ &= S_{\mathfrak{l}}, \\ \delta_V S_{\mathfrak{l}} &= \delta_V - \delta_V Q_{\mathfrak{l}} \delta_V = S_{\mathfrak{l}} \delta_V. \quad \Box \end{aligned}$$

Corollary 6.51 *The projection $S_{\mathfrak{l}}$ provides a splitting*

$$C^k(V^*, W^*) = \Lambda^k V \otimes W^* = \ker S_{\mathfrak{l}} \oplus \mathrm{im}\, S_{\mathfrak{l}},$$

with $\mathrm{im}\,(S_{\mathfrak{l}}) \subset \ker\,(d_{V^*})$ and the natural projection $\mathrm{proj} : \mathrm{im}\,(S_{\mathfrak{l}}) \to H^k(V^*, W^*)$ into cohomology is an isomorphism.

The next aim is to modify Kostant's $\mathfrak{l}$-equivariant projection from cochains to cohomology into a $\mathfrak{g}$-equivariant projection from the Verma module induced by the cochains to the Verma module induced by cohomology. In analogy we will construct a map

$$S\colon \mathrm{Verma}\,(\mathfrak{g},\mathfrak{p},C^k(V^*,W^*)) \to \mathrm{Verma}\,(\mathfrak{g},\mathfrak{p},C^k(V^*,W^*)),$$

with the following properties: S maps to the kernel of the coboundary operator d_{V^*}, it vanishes on the image of d_{V^*}, it is an extension of the canonical projection from the kernel of d_{V^*} onto the cohomology of d_{V^*}, and it commutes with the deRham homomorphism δ.

Remark 6.52 Recall that given a linear map between $\mathfrak{p}$-representations $\phi_0\colon F^* \to E^*$ we have an induced $\mathfrak{g}$-homomorphism $\phi := id \otimes \phi_0\colon \mathfrak{U}(\mathfrak{g}) \otimes F^* \to \mathfrak{U}(\mathfrak{g}) \otimes E^*$. This descends to the corresponding Verma modules iff ϕ_0 is $\mathfrak{p}$-equivariant. Therefore the boundary operator $d_{V^*}: C^k(V^*,W^*) \to C^{k+1}(V^*,W^*)$ gives a $\mathfrak{g}$-homomorphism between Verma modules, whereas the coboundary operator $\delta_V : C^k(V^*,W^*) \to C^{k-1}(V^*,W^*)$ does not since δ_V is only $\mathfrak{l}$-equivariant. Therefore Kostant's quabla operator $\Box_{\mathfrak{l}}\colon C^k(V^*,W^*) \to C^k(V^*,W^*)$ induces a $\mathfrak{g}$-equivariant homomorphism: $\Box_{\mathfrak{l}}\colon \mathfrak{U}(\mathfrak{g}) \otimes C^k(V^*,W^*) \to \mathfrak{U}(\mathfrak{g}) \otimes C^k(V^*,W^*)$, but this homomorphism does not descend to the Verma module.

Definition 6.53 The anti commutator of the twisted deRham homomorphism δ and the coboundary operator d_{V^*} defines a quabla operator

$$\Box := d_{V^*}\circ\delta + \delta\circ d_{V^*} \quad : \quad \mathfrak{U}(\mathfrak{g}) \otimes C^k(V^*,W^*) \to \mathfrak{U}(\mathfrak{g}) \otimes C^k(V^*,W^*),$$

which descends to the corresponding Verma modules. Clearly $\Box$ commutes with d_{V^*}, so it maps the image of d_{V^*} to itself:

$$\Box\colon \mathfrak{U}(\mathfrak{g}) \otimes B^k(V^*,W^*) \to \mathfrak{U}(\mathfrak{g}) \otimes B^k(V^*,W^*).$$

Theorem 6.54 *The difference between the two quabla operators $A_{\mathfrak{l}} := \Box - \Box_{\mathfrak{l}}$ commutes with d_{V^*}, so also $A_{\mathfrak{l}}$ maps $\mathfrak{U}(\mathfrak{g}) \otimes B^k(V^*,W^*)$ to itself. $A_{\mathfrak{l}}$ is given by the action of V^* on cochains as:*

$$A_{\mathfrak{l}} = \Box - \Box_{\mathfrak{l}} = \sum_i r_{t_i} \otimes \theta^i.$$

For a finite dimensional W^* the composite $({\Box_{\mathfrak{l}}}^{-1}\circ A_{\mathfrak{l}})^m(1\otimes c)$ vanishes for a large $m \in \mathbb{N}$ depending on $c \in B^k(V^*,W^*)$ and $\Box^{-1}$ defined by the following Neumann series

$$\Box^{-1} := \sum_{m=0}(-1)^m({\Box_{\mathfrak{l}}}^{-1}\circ A_{\mathfrak{l}})^m \circ {\Box_{\mathfrak{l}}}^{-1},$$

gives a two sided inverse of $\Box$ which leaves the ideal $I(\mathfrak{g},\mathfrak{p},B^k(V^*,W^*))$ invariant and hence descends to an endomorphism of the corresponding Verma module $\mathrm{Verma}\,(\mathfrak{g},\mathfrak{p},B^k(V^*,W^*))$.

Proof: We will use the symbol $\{,\}$ for the anti commutator of two maps. The difference $A_{\mathfrak{l}}$ is given by the following:

$$A_{\mathfrak{l}} \;=\; \{d_{V^*},\delta\} - \{d_{V^*},\delta_V\}$$

$$
\begin{aligned}
&= \{d_{V^*}, (\delta - \delta_V)\} \\
&= \sum_i t_i \otimes \{d_{V^*}, (\theta^i \lrcorner \otimes id_{W^*})\} \\
&= \sum_i t_i \otimes \theta^i. \quad ,
\end{aligned}
$$

where we used Cartan's identity in the last equality. The action of V^* on $C^k(V^*, W^*)$ lowers the $\mathfrak{Z}(\mathfrak{l})$ weight in $C^k(V^*, W^*)$. Now $\Box_{\mathfrak{l}}{}^{-1}$ restricted to $B^k(V^*, W^*) = \operatorname{im} d_{V*}$ is $\mathfrak{l}$-equivariant, and hence when applied leaves the weight invariant. Consequently $\Box_{\mathfrak{l}}{}^{-1} \circ A_{\mathfrak{l}}$ also lowers the weight. Since $\Lambda^k V \otimes W^*$ is a finite sum of lowest weight modules it follows that $(\Box_{\mathfrak{l}}{}^{-1} \circ A_{\mathfrak{l}})^m (1 \otimes c)$ vanishes for large enough $m \in \mathbb{N}$ for all $c \in B^k(V^*, W^*)$. Obviously $\Box^{-1}$ maps $\mathfrak{U}(\mathfrak{g}) \otimes B^k(V^*, W^*)$ to itself (since so do $\Box_{\mathfrak{l}}{}^{-1}$ and $A_{\mathfrak{l}}$). Then its clear that restricted to $\mathfrak{U}(\mathfrak{g}) \otimes B^k(V^*, W^*)$ we find $\Box^{-1} \circ \Box = id$ and $\Box \circ \Box^{-1} = id$. We recall the commutators between the relevant operators and the induced $\mathfrak{p}$-action:

