CONFORMAL SUBMANIFOLD GEOMETRY I–III

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Abstract. In Part I, we develop the notions of a Möbius structure and a conformal Cartan geometry, establish an equivalence between them; we use them in Part II to study submanifolds of conformal manifolds in arbitrary dimension and codimension. We obtain Gauß–Codazzi–Ricci equations and a conformal Bonnet theorem characterizing immersed submanifolds of S^n. These methods are applied in Part III to study constrained Willmore surfaces, isothermic surfaces, Guichard surfaces and conformally-flat submanifolds with flat normal bundle, and their spectral deformations, in arbitrary codimension. The high point of these applications is a unified theory of Möbius-flat submanifolds, which include Guichard surfaces and conformally flat hypersurfaces.

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Introduction

Our purpose in this work is to present a comprehensive, uniform, invariant, and self-contained treatment of the conformal geometry of submanifolds. This paper begins the study in three parts. In the first two parts, we present the general theory of conformal submanifolds: in Part I we develop an intrinsic theory of conformal manifolds; then in Part II, we investigate the geometry induced on an immersed submanifold of a conformal manifold. The central result is an analogue in conformal geometry of the theorem of O. Bonnet which characterizes submanifolds of euclidean space in terms of geometric data satisfying Gauß–Codazzi–Ricci equations. We demonstrate, in Part III, how this theory gives a systematic description of three integrable systems arising in conformal submanifold geometry, which describe three classes of submanifolds and their spectral deformations: constrained Willmore surfaces, isothermic surfaces, and 'Möbius-flat' submanifolds, these last being Guichard or channel surfaces in dimension two (and codimension one), and conformally flat submanifolds in higher dimension.

The conformal geometry of submanifolds has been of great interest to differential geometers for more than a century, and our contribution is far from being the first treatment of the conformal Bonnet theorem: the ideas go back at least to E. Cartan's beautiful 1923 paper [37]. Since then, there have been many attempts to understand conformal submanifold geometry, either by authors wishing to build on Cartan's work, or by those unaware of it. In particular, the subject was taken up in the 1940's by K. Yano [82, 83, 84, 85, 86, 87, 88], who showed (together with Y. Mutô) that m-dimensional conformal submanifolds of S^n could be characterized, for $m \geqslant 3$, by tensors g_{jk}, M_{jkP} and L_{PQk} satisfying five equations, three of which are analogues of the Gauß–Codazzi–Ricci equations in euclidean geometry—these last three are also discussed more recently by L. Ornea and G. Romani [65]. Another extensive account, from Cartan's point of view, was developed in the 1980's by C. Schiemangk and R. Sulanke [68, 71, 72], who obtained a conformal Bonnet theorem at least for the generic case of submanifolds with no umbilic points.

The issue of umbilic points is an significant one. Recall that these are points where the tracefree part of the second fundamental form, which is a conformal invariant, vanishes. However, following ideas of T. Thomas [75], A. Fialkow [46, 47] already noticed in 1944 that in the absence of umbilic points, there is a unique metric in the conformal class on the submanifold, with respect to which the tracefree second fundamental form has unit length. This observation allowed him to obtain a treatment of generic conformal submanifold geometry in purely riemannian terms. Around the same time J. Haantjes [51, 52, 53] and J. Maeda [62] presented a conformal theory of curves and surfaces. Subsequent authors have also obtained results in varying degrees of generality: let us mention, for instance, G. Laptev [60], M. Akivis and V. Goldberg [1, 2], G. Jensen [57], C. Wang [78, 79] and the book [12] of R. Bryant *et al.* on exterior differential systems.

In view of all this work, we would be bold to claim that the main results of Parts I and II are new. However, we believe there is no complete discussion of conformal submanifold geometry in the literature with all three of the following features:

- manifest conformal invariance;
- no restriction on umbilic points;
- uniform applicability in arbitrary dimension and codimension.

Nevertheless, we have no quarrel with the Reader who would prefer to regard Parts I–II of this work as a modern gloss on Cartan's seminal paper [37]: as we shall see, recent developments in conformal geometry make such a gloss extremely worthwhile. Our approach is also greatly inspired by Sharpe's significant book [70], which presents the general theory of Cartan geometries to a modern audience, with many applications. Unfortunately, his application to conformal submanifold geometry contains a technical error, which leads him to restrict attention to the generic case only (no umbilic points) when studying surfaces.[1]

Another motivation for the present work is the recent development of a deeper understanding of the algebraic structures underlying conformal geometry, which we wish to apply. There are five aspects to this which we now explain: Möbius structures, parabolic geometries, tractor bundles, Lie algebra homology, and Bernstein–Gelfand–Gelfand operators.

A central difficulty in conformal submanifold geometry lies in the description of low dimensional submanifolds. The problem, in a nutshell, is that curves and surfaces in a conformal manifold acquire more intrinsic geometry from the ambient space than simply a conformal metric. This is closely related to the fact that in dimension one or two there are more local conformal transformations than just the Möbius transformations: in one or two dimensions, Möbius transformations are real or complex projective transformations respectively. In fact, we shall see that a curve in a conformal manifold acquires a natural real projective structure, whereas a surface acquires a (possibly) non-integrable version of a complex projective structure. We call these *Möbius structures*—the 2-dimensional ones were introduced in [24] under this name.

Our paper might more properly be called 'Möbius submanifold geometry' (cf. [68, 78]): Möbius structures provide a notion of conformal geometry which applies uniformly in all dimensions, modelled on the n-sphere S^n with its group of Möbius transformations. It is well known (as observed by Darboux [38]) that this group is isomorphic to the semisimple group $O_+(n+1,1)$ of time-oriented orthogonal transformations of an $(n+2)$-dimensional lorentzian vector space: indeed the Lorentz transformations act transitively on the projective light-cone, which is an n-sphere, and the stabilizer of a light-line is a parabolic subgroup of $O_+(n+1,1)$. (Similarly, the model for conformal geometry in signature (p,q) is the projective light-cone in $\mathbb{R}^{p+1,q+1}$; for notational convenience we restrict attention to euclidean signature, but the theory applies more generally with minor modifications.)

Cartan geometries provide a systematic way to make precise the notion of a curved manifold modelled on a homogeneous space [70]. The geometries modelled on G/P, where G is semisimple and P is a parabolic subgroup, are called *parabolic geometries* [25, 31, 33, 35]. Hence (Möbius) conformal geometry is a parabolic geometry.

One way to provide a uniform description of conformal submanifold geometry—and this is the approach taken by Sharpe—is to use Cartan connections throughout. However this raises a subtle question of philosophy, concerning the distinction between 'intrinsic' and 'extrinsic' geometry. The problem is that there is a lot of 'room' in a Cartan connection to hide extrinsic data: even in higher dimensions a conformal Cartan connection can contain much more information than just a conformal metric. However, it is well-known—and due to Cartan [37] of course—that there is a preferred class of Cartan connections, the *normal* Cartan connections which, in any dimension $n \geqslant 3$, correspond bijectively (up to isomorphism) with conformal metrics. We extend this result to dimensions one and two, by establishing, in a self-contained way, the existence and uniqueness of the normal Cartan connection for (Möbius) conformal geometry in all dimensions.

