Möbius structures
and two dimensional Einstein-Weyl geometry

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Abstract

Two geometric structures on a conformal 2-manifold are defined: Möbius structures, which provide a non-integrable version of the classical notion of a complex projective structure, and Einstein-Weyl structures, which have been extensively studied in higher dimensions. These structures are related and a classification of Einstein-Weyl structures on compact Riemann surfaces is given.

Conformal 2-manifolds possess a fascinatingly rich and elegant theory which can be viewed in many ways: it is the theory of Riemann surfaces in complex analysis, or of complex curves in algebraic geometry. In this paper, a purely differential geometric point of view will be taken, the aim being to introduce two geometric structures that a conformal 2-manifold might be equipped with, and to study the relationship between them. These structures are closely related to the projective and affine structures of Riemann surface theory.

The first structure can be viewed as a nonintegrable or nonholomorphic version of a complex projective structure, and will be called a *Möbius structure*. An integrable or flat Möbius structure on a conformal 2-manifold induces a complex projective structure: the manifold possesses an atlas whose transition functions are complex Möbius transformations. However, contrary to common usage [11], the Möbius structures discussed herein are not necessarily integrable: they possess a curvature, analogous to the Cotton-York tensor of a conformal 3-manifold, whose vanishing is equivalent to integrability. Möbius structures are also different from real projective structures, in much the same way as conformal and real projective structures differ in higher dimensions. (In one dimension, Möbius and real projective structures do coincide and are always integrable.)

The other topic of interest here is Einstein-Weyl geometry [3, 10, 16]. This is the geometry of a conformal manifold equipped with a compatible (or conformal) torsion free connection, such that the symmetric tracefree part of the Ricci tensor of this connection vanishes. These manifolds generalise Einstein manifolds in a natural way, and have been investigated in some detail recently (see [2, 6, 10] and references therein). In [13], Pedersen and Tod posed the problem of classifying compact two dimensional Einstein-Weyl manifolds—the possible geometries of compact *three* dimensional Einstein-Weyl manifolds have been classified (locally) by Tod [15]. However, the definition just given of an Einstein-Weyl manifold is vacuous in the two dimensional case and Pedersen and Tod did not offer an alternative definition. One of the main goals of this paper is to give explicitly such a definition and present a classification of the compact orientable examples.

In the first section the conformal and complex analytic descriptions of surfaces are summarised and compared. In particular the Cauchy-Riemann equations on various complex line bundles are identified as conformally invariant differential operators. The aim here is both to illustrate the special features of two dimensional conformal geometry, and to relate it to the higher dimensional case. In section 2, Möbius structures are defined, together with the corresponding notion of Schwarzian derivative, which generalises a definition of Osgood and Stowe [12]. The relationship with the usual Schwarzian derivative of complex analysis is discussed and it is shown that a flat Möbius structure is a complex projective structure. In section 3, Einstein-Weyl structures on surfaces are introduced, and compared to the higher dimensional geometries. The definition is equivalent to a contracted Bianchi identity for the compatible torsion free connection. More precisely, a conformal surface with compatible covariant derivative D is Einstein-Weyl iff

$$D scal^D - 2\, div^D\, F^D = 0,$$

where $div^D = tr\, D$, $scal^D$ is the scalar curvature of D, and F^D is the Faraday 2-form of D, which is the curvature of D on an associated real line bundle.

After presenting this definition, I then show that the standard theory of the Gauduchon gauge [5, 6, 15] continues to work in two dimensions, an observation which leads to the following result.

Theorem. *Let M be a compact orientable surface which is Einstein-Weyl. Then if the genus of M is greater than one, the Einstein-Weyl structure is defined by a compatible metric of constant curvature.*

Since metrics of constant curvature on compact surfaces are well understood, it remains to classify the Einstein-Weyl structures on the sphere and the torus. This classification is carried out in the final section. The solutions come in one parameter families with the constant curvature metrics as limits.

It is a pleasure to thank Henrik Pedersen for stimulating discussions and for helpful comments on a preliminary version of this paper.

1 Natural bundles and operators on a conformal surface

The elegance of analysis on a conformal 2-manifold M is largely due to the fact that M is equivalently (if oriented) a complex 1-manifold. This will be viewed, half-jokingly, as a twistor correspondence for two dimensional conformal geometry: the twistor space of an oriented conformal 2-manifold is the manifold itself, thought of as a holomorphic curve. In the unoriented situation, the twistor space could be taken to be the oriented double cover.

In this section some of the bundles and differential operators arising naturally on a conformal surface will be discussed, together with the corresponding (i.e., the same!) objects on the twistor space. Firstly, however, the notion of a density will be reviewed, since this will be crucial in the sequel.

1.1 Definition. Let V be a real n-dimensional vector space and w any real number. Then a *density of weight w* or *w-density* on V is a map $\rho\colon (\Lambda^n V) \setminus 0 \to \mathbb{R}$ such that

$\rho(\lambda\omega) = |\lambda|^{-w/n}\rho(\omega)$ for all $\lambda \in \mathbb{R}^*$ and $\omega \in (\Lambda^n V) \setminus 0$. The space of densities of weight w is denoted $L^w = L^w(V)$.

L^w naturally carries the representation $\lambda.\rho = |\lambda|^w\rho$ of the centre of $\mathrm{GL}(V)$ or equivalently the representation $A.\rho = |\det A|^{w/n}\rho$ of $\mathrm{GL}(V)$. Note also:

- L^w is an oriented one dimensional linear space with dual space L^{-w}, and L^0 is canonically isomorphic to $\mathbb{R}$.
- The absolute value defines a map from $\Lambda^n V^*$ to L^{-n}. If V is oriented then the $(-n)$-densities can be identified with the volume forms.
- The densities of $L^{-1} \otimes V$ are canonically isomorphic to $\mathbb{R}$.