$$
\begin{aligned}
[\, d_{V^*}, \mathrm{ind}_X \,] &= 0, \\
[\, \delta, \mathrm{ind}_X \,] &= \sum_i (\mathrm{ind} - l)_{[\, X, t_i \,]_{\mathfrak{p}}} \circ (1 \otimes \theta^i \lrcorner), \\
[\, \Box, \mathrm{ind}_X \,] &= [\, \{d_{V^*}, \delta\}, \mathrm{ind}_X \,] = \{d_{V^*}, [\, \delta, \mathrm{ind}_X \,]\} \\
&= \{d_{V^*}, \sum_i (\mathrm{ind} - l)_{[\, X, t_i \,]_{\mathfrak{p}}} \circ (1 \otimes \theta^i \lrcorner)\} \\
&= \sum_i (\mathrm{ind} - l)_{[\, X, t_i \,]_{\mathfrak{p}}} \circ (1 \otimes \{d_{V^*}, \theta^i \lrcorner\}) \\
&= \sum_i (\mathrm{ind} - l)_{[\, X, t_i \,]_{\mathfrak{p}}} (1 \otimes \theta^i.), \\
[\, \Box^{-1}, \mathrm{ind}_X \,] &= -\Box^{-1} [\, \Box, \mathrm{ind}_X \,] \Box^{-1}.
\end{aligned}
$$

From the last two equations we deduce:

$$
\Box^{-1} (\mathrm{ind} - l)_X \;=\; (\mathrm{ind} - l)_X \Box^{-1} + \sum_i \Box^{-1} (\mathrm{ind} - l)_{[\, t_i, X \,]_{\mathfrak{p}}} (1 \otimes \theta^i.) \Box^{-1}.
$$

We now iterate this equation to obtain:

$$
\begin{aligned}
&\Box^{-1} (\mathrm{ind} - l)_X \\
&= (\mathrm{ind} - l)_X \Box^{-1} \\
&\quad + \sum_{m=1}^{M-1} \sum_{i_1, \ldots, i_m} (\mathrm{ind} - l)_{[\, t_{i_m}, \ldots [\, t_{i_1}, X \,]_{\mathfrak{p}} \ldots \,]_{\mathfrak{p}}} \Box^{-1} (1 \otimes \theta^{i_m}.) \ldots \Box^{-1} (1 \otimes \theta^{i_1}.) \Box^{-1} \\
&\quad + \sum_{i_1, \ldots, i_M} \Box^{-1} (\mathrm{ind} - l)_{[\, t_{i_M}, \ldots [\, t_{i_1}, X \,]_{\mathfrak{p}} \ldots \,]_{\mathfrak{p}}} (1 \otimes \theta^{i_M}.) \Box^{-1} \ldots (1 \otimes \theta^{i_1}.) \Box^{-1}.
\end{aligned}
$$

For a large M the last sum is zero, since $\mathfrak{p}$ has a highest $\mathfrak{Z}(\mathfrak{l})$ weight with respect to the adjoint V action. Hence the commutator with the induced $\mathfrak{p}$-action is:

$$
[\, \Box^{-1}, \mathrm{ind}_X \,] = \sum_{m=1} \sum_{i_1, \ldots, i_m} (\mathrm{ind} - l)_{[\, t_{i_m}, \ldots [\, t_{i_1}, X \,]_{\mathfrak{p}} \ldots \,]_{\mathfrak{p}}} \Box^{-1} (1 \otimes \theta^{i_m}.) \ldots \Box^{-1} (1 \otimes \theta^{i_1}.) \Box^{-1} \;,
$$

which maps into the ideal $I(\mathfrak{g}, \mathfrak{p}, B^k(V^*, W^*))$. $\square$

Corollary 6.55 *(Projection map) From above we have a well defined $\mathfrak{g}$-equivariant homomorphism*

$$Q := \square^{-1} \circ d_{V^*} : \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, C^{k-1}(V^*, W^*)) \to \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, C^k(V^*, W^*)).$$

The homomorphism S from $\mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, C^k(V^*, W^*))$ *to itself defined by*

$$S := id - \delta \circ Q - Q \circ \delta,$$

satisfies the following properties:

$$\begin{aligned} d_{V^*} \circ S &= 0, \\ S \circ d_{V^*} &= 0, \\ \mathrm{proj} \circ S \circ \mathrm{repr} &= id \text{ on } \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, H^k(V^*, W^*)), \end{aligned}$$

where $\mathrm{proj} : Z^k(V^*, W^*) \to H^k(V^*, W^*)$ *denotes the $\mathfrak{p}$-equivariant projection from the kernel of d_{V^*} to cohomology and* repr *means the choice of a representative of the cohomology class. Furthermore we have*

$$\begin{aligned} S \circ S &= S, \\ \delta \circ S &= S \circ \delta, \\ ev \circ S &= ev \text{ in degree } k = 0. \end{aligned}$$

Proof: The calculations are the same as in the presented version of Kostant's Hodge theory for $S_{\mathfrak{l}}$. For $d_{V^*}z = 0$ we know $Sz = z - Q \circ \delta(z)$ and Q maps into $\mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, B^k(V^*, W^*))$, hence $\mathrm{proj}\,(Sz) = \mathrm{proj}\,(z)$. For $k = 0$ note $S = id - \delta \circ Q$ and $ev \circ \delta = 0$, hence $ev \circ S = ev$. $\square$

Corollary 6.56 *The projection S provides a splitting*

$$\mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, C^k(V^*, W^*)) = \ker S \oplus \mathrm{im}\, S,$$

with $\mathrm{im}\,(S) \subset \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, Z^k(V^*, W^*))$ *and the natural projection*

$$\mathrm{proj} : \mathrm{im}\,(S) \to \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, H^k(V^*, W^*))$$

is an isomorphism onto cohomology.

Remark 6.57 The projection S can be viewed as a chain map from the twisted deRham complex to itself, since $\delta \circ S = S \circ \delta$. From the definition of S the map Q is a chain homotopy between S and id such that the induced map of S in the twisted deRham homology is the identity.

6.6 Bernstein Gelfand Gelfand resolution

Again W^* is a finite dimensional module for a semisimple $\mathfrak{g}$ and $\mathfrak{p} \subset \mathfrak{g}$ is a parabolic subalgebra with complement $V \oplus \mathfrak{p} = \mathfrak{g}$, see definition 6.14. In this context we have constructed a projection $S\colon \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, \Lambda^k V \otimes W^*) \to \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, \Lambda^k V \otimes W^*)$, see corollary 6.55, between the Verma modules, see definition 6.6, induced by the twisted forms $\Lambda^k V \otimes W^*$. On these forms $C^k(V^*, W^*) := \Lambda^k V \otimes W^*$ acts the $\mathfrak{p}$-equivariant coboundary operator d_{V^*}, with cohomology spaces denoted by $H^k(V^*, W^*)$, see paragraph 6.33. The two additional properties $d_{V^*} \circ S = 0$ and $S \circ d_{V^*} = 0$ allow to define two $\mathfrak{p}$-equivariant homomorphisms

$$\begin{aligned} \mathrm{proj} \circ S \quad &: \quad \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, C^k(V^*, W^*)) \to \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, H^k(V^*, W^*)), \text{ and} \\ S \circ \mathrm{repr} \quad &: \quad \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, H^k(V^*, W^*)) \to \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, C^k(V^*, W^*)), \end{aligned}$$

where $\mathrm{proj}\, : Z^k(V^*, W^*) \to H^k(V^*, W^*)$ denotes the $\mathfrak{p}$-equivariant projection from the kernel of d_{V^*} to cohomology and repr means the choice of a representative of the cohomology class.