To do this, we use a linear description of conformal Cartan connections, pioneered by Thomas [75] (although it is also implicit in [37]), and rediscovered independently by P. Gauduchon [48] and by T. Bailey *et al.* [3], then developed further by A. Čap and A.R. Gover [28, 29, 30]. A Cartan connection may be viewed as a connection on a principal G-bundle satisfying an open condition with respect to a reduction to P; however, it can

[1]The error is in the evaluation of the Ricci trace in the proof of [70, Proposition **7**.4.9 (a)], where Sharpe claims that a component of the Cartan connection does not influence the curvature of surface: in fact it is the case of curves, not surfaces, which is special in this respect. Hence in [70, Theorem **7**.4.29] and [70, Corollary **7**.4.30], $n \neq 2$ should be replaced by $n \neq 1$.

be more convenient to work with a connection on a vector bundle. Such vector bundles with connection, induced by a Cartan connection, are now called *tractor bundles* or (*local*) *twistor bundles* [3, 4, 29]. In conformal Cartan geometry, when $G = O_+(n+1,1)$, the conformal Cartan connection on the associated $\mathbb{R}^{n+1,1}$-bundle is easy to characterize, and, following [3, 48], this is what we do.

The existence and uniqueness of the normal Cartan connection is now known to be a characteristic feature of parabolic geometries, governed by a beautiful algebraic machine, *Lie algebra homology* [4, 31, 34, 59, 64, 73]: the normality condition means that the curvature of the Cartan connection is a 2-cycle in a chain complex for this homology theory.

It turns out that Lie algebra homology also governs the extrinsic geometry of conformal submanifolds. *A priori*, there are many ways to split an ambient Cartan connection, along a submanifold, into tangential and normal parts. However, as observed by Sharpe [70], there is a unique choice such that the associated second fundamental form is tracefree. This way of normalizing the extrinsic geometry amounts to requiring that the 'extrinsic' part of the Cartan connection is a 1-cycle in a Lie algebra homology chain complex. (Sharpe does not then normalize the induced 'intrinsic' Cartan connection, which leads him to remark [70, page 266] that "the Willmore form, while appearing to be *extrinsic* data from the Riemannian perspective, is seen to be *intrinsic* data from the perspective of the Möbius geometry induced on M." We do not agree with this viewpoint.)

The final ingredient of our approach, the *Bernstein–Gelfand–Gelfand operators*, reveals that the Gauß–Codazzi–Ricci equations also have a natural homological interpretation. This is a feature of parabolic geometry that has only been fully elucidated in the last ten years [35, 25]. Since any tractor bundle W is equipped with a connection, there is an associated twisted deRham sequence of W-valued differential forms; these bundles also form the chain complex computing W-valued Lie algebra homology (where the Lie algebra boundary operator acts in the opposite direction to the twisted deRham differential). It turns out that the twisted deRham sequence descends to give a sequence of (possibly) higher order operators between the Lie algebra homology bundles. This Bernstein–Gelfand–Gelfand sequence is a curved version of the (generalized) Bernstein–Gelfand–Gelfand complex on a (generalized) flag variety G/P which is a resolution of a G-module corresponding to W [5, 61]. The curved analogues were introduced by M. Eastwood and J. Rice [41] and R. Baston [4] in special cases, and by Čap, J. Slovak and V. Souček [35] in general. Further, it was shown by the second author and T. Diemer [25] that wedge products of tractor-valued differential forms descend to give bilinear differential operators between Lie algebra homology bundles.

This is the most novel aspect of our conformal Bonnet theorem: we show that the Gauß–Codazzi–Ricci equations may be written in terms of the manifestly Möbius-invariant linear and bilinear differential Bernstein–Gelfand–Gelfand operators defined on Lie algebra homology. This is not just an aesthetic point, but a practical one: it reveals the minimal, homological, data describing the geometry of the conformal submanifold, and shows that of the five Gauß–Codazzi–Ricci equations appearing in other treatments (such as [88]), only three are needed in each dimension—which three depends on the Lie algebra homology in that dimension. This fact is tedious to check directly, but follows easily from [25].

This treatment of conformal submanifold geometry suggests a broader context for our work: *parabolic subgeometries*. If we regard conformal submanifolds as being modelled on the conformal m-sphere inside the conformal n-sphere, then we might more generally consider geometries modelled on arbitrary (homogeneous) inclusions between generalized flag varieties. A particularly important case is the conformal n-sphere (or a hyper-quadric of any signature) in $\mathbb{R}P^{n+1}$, which is the model for hypersurfaces in projective space.

In a subsequent work [15] (see [16] for a foretaste, and some preliminary results) we shall show that our methods do indeed generalize to a wider setting, supporting the claim that our machinery is natural. In the present work, we demonstrate that it is also effective, by exploring some applications. A number of classical results follow effortlessly (once the machine is up and running), often in greater generality than was previously known.

Contents and results. Having set out our stall, let us explain in more detail the fruit on offer. As we remarked at the start of this introduction, our development of conformal submanifold geometry, begins with two parts, like those of Cartan [37]: intrinsic conformal geometry, and the geometry of submanifolds. In the first part of the paper, we present an unashamedly modern treatment of (Möbius) conformal geometry in all dimensions. We have an auxiliary goal here, to show that it is perfectly possible—and indeed desirable—to carry out computations in conformal geometry without introducing a riemannian metric in the conformal class. Indeed, when a choice must be made, we wish to show that it is actually simpler to introduce a Weyl structure, *i.e.*, a torsion-free conformal connection. This avoids the conformal rescaling arguments and logarithmic derivatives that are still prevalent in a surprisingly (to us) large portion of the literature. We believe that conformal geometry is more about conformal invariance than conformal covariance (in line with the modern coordinate free, rather than covariant, view of differential geometry).

In fact we offer the Reader *two* manifestly invariant approaches to conformal geometry: a top-down, or holistic, one, the *conformal Cartan connection*, and a bottom-up, or reductionist, one, the *Möbius structure*. We develop these in parallel in the first part of the paper: the 'A' paragraphs (§§1A–4A) concern conformal Cartan connections, while the 'B' paragraphs (§§1B–4B) concern Möbius structures. The goal of our treatment is to prove that these two notions of conformal structure are equivalent, which we reach in section 5.

Both approaches are motivated by the projective light-cone description of the conformal sphere S^n, the 'flat model' for conformal geometry. The top-down approach, which consists of a (filtered) vector bundle V equipped with a 'conformal Cartan connection', generalizes the bundle $S^n \times \mathbb{R}^{n+1,1}$ with its trivial connection. In section 1, we show how straightforward it is to prove that a manifold M with a flat conformal Cartan connection is locally isomorphic to S^n—the same idea underlies our proof of the conformal Bonnet theorem in section 10. We also show how such a conformal Cartan connection induces the most basic ingredient in the bottom-up approach: a conformal metric.

In section 2, we introduce bundles of conformal algebras: first the bundle $\mathfrak{so}(V)$ of filtered Lie algebras, then its associated graded algebra, which we identify with $TM \oplus \mathfrak{co}(TM) \oplus T^*M$. Here we discuss the most crucial elements of Lie algebra homology theory for the development. In section 3 we turn to Weyl derivatives. From the top-down approach, these are equivalently complementary subspaces to the filtration of V, whereas from the bottom-up approach, they correspond to torsion-free conformal connections on TM.

The key idea in the first part of the paper is introduced in section 4. Here we reveal extra information hidden in a conformal Cartan connection in the form of two second order differential operators. These operators motivate our definition of Möbius structure. We then define a distinguished class of 'conformal' Möbius structures, thus paving the way for our proof in section 5 of the following result.

Theorem. *There is a one-to-one correspondence, up to natural isomorphism, between Möbius structures and conformal Cartan geometries, and the conformal Cartan connection is normal if and only if the Möbius structure is conformal.*

For manifolds of dimension at least 3, similar results have been obtained in [37, 28, 48, 75], and a 2-dimensional version is sketched in [24], but the above theorem covers a wider class of Cartan connections than in these references, a generalization which is vital for our applications in submanifold geometry.

