# EINSTEIN METRICS AND COMPLEX SINGULARITIES

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Abstract. This paper is concerned with the construction of special metrics on non-compact 4-manifolds which arise as resolutions of complex orbifold singularities. Our study is close in spirit to the construction of the hyperkähler gravitational instantons, but we focus on a different class of singularities. We show that any resolution $X$ of an isolated cyclic quotient singularity admits a complete scalar-flat Kähler metric (which is hyperkähler if and only if $K_X$ is trivial), and that if $K_X$ is strictly nef, then $X$ also admits a complete (non-Kähler) self-dual Einstein metric of negative scalar curvature. In particular, complete self-dual Einstein metrics are constructed on simply-connected non-compact 4-manifolds with arbitrary second Betti number.

Deformations of these self-dual Einstein metrics are also constructed: they come in families parameterized, roughly speaking, by free functions of one real variable.

All the metrics constructed here are *toric* (that is, the isometry group contains a 2-torus) and are essentially explicit. The key to the construction is the remarkable fact that toric self-dual Einstein metrics are given quite generally in terms of *linear* partial differential equations on the hyperbolic plane.

## 1. Introduction and main theorems

Let $\Gamma \subset U(2)$ be a finite cyclic subgroup, acting on $\mathbb{C}^2$ in such a way that $\mathbb{C}^2/\Gamma$ is a complex orbifold with precisely one isolated singular point. Let $X$ be the minimal (Hirzebruch–Jung) resolution [2, §III.5] of $\mathbb{C}^2/\Gamma$. This paper is devoted to the construction of complete metrics on $X$. These metrics are all self-dual (with respect to the opposite of the complex orientation of $X$) and toric, *i.e.*, the isometry group contains a torus $SO(2) \times SO(2)$. We find a finite dimensional family of scalar-flat Kähler metrics (SFK) on $X$, and, if the canonical bundle $K_X$ is strictly nef, an infinite dimensional family of self-dual Einstein (SDE) metrics of negative scalar curvature. The SFK metrics on $X$ are all asymptotically locally euclidean (ALE), whereas the SDE metrics are asymptotically locally hyperbolic (AH), with the exception of a unique asymptotically locally *complex* hyperbolic (ACH) metric.

Our results constitute a generalization of the case where $\Gamma$ acts by scalar multiples of the identity on $\mathbb{C}^2$: if $|\Gamma| = p$, $X$ is the total space of the complex line bundle $\mathcal{O}(-p) \to \mathbb{C}P^1$. Then for each $p$ (including $p = 1$, the blow up of $\mathbb{C}^2$) there is a $U(2)$-invariant SFK metric on $X = \mathcal{O}(-p)$: this is due to Burns if $p = 1$, to Eguchi–Hanson if $p = 2$, and to LeBrun for $p > 2$ (see [22]). By the adjunction formula, $K_X$ is strictly nef if $p > 2$, in which case LeBrun's SFK metrics are known to be (locally) conformal to the SDE metrics of Pedersen [27]. We note in passing that if $p = 1$, then $K_X^{-1}$ is strictly nef and the Burns metric is locally conformal to the Fubini-Study metric (which is SDE, with positive scalar curvature) on the punctured projective plane $\mathbb{C}P^2 \smallsetminus \{\infty\}$.

If $p = 2$, then $K_X$ is trivial and the Eguchi–Hanson metric is itself Einstein—in fact, Ricci-flat (hyperkähler). This is also the first of the hyperkähler ALE gravitational instantons [12, 14, 20], which live on the minimal resolution $X$ of $\mathbb{C}^2/\Gamma$ when $\Gamma$ is a finite subgroup of $SU(2)$. Since $K_X$ is trivial when $\Gamma \subset SU(2)$, we have the following picture: if $X$ is a resolution of $\mathbb{C}^2/\Gamma$, where $\Gamma$ is cyclic, then

- $K_X^{-1}$ strictly nef $\Longrightarrow X$ carries a SDE metric of positive scalar curvature;

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- $K_X$ trivial $\Longrightarrow X$ carries complete SDE metrics of zero scalar curvature;
- $K_X$ strictly nef $\Longrightarrow X$ carries complete SDE metrics of negative scalar curvature.

Among these the first recovers only the Fubini–Study metric: $K_X^{-1}$ is strictly nef only in the case that $\Gamma = 1$ and $X$ is the blow-up of the origin of $\mathbb{C}^2$.

In order to state our results, it is convenient to describe more explicitly the Hirzebruch–Jung resolution $X$. This comes with a holomorphic map $\varpi\colon X \to \mathbb{C}^2/\Gamma$ and contains an *exceptional divisor* $E = \varpi^{-1}(0)$ such that $\varpi$ identifies $X \smallsetminus E$ biholomorphically with $\mathbb{C}^2/\Gamma \smallsetminus 0$. One has $E = S_1 \cup S_2 \cup \ldots \cup S_k$, where each $S_j$ is a smoothly embedded rational curve (Riemann sphere) and these curves intersect each other according to the matrix

$$(S_i \cdot S_j) = \begin{bmatrix} -e_1 & 1 & 0 & \cdots & 0 \\ 1 & -e_2 & 1 & \cdots & 0 \\ 0 & 1 & -e_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & -e_k \end{bmatrix} \tag{1.1}$$

where the $e_j$ are certain integers, $e_j \geqslant 2$, determined by $\Gamma$. (See §2.1 below.)

Recall also that a complex line bundle $L$ over a complex manifold $X$ is said to be *numerically effective* (*nef*) if

$$\deg(L|_C) \geqslant 0 \text{ for every compact complex curve } C \subset X. \tag{1.2}$$

Further, $L$ is said to be *strictly nef* if equality is strict in the above for every $C$. By the adjunction formula, for the Hirzebruch–Jung resolution $X$ of $\mathbb{C}^2/\Gamma$, $K_X$ is always nef, and is strictly nef if and only if $e_j \geqslant 3$ for each $j$.

Our first result concerns scalar-flat Kähler metrics:

**Theorem A.** *Let $\Gamma \subset U(2)$ be a finite cyclic subgroup such that $0 \in \mathbb{C}^2$ is the only fixed point, and let $X$ be a toric resolution of $\mathbb{C}^2/\Gamma$. Then $X$ admits a finite-dimensional family of asymptotically locally euclidean scalar-flat Kähler metrics.*

*Remarks.* First of all, the Hirzebruch–Jung resolution $X_0$ is a toric resolution, and any other such resolution $X$ is an iterated blow-up of $X_0$, the centres of the blow-ups being at fixed-points of the torus action. The SFK metrics of this theorem are given by *explicit formulae* and, as we have already indicated, admit an isometric action of the 2-torus. The result is implicit in the work of Joyce [16, 17] and follows easily from it.

Second, if $\Gamma = \{1\}$, this result yields asymptotically euclidean SFK metrics on certain iterated blow-ups of $\mathbb{C}^2$. A large class of these metrics were found earlier by LeBrun [23], who, more generally, allowed the centres of the blow-ups to lie on a complex line, so that $X$ carries an $S^1$ action, but is not necessarily toric. The case of a single blow-up, as we have mentioned, is the Burns metric. By the gluing theorems of Kovalev and the second author [19], there exist complete scalar-flat Kähler metrics on *any* iterated blow-up of $X_0$, but explicitness is then lost.

In our next result we find a conformally related non-Kähler SDE metric, with negative scalar curvature.

**Theorem B.** *Let $\Gamma$ and $X$ be as in Theorem A and suppose that $K_X$ is strictly nef. Then there is a function $F$ on $X$ with the property that*

$$dF \neq 0 \text{ on } Y := \{F = 0\}, \quad E \subset X_+ := \{F > 0\},$$

*and for* one *of the metrics $g$ in Theorem A,*

$$g_+ := F^{-2} g|_{X_+} \tag{1.3}$$

*is a complete self-dual Einstein metric with negative scalar curvature (with respect to the opposite orientation). Moreover, if $X_- = \{F < 0\}$, we have that*

$$g_- := F^{-2} g|_{X_-} \tag{1.4}$$

*is also self-dual Einstein metric with negative scalar curvature, which extends to a complete orbifold metric on the one-point compactification of $X_-$.*

The zero-set $Y$ of $F$ is a smoothly embedded lens space, diffeomorphic to the link of the singularity at the origin in $\mathbb{C}^2/\Gamma$, and $X_+$ is diffeomorphic to $X$ (Figure 1). The one-point compactification referred to here corresponds to the addition of a point at infinity of $X$. This picture was well known in the special case of the Pedersen–LeBrun metrics when $X$ is the total space of $\mathcal{O}(-p)$ (see e.g. [15]).

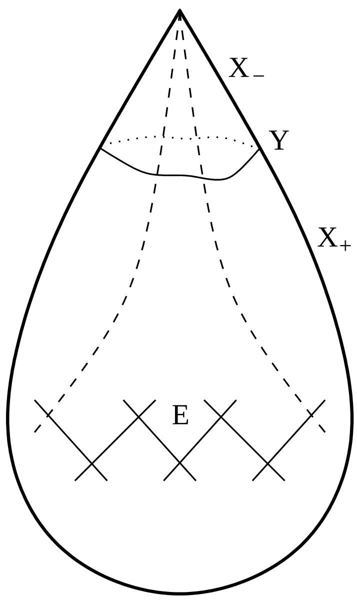


FIGURE 1. The one-point compactification of $X$.

*Remarks.* The constraint $K_X$ strictly nef does not give a restriction on the size of $b_2(X)$, so this theorem supplies explicit complete Einstein metrics on 4-manifolds with arbitrarily large second Betti number.

The metric in this theorem can also be realized as a quaternion-Kähler quotient; see Remark 4.2.1.

The given formulation of Theorem B is designed to fit naturally with the perspective of *asymptotically hyperbolic* metrics. This is a class of complete metrics generalizing the classical relation

$$g_{\text{hyp}} = \frac{4}{(1-r^2)^2} g_{\text{euc}}$$

between the hyperbolic metric $g_{\text{hyp}}$ on the open ball $\{r < 1\}$ and the euclidean metric $g_{\text{euc}}$ on its closure. The key point here is that after multiplication by the square of the defining function $(1-r^2)/2$, $g_{\text{hyp}}$ extends to a riemannian metric on the boundary $\{r = 1\}$. The relation (1.3) is an example of the same kind, for $g$ extends smoothly (as a riemannian metric) to $X$.

More generally, if $M$ is a compact manifold with boundary $N$, we say that a riemannian metric $g$ in the interior $M^o$ of $M$ is *asymptotically (locally) hyperbolic* (AH) with *conformal infinity* $(N, c)$, if for any boundary-defining function $u$, $u^2 g$ extends smoothly to $N$ and $u^2 g$ is in the conformal class $c$. In this situation, we also say that $(M, g)$ is a *filling* of $(N, c)$ or that $(N, c)$ *bounds* $(M, g)$. Notice that the freedom to multiply defining functions by any function smooth and positive near $N$ is absorbed precisely by the specification of a conformal class rather than a metric on $N$. These ideas suggest natural boundary-value problems, such as: given $(N, c)$, does there exist a filling $(M, g)$ with $g$ an Einstein metric? If so, is $g$ unique? Following work of Fefferman–Graham, Graham–Lee, Biquard and Anderson [11, 13, 7, 1], one is beginning to have a good understanding of this problem: at least if $c$ has positive Yamabe constant, it seems that 'generically' the Einstein filling exists and is unique up to diffeomorphism, so the problem is well-posed.

If $N$ is of dimension 3, there is another boundary-value problem, namely that of filling $c$ by a self-dual or anti-self-dual Einstein metric. In so far as the Einstein problem is well-posed, this problem must be over-determined. An important step in the study of this problem is Biquard's recent proof [6] of the positive-frequency conjecture of LeBrun [21, 24]. This asserts (roughly speaking) the existence of a decomposition $c = c_- + c_0 + c_-$ (if $c$ is close to the round conformal structure $c_0$ on $S^3$) with the property that $c_+ + c_0$ bounds a self-dual Einstein metric on the ball and $c_- + c_0$ bounds an anti-self-dual Einstein metric on the ball—equivalently $c_+ + c_0$ and $c_- + c_0$ bound SDE metrics on the ball inducing opposite orientations on $S^3$. This is a nonlinear version of the decomposition of a function on the circle into positive and negative Fourier modes.

In this language, the SDE metrics $g_\pm$ of Theorem B are both AH metrics, with a common conformal infinity $(Y, c)$, where $c$ is the conformal class of $g$. In this respect, $(Y, c)$ behaves quite analogously to the standard round metric on $S^3$. It seems that there are many new phenomena worthy of exploration here, to which we hope to return.

Our next result belongs to this circle of ideas. To state it, we recall that the link of the singularity in $X$ is a lens space.

**Theorem C.** *Let $X$ be as in Theorem B, with $K_X$ strictly nef, and let $N$ be the corresponding lens space. Then there is an infinite-dimensional family of conformal structures on $N$ that bound complete, asymptotically hyperbolic self-dual Einstein metrics on connected neighbourhoods $X_+$ of the exceptional divisor $E$ of $X$.*

These metrics are not quite as explicit as the one in Theorem B, since they involve functions given by integral formulae over the boundary of the hyperbolic plane. They are parameterized (roughly speaking) by distributions in one variable with compact support in $(-\infty, 0)$. Note also that the generic conformal structure in Theorem C will *not* bound an SDE orbifold metric on $X_-$. Indeed, we show that there are examples where the conformal structure on $X_+$ does not extend, as a self-dual conformal structure, at all into $X_-$.