In the dual geometric picture Čap, Slovák and Souček in [CSS99] constructed the homomorphism $\mathrm{proj} \circ S$ even in curved rather then homogeneous parabolic geometry. Their construction of this map is given by an inductive process. Our construction involves a Neumann series. Moreover to construct bilinear pairings we need the full S rather then just $\mathrm{proj} \circ S$.

Definition 6.58 (*Bernstein Gelfand Gelfand homomorphisms*) From the twisted deRham resolution, see proposition 6.47,

$$0 \leftarrow W^* \overset{ev}{\leftarrow} \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, W^*) \overset{\delta}{\leftarrow} \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, V \otimes W^*) \overset{\delta}{\leftarrow} \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, \Lambda^2 V \otimes W^*) \overset{\delta}{\leftarrow} \ldots,$$

we can now define $\mathfrak{g}$-homomorphisms between the induced Verma modules corresponding to the cohomology spaces:

$$\delta_H := \mathrm{proj}\, \circ S \circ \delta \circ S \circ \mathrm{repr}\, : \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, H^k(V^*, W^*)) \to \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, H^{k-1}(V^*, W^*)).$$

The zero cohomology $H^0(V^*, W^*) = W^{*V^*}$ is the space of invariants in W^* with respect to V^*. This is a $\mathfrak{p}$-invariant subspace, so that the inclusion $H^0(V^*, W^*) \hookrightarrow W^*$ is $\mathfrak{p}$-equivariant, inducing a $\mathfrak{g}$-equivariant homomorphism:

$$ev_H : \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, H^0(V^*, W^*)) \to W^*,$$

which could also be viewed as $ev_H = ev \circ \mathrm{repr}$.

Theorem 6.59 *(Parabolic Bernstein Gelfand Gelfand resolution) The resulting sequence*

$$0 \leftarrow W^* \overset{ev_H}{\longleftarrow} \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, H^0(V^*, W^*)) \overset{\delta_H}{\leftarrow} \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, H^1(V^*, W^*)) \overset{\delta_H}{\leftarrow} \ldots,$$

defines a complex $\delta_H \circ \delta_H = 0$, $ev_H \circ \delta_H = 0$ and ev_H is surjective. The two homomorphisms $S \circ \mathrm{repr}$ and $\mathrm{proj} \circ S$ are chain maps going back and forth between the twisted deRham complex and the Bernstein Gelfand Gelfand complex (on W^* the chain map is given by the identity). These chain maps induce an isomorphism between the homologies of the two complexes. Since twisted deRham is a resolution, so is the BGG complex.

Proof: Unraveling the definitions and using $S^2 = S$, $[S, \delta] = 0$, $ev \circ S = ev$ and $ev \circ \delta = 0$ gives:

$$\begin{aligned}
\delta_H \circ \delta_H &= (\mathrm{proj} \circ S \circ \delta \circ S \circ \mathrm{repr}) \circ (\mathrm{proj} \circ S \circ \delta \circ S \circ \mathrm{repr}) \\
&= (\mathrm{proj} \circ S \circ \delta \circ S \circ \delta \circ S \circ \mathrm{repr}) \\
&= (\mathrm{proj} \circ S \circ \delta \circ \delta \circ S \circ \mathrm{repr}) = 0, \\
ev_H \circ \delta_H &= (ev \circ \mathrm{repr}) \circ (\mathrm{proj} \circ S \circ \delta \circ S \circ \mathrm{repr}) \\
&= ev \circ S \circ \delta \circ S \circ \mathrm{repr} \\
&= ev \circ \delta \circ S \circ \mathrm{repr} = 0.
\end{aligned}$$

The evaluation $ev \colon \mathrm{Verma}(\mathfrak{g}, \mathfrak{p}, W^*) \to W^*$ is surjective and $ev_H \circ \mathrm{proj} \circ S = ev$, hence so is ev_H. The two homomorphisms $S \circ \mathrm{repr}$ and $\mathrm{proj} \circ S$ are chain maps since

$$\begin{aligned}
\delta_H \circ (\mathrm{proj} \circ S) &= (\mathrm{proj} \circ S \circ \delta \circ S \circ \mathrm{repr}) \circ (\mathrm{proj} \circ S) \\
&= \mathrm{proj} \circ S \circ \delta \circ S \\
&= (\mathrm{proj} \circ S) \circ \delta, \\
\mathrm{ev}_H \circ (\mathrm{proj} \circ S) &= id \circ \mathrm{ev}, \\
(S \circ \mathrm{repr}) \circ \delta_H &= (S \circ \mathrm{repr}) \circ (\mathrm{proj} \circ S \circ \delta \circ S \circ \mathrm{repr}) \\
&= S \circ \delta \circ S \circ \mathrm{repr} \\
&= \delta \circ (S \circ \mathrm{repr}), \\
id \circ \mathrm{ev}_H &= \mathrm{ev} \circ (S \circ \mathrm{repr}).
\end{aligned}$$

These chain maps induce an isomorphism between the homologies of the two complexes, since $(\mathrm{proj} \circ S) \circ (S \circ \mathrm{repr}) = \mathrm{proj} \circ S \circ \mathrm{repr} = id$. □

Conjecture 6.60 The complex constructed above is equivalent to the resolution of W^* in terms of Verma modules, which was originally constructed by Verma [Ver68] and Bernstein Gelfand Gelfand [BGG71] in the case where $\mathfrak{p} \subset \mathfrak{g}$ is a Borel subalgebra and later was generalized by Lepowsky [Lep77] to the case where $\mathfrak{p}$ is a parabolic subalgebra.

Theorem 6.61 *(Co-Leibniz rule) Let W_1^*, W_2^* and W_3^* be three $\mathfrak{g}$-representations which are finite sums of lowest weight modules. Let $F^* \colon W_3^* \to W_1^* \otimes W_2^*$ be a (nontrivial) $\mathfrak{g}$-equivariant homomorphism. Then F^* together with the wedge coproduct $\triangle^{kl} \colon \Lambda^{k+l}V \to \Lambda^k V \otimes \Lambda^l V$ and the projection S define a $\mathfrak{g}$-equivariant coproduct on the Verma modules from cohomology*

$$\triangle_H^{kl} \colon \mathrm{Verma}(\mathfrak{g}, \mathfrak{p}, H^{k+l}(V^*, W_3^*)) \to \mathrm{Verma}(\mathfrak{g}, \mathfrak{p}, H^k(V^*, W_1^*)) \otimes \mathrm{Verma}(\mathfrak{g}, \mathfrak{p}, H^l(V^*, W_2^*)),$$

given by

$$\triangle_H^{kl} := (\mathrm{proj} \circ S \otimes \mathrm{proj} \circ S) \circ (\triangle^{kl} \otimes F^*) \circ (S \circ \mathrm{repr}).$$

This is an extension of F^* in the sense that for $k = l = 0$ this coproduct satisfies

$$F^* \circ \mathrm{ev}_H = (\mathrm{ev}_H \otimes \mathrm{ev}_H) \circ \triangle_H^{00}.$$