An immediate consequence of this theorem is a canonical way to normalize a non-normal conformal Cartan geometry, a procedure which will be very important in the rest of the paper. We discuss some further ramifications in sections 6–7: we provide a more explicit analysis of Möbius structures for curves and surfaces, relating them to the schwarzian derivative; we discuss the gauge theory and moduli of conformal Cartan connections; and we revisit the flat model to discuss spaceform geometries and symmetry breaking.

In the second part of the paper, we apply the formalism of Part I to submanifolds of conformal manifolds. We emphasise a gauge-theoretic point of view, but use vector bundles and connections rather than frames and matrices of 1-forms. Although the latter are more elementary, the choice of frame can obscure the geometry and lead to large and complicated matrices in applications. Without frames, the theory has conceptual simplicity: some investment is needed to master the abstraction, but the payoff is a uniform and efficient calculus which adapts geometrically to almost any application.

As in Part I, we give two approaches, one emphasising bundles and connections, the other, more primitive homological data, and we relate these points of view. In sections 8–9, we show how such a submanifold inherits a conformal Möbius structure from the ambient geometry. We first study how Möbius structures on the submanifold are induced by a choice of 'Möbius reduction', then we show that Lie algebra homology distinguishes a canonical Möbius reduction (cf. [70]) and hence obtain a canonical induced conformal Möbius structure on the submanifold. This leads to canonical Gauß–Codazzi–Ricci equations relating the curvature of these induced Möbius structures to the ambient space.

In the case of submanifolds of the conformal sphere S^n, a Möbius reduction is the same thing as a sphere congruence enveloped by the submanifold, which are of great interest in their own right, although we postpone a detailed studied of sphere congruences *per se* to a sequel to this paper [14]. The canonical Möbius reduction in this case is the central sphere congruence of W. Blaschke and G. Thomsen [8] or the conformal Gauß map of Bryant [10, 11]. We show that the minimal data characterizing the submanifold (locally, up to Möbius transformation) are certain Lie algebra homology classes satisfying natural homological Gauß–Codazzi–Ricci equations. These equations encode the flatness of the ambient conformal Cartan connection of S^n, meaning that the immersion can be reconstructed (locally, up to Möbius transformation) from the Gauß–Codazzi–Ricci data. Thus we obtain, in section 10, an analogue of the classical Bonnet theorem.

Theorem. *An m-manifold Σ can be locally immersed in S^n with a given induced conformal Möbius structure, connection on the weightless normal bundle and tracefree second fundamental form* (*or conformal acceleration for $m = 1$*) *if and only if these data satisfy the homological Gauß–Codazzi–Ricci equations. Moreover, in this case, the immersion is unique up to a Möbius transformation of S^n.*

In section 11 we introduce ambient Weyl structures along a submanifold and use them to give explicit formulae for the operators entering into the homological Gauß–Codazzi–Ricci equations. Then, in section 12, we specialize this result first to curves, for which we compute some Möbius invariants, and then to surfaces, where we relate our conformal Bonnet theorem to the explicit approach of [21], developed by the first author and his coworkers at the same time as our general theory. Finally we sketch the relation of our approach to the quaternionic formalism for surfaces in $S^4 \cong \mathbb{H}P^1$ expounded in [18].

The third part of the paper concerns applications, with a view to demonstrating the speed and effectiveness of our formalism once the machinery is in place. In section 13, we show how the most basic aspects of conformal submanifold geometry, such as totally umbilic submanifolds and channel submanifolds, have straightforward treatments in our approach. We also discuss symmetry breaking and give an almost computation-free proof of Dupin's Theorem on orthogonal coordinates.

In sections 14–16, we apply our methods to constrained Willmore surfaces, isothermic surfaces, Guichard surfaces and conformally flat submanifolds with flat normal bundle, which all find their natural home in conformal Möbius geometry and have in common an integrable systems interpretation: the Gauß–Codazzi–Ricci data come in one parameter families, leading to spectral deformations and a family of flat connections, which we derive straightforwardly in arbitrary codimension. For Willmore and constrained Willmore surfaces, we show that the well-known relation with harmonicity of the central sphere congruence has a homological interpretation. We also demonstrate easily that products of curves, constant mean curvature surfaces, generalized H-surfaces, and quadrics are all isothermic, and we compute the spectral deformation for products of curves.

In section 16, we turn to a class of submanifolds in in arbitrary dimension and codimension, which we call *Möbius-flat*: if the dimension is 3 or more and the codimension is one, this is the well known theory of conformally-flat hypersurfaces, and in higher codimension, we simply add flatness of the normal bundle as an extra hypothesis. However, we also present a new 2-dimensional theory, which unifies conformally flat submanifolds with channel surfaces and the surfaces of Guichard [49, 23]. In this theory, we suppose that the surface (with flat normal bundle) envelopes a sphere congruence for which the induced normal Cartan connection (or equivalently, the induced conformal Möbius structure) is flat. When the sphere congruence is the central sphere congruence, this means that the canonically induced Möbius curvature of the surface (which is an analogue of the Cotton–York curvature of conformal 3-manifolds) is zero, and we say the surface is *strictly Möbius-flat*, whereas in general, this curvature is the exterior derivative of a quadratic differential commuting with the shape operators, which we call a *commuting Cotton–York potential*. In codimension one, the strictly Möbius-flat surfaces are Dupin cyclides (orbits of a two dimensional abelian subgroup of the Möbius group), and we obtain a transparent conformally-invariant derivation of their properties and classification. Möbius-flat surfaces in general have a much richer theory: in particular, as we prove, since they include the Guichard surfaces, they also include the surfaces of constant gaussian curvature in a spaceform, just as higher dimensional submanifolds of constant gaussian (sectional) curvature in a spaceform are conformally-flat. In a subsequent paper we shall show that the classical transformation theory of Guichard surfaces extends to Möbius-flat submanifolds in arbitrary dimension and codimension.

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A notational apology. Notation presents significant difficulties in this work, since there are many different geometric objects which can be constructed from each other in various ways. In this paper the Reader will find the following constructions:

- a conformal Cartan connection $\mathfrak{D}^V$ from an enveloped sphere congruence V;

- a Möbius structure $\mathcal{M}^{\mathfrak{D}}$ from a conformal Cartan connection $\mathfrak{D}$;
- a normalized Ricci curvature $r^{D,\mathcal{M}}$ from a Möbius structure $\mathcal{M}$ and a Weyl derivative D.

Combining these constructions could lead to unreadable superscripts, so we have adopted the policy of omitting intermediate steps whenever this causes no confusion, leading to notations such as $\mathcal{M}^V$ and $r^{D,V}$. We hope that our preference for less decorated notation is more a help to the Reader than a hindrance.

Part I. Conformal Möbius Geometry

In the first part of our paper, we provide a complete theory of the intrinsic conformal geometry that we shall use to study conformal submanifold geometry in the later parts. Although conformal geometry is well-established, our theory has many novelties, in particular the notion of a Möbius structure, which allows us to describe, in intrinsic terms, the geometry of the large class of conformal Cartan connections that arise in submanifold geometry. We also hope that it is a helpful treatment for several kinds of Reader, from the representation theorist to the riemannian geometer, because of the complementary approaches provided by the A and B sections. Let us describe their contents in more detail.

In §1A, we introduce conformal Cartan connections $(V, \Lambda, \mathfrak{D})$ as curved versions of the model for conformal geometry, the celestial sphere or projective lightcone in Minkowski space. The conformal structure induced by such connections motivate the first ingredient of the bottom-up approach, the conformal metric, which we discuss in §1B, together with the disarmingly simple notion of densities, which render unnecessary the use of riemannian metrics in the conformal class.