The AH condition is not the only boundary condition we might consider: Biquard [7] also considers metrics which are asymptotic to the Bergman metric of complex hyperbolic space. This may be viewed as a limiting case in which the conformal structure on $N$ degenerates into a (pseudoconvex) CR structure. Recall that the latter is is given by a contact distribution $\mathcal{H} \subset TY$ and a conformal structure $c$ on $\mathcal{H}$ which is compatible with the Levi form. This means that there is a (uniquely determined) almost complex structure $J$ on $\mathcal{H}$, such that for any contact 1-form $\theta$ (for $\mathcal{H}$), $d\theta|_{\mathcal{H}}$ is the Kähler form, with respect to $J$, of a metric $h$ in $c$.

Now if $M$ is a compact manifold with boundary $N$, we say that a riemannian metric $g$ in the interior $M^o$ of $M$ is *asymptotically* (*locally*) *complex hyperbolic* (ACH) with *CR infinity* $(N, \mathcal{H}, c)$ if for any boundary-defining function $u$, there is a 1-form $\theta$ on $M$ such that $u^2 g - u^{-2}\theta^2$ extends to a smooth and degenerate metric $h$ on $M$, and such that the pull-back of $(\theta, h)$ to $N$ is a contact metric structure compatible with $(\mathcal{H}, c)$ on $N$. Note again that the CR structure is independent of the choice of boundary defining function (and the choice of the 1-form $\theta$ on $M$).

The ACH boundary-value problem for Einstein metrics is studied by Biquard [7]. As with the AH boundary condition, when $\dim N = 3$, we can strengthen the Einstein equation to the self-dual Einstein equation. To describe the explicit examples we obtain, we recall [3] that a CR manifold $N$ is said to be *normal* if it admits a Reeb vector field which generates CR automorphisms of $N$, and in addition *quasiregular* if the orbit space is an orbifold Riemann surface.

**Theorem D.** *Let $X$ be as in Theorem B, with $K_X$ strictly nef, and let $N$ be the corresponding lens space. Then there is an asymptotically complex hyperbolic self-dual Einstein metric on $X$ whose CR infinity is a quasiregular normal CR structure on $N$.*

In contrast to the AH boundary condition, we obtain only one SDE metric in each case, *i.e.*, we do not find deformations of the CR structure on $N$ which still bound ACH SDE metrics on manifolds diffeomorphic to $X$.

Our results, as summarized at the beginning of this introduction, provide an intriguing analogue of the Kähler–Einstein trichotomy: recall that a compact Kähler manifold $M$ can only admit a Kähler–Einstein metric if the first Chern class is positive-definite, zero, or negative-definite, the sign of the scalar curvature being respectively positive, zero or negative. (Moreover, thanks to the well-known work of Aubin, Calabi and Yau, the necessary conditions are also sufficient, if $c_1(M) \leqslant 0$.) If $L$ is a holomorphic line bundle and $M$ is a compact complex surface, then Nakai's criterion states that $L$ is ample (so that $c_1(L)$ can be represented by a Käher form) if and only if $c_1^2(L) > 0$ and $L$ is strictly nef [2, §IV.5]. The hypothesis '$K_X$ strictly nef' that we have used in our statements, should therefore be viewed as a replacement (which makes better sense in the present non-compact context) for the hypothesis '$c_1(X) < 0$'.

We now give two results which explore the condition '$K_X$ strictly nef'. On the positive side, we first note that the condition can be relaxed if we consider orbifolds. If the resolution $X$ has $K_X$ nef but not strictly nef, then our construction yields SDE orbifold metrics which are non-trivial if $K_X$ is non-trivial. Let $\check{X}$ denote the complex orbifold[1] obtained by contracting all chains of $(-2)$ curves contained in $E$. Then $\check{X}$ has a finite number of $A_n$-type singularities.

**Theorem E.** *Let $\check{X}$ be the orbifold described above. Then theorems B, C, and D apply to $\check{X}$, yielding complete self-dual Einstein orbifold metrics of negative scalar curvature.*

On the other hand, the following result shows that the condition that $K_X$ is strictly nef is necessary for the existence of toric SDE metrics on the smooth resolution $X$ itself.

**Theorem F.** *Let $(U, g)$ be a self-dual Einstein manifold of negative scalar curvature. Suppose that $\Sigma \subset U$ is a smoothly embedded, compact, totally geodesic surface. Then $\chi(\Sigma) - \Sigma \cdot \Sigma < 0$, where $\chi(\Sigma)$ is the Euler characteristic of $\Sigma$.*

Suppose that $U = X$ with the opposite of the complex orientation and a toric SDE metric $g$ on $X$, and that $\Sigma = S_j$, a rational curve in the exceptional divisor $E$ of $X$. Since $S_j$ is a component of the fixed point set of an isometry, it is totally geodesic, and so we can apply the above theorem to get $2 - e_j < 0$. We deduce that $e_j \geqslant 3$ for all $j$, which, as we have seen, is equivalent to $K_X$ being strictly nef. In particular, if $\Gamma \subset SU(2)$ and $X$ is the canonical resolution, all $e_j$ are equal to 2, so there can exist no toric SDE metric of negative scalar curvature on $X$.

The methods of this paper are essentially explicit and depend crucially on the classification of toric SDE metrics of nonzero scalar curvature by Pedersen and the first author [9]. It is indeed remarkable that the torus symmetry reduces the SDE equations to a standard linear

[1] This is the algebraic geometers' 'canonical model' of $\mathbb{C}^2/\Gamma$, cf. [29]

partial differential equation in the hyperbolic plane, which can be studied relatively easily. This may be contrasted with the case of $SU(2)$ or $SO(3)$ symmetry which leads to a nonlinear ordinary differential equation [15, 31]. Solutions of the latter generally have to be written down in terms of $\vartheta$-functions.

In the first two sections we review some essential background: the geometry of resolutions of cyclic singularities from the complex and smooth points of view; and the local construction of $T^2$-invariant self-dual and SDE metrics. Then we move on to the proofs of the main theorems giving examples along the way. An appendix is devoted to a brief exposition of some aspects of the geometry of hyperbolic space that were suppressed in the body of the paper.

**A note on orientations.** If $X$ is a complex surface, then with the standard complex orientation, a scalar-flat Kähler metric is anti-self-dual. The Fubini–Study metric on $\mathbb{C}P^2$, again with the standard complex orientation, is self-dual (and Einstein). In this paper are primary concern is (anti-)self-dual Einstein metrics, which generalize the Fubini–Study metric in a natural sense. We have therefore decided to state our results for self-dual metrics, which implies a reversal of the complex orientation of the complex surface $X$ in the above. We hope the reader will not find this change of orientations too tiresome.

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The diagrams were produced using Xfig, MAPLE and Paul Taylor's commutative diagrams package.

## 2. Toric topology

2.1. **Hirzebruch-Jung strings and cyclic surface singularities.** Let $p$ and $q$ be coprime integers with $p > q > 0$. Let $\Gamma$ be the cyclic subgroup of $U(2)$ generated by the matrix

$$\begin{bmatrix} \exp(2\pi i/p) & 0 \\ 0 & \exp(2\pi iq/p) \end{bmatrix}. \tag{2.1}$$

The quotient $\mathbb{C}^2/\Gamma$ is a complex orbifold, with an isolated singular point corresponding to the origin. The resolution of this singularity by a Hirzebruch–Jung string is well known in algebraic geometry, cf. [2, §III.5], especially [2, Theorem 5.1 and Proposition 5.3]. In outline, the story is as follows. There is a *minimal*[2] resolution $\pi : X \to \mathbb{C}^2/\Gamma$ with the properties

(i) $X$ is a smooth complex surface;
(ii) there is an *exceptional divisor* $E \subset X$ such that $\pi(E) = \{0\}$, but the restriction of $\pi$ is a biholomorphic map $X \smallsetminus E \to (\mathbb{C}^2/\Gamma) \smallsetminus \{0\}$;
(iii) $E = S_1 \cup S_2 \cup \cdots \cup S_k$ where the $S_j$ are holomorphically embedded smooth 2-spheres and the intersection matrix of the $S_j$ has the form

$$(S_i \cdot S_j) = \begin{bmatrix} -e_1 & 1 & 0 & \cdots & 0 \\ 1 & -e_2 & 1 & \cdots & 0 \\ 0 & 1 & -e_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & -e_k \end{bmatrix} \tag{2.2}$$

[2] *i.e.*, any other resolution is some blow-up of this one, or equivalently, there are no $(-1)$-curves

where all $e_j \geqslant 2$.

The integers $e_j$ are determined by $p$ and $q$ through the continued-fraction expansion

$$\frac{q}{p} = \cfrac{1}{e_1 - \cfrac{1}{e_2 - \cdots \cfrac{1}{e_k}}} \tag{2.3}$$

or equivalently by the euclidean algorithm in the form:

$$r_{-1} := p, \qquad r_0 := q, \qquad r_{j-1} = r_j e_{j+1} - r_{j+1}, \quad \text{where} \quad 0 \leqslant r_{j+1} < r_j. \tag{2.4}$$

Note that in this version of the division algorithm, the quotients $e_j$ are overestimated and since $r_{j+1} < r_j$, we have $e_j \geqslant 2$ for $j = 1, \dots k$. We remark also that the effect of reversing the order of the $e_j$ in this continued fraction is to replace $q/p$ by $\tilde{q}/p$, where $q\tilde{q} \equiv 1 \mod p$. This replacement in the action (2.1) does not change the orbifold singularity, just as one would expect.

By the adjunction formula, $K_X$ is nef; and we have that $K_X$ is strictly nef if all $e_j > 2$ and $K_X$ is trivial if all $e_j = 2$. In the latter case, $q = p - 1$, $\Gamma \subset SU(2)$ and $X$ is the canonical resolution of the $A_{p-1}$ singularity.

2.2. **Toric differential topology.** We turn now to a discussion of 'toric' 4-manifolds and orbifolds. These are a class of smooth 4-manifolds (orbifolds) with a given smooth action of the 2-torus $T^2 = S^1 \times S^1 = \mathbb{R}^2/2\pi\mathbb{Z}^2$ with the property that the generic orbit is a copy of $T^2$. The source for this material is the paper of Orlik and Raymond [26]; the summary in [18] is also useful.

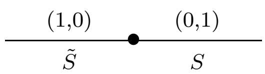
We begin with the standard action of $T^2$ on $\mathbb{C}^2 = \mathbb{R}^4$

$$(t_1, t_2) \cdot (z_1, z_2) = (e^{it_1} z_1, e^{it_2} z_2) \tag{2.5}$$

where $(t_1, t_2)$ are standard coordinates on $T^2$. The quotient space $\mathbb{R}^4/T^2$ is easily described by introducing polar coordinates $z_j = r_j e^{i\theta_j}$. Then it is clear that $\mathbb{R}^4/T^2$ is the quarter-space $Q = \{(r_1, r_2) : r_1 \geqslant 0, r_2 \geqslant 0\}$ and that the orbit corresponding to any interior point of $Q$ is a copy of $T^2$, while the orbits corresponding to boundary points have non-trivial stabilizers inside $T^2$. Since the origin is the only fixed point of this $T^2$-action, the stabilizer of the corner $(0, 0) \in Q$ is the whole of $T^2$, while the stabilizer of orbits corresponding to the positive $r_1$ and $r_2$ axes are certain circle subgroups of $T^2$. Such circle subgroups are essential in what follows and we adopt the description of them used by Orlik and Raymond.

2.2.1. *Definition.* For any pair of coprime integers $(m, n)$, $G(m, n) = \{mt_1 + nt_2 = 0\} \subset T^2$. (It is clear that $G(m, n) = G(\tilde{m}, \tilde{n})$ if and only if $(\tilde{m}, \tilde{n}) = \pm(m, n)$.)

With these conventions the stabilizer of the positive $r_2$-axis in $\mathbb{R}^4$ is $G(1, 0)$ and that of the positive $r_1$-axis is $G(0, 1)$. A complete description of the quotient space consists of the quarter space, with its two edges labelled by their stabilizers:




Here we have opened out the quarter-space in order to simplify the diagram and have labelled the edges (which correspond to the special orbits with circle-stabilizers) $S$ and $\tilde{S}$. It is to be understood with such diagrams that the generic orbits correspond to the upper half-space.

This $T^2$-action on $\mathbb{R}^4$ extends smoothly to the one-point compactification $S^4$, the point at infinity being another fixed point. The quotient $S^4/T^2$ is obtained as the one-point compactification of $Q$—it is a di-gon with two vertices and two edges labelled by $(1, 0)$ and $(0, 1)$.

Now consider the orbifold $\mathbb{R}^4/\Gamma$. Since the action of the finite group $\Gamma$ in (2.1) commutes with the action of $T^2$ on $\mathbb{R}^4$ it follows that $\mathbb{R}^4/\Gamma$ is a toric orbifold. The quotient map $\mathbb{R}^4 \to \mathbb{R}^4/\Gamma$ induces a map $Q \times T^2 \to Q \times T^2$, which must be the identity on $Q$. The required map $T^2 \to T^2$ must be surjective with kernel $\mathbb{Z}(1/p, q/p)$ so can be taken as

$$(t_1, t_2) \mapsto (pt_1, qt_1 - t_2).$$

The labels $(1,0)$ and $(0,1)$ map to $(p,q)$ and $(0,-1)$ by this map, so the combinatorial picture of $\mathbb{R}^4/\Gamma$ is as follows:

$$\frac{\quad(p,q)\quad}{\tilde{S}}\bullet\frac{\quad(0,-1)\quad}{S}$$

The compactification $S^4/\Gamma$ is an orbifold with two isolated singular points and the corresponding quotient is a di-gon with edges labelled by $(p,q)$ and $(0,-1)$.