Suppose M is any manifold. Then the *density line bundle* $L^w = L^w_{TM}$ of M is defined to be the bundle whose fibre at $x \in M$ is $L^w(T_xM)$. Equivalently it is the associated bundle $\mathrm{GL}(M) \times_{\mathrm{GL}(n)} L^w(n)$ where $\mathrm{GL}(M)$ is the frame bundle of M and $L^w(n)$ is the space of w-densities of $\mathbb{R}^n$. The density bundles are oriented (hence trivialisable) real line bundles, but there is no preferred trivialisation.

Sections of $L = L^1$ may be thought of as scalar fields with dimensions of length. This geometric dimensional analysis may also be applied to tensors:

1.2 Definition. The tensor bundle $L^w \otimes (TM)^j \otimes (T^*M)^k$ (and any subbundle, quotient bundle, element or section) will be said to have *weight* $w + j - k$, or *dimensions* of $[\text{length}]^{w+j-k}$.

The notion of density also allows one to define a conformal structure, not as an equivalence class of inner products, but a genuine inner product:

1.3 Definition. A *conformal structure* on a manifold M is an L^2 valued inner product on TM. More precisely it is a section $\mathsf{c} \in \mathrm{C}^\infty(M, L^2 \otimes S^2T^*M)$ which is everywhere positive definite. It will always be assumed that c is *normalised* in the sense that $|\det \mathsf{c}| = 1$.

If c is viewed as a metric on the *weightless tangent bundle* $L^{-1} \otimes TM$, then the normalisation condition means that the trivialisation of the densities of $L^{-1} \otimes TM$ determined by the metric equals the canonical one. The conformal inner product of tangent vector fields X, Y will be denoted $\langle X, Y\rangle$, and is a section of L^2.

Now let M be an oriented conformal 2-manifold. The isomorphism $\mathrm{SO}(2) \cong \mathrm{U}(1)$ means that M has a preferred almost complex structure J, namely the section of $\mathrm{SO}(TM)$ given by a positive rotation of $\pi/2$ in each tangent plane. Equivalently, the unique positive normalised weightless 2-form on M is $\omega(X, Y) = \langle JX, Y\rangle$, and $\mathsf{c} + i\omega$ is a Hermitian form on TM with values in $L^2 \otimes \mathbb{C}$.

As is well known, any conformal 2-manifold is conformal flat (the existence of isothermal coordinates), so the almost complex structure J is integrable and TM is a holomorphic line bundle.

1.4 Notation. When tensoring a vector bundle with some L^w (over $\mathbb{R}$), the tensor product sign will often be omitted. The tensor product of complex line bundles $\mathcal{L}_1, \mathcal{L}_2$ over $\mathbb{C}$ will be denoted $\mathcal{L}_1 \cdot \mathcal{L}_2$, and the conjugate line bundle to $\mathcal{L}$ will be denoted $\overline{\mathcal{L}}$ as usual. Note that $L^2 = L \otimes L$ (over $\mathbb{R}$) whereas for complex line bundles $\mathcal{L}$, $\mathcal{L}^2$ will denote $\mathcal{L} \cdot \mathcal{L}$.

Sections of the complex line bundle $TM^{-p} \cdot \overline{TM}^{-q} = T^*M^p \cdot \overline{T^*M}^q$ are often called (p,q)-*differentials* (for $p, q \in \mathbb{Z}$). If $q = 0$ there is a $\overline{\partial}$ operator $T^*M^p \to T^*M^p \cdot \overline{T^*M}$ whose kernel consists of the holomorphic sections.

The Hermitian form $\mathsf{c} + i\omega$ is easily seen to be an isomorphism $TM \cdot \overline{TM} \to L^2 \otimes \mathbb{C}$ (and so $TM \cong L^2 \overline{T^*M}$). More generally there are the following correspondences between complex and conformal geometric objects:

$T^*M^0 \cdot \overline{T^*M}^0 = \mathbb{C}$		$\mathbb{R} \oplus L^{-2}\Lambda^2 TM \cong \mathbb{R} \oplus L^2 \Lambda^2 T^*M \cong \mathfrak{co}(TM)$
$T^*M \cdot \overline{T^*M}$	Area differentials	$L^{-2} \oplus \Lambda^2 T^*M$
$TM \cdot \overline{T^*M}$	Beltrami differentials	$\mathrm{Sym}_0\, TM \cong L^{-2} S^2_0 TM$
T^*M	Abelian differentials	T^*M
T^*M^2	Quadratic differentials	$S^2_0 T^*M$
$T^*M^p = TM^{-p}$		$S^p_0 T^*M = S^{-p}_0 TM$

Here S^p_0 denotes the subbundle of the pth symmetric power consisting of tensors which are tracefree with respect to the conformal structure, and $\mathrm{Sym}_0\, TM$ denotes the bundle of symmetric tracefree endomorphism of TM.

The holomorphic $(p, 0)$-differentials correspond to solutions of conformally invariant first order differential equations on M. For the Abelian differentials, the $\overline{\partial}$ operator is given by the $d + \delta$ operator, taking values in the area differentials.

More generally, to understand natural differential operators on a conformal manifold M of any dimension, it is convenient to introduce the class of torsion free covariant derivatives which are compatible with the conformal structure. These are characterised by the following result.

1.5 The Fundamental Theorem of Conformal Geometry. [16] *On a conformal manifold M there is an affine bijection between covariant derivatives on L^1 and torsion free connections on TM preserving the conformal structure. More explicitly, the covariant derivative on TM is determined from the one on L^1 by the Koszul formula*

$$\begin{aligned} 2\langle D_X Y, Z\rangle = {} & D_X \langle Y, Z\rangle + D_Y \langle X, Z\rangle - D_Z \langle X, Y\rangle \\ & + \langle [X,Y], Z\rangle - \langle [X,Z], Y\rangle - \langle [Y,Z], X\rangle, \end{aligned}$$

where X, Y, Z are vector fields and the conformal inner products are sections of L^2.