Algebra in conformal geometry is more subtle than in riemannian geometry, because the stabilizer of a point (a lightline) in the model is not reductive, but a parabolic subgroup of the conformal (or Möbius) group, which is essentially the isometry group of Minkowski space. Fortunately there is an algebraic machine, called *Lie algebra homology*, which relates representations of this parabolic group to its reductive part, the *Levi factor*, which is here the conformal linear group. We discuss this for the filtered Lie algebra $\mathfrak{so}(V)$ in §2A, then for its associated graded algebra $TM \oplus \mathfrak{co}(TM) \oplus T^*M$ in §2B.

We then introduce *Weyl derivatives*, *i.e.*, covariant derivatives on density line bundles. In §3A, we show that they induce splittings of the filtration of V and hence a curvature decomposition and a gauge theory. In §3B, we see that they induce torsion-free conformal connections and curvature, whose dependence on the Weyl derivative we describe.

In §4A, we introduce a natural differential lift from 1-densities to sections of V and use it to define two second order *Möbius operators*, one linear, one quadratic, associated to a conformal Cartan connection. In §4B, we show that similar operators exist naturally on any conformal manifold of dimension $n \geqslant 3$, and introduce *Möbius structures*, which cover such operators in generality as well as allowing us to describe the natural ones.

We combine our two threads in section 5, where we construct a conformal Cartan connection from a Möbius structure and prove that this leads to an equivalence, so that the natural affine structure on the space of Möbius structures can be lifted to the space of conformal Cartan connections, providing a way to 'normalize' such a connection.

In section 6, we discuss how to compute the Möbius structure from the Cartan connection, particularly in dimensions one and two, where the Möbius structure can be reinterpreted as a real or (perhaps non-holomorphic) complex projective structure, and hence related to schwarzian derivatives, cf. [21, 24].

Section 7 is concerned with further consequences of our main result. In §7.1 we describe the moduli space of conformal Cartan geometries inducing a given conformal metric. In §7.2 we revisit the model geometry of S^n to relate it to the flat Möbius structure of euclidean space and other spaceform geometries. This sort of symmetry breaking often shows up in submanifold geometry, and in §7.3 we end Part I with a general version of such phenomena.

1. Conformal geometry

1A. The sphere and conformal Cartan connections. The flat conformal structure on the n-sphere arises most naturally by viewing it not as the round sphere $\mathcal{S}^n$ in the euclidean space $\mathbb{R}^{n+1}$, but as the celestial sphere S^n in the lorentzian spacetime $\mathbb{R}^{n+1,1}$ [38]. Recall that $\mathbb{R}^{n+1,1}$ is real vector space of dimension $n+2$ with a nondegenerate inner product $(\cdot\,,\cdot)$ of signature $(n+1,1)$. Inside $\mathbb{R}^{n+1,1}$, we distinguish the *light-cone* $\mathcal{L}$:

$$\mathcal{L} = \{v \in \mathbb{R}^{n+1,1} \smallsetminus \{0\} : (v,v) = 0\},$$

which is a submanifold of $\mathbb{R}^{n+1,1}$.

Clearly, if $v \in \mathcal{L}$ and $r \in \mathbb{R}^\times$ then $rv \in \mathcal{L}$, so that $\mathbb{R}^\times$ acts freely on $\mathcal{L}$ and we may take the quotient $P(\mathcal{L}) \subset P(\mathbb{R}^{n+1,1})$:

$$P(\mathcal{L}) = \mathcal{L}/\mathbb{R}^\times \cong \{U \subset \mathbb{R}^{n+1,1} : U \text{ is a 1-dimensional null subspace}\},$$

which is a smooth n-sphere. The projection $q\colon \mathcal{L} \to P(\mathcal{L})$ is an $\mathbb{R}^\times$-bundle over $P(\mathcal{L})$, which may be viewed as the nonzero vectors in a tautological line bundle Λ over $P(\mathcal{L})$: Λ is defined to be the subbundle of the trivial bundle $P(\mathcal{L}) \times \mathbb{R}^{n+1,1}$ whose fibre at $U \in P(\mathcal{L})$ is U itself, viewed as a null line in $\mathbb{R}^{n+1,1}$. It is convenient to orient Λ, by choosing one of the components $\mathcal{L}^+$ of $\mathcal{L}$: this choice is equivalently a 'time-orientation' of $\mathbb{R}^{n+1,1}$.

Note that $T\mathcal{L} \cong q^*\Lambda^\perp \subset \mathcal{L} \times \mathbb{R}^{n+1,1}$ with vertical bundle $q^*\Lambda$. Hence the differential $d\sigma$ of a section σ of $\mathcal{L}^+ \subset \Lambda$ can be viewed $\Lambda^\perp$-valued 1-form on $P(\mathcal{L})$ such that $d\sigma \mathbin{mod} \Lambda$ is an isomorphism $TP(\mathcal{L}) \to \Lambda^\perp/\Lambda$. This isomorphism is clearly algebraic and linear in σ, so there is a canonical isomorphism $TP(\mathcal{L})\,\Lambda \cong \Lambda^\perp/\Lambda$. Here, and in the following, we omit tensor product signs when tensoring with line bundles.

Since $\Lambda^\perp/\Lambda$ inherits a metric from $\mathbb{R}^{n+1,1}$, each positive section σ defines a metric on $P(\mathcal{L})$ by $X, Y \mapsto (d_X\sigma, d_Y\sigma)$. Rescaling σ evidently gives a conformally equivalent metric, and this equips $P(\mathcal{L})$ with a conformal structure. In particular, by considering the conic sections of $\Lambda \subset P(\mathcal{L}) \times \mathbb{R}^{n+1,1}$ given by $(v,\sigma) = k$ for fixed $v \in \mathbb{R}^{n+1,1}$ and $k \in \mathbb{R}^\times$, one sees that this conformal structure is flat (cf. §7.2).

The beauty of this model is that it *linearizes* conformal geometry. Let $O_+(n+1,1)$ be the group of Lorentz transformations of $\mathbb{R}^{n+1,1}$ preserving the time orientation. Then the linear action of $O_+(n+1,1)$ on $\mathbb{R}^{n+1,1}$ preserves $\mathcal{L}$ and so descends to an action on $P(\mathcal{L})$. Since the metric on $\Lambda^\perp/\Lambda$ is preserved, $O_+(n+1,1)$ acts by conformal diffeomorphisms on $P(\mathcal{L})$ and this gives an isomorphism between $O_+(n+1,1)$ and the group $M\ddot{o}b(n)$ of *Möbius transformations* of $S^n = P(\mathcal{L})$ (which are the global conformal diffeomorphisms for $n \geqslant 2$ and the projective transformations of $S^1 \cong \mathbb{R}P^1$ for $n=1$). $M\ddot{o}b(n)$ acts transitively on $P(\mathcal{L})$ with stabilizer a parabolic subgroup $P \cong CO(n) \ltimes \mathbb{R}^{n*}$, which identifies S^n with $M\ddot{o}b(n)/P$, a generalized flag variety. Note that signature $(k+1,1)$ linear subspaces of $\mathbb{R}^{n+1,1}$ give the conformal k-spheres in S^n by intersecting with $\mathcal{L}$—see §8.3 for more details.

A conformal manifold is a 'curved version' of S^n: curved versions of homogeneous spaces are usually defined as Cartan connections on principal bundles; however, since we have described the flat model linearly, using flat differentiation on $P(\mathcal{L}) \times \mathbb{R}^{n+1,1}$, we shall also describe conformal Cartan connections linearly, cf. [3, 29, 48, 75].