For general k, l the coproduct satisfies the following Co-Leibniz rule with the Bernstein Gelfand Gelfand homomorphisms:

$$\triangle_H^{kl} \circ \delta_H = (\delta_H \otimes id) \circ \triangle_H^{k+1,l} + (-1)^k (id \otimes \delta_H) \circ \triangle_H^{k,l+1}.$$

Proof: The result demonstrates the flexibility of using S, since it follows directly from the Co-Leibniz rule of the twisted deRham complex:

$$\begin{aligned}(ev_H \otimes ev_H) \circ \triangle_H^{00} &= (ev \otimes ev) \circ (S \otimes S) \circ (\triangle^{00} \otimes F^*) \circ (S \circ \mathrm{repr}\,) \\ &= (ev \otimes ev) \circ (\triangle^{00} \otimes F^*) \circ (S \circ \mathrm{repr}\,) \\ &= F^* \circ \mathrm{ev} \circ (S \circ \mathrm{repr}\,) \\ &= F^* \circ \mathrm{ev} \circ \mathrm{repr} \\ &= F^* \circ \mathrm{ev}_H.\end{aligned}$$

Similarly:

$$\begin{aligned}&(\delta_H \otimes id)\triangle_H^{k+1,l} + (-1)^k (id \otimes \delta_H)\triangle_H^{k,l+1} \\ &= (\mathrm{proj}\, S\delta S\, \mathrm{repr}\ \otimes id)(\mathrm{proj}\, S \otimes \mathrm{proj}\, S)\triangle^{k+1,l}(S\, \mathrm{repr}\,) \\ &\quad +(-1)^k (id \otimes \mathrm{proj}\, S\delta S\, \mathrm{repr}\,)(\mathrm{proj}\, S \otimes \mathrm{proj}\, S)\triangle^{k,l+1}(S\, \mathrm{repr}\,) \\ &= (\mathrm{proj}\, S \otimes \mathrm{proj}\, S)(\delta \otimes id)\triangle^{k+1,l}(S\, \mathrm{repr}\,) \\ &\quad +(-1)^k(\mathrm{proj}\, S \otimes \mathrm{proj}\, S)(id \otimes \delta)\triangle^{k,l+1}(S\, \mathrm{repr}\,) \\ &= (\mathrm{proj}\, S \otimes \mathrm{proj}\, S)\triangle^{k,l}\delta(S\, \mathrm{repr}\,) \\ &= (\mathrm{proj}\, S \otimes \mathrm{proj}\, S)\triangle^{k,l} S\, \mathrm{repr}\ \mathrm{proj}\, S\delta(S\, \mathrm{repr}\,) \\ &= \triangle_H^{k,l}\delta_H. \quad \square\end{aligned}$$

6.7 Adjoint homomorphisms

For a left $\mathfrak{p}$-representation E^* we have a $\mathfrak{p}$-equivariant map $\Lambda^n V \to E^* \otimes (E \otimes \Lambda^n V)$ induced by the identity: if e_m, η^m is a dual basis of E then $\Lambda^n V \ni a \mapsto \sum_m \eta^m \otimes (e_m \otimes a)$. The coproduct $\triangle : \mathfrak{U}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g})$ then induces a coproduct

$$\triangle \otimes id_{E\otimes\Lambda^n V} : \mathfrak{U}(\mathfrak{g}) \otimes \Lambda^n V \to (\mathfrak{U}(\mathfrak{g}) \otimes E^*) \otimes (\mathfrak{U}(\mathfrak{g}) \otimes (E \otimes \Lambda^n V)),$$

which descends to the corresponding Verma modules.

Definition 6.62 Let E^*, F^* be left $\mathfrak{p}$-representations and

$$\phi : \mathfrak{U}(\mathfrak{g}) \otimes F^* \to \mathfrak{U}(\mathfrak{g}) \otimes E^*$$

be a $\mathfrak{g}$-homomorphism. A $\mathfrak{g}$-homomorphism

$$\phi^* : \mathfrak{U}(\mathfrak{g}) \otimes (E \otimes \Lambda^n V) \to \mathfrak{U}(\mathfrak{g}) \otimes (F \otimes \Lambda^n V)$$

is called an *adjoint to* ϕ if there is a $\mathfrak{g}$-equivariant coproduct

$$X_\phi : \mathfrak{U}(\mathfrak{g}) \otimes \Lambda^{n-1}V \to (\mathfrak{U}(\mathfrak{g}) \otimes E^*) \otimes (\mathfrak{U}(\mathfrak{g}) \otimes (F \otimes \Lambda^n V)),$$

such that with the deRham homomorphism

$$\delta^{dR} : \mathfrak{U}(\mathfrak{g}) \otimes \Lambda^n V \to \mathfrak{U}(\mathfrak{g}) \otimes \Lambda^{n-1}V$$

the following codivergence formula holds:

$$X_\phi \circ \delta^{dR} = (\phi \otimes id) \circ (\triangle \otimes id_{F\otimes\Lambda^n V}) - (id \otimes \phi^*) \circ (\triangle \otimes id_{E\otimes\Lambda^n V}).$$

A similar definitions holds for $\mathfrak{g}$-homomorphisms descending to the corresponding Verma modules.

Remark 6.63 In view of the definition of adjoint differential operators in paragraph 1.18 we remark that adjoint Verma module homomorphisms have corresponding differential operators on the homogeneous space G/P which are adjoint after one of them is twisted by the bundle of pseudoscalars $L^{-n} \otimes \Lambda^n V$. The Lie algebra $\mathfrak{p}$ acts trivially on $L^{-n} \otimes \Lambda^n V$, so we used $\Lambda^n V$ in the above algebraic definition rather then L^n.

Example 6.64 A *derivative* $D : \mathfrak{U}(\mathfrak{g}) \otimes (E^* \otimes V) \to \mathfrak{U}(\mathfrak{g}) \otimes E^*$ is defined to be generated by

$$D := \sum_i r_{t_i} \otimes id_{E^*} \otimes \langle \theta^i, \ \rangle.$$

It has an adjoint homomorphism $D^* : \mathfrak{U}(\mathfrak{g}) \otimes (E \otimes \Lambda^n V) \to \mathfrak{U}(\mathfrak{g}) \otimes (E \otimes V^* \otimes \Lambda^n V)$ given by

$$D^* := -\sum_i r_{t_i} \otimes (id_E \otimes \theta^i \otimes id_{\Lambda^n V}) - \sum_i 1 \otimes (id_E \otimes \theta^i \otimes t_i \wedge \delta_V).$$

The coproduct X_D is generated by the wedge product $\Lambda^{n-1}V \mapsto (E^*) \otimes (E \otimes V^* \otimes \Lambda^n V)$, as $X_D := \triangle \circ \sum_i 1 \otimes (id_E \otimes \theta^i \otimes t_i \wedge)$. For the composite $X_D \circ \delta^{dR}$ note $t \wedge (\theta \lrcorner) = \theta(t)$ on $\Lambda^n V$ and $\triangle \circ r_t = (r_t \otimes 1)\triangle + (1 \otimes r_t)\triangle$, hence:

$$\begin{aligned}
X_D \circ \delta^{dR} &= \sum_{i,j} \triangle \circ (1 \otimes id_E \otimes \theta^i \otimes t_i \wedge) \circ (r_{t_j} \otimes \theta^j \lrcorner + \delta_V) \\
&= \sum_i (r_{t_i} \otimes 1)\triangle(id_E \otimes \theta^i \otimes id_{\Lambda^n V}) + (1 \otimes r_{t_i})\triangle(id_E \otimes \theta^i \otimes id_{\Lambda^n V}) \\
&\quad + \sum_i (1 \otimes \theta^i \otimes t_i \wedge \delta_V)\triangle id_E \\
&= (D \otimes 1)\triangle \otimes id_{E\otimes V^*\otimes\Lambda^n V} - (1 \otimes D^*)\triangle \otimes id_{E\otimes\Lambda^n V}.
\end{aligned}$$