According to Orlik and Raymond, the general picture of a smooth 4-orbifold $M$ with isolated singular points and smooth $T^2$-action will be modelled by the examples we have just described. More precisely, if we assume that $M$ is simply connected, then the quotient $M/T^2$ is topologically a closed polygon (simply connected 2-manifold with corners). A typical edge $S_j$ is labelled by the coprime pair $(m_j, n_j)$ such that $G(m_j, n_j)$ is the corresponding isotropy group. It is convenient also to label the vertex $S_j \cap S_{j+1}$ by the number

$$\varepsilon_j = m_j n_{j+1} - m_{j+1} n_j \tag{2.6}$$

so that the full picture of the boundary of $M/T^2$ is as follows:

$$\cdots\frac{\quad(m_{j+1},n_{j+1})\quad}{S_{j+1}}\overset{\varepsilon_j}{\bullet}\frac{\quad(m_j,n_j)\quad}{S_j}\overset{\varepsilon_{j-1}}{\bullet}\frac{\quad(m_{j-1},n_{j-1})\quad}{S_{j-1}}\cdots$$

The union of orbits corresponding to the edge $S_j$ is an embedded 2-sphere in $M$ and the vertex $S_j \cap S_{j+1}$ is a smooth point of $M$ if $\varepsilon_j = \pm 1$ and is more generally an orbifold point with isotropy of order $|\varepsilon_j|$. Indeed the precise quotient singularity can be discovered by transforming $(m_j, n_j)$ to $(0,-1)$ and $(m_{j+1}, n_{j+1})$ to $(p,q)$, $p > q > 0$, by an element of $SL(2,\mathbb{Z})$, and comparing with the model described above. One of the crucial points of the work of Orlik-Raymond is the converse to the above description, namely that *any* diagram of the above type gives rise to a unique 4-orbifold with a given action of $T^2$.

The topological properties of $M$ are encoded in diagrams of the above kind. For example, let

$$e_j := \varepsilon_{j-1}\varepsilon_j(m_{j-1}n_{j+1} - m_{j+1}n_{j-1}). \tag{2.7}$$

Then if $|\varepsilon_{j-1}| = |\varepsilon_j| = 1$ (and with a suitable orientation convention),

$$S_j \cdot S_j = e_j. \tag{2.8}$$

Notice that here we have blurred the distinction between the edge $S_j$ and the corresponding sphere in $M$ which lies over this edge. We shall continue with this abuse wherever convenient.

2.3. **Toric resolutions of singularities.** In order to resolve the singularity at the origin of $\mathbb{R}^4/\Gamma$

$$\frac{\quad(p,q)\quad}{\tilde{S}}\overset{p}{\bullet}\frac{\quad(0,-1)\quad}{S}$$

we must replace the vertex by a chain of vertices and edges of the following kind

$$\frac{\quad(p,\tilde{q})\quad}{\tilde{S}=S_{k+1}}\overset{1}{\bullet}\frac{\quad(m_k,n_k)\quad}{S_k}\overset{1}{\bullet}\frac{\qquad\qquad}{}\cdots\frac{\qquad\qquad}{}\overset{1}{\bullet}\frac{\quad(m_2,n_2)\quad}{S_2}\overset{1}{\bullet}\frac{\quad(1,0)\quad}{S_1}\overset{1}{\bullet}\frac{\quad(0,-1)\quad}{S=S_0}$$

where $\tilde{q} \equiv q \pmod{p}$ and we have used up the freedom to change bases by choosing

$$(m_0, n_0) = (0,-1), \quad (m_1, n_1) = (1,0).$$

(This normalization, which will be convenient later, forces us to allow $q$ to be replaced by $\tilde{q}$ as above. We have also chosen all $\varepsilon_j = +1$, which can always be done by playing with the sign ambiguity of the $(m_j, n_j)$.)

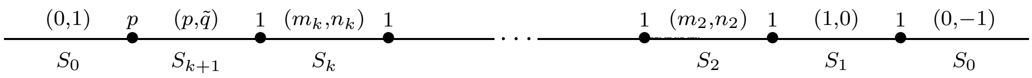
For the rest of this paper we shall assume also that $m_j > 0$ for $j = 1, \dots k+1$. This has the effect of making the toric resolution have semi-definite intersection-form, as follows from (2.8). Thus we make the following definition.

2.3.1. *Definition.* A sequence of coprime integers $(m_j, n_j)$ is called *admissible* if

(i) $(m_0, n_0) = (0, -1)$, $(m_1, n_1) = (1, 0)$, $(m_{k+1}, n_{k+1}) = (p, \tilde{q})$, $\tilde{q} \equiv q \pmod p$;
(ii) $m_j > 0$ for $j = 1, 2, \dots k+1$;
(iii) $m_j n_{j+1} - m_{j+1} n_j = 1$ for $j = 0, 1, \dots k$.

Then there is a one to one correspondence between admissible sequences and toric resolutions of $\mathbb{R}^4/\Gamma$ with semi-definite intersection form.

We note also that the compactified version of this resolution is given by the diagram




where the point labelled $p$ is the orbifold point at infinity.

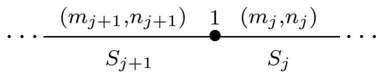
We observe that for an admissible sequence,

$$\frac{n_{j+1}}{m_{j+1}} = \cfrac{1}{e_1 - \cfrac{1}{e_2 - \cdots \cfrac{1}{e_j}}} \quad \text{for} \quad j = 1, 2, \dots k,$$

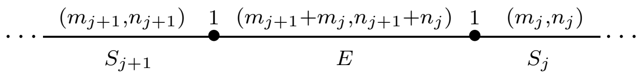
where the $e_j$ are defined by (2.7). Conversely any continued fraction expansion (2.3) (*i.e.*, any sequence of integers $e_1, e_2 \dots e_k$ with $e_j \geqslant 2$) defines the *minimal admissible sequence*

$$\begin{gathered}(m_2, n_2) = (e_1, 1), \quad (m_3, n_3) = (e_1 e_2 - 1, e_2), \quad \dots \\ (m_{j+1}, n_{j+1}) = e_j(m_j, n_j) - (m_{j-1}, n_{j-1})\end{gathered} \tag{2.9}$$

and this is the description, in toric differential topology, of the Hirzebruch-Jung string. Observe that the sequence $(m_j)$ is strictly increasing. It is not hard to check that any admissible sequence arises from the minimal one by blow-up, (*i.e.*, by connected sum with $\overline{\mathbb{C}P^2}$) which corresponds to the replacement of




by




where we have labelled the exceptional divisor $E$ and have abused notation by labelling the proper transforms of $S_j$ and $S_{j+1}$ by the same symbols. In terms of continued fractions, this process of blowing up corresponds to the identity

$$\cdots \cfrac{1}{a - \cfrac{1}{b - \cdots}} = \cdots \cfrac{1}{a + 1 - \cfrac{1}{1 - \cfrac{1}{b + 1 - \cdots}}},$$

which yields a new sequence of integers $e_j \geqslant 1$, the corresponding admissible sequence being still defined by (2.9). If a blow-up is performed at either end of the sequence, then $q$ will be changed by a multiple of $p$.

## 3. Local constructions of half-flat conformal structures

In this section we review the local construction of half-flat conformal structures admitting two commuting Killing vectors. For more details, please see [9, 18].

### 3.1. **Joyce's construction.**

3.1.1. *Data.*

- $(B, h)$ is a spin 2-manifold with metric $h$ of constant curvature $-1$;
- $\mathcal{W} \to B$ is the spin-bundle, viewed as a real vector bundle of rank 2 equipped with the induced metric, also denoted $h$;
- $\mathbb{V}$ is a given 2-dimensional vector space, with symplectic structure $\varepsilon(\cdot,\cdot)$ and $M = B \times \mathbb{V}$ is the corresponding product bundle.

Note that $\mathbb{V}$ acts in the obvious way by translations on $M$.

Given data as above, we define a $\mathbb{V}$-invariant riemannian metric on $M$ associated to any bundle isomorphism $\Phi\colon \mathcal{W} \to B \times \mathbb{V}$. Indeed, given $\Phi$, we define a family of metrics on $\mathbb{V}$, parameterized by $B$,

$$(v, \tilde{v})_\Phi = h(\Phi^{-1}(v), \Phi^{-1}(\tilde{v}))$$

and then a metric on the total space

$$g_\Phi = \Omega^2(h + (\cdot,\cdot)_\Phi), \tag{3.1}$$

where the conformal factor $\Omega > 0$ is in $C^\infty(B)$. It is clear that such a metric is invariant under the action of $\mathbb{V}$ on $M$. Any such metric also descends to the quotient $B \times (\mathbb{V}/\Lambda)$ where $\Lambda \subset \mathbb{V}$ is any lattice.

The remarkable observation of Joyce is that $g_\Phi$ is conformally half-flat if $\Phi$ satisfies a linear differential equation that we shall call here the Joyce equation. The Joyce equation essentially makes $\Phi$ an eigenfunction of the Dirac operator. Here we shall content ourselves with an explicit form of this equation, suitable for our later purposes. (See the appendix for a sketch of the underlying geometry.)

3.1.2. *The Joyce equation.* If $B \subset \mathcal{H}^2$ we can introduce half-space coordinates $(\rho, \eta)$, with $\rho > 0$, so that

$$h = \frac{d\rho^2 + d\eta^2}{\rho^2}$$

and an orthonormal frame $\lambda_1$, $\lambda_2$ of $\mathcal{W}^*$ such that

$$\lambda_1 \otimes \lambda_1 - \lambda_2 \otimes \lambda_2 = d\rho/\rho, \quad \lambda_1 \otimes \lambda_2 + \lambda_2 \otimes \lambda_1 = d\eta/\rho.$$

(From the complex point of view, $\lambda_1 + i\lambda_2 = \sqrt{(d\rho + i d\eta)/\rho}$.)

Since $\Phi \in C^\infty(B, \mathcal{W}^* \otimes \mathbb{V})$ we can write

$$\Phi = \lambda_1 \otimes v_1 + \lambda_2 \otimes v_2$$

where $v_1$ and $v_2$ are in $C^\infty(B, \mathbb{V})$. Then the Joyce equation is the system

$$\rho\partial_\rho v_1 + \rho\partial_\eta v_2 = v_1, \quad \rho\partial_\eta v_1 - \rho\partial_\rho v_2 = 0. \tag{3.2}$$

Clearly $\Phi$ defines a bundle isomorphism if $\varepsilon(v_1, v_2) \neq 0$ and then

$$\Phi^{-1} = \frac{\varepsilon(v_1,\cdot) \otimes \ell_1 - \varepsilon(v_2,\cdot) \otimes \ell_2}{\varepsilon(v_1, v_2)},$$

where $\ell_1, \ell_2$ is the dual orthonormal frame of $\mathcal{W}$.

Let us summarize these observations.

3.1.3. **Theorem.** [18] *With the above notation, if*

$$\varepsilon(v_1, v_2) \neq 0 \text{ in } B \tag{3.3}$$

*and $v_1$ and $v_2$ satisfy* (3.2)*, then for any conformal factor $\Omega$, the metric*

$$g_\Phi = \Omega^2\Big(\frac{d\rho^2 + d\eta^2}{\rho^2} + \frac{\varepsilon(v_1,\cdot)^2 + \varepsilon(v_2,\cdot)^2}{\varepsilon(v_1,v_2)^2}\Big) \tag{3.4}$$

*is a $\mathbb{V}$-invariant conformally half-flat metric on $M$.*

3.1.4. *Scalar-flat Kähler representatives.* For each choice of a point $y$ on the circle at infinity of $\mathcal{H}^2$, there is a scalar-flat Kähler metric conformal to $g_\Phi$:

$$g_{SFK} = \frac{\rho|\varepsilon(v_1,v_2)|}{\rho^2 + (\eta - y)^2}\Big(\frac{d\rho^2 + d\eta^2}{\rho^2} + \frac{\varepsilon(v_1,\cdot)^2 + \varepsilon(v_2,\cdot)^2}{\varepsilon(v_1,v_2)^2}\Big). \tag{3.5}$$

The fact that $g_{SFK}$ is a scalar-flat Kähler metric follows from [18, Proposition 2.4.4], or alternatively, as explained in [9], from the work of LeBrun [23].

3.1.5. *Basic solutions of the Joyce equation.* There is a basic solution of the Joyce equation (3.2)—or rather of the corresponding equation where the coefficients $v_1$ and $v_2$ are ordinary functions rather than $\mathbb{V}$-valued functions—associated to any given point $(0, y)$ of the boundary of $\mathcal{H}^2$:

$$\phi(\rho,\eta;y) = \frac{\rho\lambda_1 + (\eta - y)\lambda_2}{\sqrt{\rho^2 + (\eta - y)^2}}. \tag{3.6}$$

In [18] solutions of (3.2) given by finite linear combinations of the form $\phi(\rho,\eta;y)\otimes v$ were used to construct conformally half-flat metrics on connected sums of the complex projective plane. This idea will be used in this paper too, though we shall also use suitable infinite linear combinations of these solutions to construct infinite-dimensional families of metrics.

3.1.6. *Compactification.* We now explain how this construction of conformally half-flat metrics is combined with the previous description of toric 4-manifolds. The main point is to give sufficient conditions for the smooth extension of the metric $g_\Phi$ over the special orbits at the boundary. The following sufficient conditions were given by Joyce. Consider a combinatorial diagram with labelled edges and vertices as in §2.2 and consider the boundary of $M/T^2$ to be identified with the boundary $\rho = 0$ of $\mathcal{H}^2$. Note that $\mathbb{V}$ is the Lie algebra of $T^2$, which we identify with $\mathbb{R}^2$. Then if $B$ is a neighbourhood in $\mathcal{H}^2$ of a point in the interior of an edge labelled $(m, n)$, the metric (3.4) extends smoothly to the special orbits over $S \cap B$ provided that

$$v_1 = O(\rho), \quad v_2 = (m,n) + O(\rho^2) \text{ as } \rho \to 0 \tag{3.7}$$

and

$$\rho^{-2}\Omega^2 \text{ is smooth and positive in } B. \tag{3.8}$$

Similarly, if $B$ is now a neighbourhood of a corner at $\eta = \eta_0$, say, then (3.4) extends smoothly to the fixed-point if in addition to the boundary conditions (3.7) and (3.8), the conformal factor is chosen so that

$$\frac{\sqrt{\rho^2 + (\eta - \eta_0)^2}}{\rho^2}\Omega^2 \tag{3.9}$$

is smooth near the corner at $\eta = \eta_0$.