Definition 1.1. A *conformal Cartan connection* on an n-manifold M is defined by:

- the *Cartan vector bundle* V, a rank $n+2$ vector bundle with a signature $(n+1,1)$ lorentzian metric on each fibre;
- the *tautological line* Λ, an oriented (so trivializable) null line subbundle of V;

- the *Cartan connection* $\mathfrak{D}$, a metric connection on V satisfying the following Cartan condition: the *soldering form* $\beta^{\mathfrak{D}}\colon TM \to \mathit{Hom}(\Lambda, \Lambda^{\perp}/\Lambda)$, defined by

$$\beta^{\mathfrak{D}}_X \sigma = -\mathfrak{D}_X \sigma \mod \Lambda \tag{1.1}$$

  for any section σ of Λ and any vector field X, is a bundle isomorphism. (This makes sense because the right hand side is algebraic in $\sigma \otimes X$, as it is for $M = P(\mathcal{L})$, $\mathfrak{D} = d$.)

Our data equip V and M with structures intertwined by the soldering form $\beta^{\mathfrak{D}}$. Our first observation is that the lorentzian metric on V induces a conformal metric on M, generalizing the case $M = P(\mathcal{L})$. For this, we denote by $\mathcal{L}^+$ the positive ray subbundle of Λ, and a section σ of $\mathcal{L}^+$ (*i.e.*, a positive section of Λ) will be called a *gauge*. We then obtain a conformal class of metrics on M defined by $X, Y \mapsto (\mathfrak{D}_X\sigma, \mathfrak{D}_Y\sigma)$ for each gauge σ. In more invariant terms, we can write $(\mathfrak{D}_X\sigma, \mathfrak{D}_Y\sigma) = \langle X, Y\rangle\sigma^2$, where $\langle X, Y\rangle$ is a fibrewise inner product on TM with values in $(\Lambda^*)^2$ which is independent of the choice of gauge.

Proposition 1.2. [37] *The celestial n-sphere $S^n = P(\mathcal{L})$ is naturally equipped with a flat conformal Cartan connection. Conversely, if M is equipped with a flat conformal Cartan connection, then M is locally isomorphic to S^n.*

Proof. We take $V = P(\mathcal{L}) \times \mathbb{R}^{n+1,1}$, with the constant lorentzian metric, the tautological null line subbundle, and flat differentiation. We have already seen that the Cartan condition holds, and $R^{\mathfrak{D}} = 0$ by definition.

Conversely, if $\mathfrak{D}$ is flat, then the inclusion $\Lambda \to V$ defines a map Φ from each simply connected open subset Ω of M to the space of parallel null lines in $V|_{\Omega}$, which is diffeomorphic to S^n. By the Cartan condition, Φ is a local diffeomorphism. Furthermore Φ clearly identifies Λ with the tautological line bundle over S^n, and $\mathfrak{D}$ with the trivial connection on $S^n \times \mathbb{R}^{n+1,1}$. In particular Φ is conformal. $\square$

The same argument will be used later to obtain an immersion into S^n in the proof of the Bonnet theorem for conformal submanifolds.

Remark 1.3. A Cartan connection in the usual sense (see [26, 70]), modelled on a homogeneous space G/P, is a principal G-bundle $\mathcal{C}$ equipped with a principal G-connection ω and a reduction $\mathcal{G} \subset \mathcal{C}$ to a principal P-bundle such that $\omega|_{\mathcal{G}} \mod \mathfrak{p}\colon T\mathcal{G} \to \mathfrak{g}/\mathfrak{p}$ is an isomorphism on each tangent space. When $G = \mathit{M\ddot{o}b}(n)$ and $G/P = S^n$, it is easy to see that this induces a conformal Cartan connection on the bundle associated to the standard representation $\mathbb{R}^{n+1,1}$ of $G \cong O_+(n+1, 1)$, and this linear representation suffices to recover the original Cartan connection.

1B. Densities and conformal metrics. In §1A, we obtained the conformal class of riemannian metrics on S^n from gauges, *i.e.*, sections of $\mathcal{L}^+ \subset \Lambda$. This is an entirely general phenomenon: a conformal structure on a manifold M is an oriented line subbundle of S^2T^*M whose positive sections are riemannian metrics (*i.e.*, positive definite). One small difficulty with such a definition is that each conformal structure has its own line bundle. However, we can resolve this by relating any such line bundle to the density bundle of M.

If E is a real n-dimensional vector space and w any real number, then the oriented one dimensional linear space $L^w = L^w_E$ carrying the representation $A \mapsto |\det A|^{w/n}$ of $GL(E)$ is called the space of *densities of weight w* or *w-densities.* This space can be constructed canonically from E as the space of maps

$$\lambda\colon (\wedge^n E) \smallsetminus \{0\} \to \mathbb{R} \;\text{ such that }\; \lambda(rA) = |r|^{-w/n}\lambda(A) \;\text{ for all }\; r \in \mathbb{R}^\times,\ A \in (\wedge^n E) \smallsetminus \{0\}.$$

Note that the modulus of $\omega \in \wedge^n E^*$ is an element $|\omega|$ of L^{-n} defined by $|\omega|(A) = |\omega(A)|$.

The *density line bundle* $L^w = L^w_{TM}$ on an n-dimensional manifold M is the bundle whose fibre at $x \in M$ is $L^w_{T_xM}$. This is the associated bundle $GL(M) \times_{GL(n)} L^w(n)$ to the frame bundle $GL(M)$, where $L^w(n)$ is the space of w-densities of $\mathbb{R}^n$. The density bundles are oriented real line bundles, but there is no preferred trivialization unless $w = 0$.

Proposition 1.4. *Let $\Lambda \otimes \Lambda$ be a line subbundle of S^2T^*M, with Λ oriented, whose positive sections are nondegenerate. Then there is a canonical oriented isomorphism $\Lambda \to L^{-1}$.*

Proof. The inclusion of $\Lambda^2 := \Lambda \otimes \Lambda$ into S^2T^*M is a section of $S^2T^*M\,(\Lambda^2)^* = S^2(TM\,\Lambda)^*$, and this defines a nondegenerate metric on $TM\,\Lambda$ by assumption. The modulus of the volume form of this metric is a positive section of $L^{-n}\,(\Lambda^*)^n$, and the positive nth root of this, a section of $L^{-1}\,\Lambda^* \cong \mathit{Hom}(\Lambda, L^{-1})$, is the isomorphism we seek. □

This proposition tells us that we may view the conformal structure as a section of $S^2T^*M\,L^2$. The sections we obtain in this way are not arbitrary, but *normalized* by the condition that the modulus of their volume form is the canonical section 1 of $L^{-n}\,L^n = M \times \mathbb{R}$.

Definition 1.5. [75] A *conformal metric* on M is a normalized metric c on the *weightless tangent bundle* $TM\,L^{-1}$. It induces an inner product $\langle\cdot\,,\cdot\rangle$ on TM with values in L^2.

The use of densities allows a simple geometric dimensional analysis for tensors [40, 80]. Sections of $L = L^1$ may be thought of as scalar fields with dimensions of length: a positive section ℓ defines a *length scale*. The tensor bundle $L^w \otimes (TM)^j \otimes (T^*M)^k$ (and any subbundle, quotient bundle, element or section) will be said to have *weight* $w + j - k$.

A conformal metric c is weightless, and we shall use it freely to 'raise and lower indices' with the proviso that the weight of tensors is preserved. For example, a 1-form γ is identified in this way with a vector field of weight -1, that is, a section of $TM\,L^{-2}$. Where necessary for clarity, we denote this isomorphism and its inverse by $\sharp$ and $\flat$ in the usual way.