Example 6.65 (*Adjoints of deRham homomorphisms*) Notice once more the canonical isomorphism $\lrcorner : \Lambda^k V^* \otimes \Lambda^n V \to \Lambda^{n-k}V$. Consequently, for the identity $id_{\Lambda^k V^* \otimes \Lambda^n V}$ we have with the cowedge product: $\lrcorner \circ id_{\Lambda^k V^*\otimes\Lambda^n V} = \triangle^{k,n-k} : \Lambda^n V \to \Lambda^k V \otimes \Lambda^{n-k}V$. The Co-Leibniz rule shows that the adjoint of the deRham homomorphism

$$\begin{aligned}
\delta^{dR} &: \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, \Lambda^k V) \to \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, \Lambda^{k-1}V) \\
\delta^{dR*} &: \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, \Lambda^{k-1}V^* \otimes \Lambda^n V) \to \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, \Lambda^k V^* \otimes \Lambda^n V)
\end{aligned}$$

satisfies $\lrcorner\,\delta^{dR^*} = (-1)^k \delta^{dR} \circ \lrcorner$ with the deRham homomorphism acting on degree $(n-k+1)$ since

$$\triangle^{k-1,n-k} \circ \delta^{dR} \;=\; (\delta^{dR} \otimes id) \circ \triangle^{k,n-k} + (-1)^{k-1}(id \otimes \delta^{dR}) \circ \triangle^{k-1,n-k+1}.$$

Example 6.66 For the twisted deRham homomorphism we have $F^* = \Lambda^k V \otimes W^*$ and $F \otimes \Lambda^n V = \Lambda^{n-k} \otimes W$. The twisted Co-Leibniz rule applied to the three $\mathfrak{g}$-representations $\mathbb{R} \to W^* \otimes W$ shows that δ on $\Lambda^k V \otimes W^*$ is adjoint to $(-1)^k \delta$ on $\Lambda^{n-k} V \otimes W$.

Example 6.67 (*Adjoints of Bernstein Gelfand Gelfand homomorphisms*) Poincaré duality applied to $F^* = H^k(V^*, W^*)$ gives $F \otimes \Lambda^n V = H^{n-k}(V^*, W)$. The Co-Leibniz rule applied to the three $\mathfrak{g}$-representations $\mathbb{R} \to W^* \otimes W$ shows that δ_H on $H^k(V^*, W^*)$ is adjoint to $(-1)^k \delta_H$ on $H^{n-k}(V^*, W)$.

Remark 6.68 Let G^* be another left $\mathfrak{p}$-representations and

$$\psi\colon \mathfrak{U}(\mathfrak{g}) \otimes G^* \to \mathfrak{U}(\mathfrak{g}) \otimes F^*$$

another $\mathfrak{g}$-homomorphism with adjoint

$$\psi^*\colon \mathfrak{U}(\mathfrak{g}) \otimes (F \otimes \Lambda^n V) \to \mathfrak{U}(\mathfrak{g}) \otimes (G \otimes \Lambda^n V),$$

then $\phi \circ \psi$ has an adjoint given by $\psi^* \circ \phi^*$ with the coproduct

$$X_{\phi\psi}\colon \mathfrak{U}(\mathfrak{g}) \otimes \Lambda^{n-1} V \to (\mathfrak{U}(\mathfrak{g}) \otimes E^*) \otimes (\mathfrak{U}(\mathfrak{g}) \otimes (G \otimes \Lambda^n V)),$$

given by

$$X_{\phi\psi} := (\phi \otimes id) \circ X_\psi + (id \otimes \psi^*) \circ X_\phi.$$

Remark 6.69 The above remark indicates that the projection

$$S\colon \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, C^k(V^*, W^*)) \to \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, C^k(V^*, W^*)),$$

as composite of operators which all have adjoints, also has an adjoint. A candidate for such an adjoint is the projection S induced by W in complementary degree:

$$S\colon \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, C^{n-k}(V^*, W)) \to \mathrm{Verma}\,(\mathfrak{g}, \mathfrak{p}, C^{n-k}(V^*, W)).$$

Chapter 7

Appendix: Elementary representation theory

In this appendix we summarize basic notions of the representations theory of Lie algebras. It is taken from [FH91].

Definition 7.1 (*Lie algebras*) Let $\mathbb{F}$ be a field (we are interested in the cases $\mathbb{F} = \mathbb{R}$ and $\mathbb{F} = \mathbb{C}$). A finite dimensional vector space $\mathfrak{g}$ over $\mathbb{F}$ with a skew symmetric bilinear multiplication $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, denoted on $X, Y \in \mathfrak{g}$ by $X \otimes Y \mapsto [\,X, Y\,]$ is called a *Lie algebra*, if it satisfies the *Jacobi identity*: for all $X, Y, Z \in \mathfrak{g}$ we have $[\,X, [\,Y, Z\,]\,] + [\,Y, [\,Z, X\,]\,] + [\,Z, [\,X, Y\,]\,] = 0$.

If $\mathfrak{p} \subset \mathfrak{g}$ is a subspace, such that $X, Y \in \mathfrak{p}$ implies $[\,X, Y\,] \in \mathfrak{p}$, then $\mathfrak{p}$ is called a *subalgebra* of $\mathfrak{g}$. A linear map $\phi\colon \mathfrak{g} \to \mathfrak{h}$ between two Lie algebras is called a *Lie algebra homomorphism*, if $\phi([\,X, Y\,]) = [\,\phi(X), \phi(Y)\,]$ for all $X, Y \in \mathfrak{g}$ (this condition is nonlinear in ϕ similar to the orthogonality condition for a linear transformation).

Example 7.2 The space of endomorphisms $\mathrm{End}\,(V) = \mathfrak{gl}(V)$ of a n-dimensional vector space V over the field $\mathbb{F}$ carries a natural Lie algebra structure by $[\,A, B\,] := A \circ B - B \circ A$ and is called the *general linear Lie algebra*. It has dimension $\dim \mathfrak{gl}(V) = n^2$

Lie algebras which are isomorphic to subalgebras of some $\mathfrak{gl}(V)$ are called *linear Lie algebras*. Lie algebras with trivial Lie bracket are called *Abelian Lie algebras*. One dimensional Lie algebras are necessarily Abelian.

Definition 7.3 (*Modules*) If $\mathfrak{g}$ denotes a Lie algebra over $\mathbb{F}$, a vector space W (finite or infinite dimensional) is said to carry a *right* $\mathfrak{g}$*-module structure* or W is called a *right* $\mathfrak{g}$*-representation space*, if there is a linear map $\theta\colon W \otimes \mathfrak{g} \to W$ denoted by $\theta(w \otimes X) =: w.X$ such that for $Y \in \mathfrak{g}$ we have $w.[\,X, Y\,] = w.X.Y - w.Y.X$.