We note, further, that smooth orbifold metrics can be obtained by a very simple modification of these conditions. If, for example, the coprime pair $(m, n)$ is replaced by $(km, kn)$ in (3.7), then after changing to an index-$k$ sub-lattice of $\mathbb{Z} \oplus \mathbb{Z}$, we can apply the preceding discussion to get a smooth metric relative to this sub-lattice. Passing to this sub-lattice corresponds geometrically to passing to a $k$-fold cover branched over the special orbits over $S \cap B$. This is exactly the definition of a smooth orbifold metric with local isotropy $\mathbb{Z}/k$. Similarly, if

$$v_2 = (m,n) + O(\rho^2) \text{ for } \eta > \eta_0, \quad v_2 = (m',n') + O(\rho^2) \text{ for } \eta < \eta_0 \tag{3.10}$$

and $v_1 = O(\rho)$ as $\rho \to 0$ and $(m,n)$, $(m',n')$ generate a sub-lattice of index $k$, then by the same argument we see that the metric (3.4) extends smoothly to a $k$-fold cover of the fixed-point and so defines a smooth orbifold metric near that point.

3.1.7. *Boundary behaviour of the basic solution.* The basic solution $\phi(\rho,\eta;y)$ has very simple boundary behaviour, so that the preceding sufficient conditions are easily checked for linear combinations of them:

$$\text{if } \eta \neq y, \quad \phi(\rho,\eta;y) = O(\rho)\lambda_1 + \big(\mathit{sign}(\eta - y) + O(\rho^2)\big)\lambda_2 \text{ for small } \rho > 0. \tag{3.11}$$

3.2. **Self-dual Einstein metrics.** Return now to the local considerations. Suppose that $F \in \mathrm{C}^\infty(B)$ is an eigenfunction with eigenvalue $3/4$ of the hyperbolic laplacian

$$\Delta F = \frac{3}{4} F. \tag{3.12}$$

Such a function is a *potential* for a Joyce matrix in the following sense. Set

$$f(\rho,\eta) = \sqrt{\rho} F(\rho,\eta), \quad v_1 = (f_\rho, \eta f_\rho - \rho f_\eta), \quad v_2 = (f_\eta, \rho f_\rho + \eta f_\eta - f), \tag{3.13}$$

regarded as $\mathbb{V}$-valued functions on $B$. Then, as is easily verified,

$$\Phi = \frac{1}{2}(\lambda_1 \otimes v_1 + \lambda_2 \otimes v_2)$$

is a solution of the Joyce equation. The significance of these special solutions of Joyce's equation is as follows.

3.2.1. **Theorem.** [9] *Let $F$ be as in* (3.12) *and $v_1$ and $v_2$ as in* (3.13). *Suppose further that*

$$F^2 \neq 4|dF|^2 \tag{3.14}$$

*in $B$. Let*

$$D_+ = \{F > 0\}, \quad Z = \{F = 0\}, \quad D_- = \{F < 0\}.$$

*Then the metric*

$$g_F = \frac{\big|F^2 - 4|dF|^2\big|}{4F^2} \left( \frac{d\rho^2 + d\eta^2}{\rho^2} + \frac{\varepsilon(v_1,\cdot)^2 + \varepsilon(v_2,\cdot)^2}{\varepsilon(v_1,v_2)^2} \right) \tag{3.15}$$

*is a $\mathbb{V}$-invariant self-dual Einstein metric in $D_+ \times \mathbb{V}$ and $D_- \times \mathbb{V}$* (*if non-empty*). *The scalar curvature has the same sign as $F^2 - 4|dF|^2$, and conversely any $\mathbb{V}$-invariant SDE metric of nonzero scalar curvature arises* (*locally, and up to homothety*) *in this way.*

Notice that if $Z$ is non-empty then the scalar curvature must be negative and that (3.14) ensures that $F$ is a defining function for $Z$. Since $F^2 g_F$ extends to a smooth metric in $B \times \mathbb{V}$, it is clear that $g_F$ is 'locally conformally compact' with $Z \times \mathbb{V}$ as its conformal infinity.

3.2.2. *Basic solutions.* For each $y$, the function

$$F(\rho,\eta;y) = \frac{\sqrt{\rho^2 + (\eta - y)^2}}{\sqrt{\rho}} \tag{3.16}$$

is an eigenfunction satisfying (3.12) and it is easy to check that $F(\rho,\eta;y)$ is a potential for $\phi(\rho,\eta;y) \otimes (1,y)$.

3.2.3. *Remark.* The role of $F$ as a potential for $\Phi$ relies on some geometry which we outline in the appendix. The idea is that $\mathcal{W}$ is being identified with $\mathcal{H}^2 \times \mathbb{V}$; the derivative $dF$ of $F$ is then a section of $\mathcal{W}^* \otimes \mathbb{V}^*$, which gives a section of $\mathcal{W}^* \otimes \mathbb{V}$ using the symplectic form $\varepsilon(\cdot,\cdot)$. Explicitly, as a section of $S^2\mathcal{W}^*$, $\Phi$ is given by

$$\Phi = (\tfrac{1}{2}F + \rho F_\rho)\lambda_1 \otimes \lambda_1 + (\rho F_\eta)(\lambda_1 \otimes \lambda_2 + \lambda_2 \otimes \lambda_1) + (\tfrac{1}{2}F - \rho F_\rho)\lambda_2 \otimes \lambda_2.$$

The identification of $\mathcal{W}^*$ with $\mathcal{H}^2 \times \mathbb{V}$ is then obtained by setting

$$\lambda_1 = (1/\sqrt{\rho}, \eta/\sqrt{\rho}), \quad \lambda_2 = (0, -\sqrt{\rho}).$$

Note that $\det \Phi = \frac{1}{4}F^2 - |dF|^2$ which is used in (3.14).

3.2.4. *Example: hyperbolic space.* The hyperbolic metric on the unit ball in $\mathbb{R}^4$ takes the form

$$g = (1 - r_1^2 - r_2^2)^{-2}(dr_1^2 + dr_2^2 + r_1^2 d\vartheta_1^2 + r_2^2 d\vartheta_2^2)$$

in coordinates $(r_1, \vartheta_1)$, $(r_2, \vartheta_2)$ adapted to the standard action of $T^2$. This metric arises in the above construction from the function

$$F = \frac{1}{2}\big(F(\rho, \eta; -1) - F(\rho, \eta; 1)\big).$$

To make this explicit is a matter of direct computation, using the formula

$$(r_1 + ir_2)^2 = \frac{\eta - 1 + i\rho}{\eta + 1 + i\rho}.$$

The reader may care to verify that

$$\det \Phi = -\frac{\rho}{\sqrt{\rho^2 + (\eta-1)^2}\sqrt{\rho^2 + (\eta+1)^2}}$$

and that the fibre metric (including the conformal factor) is

$$\frac{\sqrt{\rho^2 + (\eta-1)^2}\sqrt{\rho^2 + (\eta+1)^2}}{\big(\sqrt{\rho^2 + (\eta-1)^2} - \sqrt{\rho^2 + (\eta+1)^2}\,\big)^2}\begin{bmatrix}(1-u)/2 & 0 \\ 0 & (1+u)/2\end{bmatrix}$$

where

$$u = \frac{\rho^2 + \eta^2 - 1}{\sqrt{\rho^2 + (\eta-1)^2}\sqrt{\rho^2 + (\eta+1)^2}}.$$

Multiplying out and changing from $(\rho, \eta)$ to $(r_1, r_2)$ yields the hyperbolic metric.

## 4. The SFK metrics and canonical SDE metrics

In the following three sections, we shall prove the theorems stated in the introduction. In this section we shall give rather brief indications of the proofs of Theorems A and B (the theorems concerning the scalar-flat Kähler metrics and canonical SDE metrics associated to Hirzebruch–Jung strings). Further details are given, in a more general setting, in the next section. Indeed the results given here can be read as a guide to the next section, as an extended set of examples.

4.1. **ALE scalar-flat Kähler metrics.** In order to prove Theorem A, we take up the toric description of Hirzebruch–Jung strings (and blow-ups of these strings) from §2.2. Thus we suppose given an admissible sequence $(m_j, n_j)$ for $j = 0, \ldots k+1$, where

$$(m_{k+1}, n_{k+1}) = (p, \tilde{q}), \quad \tilde{q} \equiv q \pmod{p}.$$

We supplement this by setting

$$(m_{k+2}, n_{k+2}) = -(m_0, n_0) = (0, 1).$$

Since

$$m_j n_{j+1} - m_{j+1} n_j = 1 \quad (j \neq k+1), \quad m_{k+1} n_{k+2} - m_{k+2} n_{k+1} = p$$

we see that the sequence of rationals $n_j/m_j$ enjoys a *monotonicity property*

$$\frac{n_{j+1}}{m_{j+1}} > \frac{n_j}{m_j} \text{ all } j.$$

4.1.1. *Notation.* Throughout this section we shall denote by $X$ the complex manifold corresponding to the data $(m, n)$—as we have seen, it is a resolution of $\mathbb{C}^2/\Gamma$ obtained from the minimal one by blowing up at fixed points of the $T^2$-action. We denote by $\overline{X}$ the one-point compactification (the added point will be denoted $\infty$).

Given an admissible sequence, pick real numbers $y_0 > y_1 > \cdots > y_{k+1}$ and define

$$\Phi(\rho, \eta) = \frac{1}{2}\sum_{j=0}^{k+1} \phi(\rho, \eta; y_j) \otimes (m_j - m_{j+1}, n_j - n_{j+1}). \tag{4.1}$$

As a finite sum of the basic solutions $\phi(\rho, \eta; y)$, it is clear that $\Phi$ satisfies the Joyce equation and hence the metric $g_\Phi$ of (3.1) is a toric conformally half-flat metric wherever $\det \Phi \neq 0$. We re-sum (4.1) in the form

$$\Phi(\rho, \eta) = \frac{1}{2}\sum_{j=0}^{k+1} \big(\phi(\rho, \eta; y_j) - \phi(\rho, \eta; y_{j-1})\big) \otimes (m_j, n_j) \tag{4.2}$$

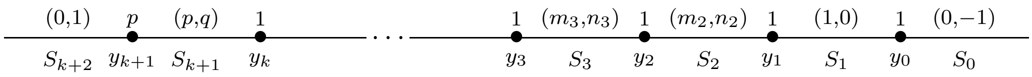
where we decree

$$\phi(\rho, \eta; y_{-1}) = -\phi(\rho, \eta; y_{k+1});$$

it then follows from [18, Lemma 3.3.2] and the monotonicity of the rationals $n_j/m_j$ that $\det \Phi < 0$ in $\mathcal{H}^2$. (See also Proposition 5.1.6 below.) So $g_\Phi$ defines a $T^2$-invariant conformally half-flat metric over the whole of $T^2 \times \mathcal{H}^2$. Now it follows easily from the definitions and the known boundary behaviour of $\phi(\rho, \eta; y)$ that if $y_j < \eta < y_{j-1}$, then

$$\Phi(\rho, \eta) = \lambda_1 \otimes O(\rho) + \lambda_2 \otimes \big((m_j, n_j) + O(\rho^2)\big).$$

This is indicated by the following diagram




where the labelling conventions are as in §2.2. In particular, all points save $y_{k+1}$ are smooth points of the toric manifold, while $y_{k+1}$ is an orbifold point.

It follows from this discussion and the results summarized from Joyce [18] in §3.1.6 that the conformal class defined by $g_\Phi$ extends as a smooth orbifold metric to $\overline{X}$. Next one checks that the 'scalar-flat Kähler' conformal factor

$$\Omega^2 = \frac{\rho |\det \Phi|}{\rho^2 + (\eta - y_{k+1})^2} \tag{4.3}$$

has precisely the correct boundary behaviour so that $g_{SFK}$ of (3.5) extends smoothly to $X$.

Finally we observe (as remarked by Joyce after the proof of [18, Theorem 3.3.1]) that the conformal factor (4.3) ensures that the metric is asymptotic near $y_{k+1}$ to the flat metric on $\mathbb{R}^4$ near infinity. Since this is an orbifold singularity of $\overline{X}$ in general, the metric is ALE. (When there is no singularity, we recover Joyce's half conformally flat metrics on $k\mathbb{C}P^2$.) The complex structure preserves the special orbits, so it is associated to a realization of $X$ as a resolution of $\mathbb{C}^2/\Gamma$. This proves Theorem A.

This result effectively gives a $k - 1$ dimensional family of ALE scalar-flat Kähler metrics, because an equivalent metric is obtained under a projective transformation of the points $y_0, y_1, \dots y_{k+1}$. We could equally well have assumed that the points are increasing (as does Joyce), but to assume them decreasing is more natural in the next section.

4.2. **Self-dual Einstein metrics.** We shall prove Theorem B using the results of [9] summarized in Theorem 3.2.1. We shall show that if all $e_j \geqslant 3$, then the Joyce matrix $\Phi$ admits a potential, for a particular choice of the $y_j$. For any set of real numbers $w_j$, the eigenfunction

$$F = \sum_{j=0}^{k+1} w_j F(\cdot; y_j) \tag{4.4}$$

is a potential for the Joyce matrix

$$\Phi = \frac{1}{2} \sum_{j=0}^{k+1} \phi(\cdot; y_j) \otimes (w_j, y_j w_j) \tag{4.5}$$

and this has the form (4.1) if

$$(m_j - m_{j+1}, n_j - n_{j+1}) = (w_j, y_j w_j).$$

Reading this the other way, we see that the SFK metric on $X$ has (locally) a SDE representative $g_F$ in its conformal class, with potential given by (4.4) if and only if

$$w_j = m_j - m_{j+1},$$

and the sequence $y_0, \dots y_{k+1}$ is given by

$$y_j = \frac{n_{j+1} - n_j}{m_{j+1} - m_j}.$$

We require that this sequence is monotonic. We first recall that if all $e_j \geqslant 2$ (*i.e.*, for a minimal resolution) then the sequence $(m_j)$ is strictly increasing for $j = 1, \dots k+1$, and we note that we automatically have $y_{k+1} < y_k$. It remains then to impose the condition that $y_j < y_{j-1}$ for $j = 1, \dots k$. Using the definition (2.7) of $e_j$ we obtain

$$y_j - y_{j-1} = \frac{2 - e_j}{(m_{j+1} - m_j)(m_j - m_{j-1})},$$

and so it follows that if all $e_j \geqslant 3$, then the sequence $(y_j)$ is strictly decreasing.