2. Conformal algebra

2A. The filtered Lie algebra bundle. Let $(V, \Lambda, \mathfrak{D})$ be a conformal Cartan connection on M. The filtration $0 \subset \Lambda \subset \Lambda^\perp \subset V$ induces a filtration of the Lie algebra bundle $\mathfrak{so}(V)$:

$$0 \subset \mathfrak{so}(V)_{-1} \subset \mathfrak{so}(V)_0 \subset \mathfrak{so}(V)_1 = \mathfrak{so}(V),$$

where

$$\mathfrak{so}(V)_{-1} = \{S \in \mathfrak{so}(V) \colon S|_\Lambda = 0,\, S(\Lambda^\perp) \subseteq \Lambda\},$$
$$\mathfrak{so}(V)_0 = \{S \in \mathfrak{so}(V) \colon S(\Lambda) \subseteq \Lambda\}.$$

Hence $\mathfrak{so}(V)_0$ is the stabilizer $\mathfrak{stab}(\Lambda)$ of Λ, which is a bundle of parabolic subalgebras, whereas $\mathfrak{so}(V)_{-1}$ is the Killing annihilator $\mathfrak{stab}(\Lambda)^\perp$, which is the bundle of (abelian) nilradicals of $\mathfrak{stab}(\Lambda)$. By definition, for $S \in \mathfrak{stab}(\Lambda)^\perp$, $S|_{\Lambda^\perp}$ has image and kernel Λ, and it is easy to see that this restriction defines an isomorphism $\mathfrak{stab}(\Lambda)^\perp \cong \mathit{Hom}(\Lambda^\perp/\Lambda, \Lambda)$.

These filtrations give rise to a bundle

$$\mathfrak{so}(V)_{\mathrm{gr}} = \mathfrak{so}(V)_{-1} \oplus \mathfrak{so}(V)_0/\,\mathfrak{so}(V)_{-1} \oplus \mathfrak{so}(V)_1/\,\mathfrak{so}(V)_0$$

of graded Lie algebras acting on the bundle $V_{\mathrm{gr}} = \Lambda \oplus \Lambda^\perp/\Lambda \oplus V/\Lambda^\perp$ of graded lorentzian vector spaces. Only $\mathfrak{so}(V)_0/\,\mathfrak{so}(V)_{-1}$ preserves the grading, and this defines an isomorphism $\mathfrak{so}(V)_0/\,\mathfrak{so}(V)_{-1} \cong \mathfrak{so}(\Lambda \oplus V/\Lambda^\perp) \oplus \mathfrak{so}(\Lambda^\perp/\Lambda)$. On the other hand, restriction to Λ defines an isomorphism from $\mathfrak{so}(V)/\,\mathfrak{so}(V)_0$ to $\mathit{Hom}(\Lambda, \Lambda^\perp/\Lambda)$.

The key point now is that the soldering form allows us to identify each graded component of V_{gr}, and hence of $\mathfrak{so}(V_{\mathrm{gr}})$, with a natural vector bundle on M. For this, begin by noting that $\beta^{\mathfrak{D}} \colon TM \to \mathit{Hom}(\Lambda, \Lambda^\perp/\Lambda)$ naturally induces an isomorphism $TM\,\Lambda \cong \Lambda^\perp/\Lambda$. As in §1B (cf. Proposition 1.4) the conformal metric induces an identification $\Lambda \cong L^{-1}$, which we shall use freely. We thus have a projection $\pi \colon \Lambda^\perp \to TM\,L^{-1}$, with kernel Λ, given by

$$\pi\mathfrak{D}_X\sigma = -X \otimes \sigma. \tag{2.1}$$

We next use minus the metric on V to identify $V/\Lambda^\perp$ with $\Lambda^* \cong L$ and so obtain a projection $p\colon V \to L$, with kernel $\Lambda^\perp$, such that for $v \in V$, $\sigma \in \Lambda$, we have

$$\langle p(v), \sigma\rangle = -(v, \sigma). \tag{2.2}$$

(We have introduced minus signs in the definitions of $\beta^{\mathfrak{D}}$ and p so that derivatives appear positively in components of jets later on: see §5.1.)

Dualizing the soldering form gives an isomorphism of T^*M with $\mathit{Hom}(\Lambda^\perp/\Lambda, \Lambda)$, and hence with $\mathfrak{stab}(\Lambda)^\perp$. We have therefore realized T^*M as an abelian subalgebra of $\mathfrak{so}(V)$ with action on V determined by

$$\gamma \cdot \sigma = 0, \qquad \gamma \cdot \mathfrak{D}_X \sigma = -\gamma(X)\sigma, \tag{2.3}$$

for $\gamma \in T^*M$, $X \in TM$, and $\sigma \in \Lambda$.

The soldering form also identifies $\mathfrak{so}(V)/\mathfrak{so}(V)_0$ with TM (restrict to $\Lambda \cong L^{-1}$) and $\mathfrak{so}(V)_0/\mathfrak{so}(V)_{-1}$ with $\mathfrak{co}(TM) := \langle id\rangle \oplus \mathfrak{so}(TM)$ (use $TM \cong (TM\,L^{-1}) \otimes L \cong \Lambda^\perp/\Lambda \otimes V/\Lambda^\perp$). The projections p and π onto graded pieces, which we introduced for V, have analogues for $\mathfrak{so}(V)$ making these identifications explicit: $p\colon \mathfrak{so}(V) \to TM$, with kernel $\mathfrak{stab}(\Lambda)$ is defined by $(pS) \otimes \sigma = \pi(S\sigma)$ for $S \in \mathfrak{so}(V)$; $\pi\colon \mathfrak{stab}(\Lambda) \to \mathfrak{co}(TM)$, with kernel $\mathfrak{stab}(\Lambda)^\perp$, is defined $(\pi S) \circ p = p \circ S$ and $(\pi S) \circ \pi = \pi \circ S|_{\Lambda^\perp}$ for $S \in \mathfrak{stab}(\Lambda)$. In particular:

$$\pi(S(\mathfrak{D}_X \sigma)) = -(\pi S)(X \otimes \sigma). \tag{2.4}$$

These constructions lead to several other useful formulae. First we have $(\mathfrak{D}_X\sigma, \mathfrak{D}_Y\sigma) = -(\sigma, \mathfrak{D}_X\mathfrak{D}_Y\sigma)$, yielding

$$p(\mathfrak{D}_X\mathfrak{D}_Y\sigma) = \langle X, Y\rangle\sigma. \tag{2.5}$$

It will often be convenient to use the metric on $TM\,L^{-1}$ to view π as a map $\Lambda^\perp \to T^*M\,L$. With v any section of $\Lambda^\perp$, (2.5) can then be rewritten as:

$$p(\mathfrak{D}v) = -\pi v. \tag{2.6}$$

We also have, for any $v \in V$, $(\pi(\gamma\cdot v), X \otimes \sigma) = (\gamma \cdot v, \beta^{\mathfrak{D}}_X \sigma) = -(v, \gamma \cdot \beta^{\mathfrak{D}}_X\sigma) = -\gamma(X)(v, \sigma)$ so that

$$\pi(\gamma \cdot v) = \gamma \otimes p(v). \tag{2.7}$$

We will have frequent recourse hereafter to the Lie algebra homology of the bundle of abelian Lie algebras T^*M. For this, let W be any vector bundle carrying a fibrewise representation of T^*M. Define, for each k, $\partial\colon \wedge^{k+1}T^*M \otimes W \to \wedge^k T^*M \otimes W$ by

$$\partial\alpha = \textstyle\sum_i \varepsilon_i \cdot (e_i \lrcorner \alpha), \qquad \textit{i.e.,} \qquad \partial\alpha_{X_1,\dots X_k} = \sum_i \varepsilon_i \cdot \alpha_{e_i, X_1,\dots X_k}. \tag{2.8}$$

Here ε_i, e_i are dual local frames of T^*M and TM. Now if $\alpha = \partial\beta$, then

$$\partial\alpha_{X_1,\dots X_k} = \sum_{i,j} \varepsilon_i \cdot \varepsilon_j \cdot \beta_{e_j, e_i, X_1,\dots X_k} = \sum_{i<j} [\varepsilon_i, \varepsilon_j] \cdot \beta_{e_j,e_i,X_1,\dots X_k} = 0$$

since T^*M is abelian. Hence $(\wedge^\bullet T^*M \otimes W, \partial)$ is a chain complex, and we denote its cycles by $Z_\bullet(T^*M, W)$ and its homology, called *Lie algebra homology*, by $H_\bullet(T^*M, W)$.