A $\mathfrak{g}$-representation from the left $\theta\colon \mathfrak{g} \otimes W \to W$ will be denoted by $\theta(X \otimes w) = X.w$ and satisfies $[\,X, Y\,].w = X.Y.w - Y.X.w$. Every left action has an associated right action defined by $w.X = -X.w$ and vice versa. If W is a right module then W^* is a left module defined (without signs) by $\langle X.\omega, w\rangle = \langle \omega, w.X\rangle$ for $\omega \in W^*$ and all $w \in W$.

Example 7.4 Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$ and $\mathfrak{g} \subset \mathfrak{gl}(V)$ a Lie sub algebra. Clearly, V is a $\mathfrak{g}$-representation, also its dual V^* and all tensor products $L^k \otimes V^i \otimes V^{*j}$ including densities L^k for $i, j \in \mathbb{N}$ and $k \in \mathbb{R}$.

Example 7.5 (*Adjoint representation*) For any Lie algebra $\mathfrak{g}$ the Jacobi identity shows, that the Lie bracket itself induces a left representation on $\mathfrak{g}$ itself, called *adjoint representation* $\mathrm{ad}\colon \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, $\mathrm{ad}(X)Y := [\,X, Y\,]$. The kernel of the adjoint representation is given by the centre $\ker \mathrm{ad} = \mathfrak{Z}(\mathfrak{g})$. The *coadjoint representation* $\mathrm{coad}\colon \mathfrak{g} \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ is given by: $\mathrm{coad} := -\mathrm{ad}^*$.

Definition 7.6 (*Enveloping algebra*) Let $\mathfrak{g}$ be a Lie algebra over the field $\mathbb{F}$. An *enveloping algebra* of $\mathfrak{g}$ is a pair $(\mathfrak{U}, i)$ consisting of an associative algebra $\mathfrak{U}$ over $\mathbb{F}$ with 1 and a linear map $i\colon \mathfrak{g} \to \mathfrak{U}$ satisfying $i([\,X, Y\,]) = i(X)i(Y) - i(Y)i(X)$ for all $X, Y \in \mathfrak{g}$.

The *universal enveloping algebra* $\mathfrak{U}(\mathfrak{g})$ for $\mathfrak{g}$ satisfies the following property: for any enveloping algebra $\mathfrak{A}, j$ there exists a unique homomorphism of algebras $\phi\colon \mathfrak{U}(\mathfrak{g}) \to \mathfrak{A}$ (sending 1 to 1) such that $\phi \circ i = j$. The universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ can be realized as $\mathfrak{U}(\mathfrak{g}) := T(\mathfrak{g})/I$, where $T(\mathfrak{g})$ is the graded tensor algebra and I is the two sided $T(\mathfrak{g})$ ideal generated by $<X \otimes Y - Y \otimes X - [\,X, Y\,] \,|\, X, Y \in \mathfrak{g}>$. This ideal respects the filtration of $T(\mathfrak{g})$ which induces a filtration of the quotient $\mathfrak{U}(\mathfrak{g}) = T(\mathfrak{g})/I$. If W^* is a left $\mathfrak{g}$-representation, then $\mathfrak{U}(\mathfrak{g})$ acts as algebra on W^*: for $XY \in \mathfrak{U}(\mathfrak{g})$ we define $(XY).\omega := X.Y.\omega$ with $X, Y \in \mathfrak{g}$ and $\omega \in W^*$.

Definition 7.7 (*Invariant submodules*) A subspace $W' \subset W$ of a $\mathfrak{g}$-representation $\theta\colon \mathfrak{g} \to \mathfrak{gl}(W)$ is called $\mathfrak{g}$-invariant, if $w' \in W'$ and $X \in \mathfrak{g}$ implies $\theta(X)(w') \in W'$. A vector $w \in W$ is called $\mathfrak{g}$-fix or $\mathfrak{g}$-invariant, if for all $X \in \mathfrak{g}$ we have $\theta(X)(w) = 0$. The set of $\mathfrak{g}$-invariant vectors forms a $\mathfrak{g}$-invariant subspace denoted by $W^{\mathfrak{g}} \subset W$.

Application 7.8 (*Linear Lie algebras defined by invariant tensors*) Let V be a vector space over $\mathbb{F}$ and $T \in V^i \otimes V^{*j}$ a nonzero tensor. Then $\mathfrak{gl}(V)$ acts on T as an element of the representation space $V^i \otimes V^{*j}$, hence T induces a linear map $\mathfrak{gl}(V) \to V^i \otimes V^{*j}$. Define the kernel of that map to be $\mathfrak{g}(V, T) := \{A \in \mathfrak{gl}(V) \,|\, T.A = 0\}$. This clearly defines a Lie subalgebra. The representation space V is then called the *defining representation* $\mathfrak{g}(V, T) \to \mathfrak{gl}(V)$.

Let V be a n-dimensional vector space over $\mathbb{F}$. The following Lie algebras defined by invariant tensors are called *classical Lie algebras*:

Definition 7.9 (*Special linear Lie algebra*) Let $\alpha \in \Lambda^n V^*$ denote a nonzero volume form, then define $\mathfrak{sl}(V, \alpha) := \{A \in \mathfrak{gl}(V) \,|\, 0 = \alpha.A = \alpha(A\ , \ldots,) + \ldots + \alpha(\ldots, A\ ) = \mathrm{tr}(A)\alpha\}$. This space of trace-free endomorphisms is clearly independent of the choice of α hence $\mathfrak{sl}(V) := \mathfrak{sl}(V, \alpha)$ is called the *special linear Lie algebra* and has dimension $\dim \mathfrak{sl}(V) = n^2 - 1$.

Definition 7.10 (*Symplectic Lie algebra*) If n is even and $\omega \in \Lambda^2 V^*$ denotes a nondegenerated bilinear skew form then $\mathfrak{sp}(V, \omega) := \{A \in \mathfrak{gl}(V) \,|\, \omega.A = \omega(A\ ,\ ) + \omega(\ , A\ ) = 0\}$ is called *symplectic Lie algebra* and has dimension $\dim \mathfrak{sp}(V, \omega) = n(n+1)/2$. Obviously, for $n = 2$ we have $\mathfrak{sl}(V) = \mathfrak{sp}(V)$.

Definition 7.11 (*Orthogonal Lie algebra*) If $g \in \mathrm{Sym}^2 V^*$ denotes a nondegenerate bilinear symmetric form then $\mathfrak{so}(V,g) := \{A \in \mathfrak{gl}(V) \,|\, g.A = g(A\ ,\ ) + g(\ ,A\ ) = 0\}$ is called the *orthogonal Lie algebra* and has dimension $\dim \mathfrak{so}(V,g) = n(n-1)/2$.

Remark 7.12 The special linear group $\mathfrak{sl}(n)$ is (n^2-1)-dimensional. On $\mathfrak{sl}(n)$ the following symmetric bilinear form g is invariant and non degenerated: $g(A,B) := \mathrm{tr}\, A \circ B$. Hence the adjoint representation induces a map $\mathrm{ad}\colon \mathfrak{sl}(n) \to \mathfrak{so}(\mathfrak{sl}(n), g)$ which defines an isomorphism for $n = 2$ when $\mathfrak{sl}(2)$ is three dimensional.