The SDE representative is defined only where $F \neq 0$, so the next step is to analyse the zero-set of $F$. Since

$$f(\rho, \eta) := \sqrt{\rho} F(\rho, \eta) \tag{4.6}$$

is continuous up to the boundary we may define

$$Z = \{f = 0\}, \quad D_+ = \{f > 0\} \quad \text{and} \quad D_- = \{f < 0\}$$

as subsets of the conformal compactification $\overline{\mathcal{H}}^2 := \{\rho \geqslant 0, \eta \in \mathbb{R}\} \cup \infty$ of $\mathcal{H}^2$.

Notice that at any interior point of $Z$,

$$|dF|^2 = -\det \Phi > 0 \tag{4.7}$$

so that $F$ is a defining function and $Z \cap \mathcal{H}^2$ is a closed submanifold of $\mathcal{H}^2$. We claim now that $Z$ is a simple arc joining a certain point $(0, \eta_0)$ to $\infty$ and intersecting no other point on the boundary of $\mathcal{H}^2$. First we rule out the possibility that $Z$ contains a closed $C$ curve in the interior of $\mathcal{H}^2$. Because $F$ is a defining function, such a curve is a smooth submanifold, and hence is *simple*. By the Jordan curve theorem, $C$ has an interior $U$, say and because $F$ is continuous on the compact set $U \cup C$, and is zero on the boundary, it is either identically zero or achieves a positive maximum or a negative minimum in $U$. But as a solution of $\Delta F = (3/4)F$, $F$ satisfies a strong maximum principle which rules out the last two possibilities. Hence $F = 0$ in $U$, contradicting (for example) (4.7). We conclude that $Z$ consists of one or more disjoint simple arcs joining boundary points of $\mathcal{H}^2$.

In order to determine the end-points of $Z$, we must calculate the zeros of $f(0,\eta)$. It can be checked (see also Proposition 5.1.1) that if $y_j < \eta < y_{j-1}$, then

$$f_0(\eta) := f(0,\eta) = m_j\eta - n_j. \tag{4.8}$$

Since $m_s \geqslant 0$, $f_0(\eta)$ is non-decreasing in $(y_{s+1}, y_s)$ and since $f_0(\eta)$ is continuous, it follows that $f_0(\eta)$ is a non-decreasing function of $\eta \in \mathbb{R}$. Since $f_0(\pm\infty) = \pm 1$ and $f_0$ is strictly increasing in $[y_{k+1}, y_0]$, $f_0(\eta)$ has a unique zero at $\eta_0$, say (Figure 2).

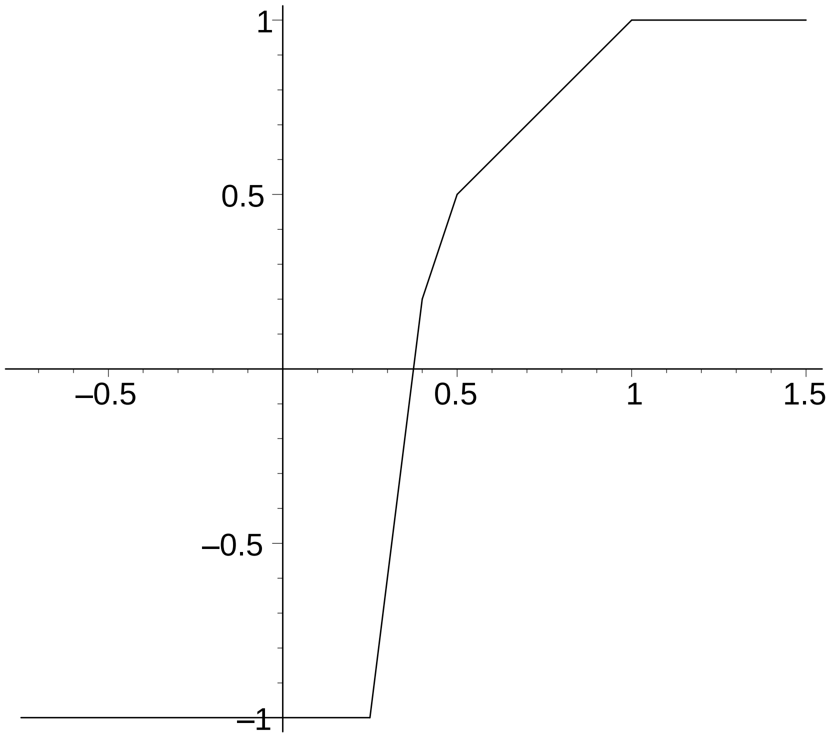



FIGURE 2. The graph of $f_0$ when $q/p = 3/8$. The zero is at $3/8$.

In fact, since

$$y_{k+1} = \frac{n_{k+2} - n_{k+1}}{m_{k+2} - m_{k+1}} = \frac{q-1}{p} < \frac{q}{p} < \frac{q - n_k}{p - m_k} = \frac{n_{k+1} - n_k}{m_{k+1} - m_k} = y_k$$

(note that $n_k/m_k < q/p$), it follows that $\eta_0 = q/p$ and this point lies in the interval $(y_{k+1}, y_k)$. Hence the boundary of $D_+$ is equal to $(q/p, \infty)$.

Since $f$ is smooth up to the boundary near $(0, \eta_0)$, $Z$ must consist of a single, simple arc joining this point to $\infty$. Further, $\partial_\rho f(0,\eta_0) = 0$ so $Z$ cuts the $\eta$-axis orthogonally. To complete the proof, we simply define $X_\pm$ to be the set of orbits of $X$ lying over $D_\pm$ and $Y$ to be the set of orbits lying over $Z$. Then the pull-back of $f$ to $X$ is a defining function for $Y$ and the restriction $g_+$ of $g_F$ to $X_+$ is the desired complete SDE metric.

The metric $g_F$ extends to the special orbits because $g_{SFK}$ has this property and the ratio of the SDE and SFK conformal factors is

$$\frac{\rho F^2}{\rho^2 + (\eta - (q-1)/p)^2}$$

which has a continuous positive limit as $\rho \to 0$ if $\eta > q/p$. This proves Theorem B.

4.2.1. *Remark.* The metrics of Theorem B can be realized as quaternion Kähler quotients of $(4k+4)$-dimensional quaternionic hyperbolic space $\mathbb{H}\mathcal{H}^{k+1}$ by a $k$-dimensional sub-torus of a maximal ($(k+2)$-dimensional) torus in $\mathrm{Sp}(1,k+1)$. For analogous quotients of quaternionic projective space $\mathbb{H}P^{k+1}$, the relation with the hyperbolic eigenfunction construction was obtained in [9, §9] by comparing the hyperkähler quotient of $\mathbb{H}^{k+2}$ (as described by Bielawski–Dancer [5]) with the Swann bundle of the toric self-dual Einstein metric using the generalized Gibbons–Hawking (GH) Ansatz of Pedersen–Poon [28].

As remarked in [9], the identification of the metrics in the present context is a straightforward analytic continuation of the arguments given there. Indeed, if we write the standard semi-hyperkähler metric on $\mathbb{H}^{1,k+1}$ in the momentum coordinates $\boldsymbol{x}_i \in \mathrm{Im}\,\mathbb{H}$ $(i = 0, 1 \dots k+1)$ of the action of the standard maximal torus (the $i = k+1$ direction being space-like, the rest time-like), then it takes the form

$$g = \frac{|d\boldsymbol{x}_{k+1}|^2}{|\boldsymbol{x}_{k+1}|} + |\boldsymbol{x}_{k+1}|\theta_{k+1}^2 - \sum_{j=0}^{k}\Big(\frac{|d\boldsymbol{x}_j|^2}{|\boldsymbol{x}_j|} + |\boldsymbol{x}_j|\theta_j^2\Big) \tag{4.9}$$

where the $\theta_j$ are the connection 1-forms of the torus action. A $k$-dimensional sub-torus of the maximal torus $T^{k+2}$ may be described by giving its Lie algebra as the kernel of a linear map $\mathbb{R}^{k+2} \to \mathbb{R}^2$, which in turn is determined by the images $\alpha_j \in \mathbb{R}^2$ of the standard basis elements of $\mathbb{R}^{k+2}$. Following Bielawski–Dancer [5], the image, by the momentum map of $T^{k+2}$, of the the zero set of the momentum map of the sub-torus, may be parameterized by the straightforward substitution $\boldsymbol{x}_j = \sum_{r=1}^{2}(\alpha_j)_r \boldsymbol{y}_r$ for the momentum map $(\boldsymbol{y}_1, \boldsymbol{y}_2)$ of the quotient torus. In these coordinates, the semi-hyperkähler quotient may be written in the form of the generalized GH Ansatz:

$$\sum_{r,s=1}^{2}\Big(\Phi_{rs}\langle d\boldsymbol{y}_r, d\boldsymbol{y}_s\rangle + {\Phi^{-1}}_{rs}\eta_r\eta_s\Big) \tag{4.10}$$

for some connection 1-forms $\eta_r$, where

$$\Phi_{rs} = \frac{(\alpha_{k+1})_r(\alpha_{k+1})_s}{R_{k+1}} - \sum_{j=0}^{k}\frac{(\alpha_j)_r(\alpha_j)_s}{R_j} \tag{4.11}$$

and $R_j = |(\alpha_j)_1\boldsymbol{y}_1 + (\alpha_j)_2\boldsymbol{y}_2|$. Now from the formula for the Swann bundle of a toric self-dual Einstein metric given in [9, §8], we deduce that the quaternion Kähler quotient metric is determined by the hyperbolic eigenfunction

$$F(\rho,\eta) = \frac{\sqrt{a_{k+1}\rho^2 + (a_{k+1}\eta - b_{k+1})^2}}{\sqrt{\rho}} - \sum_{j=0}^{k}\frac{\sqrt{a_j\rho^2 + (a_j\eta - b_j)^2}}{\sqrt{\rho}} \tag{4.12}$$

with $a_j = -(\alpha_j)_2$, $b_j = (\alpha_j)_1$. We remark that this is the function induced on the hyperbolic plane by the quadratic form $\langle u, u\rangle$, $u \in \mathbb{H}^{1,k+1}$.

We now claim that any metric of Theorem B is of this form for suitable $\alpha_j$. This follows, by writing $y_j = b_j/a_j$, simply because $w_0, \dots w_k$ are negative and $w_{k+1}$ is positive.

The restriction $g_+$ of $g_F$ to $X_+$ corresponds to the quaternion Kähler quotient of $\mathbb{H}\mathcal{H}^{k+1}$, the projectivized subset of $u \in \mathbb{H}^{1,k+1}$ on which $\langle u, u\rangle$ is positive, and with a little more work one can identify $(X_+, g_+)$ *globally* with the quaternion Kähler quotient of $\mathbb{H}\mathcal{H}^{k+1}$.

As mentioned in the introduction, the restriction $g_-$ of $g_F$ to $X_-$ is also of interest. It is not difficult to check that $g_-$ extends to a smooth orbifold metric on $X_-$, and passing to the universal orbifold cover we obtain a smooth AH $\Gamma$-invariant SDE metric on the 4-ball. From the quotient point of view, this is the quaternionic Kähler quotient of $\mathbb{H}\mathcal{H}^{k,1}$, the projectivized subset of $\mathbb{H}^{1,k+1}$ on which $\langle u, u\rangle$ is negative.

This picture was well known for the special case when $X$ reduces to $\mathcal{O}(-p)$. Let us now consider these Pedersen–LeBrun metrics in more detail from this point of view.

4.2.2. *Example.* Here $k = 1$ and there is only one parameter $p = e_1 > 0$. We have

$$(m_0, n_0) = (0, -1), \quad (m_1, n_1) = (1, 0), \quad (m_2, n_2) = (p, 1), \quad (m_3, n_3) = (0, 1)$$

and

$$y_0 = 1, \quad y_1 = 1/(p-1), \quad y_2 = 0.$$

We see at once that we must take $p \neq 2$ if the $y_j$ are to be distinct. This fits with the fact that when $p = 2$ the scalar-flat Kähler metric is hyperkähler.

If $p > 2$, then we get examples of the type discussed here, with

$$F = -F(\cdot; 1) - (p-1)F(\cdot; 1/(p-1)) + pF(\cdot; 0).$$

The case $p = 1$ is slightly awkward from this point of view, but by applying an orientation-preserving projective transformation, we see that we should regard the points $1, \infty, 0$ as increasing: indeed they are increasing after a cyclic permutation. The Einstein metric in this case has positive scalar curvature: it is the Fubini–Study metric on $\mathbb{C}P^2$ defined by

$$F = F(\cdot; 1) + F(\cdot; \infty) + F(\cdot; 0)$$

where $F(\rho, \eta; \infty) := 1/\sqrt{\rho}$.

The results of this section show that exactly the same phenomenon occurs in general, except that there are no further examples with positive scalar curvature, and in the negative scalar curvature case we need $e_j > 2$ for all $j$.