We shall only be interested in the cases $W = V$ and $W = \mathfrak{so}(V)$. In particular let us compute $Z_1(T^*M, V)$: if $\alpha \in T^*M \otimes V$ with $\partial\alpha = 0$ then, in particular, from (2.7),

$$0 = \pi\partial\alpha = \textstyle\sum_i \varepsilon_i \otimes p(\alpha_{e_i})$$

whence α takes values in $\Lambda^\perp$. Now we deduce from (2.3) that

$$Z_1(T^*M, V) = \{\alpha \in T^*M \otimes \Lambda^\perp : \textstyle\sum_i \langle \varepsilon_i, \pi\alpha_{e_i}\rangle = 0\}. \tag{2.9}$$

2B. The graded Lie algebra bundle. On any manifold M, the Lie algebra bundle of endomorphisms $\mathfrak{gl}(TM)$ acts fibrewise on the density bundle L via $A\cdot\ell = (\frac{1}{n}\,tr\,A)\ell$ and hence also on $S^2T^*M\,L^2$. The stabilizer of a conformal metric c is the bundle of conformal linear Lie algebras $\mathfrak{co}(TM)$. Thus $A\in\mathfrak{co}(TM)$ if and only if

$$\langle AX,Y\rangle + \langle X,AY\rangle = \tfrac{2}{n}(tr\,A)\langle X,Y\rangle.$$

The Lie bracket on $\mathfrak{co}(TM)$ can be extended to one on $T^*M\oplus\mathfrak{co}(TM)\oplus TM$: declare TM and T^*M to be abelian subalgebras and, for $(\gamma,A,X)\in T^*M\oplus\mathfrak{co}(TM)\oplus TM$, set

$$[\![A,X]\!] = AX, \qquad [\![\gamma,A]\!] = \gamma\circ A; \tag{2.10}$$

finally define $[\![X,\gamma]\!] = -[\![\gamma,X]\!]\in\mathfrak{gl}(TM)$ by

$$[\![X,\gamma]\!]\cdot Y = -[\![\gamma,X]\!]\cdot Y = \gamma(X)Y + \gamma(Y)X - \langle X,Y\rangle\gamma^\sharp. \tag{2.11}$$

Observe that $[\![X,\gamma]\!]\in\mathfrak{co}(TM)$ with $tr[\![X,\gamma]\!] = n\gamma(X)$ and $[\![X,\gamma]\!]\cdot Y = [\![Y,\gamma]\!]\cdot X$.

Define an inner product of signature $(n+1,1)$ on $L^{-1}\oplus T^*M\,L\oplus L$ by setting

$$(v,v) = \langle\theta,\theta\rangle - 2\sigma\ell,$$

for $v=(\sigma,\theta,\ell)$. An action of $(\gamma,A,X)\in T^*M\oplus\mathfrak{co}(TM)\oplus TM$ on $(\sigma,\theta,\ell)\in L^{-1}\oplus T^*ML\oplus L$ is given as follows: $A\in\mathfrak{co}(TM)$ acts in the natural way on each component, and we set

$$\begin{aligned}
&X\cdot\sigma = X\otimes\sigma\in TM\,L^{-1}\cong T^*M\,L &&\gamma\cdot\sigma = 0\\
&X\cdot\theta = \theta(X)\in L &&\gamma\cdot\theta = \langle\gamma,\theta\rangle\in L^{-1}\\
&X\cdot\ell = 0 &&\gamma\cdot\ell = \gamma\otimes\ell\in T^*M\,L.
\end{aligned} \tag{2.12}$$

This action is skew with respect to the inner product, giving a Lie algebra isomorphism $T^*M\oplus\mathfrak{co}(TM)\oplus TM\cong\mathfrak{so}(L^{-1}\oplus T^*ML\oplus L)$: the Lie bracket of $S,T\in T^*M\oplus\mathfrak{co}(TM)\oplus TM$ can be computed via $[\![S,T]\!]\cdot v = S\cdot(T\cdot v) - T\cdot(S\cdot v)$ for any $v\in L^{-1}\oplus T^*M\,L\oplus L$.

Of course, we have not plucked this Lie algebra action out of thin air: if the conformal metric c is induced by a conformal Cartan connection $(V,\Lambda,\mathfrak{D})$, then the identifications of the previous paragraph induce isomorphisms of V_{gr} with $L^{-1}\oplus T^*M\,L\oplus L$ and of $\mathfrak{so}(V)_{\mathrm{gr}}$ with $T^*M\oplus\mathfrak{co}(TM)\oplus TM$ which intertwine the natural action of $\mathfrak{so}(V)_{\mathrm{gr}}$ on V_{gr} with the above action of $T^*M\oplus\mathfrak{co}(TM)\oplus TM$ on $L^{-1}\oplus T^*M\,L\oplus L$. It follows that the above graded Lie algebra structure on $T^*M\oplus\mathfrak{co}(TM)\oplus TM$ is precisely that induced by the filtered Lie algebra structure on $\mathfrak{so}(V)$. Hence for all $S_{-1}\in\mathfrak{so}(V)_{-1} = \mathfrak{stab}(\Lambda)^\perp$, $S_0,T_0\in\mathfrak{so}(V)_0 = \mathfrak{stab}(\Lambda)$ and $T\in\mathfrak{so}(V)$, we have $p[S_0,T] = [\![\pi S_0,pT]\!]$, $\pi[S_0,T_0] = [\![\pi S_0,\pi T_0]\!]$, $p[S_{-1},T] = [\![S_{-1},pT]\!]$ and $\pi[S_{-1},T_0] = [\![S_{-1},\pi T_0]\!]$. We shall use these formulae freely.

As in §2A, we may define Lie algebra homology, but now the operator ∂ is also graded and may be restricted to graded pieces. To analyse this, we fix a length scale ℓ and define an involution $v\mapsto v^* = (-\lambda\ell^{-2},\theta,-\sigma\ell^2)$, where $v=(\sigma,\theta,\lambda)\in L^{-1}\oplus T^*M\,L\oplus L$, so that $\langle v,v\rangle := (v^*,v) = \langle\theta,\theta\rangle + \lambda^2\ell^{-2} + \sigma^2\ell^2$ is positive definite[2] on V. We further define $S\mapsto S^* = -(X^\flat\ell^{-2},A^T,\gamma^\sharp\ell^2)$, for $S=(\gamma,A,X)\in T^*M\oplus\mathfrak{co}(TM)\oplus TM$, where the musical isomorphisms are induced by g. We compute that $(S\cdot v)^* = S^*\cdot v^*$ and hence $[\![S,T]\!]^* = [\![S^*,T^*]\!]$. Further, $\langle S,S\rangle := (S^*,S)$ is positive definite, where (S,S) is the invariant inner product of signature $(\frac{1}{2}n(n+1),n+1)$ on $T^*M\oplus\mathfrak{co}(TM)\oplus TM$ given by

$$(S,S) = \langle A_0,A_0\rangle - \mu^2 - 2\gamma(X)$$

for $S=(\gamma,A,X)$ and $A=\mu\,id+A_0$ with $A_0\in\mathfrak{so}(TM)$. It follows that $S\mapsto S^*$ is a Cartan involution of the graded Lie algebra, as in Kostant's celebrated paper [59]. Following this