Remark 7.13 Let V be a $n = 2l$ even dimensional vector space with nondegenerated skew bilinear form $\omega \in \Lambda^2 V^*$. The space of trace-free bivectors (with respect to ω) $\Lambda^2_0 V$ forms a $((n(n-1)-2)/2)$- dimensional space, which carries a nondegenerated symmetric bilinear form g induced by:

$$g(v_1 \wedge v_2, w_1 \wedge w_2) := \omega(v_1, w_1)\omega(v_2, w_2) - \omega(v_1, w_2)\omega(v_2, w_1).$$

The defining representation V of $\mathfrak{sp}(n)$ clearly leaves g-invariant. For $n = 4$ the space $\Lambda^2_0 V$ is five dimensional and this defines an isomorphism $\mathfrak{sp}(V,\omega) = \mathfrak{so}(\Lambda^2_0 V, g)$.

Remark 7.14 Let V be a $n = 2l$ even dimensional vector space with nonzero volume form $\alpha \in \Lambda^n V^*$. The wedge product induces a non degenerated symmetric bilinear form g on $\Lambda^l V$: $g(a,b) := \alpha(a \wedge b)$ which is invariant under $\mathfrak{sl}(V)$. The defining representation induces a map $\mathfrak{sl}(V) \to \mathfrak{so}(\Lambda^l V, g)$. This is an isomorphism for $n = 4$ when $\dim \Lambda^l V = 6$.

Definitions 7.15 (*Automorphisms and derivations*) A linear isomorphism $\phi\colon \mathfrak{g} \to \mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ such that $\phi([\,X,Y\,]) = [\,\phi(X), \phi(Y)\,]$ for all $X, Y \in \mathfrak{g}$ is called an *automorphism*. The set of all automorphisms builds a group $\mathbf{Aut}\,(\mathfrak{g}) \subset \mathbf{GL}\,(\mathfrak{g})$. A linear map $\delta\colon \mathfrak{g} \to \mathfrak{g}$ is called a *derivation*, if $\delta([\,X,Y\,]) = [\,\delta(X), Y\,] + [\,X, \delta(Y)\,]$ for all $X, Y \in \mathfrak{g}$ (this is the linearized automorphism condition). The set of all derivations builds a Lie algebra $\mathfrak{Der}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$. The kernel of a derivation defines a subalgebra.

Definition 7.16 (*Ideals*) A sub space $I \subset \mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ over the field $\mathbb{F}$ is called an *ideal* of $\mathfrak{g}$, if $X \in \mathfrak{g}$ and $Y \in I$ implie $[\,X,Y\,] \in I$. The kernel of a Lie algebra homomorphism is an ideal. The *centre* of $\mathfrak{g}$ is defined to be $\mathfrak{Z}(\mathfrak{g}) := \{Z \in \mathfrak{g} \,|\, [\,X,Z\,] = 0, X \in \mathfrak{g}\}$ and defines a natural ideal. The *derived algebra* $[\,\mathfrak{g},\mathfrak{g}\,] := \{[\,X,Y\,] \,|\, X,Y \in \mathfrak{g}\} \subset \mathfrak{g}$ is another example of an ideal. If $I \subset \mathfrak{g}$ is an ideal then the quotient space $\mathfrak{g}/I$ inherits a natural Lie algebra structure.

Definition 7.17 (*Simple Lie algebras and ideals*) A non Abelian Lie algebra $\mathfrak{g}$ is called *simple*, if $\{0\}$ and $\mathfrak{g}$ are the only ideals of $\mathfrak{g}$. An ideal $I \subset \mathfrak{g}$ of a Lie algebra is called *simple* if $\{0\}$ and I are the only sub ideals of $\mathfrak{g}$ in I.

Proposition 7.18 *Among the complex classical Lie algebras the following are simple:* $\mathfrak{sl}(V)$ *for* $n \geq 2$, $\mathfrak{so}(V)$ *for* $n = 3$ *and* $n \geq 5$, $\mathfrak{sp}(V)$ *for* $n \geq 2$.

Definition 7.19 A finite dimensional $\mathfrak{g}$-representation $\mathfrak{g} \otimes W \to W$ is called *semisimple*, if for any $\mathfrak{g}$-invariant subspace $W' \subset W$ there is a $\mathfrak{g}$-invariant complement in W. A Lie algebra $\mathfrak{g}$ is called *semisimple*, if any finite dimensional representation is semisimple.

Definition 7.20 (*Trace form*) If $\theta\colon \mathfrak{g} \to \mathfrak{gl}(W)$ denotes a finite dimensional $\mathfrak{g}$-representation, then θ induces a bilinear symmetric form on $\mathfrak{g}$, the so called *trace form* $Tr_\theta \in \mathrm{Sym}^2(\mathfrak{g}^*)$, defined by $Tr_\theta(X,Y) := \mathrm{tr}(\theta(X) \circ \theta(Y))$.

The trace form Tr_θ is an example of a $\mathfrak{g}$-invariant vector under the representation $\mathrm{Sym}^2(-\mathrm{ad}^*)\colon \mathfrak{g} \to \mathrm{Sym}^2(\mathfrak{g}^*)$. The *radical* of T_θ is the space $\{X \in \mathfrak{g} \mid Tr_\theta(X,Y) = 0, Y \in \mathfrak{g}\}$ and defines an ideal in $\mathfrak{g}$. If the radical is zero, then the trace from is non degenerated, i.e. it defines an inner product on W and the representation takes values in $\theta\colon \mathfrak{g} \to \mathfrak{so}(W, Tr_\theta)$.

Definition 7.21 (*Killing form*) The trace from Tr_{ad} corresponding to the adjoint representation $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is called the *Killing form*.

Examples 7.22 Let V be a n-dimensional vector space over $\mathbb{F}$. The Killing form for $\mathfrak{gl}(V)$ is given by $Tr_{\mathrm{ad}}(A,B) = (2n)\,\mathrm{tr}(A \circ B) - 2\,\mathrm{tr}(A)\,\mathrm{tr}(B)$. For the (complex) classical Lie algebras, the Killing form is a multiple of the trace form of the defining representation, the later is nondegenerate: for $\mathfrak{sl}(n)$ we get $Tr_{\mathrm{ad}}(A,B) = (2n)\,\mathrm{tr}(A \circ B)$, for $\mathfrak{so}(n)$ we get $Tr_{\mathrm{ad}}(A,B) = (n-2)\,\mathrm{tr}(A \circ B)$ and for $\mathfrak{sp}(n)$ we get $Tr_{\mathrm{ad}}(A,B) = (n+2)\,\mathrm{tr}(A \circ B)$. This shows, that the Killing form for most of the classical Lie algebras is nondegenerated.

Theorem 7.23 *A Lie algebra is semisimple iff its Killing form $Tr_{\mathrm{ad}} \in \mathrm{Sym}^2(\mathfrak{g}^*)$ is nondegenerated. Hence a simple Lie algebra is semisimple.*

Definition 7.24 (*Reductive Lie algebra*) Let $\mathfrak{g}$ be a Lie algebra. $\mathfrak{g}$ is called *reductive*, if the derived algebra $[\,\mathfrak{g}, \mathfrak{g}\,]$ is semisimple and we have a decomposition $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus [\,\mathfrak{g}, \mathfrak{g}\,]$.