## 5. Infinite-dimensional families of complete SDE metrics

In this section, we shift viewpoint so that the 'labelling' of the edges of $M/T^2$ is regarded as a $\mathbb{V}$-valued step-function

$$(m(y), n(y)) = (m_j, n_j) \quad \text{for} \quad y_j < y < y_{j-1},$$

where we regard $y_{k+2} = -\infty$, $y_{-1} = +\infty$. Then the derivative $(m', n')$ of $(m, n)$ with respect to $y$ is defined in the sense of distributions and since

$$(m', n') = \sum_{j=0}^{k+1} \delta(y - y_j)(m_j - m_{j+1}, n_j - n_{j+1}),$$

we see that the formula (4.1) for the matrix $\Phi$ used in the last section can be written

$$\Phi(\rho, \eta) = \frac{1}{2} \int \phi(\rho, \eta; y) \otimes (m', n')\, dy.$$

Note that we shall use 'classical' notation for distributions (*i.e.*, $\int u(y)\phi(y)\, dy$ in place of $\langle u, \phi \rangle$ for the pairing of a distribution $u$ with a test-function $\phi$) throughout this section.

The theme of this section is the replacement of the step-function $(m, n)$ by more general distributions. This point of view gives a rather efficient proofs of the results outlined in the previous section, in a much more general setting.

### 5.1. Smeared solutions of the Joyce equation and their potentials.

Let $v$ be a $\mathbb{V}$-valued distribution on $\mathbb{R}$ with compact support. Then

$$\Phi(\rho, \eta) = \frac{1}{2} \int \phi(\rho, \eta; y) \otimes v(y)\, dy \tag{5.1}$$

defines a smooth Joyce matrix in $\mathcal{H}^2$, since $\phi(\rho, \eta; y)$ is $\mathrm{C}^\infty$ in all variables for $\rho > 0$. Similarly, if $w$ is any real valued distribution on $\mathbb{R}$ with compact support, we may define

$$F(\rho, \eta) = \int F(\rho, \eta; y) w(y)\, dy \tag{5.2}$$

and this will be a smooth eigenfunction of the hyperbolic laplacian. These are the 'smeared' solutions of the title. Moreover, exactly as in §4.2, (5.2) is a potential for (5.1) if we set

$$v(y) = (w(y), yw(y)). \tag{5.3}$$

Motivated by the introduction to this section, we wish to write $v = (\mu', \nu')$ where $\mu$ and $\nu$ are distributions on $\mathbb{R}$. Recall that a distribution $u$ satisfies $u' = 0$ (in the sense of distributions) on an open set if and only if $u$ is locally constant on this open set. Therefore $v$ has compact support if and only if

$$\mu \text{ and } \nu \text{ are } \textit{locally constant near infinity.} \tag{5.4}$$

We shall assume additionally that $\mu$ and $\nu$ are *odd at infinity*, *i.e.*,

$$\mu(\infty) = -\mu(-\infty), \qquad \nu(\infty) = -\nu(-\infty). \tag{5.5}$$

Suppose now that (5.3) also holds, *i.e.*,

$$\mu'(y) = w(y), \qquad \nu'(y) = yw(y), \tag{5.6}$$

and set

$$f_0(y) = y\mu(y) - \nu(y). \tag{5.7}$$

Then $f_0''(y) = w(y)$ and

$$\mu(y) = f_0'(y), \qquad \nu(y) = yf_0'(y) - f_0(y). \tag{5.8}$$

Conversely, if $f_0$ is a distribution on $\mathbb{R}$ such that $f_0''$ has compact support, then (5.7) holds with $\mu$ and $\nu$ defined by (5.8); $f_0$ is *affine near infinity*, and we also require, in accordance with (5.5), that the coefficients $\mu, \nu$, are not just locally constant, but odd at infinity.

These assumptions allow us to integrate by parts once in (5.1) and twice in (5.2) to get the formulae

$$\Phi(\rho, \eta) = \frac{\rho}{2} \int \frac{(y-\eta)\lambda_1 + \rho\lambda_2}{\big(\rho^2 + (\eta - y)^2\big)^{3/2}} \otimes (\mu(y), \nu(y))\, dy \tag{5.9}$$

and

$$F(\rho, \eta) = \frac{\rho^{3/2}}{2} \int \frac{f_0(y)\, dy}{\big(\rho^2 + (\eta - y)^2\big)^{3/2}}. \tag{5.10}$$

These are Poisson formulae in the following sense.

5.1.1. **Proposition.** *If $\Phi$ and $F$ are given by* (5.9) *and* (5.10) *then we have* (*in the sense of distributions*)

$$\Phi(\rho, \eta) \to \lambda_2 \otimes (\mu(\eta), \nu(\eta)) \ \text{ as } \rho \to 0 \tag{5.11}$$

*and*

$$\sqrt{\rho} F(\rho, \eta) \to f_0(\eta) \ \text{ as } \rho \to 0. \tag{5.12}$$

*Moreover, if $(\mu, \nu)$ is constant and $f_0$ is affine in a neighbourhood of $\eta = a$ then*

$$\Phi(\rho, \eta) = \lambda_1 \otimes O(\rho) + \lambda_2 \otimes \big((\mu(\eta), \nu(\eta)) + O(\rho^2)\big) \tag{5.13}$$

*and*

$$\sqrt{\rho} F(\rho, \eta) = f_0(\eta) + O(\rho^2) \tag{5.14}$$

*near $\eta = a$.*

5.1.2. *Remark.* From now on, when we say that an eigenfunction $F$ has a boundary value $f_0$, we shall always mean (5.12).

5.1.3. *Proof.* We claim that

$$\lim_{\rho\to 0} \frac{\rho^2}{2(\rho^2+y^2)^{3/2}} = \delta(y)$$

and that

$$\lim_{\rho\to 0} \frac{y}{2(\rho^2+y^2)^{3/2}} = 0.$$

These two limits yield (5.11) and (5.12) (by translation of $y$). To establish the first claim, test against a function $\phi$ and make the change of variables $y = \rho z$ to get

$$\int \frac{\rho^2}{2(\rho^2+y^2)^{3/2}}\phi(y)\,dy = \int \frac{\phi(\rho z)\,dz}{2(1+z^2)^{3/2}} \to \phi(0)$$

as $\rho \to 0$, since $\int (1+z^2)^{-3/2}\,dz = 2$. Similarly, for the second claim,

$$\int \frac{\rho y}{2(\rho^2+y^2)^{3/2}}\phi(y)\,dy = \int \frac{z\phi(\rho z)\,dz}{2(1+z^2)^{3/2}} \to 0$$

as $\rho \to 0$.

If now $(\mu,\nu)$ is constant near $\eta = a$ we can expand in powers of $\rho^2$ the denominator $\big(\rho^2 + (a-y)^2\big)^{-1/2}$ in (5.1). Such an expansion is valid provided $\rho$ is smaller than the distance from $a$ to the support of $(\mu',\nu')$. This gives (5.13). A similar argument, starting from (5.2), and expanding $F(\rho,\eta;y)$ in powers of $\rho$, yields (5.14).

5.1.4. *Remark.* $\Phi$ and $F$ have complete asymptotic expansions as $\rho \to 0$ whatever the behaviour of $(\mu,\nu)$ or $f_0$, but in general, the next term in the expansion of $\sqrt{\rho}F$ is $O(\rho^2 \log\rho)$.

5.1.5. *Remark.* It is important to ask what happens at $\infty$. It would be quite straightforward to analyse this separately, but it is more geometric to observe that all quantities are restrictions of globally defined objects on the boundary of $\mathcal{H}^2$. For example, if we set

$$\tilde\rho = \frac{\rho}{\rho^2+\eta^2}, \quad \tilde\eta = -\frac{\eta}{\rho^2+\eta^2}, \quad \tilde y = -\frac{1}{y}$$

then

$$\frac{\rho^2+(\eta-y)^2}{\rho}(dy)^{-1} = \frac{\tilde\rho^2+(\tilde\eta-\tilde y)^2}{\tilde\rho}(d\tilde y)^{-1}.$$

These formulae are standard in analysis on hyperbolic space and imply that the analysis near $\rho = \infty$ can be replaced by the analysis near $\tilde\rho = 0$ which has already been done. In particular, we see that $F(\rho,\eta;y)$ has an invariant interpretation over $\overline{\mathcal{H}}^2 \times \mathbb{R}P^1$ as a section (with singularities at a certain subset of the boundary) of $\mathcal{O}(1)\otimes L$, where $L$ is the Möbius line bundle over $\mathbb{R}P^1$. Similarly, $\phi(\rho,\eta;y)$ is a section of $\mathcal{W}^*\otimes L$. It is the trivialization of $L$ over $\mathbb{R}P^1 \smallsetminus \{\infty\}$ that corresponds to the 'odd at infinity' condition that is so prominent in this section. This global interpretation allows us to relax the assumption that our distributions have compact support on $\mathbb{R}$, so long as they are distributions at infinity, and we have already taken advantage of this in our Poisson formulae. For more details, see the appendix.

We now come to a useful sufficient condition for $\det\Phi \neq 0$.

5.1.6. **Proposition.** *If* $\Phi$ *is given by* (5.9) *and*

$$\mu(y)\nu(z) - \mu(z)\nu(y) \leqslant 0 \text{ for } y \leqslant z \tag{5.15}$$

*with strict inequality for some* $y < z$, *then* $\det\Phi < 0$ *in* $\mathcal{H}^2$.

Note that (5.15) makes good sense in terms of the tensor product of distributions even when $\mu$ and $\nu$ are not continuous functions.

5.1.7. *Proof.* Recall that with our orientation conventions, if $\Phi = \lambda_1 \otimes v_1 + \lambda_2 \otimes v_2$ then $\det \Phi = -\varepsilon(v_1, v_2)$. Hence,

$$\det \Phi(\rho,\eta) = -\frac{\rho^3}{8} \iint \frac{(y-z)(\mu(y)\nu(z) - \mu(z)\nu(y))}{\big(\rho^2 + (\eta-y)^2\big)^{3/2}\big(\rho^2+(\eta-z)^2\big)^{3/2}}\, dy\, dz. \tag{5.16}$$

The result follows at once.

5.1.8. *Remark.* If $\mu$ and $\nu$ are piecewise continuous functions with $\mu \geqslant 0$, then (5.15) is equivalent to

$$\frac{\nu(y)}{\mu(y)} \geqslant \frac{\nu(z)}{\mu(z)} \quad \text{for} \quad y \leqslant z \tag{5.17}$$

so that $\nu(y)/\mu(y)$ is a non-increasing function of $y$. This property is enjoyed by the step-function $(m,n)$ and is used in Joyce's proof of the non-vanishing of $\det \Phi$.

5.1.9. *Remark.* We have now associated to any pair of distributions $(\mu,\nu)$ on $\mathbb{R}$ that satisfy (5.4), (5.5) and (5.15) a $\mathbb{V}$-invariant conformally half-flat metric $g$ on $\mathcal{H}^2 \times \mathbb{V}$.

5.2. **Infinite-dimensional families of SDE metrics: Proof of Theorem C.** The $\mathbb{V}$-valued step function $(m,n)$ associated to a Hirzebruch–Jung resolution $X$ with $K_X$ strictly nef, gave rise to an eigenfunction $F$ with boundary data

$$f_0^{can}(\eta) = m_j\eta - n_j \quad \text{for} \quad y_j \leqslant \eta \leqslant y_{j-1}$$

where $y_{k+2} = -\infty$ and $y_{-1} = +\infty$ as before. In particular $f_0^{can}(\eta) = \pm 1$ for $\pm\eta$ sufficiently large and positive. The graph of $f_0^{can}$ is continuous, strictly increasing in $[y_{k+1}, y_0]$ and has the property that it is *convex* where it is negative and *concave* where it is positive. The unique zero of $f_0^{can}$ is at $\eta = q/p$. (Recall the example of Figure 2.)

In order to generate an infinite-dimensional family of complete SDE metrics on $X$, thereby proving Theorem C, we shall change $f_0^{can}$ in the interval $(-\infty, q/p)$. (If we change it in $(q/p,\infty)$, the result may no longer extend smoothly to the special orbits over the intervals $[y_j, y_{j-1}]$.) For our first statement, which implies Theorem C, we consider modifications $f_0$ that are not too rough.

5.2.1. **Theorem.** *Let $f_0(\eta)$ be a continuous function equal to $f_0^{can}(\eta)$ for $\eta \geqslant q/p - \delta$, for some $\delta > 0$. Suppose further that $f_0(\eta) = -1$ for all $\eta \leqslant a < q/p$ and that $f_0$ is piecewise differentiable, strictly increasing and convex on $[a, q/p]$. Then the eigenfunction $F$ with boundary value $f_0$ determines an asymptotically hyperbolic SDE metric on the neighbourhood $E \subset X_+ \subset X$ of the exceptional divisor corresponding to the domain $F > 0$ in $\mathcal{H}^2$.*

5.2.2. *Proof.* Define distributions $\mu,\nu$ by (5.8). Then $(\mu,\nu)$ determines a Joyce matrix $\Phi$ by (5.9) with the correct boundary values for smooth extension to the special orbits $S_1, S_2, \dots S_k$. Let us check that $\det\Phi$ is never zero. Since $\mu$ is defined and is positive a.e. in $[a, y_0]$, we can apply Proposition 5.1.6 in the form of (5.17). Indeed,

$$\frac{\nu(y)}{\mu(y)} = y - \frac{f_0(y)}{f_0'(y)} \qquad \text{and so} \qquad \left(\frac{\nu}{\mu}\right)' = \frac{f_0 f_0''}{(f_0')^2} \leqslant 0$$

by the assumption that $f_0$ is concave where $f_0 > 0$ and $f_0$ is convex where $f_0 < 0$. Hence $\nu/\mu$ is a non-increasing function as required. Thus the conformal class of $g_\Phi$ is defined over the whole of $T^2 \times \mathcal{H}^2$. We now proceed as in §4.2.