1.6 Definition. A covariant derivative on L^1 is called a *Weyl derivative*, and will be identified with the induced connection on TM. A Weyl derivative admitting a global nonzero parallel section is said to be *exact*. A compatible metric g induces a global section of L^1 and hence an exact Weyl derivative D^g.

There is no preferred choice of Weyl derivative and so it is important to know how different choices are related.

Suppose that D and $\tilde{D} = D + \gamma$ are two Weyl derivatives on L^1, where γ is a 1-form and suppose $\tilde{D} = D + \Gamma$ on $L^{w-1}TM$, for some $\mathfrak{co}(TM)$-valued 1-form Γ. Then Γ is given in terms of γ by the linearised Koszul formula:

$$\Gamma_X = w\gamma(X)id + \gamma \vartriangle X, \quad \text{where} \quad (\gamma \vartriangle X)(Y) = \gamma(Y)X - \langle X, Y\rangle\gamma.$$

Here free use is being made of the natural isomorphism $\sharp\colon T^*M \to L^{-2}TM$ given by the conformal structure.

One may then compute the curvatures of D and $\tilde{D}$ on $L^{w-1}TM$. They are easily seen to be related by the formula:

$$\begin{aligned}R^{\tilde{D},w}_{X,Y} = R^{D,w}_{X,Y} &+ w\, d\gamma(X,Y) id\\ &+ \big(D_X\gamma - \gamma(X)\gamma + \tfrac{1}{2}\langle\gamma,\gamma\rangle X\big) \vartriangle Y\\ &- \big(D_Y\gamma - \gamma(Y)\gamma + \tfrac{1}{2}\langle\gamma,\gamma\rangle Y\big) \vartriangle X.\end{aligned}$$

The $d\gamma$ term comes from the curvature of the Weyl derivative on L^1. This is a real 2-form F^D, which (motivated by the links with classical electromagnetism) will be called the *Faraday curvature*.

Because Weyl derivatives have Faraday curvature, taking the trace of $R^{D,w}$ does not necessarily produce a symmetric tensor: the skew part is a multiple of the Faraday curvature which depends on w. Consequently it turns out to be more natural to work with the *normalised Ricci tensor* of D (also called the *Rho tensor*):

$$r^D = r^D_0 + \tfrac{1}{2n(n-1)} scal^D id - \tfrac{1}{2} F^D.$$

Here $scal^D$ is the scalar curvature of D (which is a section of L^{-2}) and r^D_0 is the normalised symmetric tracefree Ricci tensor $r^D_0 = \frac{1}{n-2}\, sym_0 \operatorname{Ric}^D$.

The symmetric tracefree part of the Ricci tensor vanishes automatically in two dimensions and so, although r^D_0 at present *makes no sense* in dimension two, there is some hope that, once defined more carefully, it won't be infinite. This will be carried out in the next section.

The reason for the factor $1/(n-2)$ in higher dimensions is that

$$R^{D,w}_{X,Y} = W_{X,Y} + wF^D(X,Y)id - r^D(X) \vartriangle Y + r^D(Y) \vartriangle X,$$

where W is the *Weyl curvature* of the conformal structure.

The dependence of the normalised Ricci tensor on D is given as follows.

1.7 Proposition. *If D and $\tilde{D} = D + \gamma$ are Weyl derivatives on (M^n, c) then:*

$$\begin{aligned}r^{\tilde{D}}_0 &= r^D_0 - sym_0\, D\gamma + (\gamma\otimes\gamma - \tfrac{1}{n}\langle\gamma,\gamma\rangle id)\\ scal^{\tilde{D}} &= scal^D - 2(n-1)\, tr\, D\gamma - (n-1)(n-2)\langle\gamma,\gamma\rangle\\ r^{\tilde{D}} &= r^D - (D\gamma - \gamma\otimes\gamma + \tfrac{1}{2}\langle\gamma,\gamma\rangle id).\end{aligned}$$

Some examples of differential operators defined using a Weyl derivative D will now be given, and the above formulae can be used to understand the dependence of the operator on the choice of D.

Firstly, consider sections X of the bundle $L^{w-1}TM$ (for some $w \in \mathbb{R}$). Then $X \mapsto sym_0\, DX$ defines a first order differential operator on this bundle. (As before sym_0 denotes the symmetric tracefree part.) If the Weyl derivative D is replaced by $D+\gamma$, then this operator changes by the zero order term $(w-1)\, sym_0(\gamma\otimes X)$. The operator is therefore conformally invariant for $w = 1$: it is the *conformal Killing operator*

$X \mapsto \mathcal{L}_X \mathsf{c}$ and its solutions are the conformal vector fields. In two dimensions there is a "twistor correspondence": a vector field is conformal iff it is holomorphic. A similar story applies to many other first order differential operators [4].

Secondly, consider sections λ of L^w. Comparing the dependence of $tr_{\mathsf{c}}\, D^2\lambda$ on D with that of the scalar curvature $scal^D$ leads to the conclusion that for $w = (2-n)/2$ the operator $tr\, D^2 - \frac{n-2}{4(n-1)} scal^D$ is independent of the choice of D. This is the *conformal Laplacian.* In two dimensions, the scalar curvature term is not needed, since $tr\, D^2$ is invariant on ordinary functions. A function is harmonic iff its differential is holomorphic.