Bibliography

[Bac21] R. Bach: *Zur Weylschen Relativitätstheorie und der Weylschen Erweiterung des Krümmungstensorbegriffs,* Math. Z. 9 (1921), pp. 110–135.

[Bas90] R.J. Baston: *Verma modules and differential conformal invariants,* J. Differential Geometry 32 (1990), pp. 851–898.

[Bas91] R.J. Baston: *Almost Hermitian symmetric manifolds I: Local twistor theory,* Duke Math. J. 63 (1991), pp. 81–111. *Almost Hermitian symmetric manifolds II: Differential invariants,* Duke Math. J. 63 (1991), pp. 113–138.

[BE89] R.J. Baston and M.G. Eastwood: *The Penrose transform,* Oxford University Press, Oxford, (1989).

[Ber76] P. G. Bergmann: *Introduction to the theory of relativity,* Dover Publications, New York, (1976).

[BGV91] N. Berline, E. Getzler and M. Vergne: *Heat Kernels and Dirac Operators,* Springer Verlag, Berlin Heidelberg, (1991).

[BGG71] I.N. Bernstein, I.M. Gelfand and S.I. Gelfand: *Structure of representations generated by vectors of highest weight (english translation),* Functional Anal. Appl. 5 (1971), pp. 1–8.

[BGG75] I.N. Bernstein, I.M. Gelfand and S.I. Gelfand: *Differential operators on the base affine space and a study of $\mathfrak{g}$-modules,* in I.M. Gelfand, Ed.: *Lie groups and their representations,* Adam Hilger Ltd, London, (1975).

[BC85] B.D. Boe and D.H. Collingwood: *A comparison theory for the structure of induced representations I,* J. Algebra 94 (1985), pp. 511-545. *A comparison theory for the structure of induced representations II,* Math. Z. 190 (1085), pp. 1–11.

[Bra96] T. Branson: *Nonlinear phenomena in the spectral theory of geometric linear differential operators,* Proc. Symposia Pure Mathematics 59 (1996), pp. 27–65.

[Bra98] T. Branson: *Second order conformal covariants,* Proc. American Math. Soc. 126, 4 (1998), pp. 1031–1042.

[Cal96] D.M.J. Calderbank: *Clifford analysis for Dirac operators on manifolds with boundary,* Preprint Bonn MPI 96-131 (1996).

[Cal98a] D.M.J. Calderbank: *Möbius structures and two dimensional Einstein-Weyl geometry,* J. reine angewandte Math. 504 (1998), pp. 37–53.

[Cal98b] D.M.J. Calderbank: *The Faraday 2-form in Einstein-Weyl geometry,* to appear in: Math. Scand.

[CD99] D.M.J. Calderbank and T. Diemer: *Differential bilinear invariants on curved Bernstein Gelfand Gelfand sequences,* Edinburgh preprint (1999).

[CSS99] A. Čap, J. Slovák and V. Souček: *Bernstein Gelfand Gelfand sequences,* Vienna ESI 722 (1999), ftp: ftp.esi.ac.at.

[CS90] D.H. Collingwood and B. Shelton: *A duality theorem for extensions of induced highest weight modules,* Pacific J. Math. 146, no. 2 (1990), pp. 227–237.

[DH99] T. Diemer and M. Hadley: *Charge and the topology of spacetime,* to appear in: Classical and Quantum Gravity.

[Eas96] M.G. Eastwood: *Notes on conformal differential geometry,* Rendiconti del Circolo Matematico di Palermo II, 43 (1996), pp. 57–76.

[ER87] M.G. Eastwood and J.W. Rice: *Conformally invariant differential operators on Minkowski space and their curved analogues,* Commun. Math. Phys. 109 (1987), pp. 207–228.

[Fef79] C. Fefferman: *Parabolic invariant theory in complex analysis,* Adv. Math. 31 (1979), pp. 131–262.

[FG85] C. Fefferman and C.R. Graham: *Conformal invariants,* in *The mathematical heritage of Élie Cartan,* Astérisque (1985), pp. 95–116.

[Feg76] H. D. Fegan: *Conformally invariant first order differential operators,* Quat. J. Math. Oxford (2), 27 (1976), pp. 371–378.

[Fie39] M. Fierz: *Über die relativistische Theorie kräftefreier Teilchen mit beliebigem Spin,* Helv. Phys. Acta XII (1939), pp. 3–37.

[FH91] W. Fulton and J. Harris: *Representation theory,* Springer Verlag, New York-Heidelberg-Berlin, (1991).

[Gau90] P. Gauduchon: *Connexion canonique et structures de Weyl en géométrie conforme,* Preprint (1990).

[Gau91] P. Gauduchon: *Structures de Weyl et théorèmes d'annulation sur une variété conforme autoduale,* Ann. Sc. Norm. Sup. Pisa 18 (1991), pp. 563–629.

[Hit80] N.J. Hitchin: *Linear field equations on selfdual spaces,* Proc. R. Soc. London A 370 (1980), pp. 173–191.

[Kos61] B. Kostant: *Lie algebra cohomology and the generalized Borel Weil theorem,* Ann. Math. 74 (1961), pp. 329–387.

[KP88] R.S. Kulkarni and U. Pinkall (Eds.): *Conformal Geometry,* Aspects of mathematics, Max Planck Institut für Mathematik Bonn, Friedr. Viehweg & Sohn, Braunschweig-Wiesbaden, (1988).

[LP87] J.M. Lee and T.H. Parker: *The Yamabe problem,* Bull. Amer. Math. Soc. (N.S.) 17, no. 1 (1987), pp. 37–91.

[Lep77] J. Lepowsky: *A generalization of the Bernstein Gelfand Gelfand resolution,* J. Algebra 49 (1977), pp. 496–511.

[MW57] C.W. Misner and J.A. Wheeler: *Classical physics as geometry,* Annals of Physics 2 (1957), pp. 525–603.

[ONe83] B. O'Neill: *Semi-Riemannian Geometry,* Academic Press, Inc. , Orlando, (1983).

[Par87] S. Parrot: *Relativistic electrodynamics and differential geometry,* Springer Verlag, New York-Berlin-Heidelberg, (1986).

[PR84] R. Penrose and W. Rindler: *Spinors and space-time,* Cambridge University Press, Cambridge, (1984).

[Roh65] F. Rohrlich: *Classical charged particles,* Addison-Wesley Publishing, Reading, (1965).

[Soe90] W. Soergel: *The prime spectrum of the enveloping algebra of a reductive Lie algebra,* Math. Z. 204 (1990), pp. 559–581.

[Ste91] H. Stephani: *Allgemeine Relativitätstheorie,* Deutscher Verlag der Wissenschaften, Berlin, (1991).

[Thi90] W. Thirring: *Lehrbuch der mathematischen Physik,* Springer Verlag, Wien New York, (1990).

[Ver68] D.-N. Verma: *Structure of certain induced representations of complex semisimple Lie algebras,* Bull. Amer. Math. Soc. 74 (1968), pp. 160–166.

[Wei99] G. Weingart: *Geometric structures on manifolds,* in preparation (1999).

[Wey70] H. Weyl: *Raum, Zeit, Materie,* Springer Verlag, Berlin-Heidelberg, (1970).