By the maximum principle, $F$ has no interior positive maximum or negative minimum. Hence the zero-set $Z$ is a simple smooth arc joining the boundary at $\eta = q/p$ to $\infty$ and $Z$ decomposes $\overline{\mathcal{H}}^2$ into pieces

$$D_+ = \{F > 0\}, \;\; D_- = \{F < 0\}.$$

We define $X_+$ to be the union of $T^2$-orbits over $D_+$ and $Y$ to be the union of the $T^2$-orbits over $Z$. We note (as before) that $X_+$ contains $S_1 \cup \cdots \cup S_k$ (since $f_0$ is positive to the right of $\eta = q/p$). Then the metric $g_F$ defines an asymptotically hyperbolic SDE metric on $X_+$, with conformal infinity $Y$.

5.2.3. *Remark.* In the situation of Theorem 5.2.1 we can say more about the zero-set $Z$ of $F$. We claim that it meets each circular arc with end-points $(0, q/p \pm b)$ in precisely one point, for any $b > 0$. To see this, note that these arcs are the orbits of the Killing vector field

$$K = \big((\eta - q/p)^2 - b^2\big)\partial_\eta + 2(\eta - q/p)\rho\,\partial_\rho.$$

Then $K \cdot F$ will be an eigenfunction of the laplacian, with eigenvalue $3/4$ and so by the maximum principle, $K \cdot F < 0$ in $\mathcal{H}^2$ if this is true near the boundary. Hence if we prove the latter, it will follow that $F$ is strictly decreasing along these circular arcs, and since it starts positive and ends negative, there must be a unique zero. By Proposition 5.1.1,

$$\sqrt{\rho}F(\rho, \eta) \simeq \eta\mu(\eta) - \nu(\eta) \quad \text{if } \rho \text{ is small.}$$

Hence

$$\begin{aligned}\sqrt{\rho}\, K \cdot F &\simeq (\eta - q/p)^2\mu(\eta) - (\eta - q/p)(\eta\mu(\eta) - \nu(\eta))\\ &= \frac{1}{p}\big(-b^2\mu(\eta) + (\eta - q/p)(\mu(q/p)\nu(\eta) - \mu(\eta)\nu(q/p))\big) \leqslant 0\end{aligned}$$

by considering separately the cases $\eta < q/p$ and $\eta > q/p$ and using $\mu(q/p) = p$ and $\nu(q/p) = q$ as well as the monotonicity property (5.15).

Theorem 5.2.1 produces a family of SDE metrics parameterized by piecewise differentiable functions on $(-\infty, q/p)$ satisfying only monotonicity and convexity assumptions. It is clear that the space of such functions is (continuously) infinite-dimensional. Hence Theorem 5.2.1 implies Theorem C. Notice that this result is not a perturbation theorem: to the left of $q/p$, $f_0$ can be far from $f_0^{can}$. A particularly interesting example is as follows.

5.2.4. *Example.* Following an initial suggestion of Atiyah, we note that the odd extension, $f_0^{odd}$, say, of $f_0^{can}$ to the left of $\eta = q/p$ satisfies all the properties of Theorem 5.2.1—see Figure 3. More precisely, for $\eta > 0$, we put

$$f_0^{odd}(q/p - \eta) = -f_0^{can}(q/p + \eta)$$

so that that $(\mu, \nu)$ satisfy

$$\mu(q/p - \eta) = \mu(q/p + \eta), \quad \nu(q/p - \eta) = 2q\mu(q/p + \eta)/p - \nu(q/p + \eta). \tag{5.18}$$

This extension enjoys the property that $F(\rho, q/p+\eta) = -F(\rho, q/p-\eta)$ so that $Z = \{\eta = q/p\}$ and $D_+ = \{\eta > q/p\}$. In this case we can compute a representative metric $h$ for the conformal infinity by substituting $\eta = q/p$ into the formula for $F^2 g_F$. Since $F$ vanishes along $\eta = q/p$, so does its tangential derivative $F_\rho$—hence the conformal infinity is determined by the function $F_\eta(\rho, q/p)$, which is a sum of the terms

$$F_\eta(\rho, q/p; y_j) - F_\eta(\rho, q/p; 2q/p - y_j) = \frac{2(q/p - y_j)}{\sqrt{\rho}\sqrt{\rho^2 + (q/p - y_j)^2}}$$

over $j = 0, \dots k$ (where $y_j > q/p$). We now readily obtain

$$h = \left(\sum_{j=0}^{k} \frac{2(y_j - q/p)\sqrt{\rho}}{\sqrt{\rho^2 + (y_j - q/p)^2}}\right)^2 \frac{d\rho^2}{\rho^2} + \rho\, d\phi^2 - \frac{1}{\rho}(d\psi - q/p\, d\phi)^2.$$

By definition, the odd extension has an obvious symmetry about $\eta = q/p$ and so $-f_0^{odd}$ should define a toric self-dual Einstein metric on a manifold $X_-$ diffeomorphic to $X_+$. However, to do this, a different integral lattice is needed to define the torus—otherwise $X_-$ will not be smooth. More precisely, the lift of the symmetry $y \mapsto 2q/p - y$ to a matrix of determinant $-1$ in $\mathrm{GL}_2(\mathbb{R})$ is

$$\begin{pmatrix} 1 & 0 \\ 2q/p & -1 \end{pmatrix} = \begin{pmatrix} p & 0 \\ q & -1 \end{pmatrix} \begin{pmatrix} p & 0 \\ q & 1 \end{pmatrix}^{-1}$$

and the new lattice is the image of $\mathbb{Z}^2$ under this transformation. The two lattices have a common sub-lattice of index $p$, *i.e.*, away from the fixed points there is a common covering space. We deduce that for each choice of orientation, $h$ defines a conformal structure on $S^3$ with a $\mathbb{Z}_p$-quotient bounding a self-dual Einstein metric; however, the two $\mathbb{Z}_p$ actions are *not* the same!

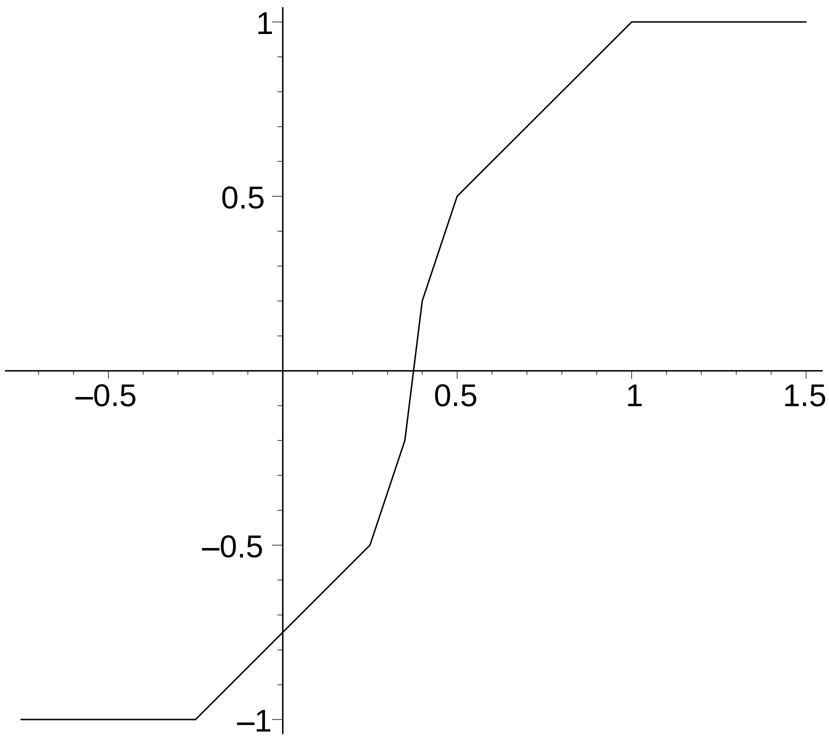



FIGURE 3. The graph of $f_0^{odd}$ when $q/p = 3/8$.

5.3. **Further results.** The metrics constructed by Theorem 5.2.1 all have the property that the underlying self-dual conformal structures extend 'a long way' into $X_-$. More precisely they extend to the complement of the special orbits (a set of codimension 2) in $X_-$. The reason for this is that we arranged $\det \Phi \neq 0$ over the whole of $\mathcal{H}^2$.

One would expect, however, that there should exist asymptotically hyperbolic SDE metrics on $X_+$, with conformal infinity at $Y$, but with the property that the self-dual conformal structure does not extend so far, or even at all into $X_-$!

5.3.1. *Perturbations* I. Let $f_0$ be a function satisfying the properties of Theorem 5.2.1 and let $u$ be any distribution with compact support contained in $(-\infty, b)$, say, where $b < q/p$. Consider, for real $t$,

$$f_0^t = f_0 + tu.$$

For small $t$, the eigenfunction $F^t$ with boundary value $f_0^t$ will be close to $F$, in the $\mathrm{C}^\infty$-topology, on any set of the form

$$\overline{\mathcal{H}}^2 \setminus \{(\rho, \eta) : 0 \leqslant \rho \leqslant \rho_0 \text{ and } \eta \leqslant b\}.$$

In particular, for all sufficiently small $t$, the zero-set of $F^t$ will be very close to that of $F$, and $\det \Phi^t \neq 0$ where $F^t \geqslant 0$. Hence $F^t$ yields a deformation of $g_F$ as an asymptotically hyperbolic SDE metric.

Note that there is now no reason why $\det \Phi^t$ should be nonzero on *all* of $\mathcal{H}^2$. The 3-pole examples given in [9] illustrate this point.

5.3.2. *Perturbations* II. A more general perturbation can be obtained from the following analytical result [25].

5.3.3. **Theorem.** *Let $Z$ be a simple closed curve dividing $\mathcal{H}^2$ into two connected components $D_\pm$. Denote by $\partial\mathcal{H}^2_\pm$ the two components of the boundary of $\mathcal{H}^2$, so that*

$$\partial D_\pm = Z \cup \partial\mathcal{H}^2_\pm.$$

*Then there exists a unique eigenfunction $F$ (with eigenvalue $3/4$) on $D_+$ with the property that $F = 0$ on $Z$ and $F$ has prescribed boundary value on $\partial\mathcal{H}^2_+$:*

$$f_0(\eta) = \lim_{\rho\to 0} \sqrt{\rho} F(\rho, \eta).$$

To use this result, let $Z_0$ be the zero-set of an eigenfunction $F_0$ with boundary value $f_0$ satisfying the conditions of Theorem 5.2.1. Let $Z$ be a small perturbation of $Z_0$ (with end-points fixed) and let $F$ be the eigenfunction, with the same boundary values as $F_0$ for $\eta \geqslant q/p$ and vanishing on $Z$. If $Z$ is sufficiently close to $Z_0$ then $F$ will be close to $F_0$ in $D_+$, and so $F^2 - 4|dF|^2$ will not vanish on $D_+$. Accordingly the metric $g_F$ will be an asymptotically hyperbolic SDE metric with conformal infinity on $Z$.

Notice that if $F$ extends beyond its zero-set $Z$, then the latter will be a real-analytic curve, since $F$ is real-analytic in the interior of its domain of definition. Hence if we choose $Z$ to be merely smooth, $F$ will not extend beyond $Z$ and the self-dual conformal structure of $g_F$ cannot be extended through $Z$. These examples fill in non-analytic conformal structures on lens spaces, cf. [6].

5.4. **Proof of Theorem D.** Consider the extension $f_0$ of $f_0^{can}$ by zero to the left of $\eta = q/p$. This does not satisfy all of the conditions of Theorem 5.2.1: in particular it is not odd at infinity. In invariant terms (over $\mathbb{R}P^1$), this means that $f_0$, though continuous, is not smooth at infinity, but has a corner. One way to handle this is to change coordinates so that a smooth point of $f_0$ is at infinity. However, it is straightforward to work directly with the given coordinates, and this is what we shall do.

It is immediate that $\det \Phi < 0$ on $\mathcal{H}^2$. Furthermore $F$ is positive on $\mathcal{H}^2$ by the Poisson formula, so the Einstein metric is smoothly defined on *all* of $\mathcal{H}^2 \times T^2$. Since $f_0$ agrees with $f_0^{can}$ on $(q/p, \infty)$, the metric extends smoothly to the special orbits over this interval. It remains to consider the behaviour of the metric as $(\rho, \eta)$ approaches the boundary segment $[-\infty, q/p]$.

We claim that this is a complete end of the self-dual Einstein metric and that it is ACH. To see this, we compare $f_0$ to the function

$$f_0^{CH} = \begin{cases} 0 & \text{if } \eta < q/p \\ p\eta - q & \text{if } q/p < \eta < (q+1)/p \\ 1 & \text{if } (q+1)/p < \eta. \end{cases}$$

We observe that $f_0$ and $f_0$ agree, except on $[y_k, y_0] \subset (q/p, \infty)$. On the other hand $f_0^{CH} = \frac{1}{2}(|p\eta - q| - |p\eta - q - 1| + 1)$ and so it generates a 3-pole solution in the sense of [9].

In fact after change of $(\rho, \eta)$ coordinates we find that

$$F^{CH}(\rho, \eta) = \sqrt{\frac{p}{2}}\Big(-\frac{1}{\sqrt{\rho}} + \frac{\sqrt{\rho^2 + (\eta+1)^2}}{2\sqrt{\rho}} + \frac{\sqrt{\rho^2 + (\eta-1)^2}}{2\sqrt{\rho}}\Big),$$

which is a hyperbolic eigenfunction generating the Bergman metric [9]. (Explicitly, with $\rho = 2\coth t \operatorname{csch} t \sin\theta$, $\eta = (2\coth^2 t - 1)\cos\theta$, we have

$$g_{CH} = 2dt^2 + \frac{1}{2}\sinh^2 t\,(d\theta^2 + (2/p)\sin^2\theta\, d\phi^2) + \frac{1}{4p}\sinh^2 2t\,(d\psi + \cos\theta\, d\phi)^2$$

defined on a $\mathbb{Z}_p$ orbifold quotient of $\mathbb{C}\mathcal{H}^2$. Rescaling $g_{CH}$ by $1/2$, and $\phi$ and $\psi$ by $\sqrt{p/2}$ gives the form of the Bergman metric given in [30].)