2 Möbius structures

The geometry of Weyl derivatives on conformal 2-manifolds is impoverished by the absense of a symmetric tracefree Ricci tensor analogous to r_0^D. The aim of this section is to repair this by introducing Möbius structures, which provide a notion of conformal structure in two dimensions more closely analogous to the higher dimensional case.

Firstly, by way of motivation, the one dimensional case will be considered. A 1-manifold automatically carries a conformal structure in the conventional sense, but it is more natural to define a conformal (or Möbius) 1-manifold as a 1-manifold equipped with a (real) projective structure. One definition of a real projective structure on an n-manifold is a $\mathrm{SL}(n+1,\mathbb{R})$-connection on $J^1 L^{n/(n+1)}$ such that the formal 1-jets of parallel sections are holonomic. This connection is equivalently given by a Hessian: a second order linear differential operator from $L^{n/(n+1)}$ to $S^2T^*M \otimes L^{n/(n+1)}$ whose symbol is the identity.

On a 1-manifold this Hessian becomes a Laplacian from $L^{1/2}$ to $L^{-3/2}$. Higher dimensional conformal manifolds automatically possess a Laplacian from $L^{(2-n)/2}$ to $L^{(-2-n)/2}$, namely the conformal Laplacian mentioned at the end of the previous section. The fact that there is no such operator in one dimension is related to the absence of scalar curvature. The solution is to introduce a choice of Laplacian as extra structure.

In two dimensions, however, the Laplacian *does* make sense, but there is another second order operator, the *conformal tracefree Hessian*, which exists in higher dimensional conformal geometry, but has no two dimensional analogue.

To find this operator, consider the tracefree Hessian $sym_0\, D^2$ defined by a Weyl derivative D. As an operator on L^w this depends upon D. More precisely, if $\tilde{D} = D + \gamma$ on L^1 and μ is a section of L^w, then

$$\begin{aligned} sym_0\, \tilde{D}^2\mu = {} & sym_0\, D^2\mu \\ & + 2(w-1)\, sym_0\, \gamma \otimes D\mu + w(sym_0\, D\gamma)\mu + w(w-2)\gamma \otimes_0 \gamma\, \mu. \end{aligned}$$

For $w = 1$ the first order term in this expression vanishes, and the zero order term is *identical* to the change in the normalised tracefree Ricci tensor r_0^D given in 1.7. The following well known fact is therefore obtained [1, 7]: the operator $sym_0\, D^2 + r_0^D$ (acting on sections of L^1 over a conformal manifold of dimension $n > 2$) is independent of the choice of the Weyl derivative D.

2.1 Definition. A *Möbius structure* on a conformal manifold M is a (smooth) second order linear differential operator $\mathcal{H}$ from L^1 to $S^2_0T^*M \otimes L^1$ such that for some Weyl derivative D, the operator $\mathcal{H} - sym_0\, D^2$ is zero order.

The calculations made above show that this condition on $\mathcal{H}$ is independent of D, that $sym_0\, D^2$ is a Möbius structure for any Weyl derivative D, and also give:

2.2 Proposition. *On a conformal manifold of dimension $n > 2$ there is a canonical Möbius structure $\mathcal{H} = sym_0\, D^2 + r^D_0$.*

On a two dimensional Möbius manifold, the difference $\mathcal{H} - sym_0\, D^2$ will be called the normalised tracefree Ricci tensor r^D_0 of D with respect to $\mathcal{H}$.

In higher dimensions, Möbius structures other than the canonical one are sometimes of interest. For example, in [12], Osgood and Stowe implicitly study the Möbius structure $sym_0(D^g)^2$ where g is a Riemannian metric on M. This is the canonical structure of the induced conformal structure iff g is Einstein. The generalisation to Einstein-Weyl geometry will be explained in the next section.

Firstly, though, these remarks require further justification, and the definition of a Möbius structure given here needs to be related to more classical notions. An initial step in this direction is the following easy fact.

2.3 Proposition. *Möbius structures (compatible with a given conformal structure on M) form an affine space modelled on the smooth sections of $S^2_0T^*M$. In two dimensions, this is the space of smooth quadratic differentials.*

Now define the *Schwarzian derivative* of a conformal vector field X or a conformal diffeomorphism θ to be the Lie derivative $\mathcal{L}_X\mathcal{H}$ or the difference $\theta^*\mathcal{H} - \mathcal{H}$. Since the Lie derivative or pullback of a density is first order in X or θ and $\mathcal{H}$ is a second order operator, it follows that the Schwarzian derivative is third order in X or θ. Also, since the conformal structure is preserved, the Schwarzian derivative is a quadratic differential. A conformal vector field or diffeomorphism with vanishing Schwarzian will be called *Möbius*.

In [12], Osgood and Stowe define a Schwarzian derivative of a conformal diffeomorphism with respect to a metric g. It is obtained by applying the operator $f \mapsto (D^g df - df \otimes df)_0$ to the conformal scale factor of the diffeomorphism (calculated with respect to g). It is elementary to check that this definition coincides with the definition given here when $\mathcal{H} = sym_0(D^g)^2$. The connection with Einstein geometry is very clear in [12]: Osgood and Stowe show, among other things, how the Schwarzian derivative provides a beautiful setting for the work of Brinkmann on conformal mappings of Einstein manifolds (see also W. Kühnel's article in [11] for an exposition of this).

Möbius structures can be used to reinterpret and generalise these results, but here the focus will be on the two dimensional case, since in higher dimensions it seems natural to work with the canonical Möbius structure, making the choice of Möbius structure less interesting.

In fact there is a canonical Möbius structure also in two dimensions, but the construction is deep and global in contrast to the simple local formula in higher dimensions. To see this, it is first necessary to look at the basic example of a Möbius structure on a domain in the Riemann sphere.

The complex affine structure on the conformal plane identifies it, up to a choice of origin and real axis, with $\mathbb{C}$. Using this it is straightforward to write down a Möbius structure as follows.