The approximation of $f_0$ by $f_0^{CH}$ is enough to ensure completeness. In particular, one can easily check directly that $\eta = q/p$ and $\eta = \pm\infty$ are 'at infinity' on the special orbits over $(q/p, y_k)$ and $(y_0, \infty)$.

We now compute the CR infinity, by taking $\eta \in (-\infty, q/p)$ and the limit $\rho \to 0$. Since $f_0 = 0$, and hence $(\mu, \nu) = (0, 0)$, on $(-\infty, q/p)$, we can make an asymptotic expansion for $\rho$ smaller than $q/p - \eta$ to obtain

$$\begin{aligned} \sqrt{\rho}F(\rho,\eta) &= \tfrac{1}{2}\rho^2 f_1(\eta) + O(\rho^4) \\ v_1 &= \rho(f_1(\eta), \eta\, f_1(\eta)) + O(\rho^3) \\ v_2 &= \tfrac{1}{2}\rho^2\big(f_1'(\eta), f_1(\eta) + \eta\, f_1'(\eta)\big) + O(\rho^4) \end{aligned}$$

where

$$f_1(\eta) = \int \frac{f_0''(y)}{|\eta - y|}dy. \tag{5.19}$$

(Recall that $f_0''$ is a sum of delta distributions.)

We let $\theta$ be the 1-form $\rho^4\varepsilon(v_1,\cdot)^2/F^2\varepsilon(v_1,v_2)$ and note that $\lim_{\rho\to 0}\theta = 2(d\psi + \eta\, d\phi)^2/f_1(\eta)^2$. Now $h = \rho^2 g - \rho^{-2}\theta^2$ is degenerate, and we compute that

$$\lim_{\rho\to 0} h = \frac{4\,(d\eta^2 + d\rho^2)\, f_1(\eta)^4 + (f_1'(\eta)d\psi + (f_1(\eta) + \eta\, f_1'(\eta))d\phi)^2}{2\, f_1(\eta)^4}.$$

Thus, after rescaling by $f_1(\eta)^2$, the pull-back of $(\theta, h)$ to $\rho = 0$ (restricting $h$ to $\ker\theta$) gives the contact metric structure

$$2(d\psi + \eta\, d\phi), \qquad 2f_1(\eta)d\eta^2 + \frac{d\phi^2}{2f_1(\eta)} \tag{5.20}$$

The Reeb field $\partial_\psi$ is CR and generates the foliation of the lens space $N$ induced by the Hopf fibration of $S^3$, and so $N$ is normal and quasiregular. The quotient metric is an $S^1$-invariant orbifold metric on $S^2$, as one can easily check directly from (5.20).

5.5. **Proof of Theorem E.** In order to prove this theorem, we simply try to repeat the arguments of the previous sections, dropping the assumption $e_j \geqslant 3$. If among the $e_j$ there is a sequence of the form

$$e_r > 2,\ e_{r+1} = e_{r+2} = \cdots = e_{r+s} = 2,\ e_{r+s+1} > 2,$$

then we have, referring back to §4.2,

$$y_{r-1} > y_r = y_{r+1} = \cdots = y_{r+s} > y_{r+s+1}.$$

Thus the piecewise linear function $f_0$ of (4.8) satisfies

$$f_0(\eta) = \begin{cases} m_r\eta - n_r, & \text{if } y_r < \eta < y_{r-1}, \\ m_{r+s+1}\eta - n_{r+s+1}, & \text{if } y_{r+s+1} < \eta < y_r, \end{cases}$$

either side of $y_r$. The integer vectors $(m_r, n_r)$ and $(m_{r+s+1}, n_{r+s+1})$ can be seen to span a sub-lattice of index $s+1$ so that the fixed-point at $y_r$ corresponds to an orbifold singularity with local isotropy $\mathbb{Z}/(s+1)$, which is, in fact, an $A_s$-singularity. This latter follows either by explicit calculation of the link of the singularity or from the fact from complex surface theory that the contraction of a chain of $s$ $(-2)$-curves always gives an $A_s$-singularity. According to the discussion at the end of §3.1.6, the metric $g_F$, where $F$ has boundary-value $f_0$, extends to define a smooth orbifold metric near the fixed-point. (The above argument is easily modified to allow for the possibilities that $e_1 = 2$ or $e_k = 2$.)

The contraction of all such sequences of $(-2)$-curves in $X$ leads precisely to the orbifold $\check{X}$ in Theorem E, and now the proofs we have given apply to $\check{X}$ with only notational changes.

## 6. Proof of Theorem F

The proof makes use of the twistor theory of SDE manifolds, for which a convenient summary is [4, Chs 13, 14], and the twistor theory of submanifolds developed in [10]. The twistor space $Z$ of $U$ is the smooth manifold fibering over $U$ whose fiber at $x \in U$ is the 2-sphere of all anti-self-dual complex structures on $T_xU$, *i.e.*, those complex structures $J$ compatible with the metric $g$ and inducing the opposite orientation to that of $U$. The Levi-Civita connection of $g$ induces a rank 4 distribution $H$ on $Z$ transverse to the fibres, and $Z$ has a natural almost complex structure, which at any point of $Z$ is the sum of the standard complex structure on the tangent space to the fibres and a tautological complex structure on $H$. Because $g$ is self-dual the almost complex structure on $Z$ is integrable. Furthermore, because $g$ is Einstein with nonzero scalar curvature $D := H^{1,0}$ defines a holomorphic contact structure on $Z$, so there is an exact sequence of holomorphic vector bundles

$$0 \longrightarrow D \longrightarrow T^{1,0}Z \longrightarrow V \longrightarrow 0$$

where $V$ is the holomorphic vertical tangent bundle of the fibration $Z \to U$, which is well known to be isomorphic to $K_Z^{-1/2}$. We thus obtain

$$K_Z^{-1} := \det T^{1,0}Z = \det D \otimes K_Z^{-1/2}$$

so that $\det D = K_Z^{-1/2}$ and

$$c_1(T^{1,0}Z) = 2c_1(D). \tag{6.1}$$

If $s < 0$, $Z$ is an indefinite Kähler–Einstein manifold, and it follows that there is a $(1,1)$ form $\alpha$, say, with $[\alpha] = c_1(Z)$ and

$$\alpha > 0 \text{ on } V, \quad \alpha < 0 \text{ on } D. \tag{6.2}$$

Now suppose that $\Sigma$ is a 2-dimensional submanifold of $U$. Using the riemannian metric, the tangent bundle of $U$ splits along $\Sigma$

$$TU|_\Sigma = T\Sigma \oplus N\Sigma. \tag{6.3}$$

There is a natural anti-self-dual almost complex structure $J$ defined on $TU|_\Sigma$: with respect to an oriented orthonormal frame $(e_1, e_2, e_3, e_4)$ adapted to the decomposition of (6.3) in the sense that $e_1, e_2 \in T\Sigma$ we define

$$Je_1 = e_2, \quad Je_3 = -e_4.$$

The introduction of $J$ on $TU|_\Sigma$ precisely defines a lift $\tilde{\Sigma}$ of $\Sigma$ into $Z$. Moreover, according to Eells and Salamon [10], if $\Sigma$ is totally geodesic with respect to $g$, then $\tilde{\Sigma}$ is holomorphic and horizontal, so that $T^{1,0}\tilde{\Sigma} \subset D$.

With this background, the proof of Theorem F is completely straightforward. Decomposing (6.3) with respect to $J$, we have

$$T^{1,0}U|_\Sigma = T^{1,0}\Sigma \oplus N^{1,0}\Sigma. \tag{6.4}$$

Now $c_1(T^{1,0}\Sigma)[\Sigma] = \chi(\Sigma)$ and $c_1(N^{1,0}\Sigma)[\Sigma] = -\Sigma \cdot \Sigma$ (because $J$ is anti-self-dual). Furthermore $T^{1,0}U|_\Sigma$ is isomorphic, as a complex vector bundle, to $D|_{\tilde{\Sigma}}$. Therefore (6.4) implies

$$\chi(\Sigma) - \Sigma \cdot \Sigma = c_1(D|_{\tilde{\Sigma}})[\tilde{\Sigma}].$$

Combining this equation with (6.1) and (6.2), using the fact that $\tilde{\Sigma}$ is holomorphic and horizontal now gives

$$\chi(\Sigma) - \Sigma \cdot \Sigma < 0 \tag{6.5}$$

which completes the proof of Theorem F.

## Appendix

In this appendix we explain the geometric origin of the basic solutions that we have used in this paper. Much of the discussion goes through for hyperbolic space of arbitrary dimension, but we shall confine ourselves to $\mathcal{H}^2$. As in [9], we shall fix a 2-dimensional symplectic vector space $\mathbb{W}$ and consider the symmetric square $S^2\mathbb{W}$, equipped with the quadratic form det as a 3-dimensional Minkowksi space. If $g \in SL(\mathbb{W})$, $x \in S^2\mathbb{W}$, then $g \cdot x = gxg^t$ is evidently an isometric action, corresponding to the double cover of the identity component of $SO(1,2)$ by $SL_2(\mathbb{R})$.

The hyperbolic plane $\mathcal{H}^2$ appears as (one sheet of) the hyperboloid $\det x = 1$ or essentially equivalently as the open subset $S_+/\mathbb{R}^+$ of the projective space $P(S^2\mathbb{W})$, where

$$S_+ = \{x \in S^2\mathbb{W} : x \text{ is positive definite}\}.$$

It is easy to check that there is a natural correspondence

$$\{f \in \mathrm{C}^\infty(S_+) : \Box f = 0,\ E \cdot f = \alpha f\} = \{g \in \mathrm{C}^\infty(\mathcal{H}^2) : \Delta g = \alpha(\alpha+1)g\}, \tag{6.6}$$

where $E$ is the Euler homogeneity operator on $S^2\mathbb{W}$ and $\Box$ is the wave operator of $S^2\mathbb{W}$. If $N$ is any nonzero null vector ($\det N = 0$), then $x \mapsto f(N \cdot x)$ is a solution of the wave equation, for any function $f$. The 'basic solutions' of the equation $\Delta F = (3/4)F$ arise in precisely this way, with the simplest possible function homogeneous of degree 1/2, namely $(N \cdot x)^{1/2}$.

If we represent $N = n \otimes n$ where $n = (1, y)$ and use the parameterization of the hyperboloid by half-space coordinates

$$x = \frac{1}{\rho}\begin{bmatrix} 1 & \eta \\ \eta & \rho^2 + \eta^2 \end{bmatrix}$$

then

$$\sqrt{N \cdot x} = \sqrt{nx^{-1}n^t} = \frac{\sqrt{\rho^2 + (\eta - y)^2}}{\sqrt{\rho}}$$

which is the 'basic solution' $F(\rho, \eta; y)$ of (3.16).

Just as eigenfunctions of the laplacian in $\mathcal{H}^2$ correspond to homogeneous solutions of the wave equation in $S_+$, so eigenfunctions of the Dirac operator in $\mathcal{H}^2$ correspond to homogeneous solutions of the Dirac equation in $S_+$. Here we can identify the product bundle $S_+ \times \mathbb{W}$ as the spin-bundle of $S_+$; its restriction to the hyperboloid is naturally isomorphic to the spin-bundle of $\mathcal{H}^2$ (though the induced metric at $x$ is given by $x^{-1}$). It is easy to check that if $N = n \otimes n$ as before, then the $\mathbb{W}$-valued function $n/\sqrt{N \cdot x}$ is a solution of the minkowskian Dirac equation. This descends to the 'basic solution' of the Joyce equation $\phi(\rho, \eta; y)$ (3.6) after replacing $n$ by the dual vector $n^*$, $\varepsilon(n, u) = n^*(u)$. (Explicitly, if $n = (1, y)$ then $n^* = (-y, 1)$.)

Over the conformal boundary $\mathbb{R}P^1$ of $\mathcal{H}^2$ there is a family of homogeneous line bundles. In terms of the parameter $n \in \mathbb{W}$, we define the bundle $\mathcal{O}(1)$ as corresponding to functions homogeneous of degree 1,

$$f(\lambda n) = \lambda f(n), \quad \lambda \neq 0,$$

and $|\mathcal{O}(1)|$ to correspond to functions homogeneous in the sense

$$f(\lambda n) = |\lambda| f(n), \quad \lambda \neq 0.$$

Notice that $\mathcal{O}(1) = |\mathcal{O}(1)| \otimes L$ where $L$ is the Möbius bundle with $L^2 = \mathcal{O}$, and that $|\mathcal{O}(1)|$ is topologically trivial, whereas $\mathcal{O}(1)$ is not. These considerations are important in trying to find the correct interpretation of the formulae for the smeared solutions in §5.1. If $M \to \mathbb{R}P^1$ is a given line bundle, then by a distributional section of $M$ we mean a continuous linear functional on the space $C^\infty(\mathbb{R}P^1, M^* \otimes \Omega)$. Here $\Omega$ is the bundle of densities, which can be identified invariantly with $\mathcal{O}(-2)$. In so far as $\sqrt{N \cdot x}$ is a section of $\mathcal{O}(1) \otimes L$ for each fixed $x$, it follows that if $u$ is a distributional section of $\mathcal{O}(-3) \otimes L$ then

$$\int u(n)\sqrt{N \cdot x}$$

is a well-defined function of $x$. This is the $SL_2(\mathbb{R})$-invariant interpretation of formula (5.2). The formula (5.1) can be interpreted similarly.

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