If $\lambda = f\mu$ is a section of L^1, where f is a function and μ is a constant section, then:

$$\mathcal{H}\lambda = \frac{d^2 f}{dz^2}\,\mu\,dz^2.$$

Here d/dz denotes the complex linear part of the derivative (which will also be denoted by a prime) and dz^2 is a constant quadratic differential.

The claim is that this Möbius structure is invariant under Möbius transformations: more precisely, the Schwarzian derivative defined above reduces to the usual Schwarzian derivative of holomorphic functions. To see this, suppose θ is a holomorphic diffeomorphism between open subsets of $\mathbb{C}$. Then $\theta_*\lambda = (|\theta'|\lambda)\circ\theta^{-1}$ and so:

$$(\theta^*\mathcal{H})\lambda = \theta^*\big(\mathcal{H}(\theta_*\lambda)\big) = |\theta'|^{-1}(\theta')^2\big((|\theta'|f)\circ\theta^{-1}\big)''\circ\theta\,\mu\,dz^2.$$

Since θ is holomorphic, $(\overline{\theta'})' = 0$ and so (after applying the chain rule) it is only necessary to compute $(\theta')^{-1/2}\theta'\big((\theta')^{-1}\big((\theta')^{1/2}f\big)'\big)'$. A calculation then gives

$$(\theta^*\mathcal{H})\lambda = \mathcal{H}\lambda + \tfrac{1}{2}S(\theta)\lambda,$$

where $S(\theta) = \big(\frac{\theta''}{\theta'}\big)' - \frac{1}{2}\big(\frac{\theta''}{\theta'}\big)^2$ is the usual Schwarzian. This proves the claim.

It now follows that any complex projective curve has a canonical Möbius structure. A Möbius structure which arises in this way will be called *integrable.* Now the universal cover of any conformal surface is conformally diffeomorphic to the sphere, the plane or the disc (The Uniformisation Theorem) and the conformal transformations of these spaces are all Möbius. It also follows that any conformal surface admits a preferred integrable Möbius structure, although this structure depends on the global properties of the surface.

Not all Möbius structures are integrable, however. They possess a curvature analogous to the Cotton-York tensor of a conformal 3-manifold.

In general the Cotton-York tensor of a Weyl derivative D on a conformal manifold is defined by the formula $C^D = d^D r^D$, where r^D is the normalised Ricci tensor viewed as a covector valued 1-form. More precisely:

$$C^D_{X,Y} = (D_X r^D)(Y,.) - (D_Y r^D)(X,.)$$

A painful computation of $d^{D+\gamma}r^{D+\gamma}$ leads to the well known fact that $C^{D+\gamma}_{X,Y} = C^D_{X,Y} - W_{X,Y}\gamma$. Note that the same calculation applies in the two dimensional case, provided r^D_0 is interpreted as the normalised tracefree Ricci tensor with respect to a fixed Möbius structure. Since there is no Weyl curvature in dimensions two and three, the Cotton-York tensor is an invariant of the Möbius or conformal structure in these dimensions.

In dimension three or more the second Bianchi identity implies that C^D is tracefree: $\sum C^D_{X,e_i}e_i = 0$, where e_i is a weightless orthonormal basis. More explicitly, this contracted Bianchi identity is given by the following.

2.4 Proposition. *On any Weyl manifold of dimension $n > 2$,*

$$div^D \left(r_0^D - \tfrac{1}{2n} scal^D id + \tfrac{1}{2} F^D \right) = 0,$$

where $div^D = tr_{\mathsf{c}} \circ D$ and in particular, $div^D F^D = \sum_i (D_{e_i} F^D)(e_i, .)$.

In two dimensions there is no such Bianchi identity and the Cotton-York tensor C^D, far from being tracefree, is necessarily tracelike. Hence the Cotton-York tensor of a Möbius 2-manifold may equivalently be viewed as a 1-form of weight -3.

2.5 Definition. Let M be a conformal surface with Möbius structure $\mathcal{H}$. Then the *(Cotton-York) curvature* of M is the section $C^{\mathcal{H}}$ of $L^{-2}T^*M \cong \overline{T^*M} \cdot T^*M^2$ defined by $C^{\mathcal{H}} = div^D \left(-r_0^D + \frac{1}{4} scal^D id - \frac{1}{2} F^D \right)$, where D is any Weyl derivative, and $r_0^D = \mathcal{H} - sym_0 D^2$. The Möbius structure is said to be *flat* iff $C^{\mathcal{H}} = 0$.

The tensor inside the divergence is $r^D(JX, JY)$.

If a Möbius structure $\mathcal{H}$ is replaced by $\mathcal{H} + \Phi$ where Φ is a quadratic differential, then for any fixed D, $r_0^D(\mathcal{H}+\Phi) = r_0^D(\mathcal{H}) + \Phi$ and so $C^{\mathcal{H}+\Phi} = C^{\mathcal{H}} - div\, \Phi$. Note that the divergence is independent of D on quadratic differentials and may be identified with the $\overline{\partial}$ operator. Therefore *flat* Möbius structures form an affine space modelled on *holomorphic* quadratic differentials.

The operator $\mathcal{H}$ is equivalently given by a subbundle $\mathcal{R}^2$ of J^2L, which are the 2-jets of formal solutions. If a Möbius structure is flat then this differential equation $\mathcal{H}$ is completely integrable. The proof of this is a straightforward exercise in the formal theory of differential equations [14] and is sketched in the appendix. The integrability in this case means that $\mathcal{R}^2$ carries a flat connection whose parallel sections correspond to local solutions of $\mathcal{H}$. This is completely analogous to the integrability, in higher dimensions, of conformal structures with vanishing Cotton-York or Weyl curvature in higher dimensions—for similar proofs in the higher dimensional case, see [1, 7]. Alternatively, Möbius structures could be defined using Cartan connections, reducing integrability questions to the method of equivalence. However, I feel that a description of a geometric structure in terms of a Cartan connection should be a theorem rather than a definition, and the construction of a connection on $\mathcal{R}^2$ essentially establishes this theorem.

It remains to check that an integrable Möbius structure induces a complex projective structure. This is a consequence of the following observations.

- The local solutions of $\mathcal{H}$ parallelise $\mathcal{R}^2$.
- $\mathcal{R}^2$ possesses a natural Lorentzian quadratic form on each fibre which is preserved by parallel transport.
- The symbol of $\mathcal{R}^2$ defines a null line in each fibre and this section of the bundle of null lines is not parallel, but in fact a local diffeomorphism into the space of parallel null lines.

The Lorentzian structure is induced by the scalar curvature: for any positive section μ of L^1, the scalar curvature of the corresponding metric defines a function $scal^{\mu} \mu^2$ on M. At each point, this function only depends on the 2-jet of μ and so makes sense as a quadratic form on J^2L. The other properties are more or less straightforward: for further details again see [1, 7]. The space of null lines in a Lorentzian vector space is the sphere with its natural Möbius structure and these local diffeomorphisms provide the required complex projective charts.

3 Einstein-Weyl geometry in two dimensions

3.1 Definition. Let (M, c, D) be a Weyl manifold of dimension greater than two. Then M is said to be *Einstein-Weyl* iff $r_0^D = 0$, i.e., the symmetric tracefree part of the Ricci tensor vanishes.

In other words, a Weyl derivative D on a conformal manifold is Einstein-Weyl iff $sym_0\, D^2$ is equal to the canonical Möbius structure. In two dimensions there is no canonical Möbius structure, but since the conformal structure is integrable, it is natural to require instead that $sym_0\, D^2$ is an *integrable* Möbius structure.

3.2 Definition. Let (M, c, D) be a Weyl manifold of dimension two. Then M is said to be *Einstein-Weyl* iff the Möbius structure $sym_0\, D^2$ is flat.

A 2-manifold is usually said to be Einstein iff it has constant scalar curvature, since this follows from the contracted Bianchi identity in higher dimensions. The above defintion generalises this to the Einstein-Weyl case.

3.3 Proposition. *Suppose M is Einstein-Weyl. Then $D scal^D - n \operatorname{div}^D F^D = 0$ (here the trace is with the first entry of F^D).*

For $n > 2$ this is immediate from the contracted Bianchi identity 2.4 (see [13, 6]), whereas in two dimensions, it is equivalent to the vanishing of the Cotton-York tensor of the Möbius structure $sym_0\, D^2$.

3.4 Corollary. *A Weyl manifold (M, c, D) of dimension two is Einstein-Weyl if and only if $D scal^D - 2 \operatorname{div}^D F^D = 0$. Equivalently, the section $\frac{1}{2} scal^D - F^D$ of $L^{-2}(\mathbb{R} \oplus L^{-2}\Lambda^2 TM) \cong L^{-2} \otimes \mathbb{C}$ is holomorphic with respect to D.*

Much of the theory of Einstein-Weyl manifolds, as described for instance in [2, 6], applies to the two dimensional case without substantial change. In some cases the two dimensional proof is simpler, an important such example being the existence of a Gauduchon gauge.

3.5 Definition. Let (M, c, D) be a Weyl manifold. Then a compatible metric g is called a *Gauduchon metric* or *gauge* iff $D = D^g + \omega^g$ with $tr_{\mathsf{c}}\, D^g \omega^g = 0$.

In two dimensions the divergence on 1-forms is conformally invariant and a Gauduchon gauge is a coclosed representative for the space of 1-forms ω with $D - \omega$ exact.

3.6 Theorem. *A compact two dimensional Weyl manifold admits a Gauduchon gauge, unique up to homothety.*

Proof. This is an immediate consequence of the Hodge decomposition for 1-forms in two dimensions. □

A remarkable feature of the Gauduchon gauge on a compact Einstein-Weyl manifold is that the Gauduchon 1-form is dual to a Killing field of the Gauduchon metric [15]. This is closely related to the existence of a *Gauduchon constant* [6] generalising the constant scalar curvature on an Einstein manifold. The two dimensional formulation is as follows.

3.7 Theorem. *Let M be a* compact *Weyl 2-manifold with $D = D^g + \omega^g$ in the Gauduchon gauge. Then $scal^D = scal^g$ and if D is Einstein-Weyl then $\sharp_g \omega^g$ is a Killing field.*

Conversely if M is any Weyl manifold with $D = D^g + \omega^g$ such that $\sharp_g \omega^g$ is a Killing field, then D is Einstein-Weyl iff the section $\kappa = scal^g - 4|\omega^g|^2$ of L^{-2} is constant (with respect to D^g).

Proof. The Cotton-York tensor is an invariant of $\mathcal{H} = sym_0 D^2$ and so computing it in terms of D and g gives:

$$\tfrac{1}{4} D scal^D - \tfrac{1}{2} \, div^D F^D = C^{\mathcal{H}} = div^g \left(-r_0^g + \tfrac{1}{4} scal^g id \right),$$

where $r_0^g = sym_0 D^2 - sym_0 (D^g)^2$ (acting on L^1). This implies

$$(*) \qquad \begin{aligned} \tfrac{1}{4} D scal^D - \tfrac{1}{2} \, div^D F^D = & 2\langle sym_0 D^g \omega^g, \omega^g \rangle + \tfrac{1}{4} D^g \left(scal^g - 4|\omega^g|^2 \right) \\ & + 2(div^g \omega^g)\omega^g - div^g (sym_0 D^g \omega^g). \end{aligned}$$

If D is Einstein-Weyl and g is a Gauduchon metric then the left hand side is zero and $div^g \omega^g = 0$. Contracting the remaining terms with ω^g leads to the equation:

$$div^g \left(\langle sym_0 D^g \omega^g, \omega^g \rangle - \tfrac{1}{4} scal^g \omega^g \right) = 2 | \, sym_0 D^g \omega^g |^2.$$

Integration then shows that $sym_0 D^g \omega^g = 0$ and so $\sharp_g \omega^g$ is Killing. Conversely, substituting $sym \, D^g \omega^g = 0$ into (*) shows that the Einstein-Weyl condition in these circumstances is equivalent to the constancy of κ. □

3.8 Corollary. *Let M be a compact orientable conformal surface and suppose D is Einstein-Weyl. Then either D is defined by a compatible metric of constant curvature or the genus of M is less than two.*

Proof. If D is not given by a compatible metric then the Gauduchon 1-form is dual to a nonzero conformal vector field. Such vector fields are precisely the holomorphic ones, and only exist globally on the Riemann sphere and each torus. □

Each conformal structure on $S^1 \times S^1$ has a family of flat Weyl structures (which are therefore Einstein-Weyl) parameterised by conformal vector fields. The only conformal vector fields on a torus are the translations with respect to the flat metric g (unique up to homothety). Any such Killing field K gives rise to a D^g-parallel 1-form $\omega = g(K, .)$ and $D = D^g + \omega^g$ is flat. Each such D induces a flat Möbius structure, equal to that induced by $D^g - \omega^g$, and so the flat Weyl structures (which may also be viewed as affine structures) form a branched 2-fold covering of the flat Möbius structures (the projective structures).

In the next section, some less classical examples will be found.

4 Classification of compact Einstein-Weyl geometries

Let M be a compact Riemann surface with a Weyl derivative D. Then $D = D^g + \omega^g$ where D^g is the Weyl derivative of a compatible metric g, and ω^g is a 1-form with $div\, \omega^g = 0$, a condition which uniquely determines ω^g from M, D (the metric g being unique up to homothety). The Einstein-Weyl equation for this structure is equivalent to the following two conditions:

- $\sharp_g \omega^g$ is a Killing field of g
- $D^g(scal^g - 4|\omega^g|^2) = 0$.

The Killing field is identically zero precisely when $D = D^g$ and g is a metric of constant curvature.

The aim of this section is to find all Einstein-Weyl structures $D = D^g + \omega^g$ with g admitting a *nontrivial* Killing field K such that $\sharp_g \omega^g = AK$ for some constant A. In this situation it is no longer necessary to assume M is compact, and constant curvature metrics with symmetry arise when $A = 0$.

The Einstein-Weyl equations may be integrated locally as follows. Suppose that K is nonvanishing on an open subset U of M diffeomorphic to a domain in $\mathbb{C}$. Since K is holomorphic, isothermal coordinates $x + it$ may be chosen locally such that $K = \partial/\partial t$. The Gauduchon metric is then of the form $g = v^2(dx^2 + dt^2)$, and $\omega^g = Av^2\, dt^2$ where v is an unknown function of x. If $v = e^f$ then

$$scal^D = scal^g = -2\, div(df) = -2f'' e^{-2f} \mu_g^{-2}$$

where μ_g is the trivialisation of L^1 determined by g. The Einstein-Weyl equation now reduces to an ordinary differential equation for f, namely

$$f'' e^{-2f} + 2A^2 e^{2f} = B,$$

which readily integrates to give

$$v^{-2} dv^2 = (-A^2 v^4 + Bv^2 + C) dx^2.$$

Therefore the Einstein-Weyl structure is locally of the form

$$g = P(v)^{-1} dv^2 + v^2 dt^2$$

$$\omega = Av^2\, dt,$$

where

$$P(v) = -A^2 v^4 + Bv^2 + C,$$

and A, B, C are arbitrary constants. Integrating the equation $v'(x)^2 = P(v)v^2$ gives the solution in the original isothermal coordinates, but it is perhaps simpler to introduce a new coordinate r by $v'(r)^2 = P(v)$. The metric is now

$$g = dr^2 + v(r)^2 dt^2$$

and $v(r)$ is an elliptic function since P is a quartic polynomial. In order to have a real solution $v(r)$ it is necessary that $P(v)$ is somewhere nonnegative. These solutions are given by the following Jacobian elliptic functions and their limits.

$$v(r) = \begin{cases} \lambda\, \mathrm{cn}(\mu r + \alpha, k) \quad \text{or} \quad \lambda\, \mathrm{sd}(\mu r + \alpha, k) \quad \text{if } C > 0 & (1) \\ \lambda\, \mathrm{dn}(\mu r + \alpha, k) \quad \text{or} \quad \lambda\, \mathrm{nd}(\mu r + \alpha, k) \quad \text{if } C < 0 & (2), \end{cases}$$

where α is a constant of integration and λ, μ, k are constants depending on A, B, C. The Einstein-Weyl structure remains unchanged if v is rescaled and r is affinely reparameterised, provided t is rescaled appropriately, and so the solutions in fact only depend (locally) on one parameter.

The two forms of the solution given in each case above are equivalent by period translation, but behave differently in the limit $k \to 1$ when the (real) period becomes infinite: cn and dn both reduce to sech, whereas sd and nd reduce to sinh and cosh. The sech solution corresponds to $C = 0$ and gives a nonexact Einstein-Weyl structure on $\mathbb{R}^2$ or $S^1 \times \mathbb{R}$, whereas the hyperbolic solutions arise when $A = 0$ and the Einstein-Weyl structure is the hyperbolic metric. In the limit $k \to 0$, cn$\to$ cos, sd$\to$ sin, while dn,nd$\to 1$. These limits therefore give the spherical metric and the flat Weyl structure.

It remains to consider when these solutions are defined on a compact surface (necessarily of genus less than two, since only these admit metrics with Killing fields). If w ranges over a half-period of cn or sd, (1) gives a one parameter family of global solutions on S^2, whereas dn and nd are periodic and nonvanishing, and so the solutions in (2) are defined on each torus $S^1 \times S^1$. The constant solutions correspond to the flat Weyl structures discussed at the end of the previous section. The non-constant solutions are more complicated, but are interesting from the point of view of Einstein-Weyl geometry, since in four or more dimensions compact Einstein-Weyl manifolds with nontrivial Faraday curvature necessarily have finite fundamental group.

5 Appendix: Integrability of Möbius structures

In this appendix, I sketch the proof that the subbundle $\mathcal{R}^2$ of J^2L, corresponding to a Möbius structure $\mathcal{H}$, carries a natural connection whose curvature is given by the Cotton-York tensor of $\mathcal{H}$. The proof is not central to the main thrust of the paper, and will perhaps seem rather obscure to readers not familiar with the formal theory of differential equations. The calculations are purely formal, because Möbius geometry is a geometry of "finite type" much like conformal geometry in higher dimensions (see [9])—indeed, to emphasise the common framework, I will only specialise to the two dimensional case at the end of the proof. It is possible to convert the following into detailed and explicit calculations bypassing this conceptual framework, but I will not attempt that here.

The *symbol* g^2 of $\mathcal{R}^2$ is the line bundle of tracelike tensors in $S^2T^* \otimes L$ and $\mathcal{R}^2$ is an extension of J^1L by g^2. The induced first order equation on L is trivial and so one sets $\mathrm{g}^1 = T^* \otimes L$.

The prolongation of $\mathcal{R}^2$ is the bundle $\mathcal{R}^3 = J^1\mathcal{R}^2 \cap J^3L \subseteq J^1J^2L$, which may be viewed as a third order differential equation on L, or a first order differential equation on the bundle $\mathcal{R}^2$. It is the latter point of view which will be pursued. The corresponding differential operator is the restriction of the *jet derivative* to $\mathcal{R}^2 \subseteq J^2L$, i.e., the first order operator from J^2L to $T^* \wedge J^2L := (T^* \otimes J^2L)\big/(S^3T^* \otimes L)$ characterising the holonomic sections of J^2L.

Since the prolongation $\mathrm{g}^3 = (T^* \otimes \mathrm{g}^2) \cap (S^2T^* \otimes \mathrm{g}^1)$ of the symbol is zero, the natural map from $T^* \otimes \mathcal{R}^2$ to $T^* \wedge J^2L$ is injective. The first obstruction to the integrability of $\mathcal{R}^2$ is the failure of the jet derivative to have its image in this bundle. This obstruction therefore takes its values in $(T^* \wedge J^2L)\big/(T^* \otimes \mathcal{R}^2)$, which may be identified with the symbol cohomology group $(T^* \wedge S^2T^* \otimes L)\big/(T^* \wedge g^2)$. If the first obstruction vanishes then $\mathcal{R}^3$ defines a connection on $\mathcal{R}^2$ and since the higher symbol cohomology groups vanish, there are no further obstructions and the connection is flat [8]. The local parallel sections are holonomic sections of $\mathcal{R}^2$ and hence $\mathcal{R}^2$ is integrable. It therefore remains to show that the first integrability obstruction is given by the Cotton-York curvature.

The computation of the integrability obstruction is most easily carried out by introducing an exact Weyl derivative D on L. Using the induced covariant derivative on T^* (which is not flat in general), D defines an isomorphism

$$J^2L \cong (S^2T^* \otimes L) \oplus (T^* \otimes L) \oplus L$$

and covariant derivatives on the summands. A covariant derivative on J^2L may then be defined by the formulae:

$$D(\phi, \alpha, \lambda) = (D\phi - R^D \cdot \alpha, D\alpha - \phi, D\lambda - \alpha),$$

where $3(R^D \cdot \alpha)_X(Y, Z) = (R^D_{X,Y}\alpha)(Z) + (R^D_{X,Z}\alpha)(Y)$. The reason for this definition is that if (ϕ, α, λ) is holonomic then is covariant derivative is $(D^3\lambda - R^D \cdot D\lambda, 0, 0)$ which is a section of $S^3T^* \otimes L$, hence in the kernel of the symbol of the jet derivative. It follows that the jet derivative in general is given by the symbol applied to $D(\phi, \alpha, \lambda)$.

Now suppose (ϕ, α, λ) is a section of $\mathcal{R}^2$. Applying the jet derivative and removing the tracelike part gives

$$\big(-D(r^D_0\lambda) - R^D.\alpha \pmod{S^3T^* \otimes L}, D\alpha - \phi, D\lambda - \alpha\big).$$

since the tracefree part of ϕ is equal to $-r^D_0 \otimes \lambda$ which is the tracefree part of $-r^D \otimes \lambda$ since $F^D = 0$. Modulo the section $((D\lambda - \alpha) \otimes (-r^D_0), D\alpha - \phi, D\lambda - \alpha)$ of $T^* \otimes \mathcal{R}^2$ this is equivalent to

$$\big(-(Dr^D_0)\lambda - \alpha \otimes r^D_0 - R^D.\alpha \pmod{S^3T^* \otimes L}, 0, 0\big).$$

This integrability obstruction depends only on the 1-jet (α, λ) and so is given by a first order operator on L. It is now straightforward to see that the symbol of this operator is the Weyl curvature, and that in two and three dimensions it reduces to the Cotton-York curvature tensor.

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