[2]For indefinite signature conformal structures, we further need a choice of maximal positive definite subbundle of TM to define such an involution.

paper, we now compute (minus) the adjoint of ∂ on $T^*M \oplus \mathfrak{co}(TM) \oplus TM$ with respect to the metrics g and $\langle , \rangle$:

$$\langle \partial\alpha, \beta\rangle = ((\partial\alpha)^*, \beta) = \bigl(\sum_i [\![\varepsilon_i^*, (e_i \lrcorner \alpha)^*]\!], \beta\bigr) = -\bigl(\alpha^*, \sum_i e_i^* \wedge [\![\varepsilon_i^*, \beta]\!]\bigr) = -\langle \alpha, [\![id \wedge \beta]\!]\rangle,$$

where $[\![id \wedge \beta]\!] = \sum_i \varepsilon_i \wedge [\![e_i, \beta]\!]$. This is a special case of the operator $w \mapsto id \wedge \cdot\, w := \sum_i \varepsilon_i \wedge (e_i \cdot w)$, defined on any representation W of $T^*M \oplus \mathfrak{co}(TM) \oplus TM$, independently of any choice of length scale. Note however, that it is *not* naturally defined on representations of the filtered Lie algebra $\mathfrak{so}(V)$.

We shall be particularly interested in the *Ricci contraction* $\partial\colon \wedge^2 T^*M \otimes \mathfrak{co}(TM) \to T^*M \otimes T^*M$ given by $\partial\colon R \mapsto ric$, with

$$ric_X(Y) = \sum_i [\![\varepsilon_i, R_{e_i,X}]\!](Y) = \sum_i \varepsilon_i(R_{e_i,X}Y),$$

and the *Ricci map* $r \mapsto [\![id \wedge r]\!]\colon T^*M \otimes T^*M \to \wedge^2 T^*M \otimes \mathfrak{co}(TM)$, where $[\![id \wedge r]\!]_{X,Y} = [\![X, r_Y]\!] - [\![Y, r_X]\!]$. As a special case of the above theory, the Ricci contraction is minus the adjoint of the Ricci map (using any fixed length scale ℓ). In particular, the kernel of the Ricci contraction and the image of the Ricci map (and vice versa) are orthogonal complements with respect to $\langle , \rangle$. Moreover, $\partial [\![id \wedge r]\!]$ is readily computed to be

$$(n-2)\, sym_0\, r + 2(n-1)\bigl(\tfrac{1}{n}\, tr_{\mathsf{c}}\, r\bigr)\mathsf{c} + \tfrac{n}{2}\, alt\, r \tag{2.13}$$

from which we see that the Ricci map is injective when $n \geqslant 3$, has kernel $S^2_0 T^*M$ when $n = 2$, and is zero for $n = 1$. (Here $(alt\, r)_X Y = r_X Y - r_Y X$.)

3. Weyl geometry

A popular approach in conformal geometry is to work with a riemannian metric in the conformal class. This amounts to fixing a length scale ℓ, or equivalently, in the context of conformal Cartan connections, a gauge σ. However, choosing a different metric leads to complicated transformation formulae, so we find it convenient to use a more general notion. For this, note that a length scale is parallel with respect to a unique connection on L.

Definition 3.1. A *Weyl derivative* on M is a connection on L. It induces a covariant derivative D on L^w for each $w \in \mathbb{R}$, whose curvature is a real 2-form wF^D, the *Faraday curvature*. If $F^D = 0$ then D is said to be *closed*: then there are local length scales ℓ with $D\ell = 0$; if such an ℓ exists globally then D is said to be *exact*. Any two Weyl derivatives differ by a 1-form so that Weyl derivatives are an affine space modelled on $\Omega^1(M, \mathbb{R})$.

In this section we relate Weyl derivatives to conformal Cartan connections and conformal metrics, cf. [25, 26, 27, 32, 40, 48, 80].

3A. Weyl structures.

Definition 3.2. Let $(V, \Lambda, \mathfrak{D})$ be a conformal Cartan connection. Then a *Weyl structure* is a null line subbundle $\hat\Lambda$ of V complementary to $\Lambda^\perp$ (*i.e.*, distinct from Λ).

Observe that $\Lambda^\perp \cap \hat\Lambda^\perp$ is a complementary subspace to Λ in $\Lambda^\perp$. Conversely given such a complement U, there is a unique null line subbundle $\hat\Lambda$ orthogonal to U and complementary to $\Lambda^\perp$. Thus a Weyl structure is given equivalently by a splitting $pr_{\hat\Lambda}$ of the inclusion $\Lambda \to \Lambda^\perp$. It follows that Weyl structures form an affine space modelled on the space of sections of $Hom(\Lambda^\perp/\Lambda, \Lambda) \cong \mathfrak{stab}(\Lambda)^\perp$, with $pr_{\hat\Lambda+\gamma}(v) = pr_{\hat\Lambda}(v) + \gamma\cdot v$. This affine structure has a gauge-theoretic interpretation that we wish to emphasise: $\hat\Lambda + \gamma = \exp(-\gamma)\hat\Lambda$.

A Weyl structure defines a covariant derivative $D = pr_{\hat\Lambda} \circ \mathfrak{D}|_\Lambda$ on Λ, and so a Weyl derivative via the usual identification $\Lambda \cong L^{-1}$ provided by the conformal structure. Conversely, given a Weyl derivative and so a covariant derivative D on Λ, we take U to be the image of $\mathfrak{D}^D i_\Lambda\colon TM \otimes \Lambda \to \Lambda^\perp$, where $i_\Lambda\colon \Lambda \to V$ is the inclusion and $\mathfrak{D}^D$ is the induced connection on $Hom(\varLambda,V)$: this is a complement to $\varLambda$ since $\pi\circ\mathfrak{D}^D i_\varLambda=-id_{TM\,\varLambda}$. Thus we have a bijection $D\mapsto\hat\varLambda^D$ between Weyl derivatives and Weyl structures which, in view of the soldering identification $T^*M\cong Hom(\varLambda^\perp/\varLambda,\varLambda)$ enunciated in (2.3), is easily seen to be affine: $\hat\varLambda^{D+\gamma}=\hat\varLambda^D+\gamma$.

Remark 3.3. There is yet another (equivalent) definition of Weyl structure that really gets to the Lie-theoretic heart of the matter. The decomposition $V=\varLambda\oplus U\oplus\hat\varLambda$ is equivalently given by an element $\varepsilon\in\mathfrak{so}(V)$ which acts by -1 on $\varLambda$, 0 on U and 1 on $\hat\varLambda$. It is easy to see that $\varepsilon\in\mathfrak{stab}(\varLambda)$ is a lift of the identity in $\mathfrak{co}(TM)\cong\mathfrak{stab}(\varLambda)/\,\mathfrak{stab}(\varLambda)^\perp$ and conversely any such lift splits the filtration of V. See [26] for more on this approach.

The decomposition $V=\varLambda\oplus U\oplus\hat\varLambda$ induced by a Weyl structure identifies V with V_{gr} so that we have an isomorphism $V\cong L^{-1}\oplus T^*M\,L\oplus L$. Hence we also obtain an isomorphism $\mathfrak{so}(V)\cong T^*M\oplus\mathfrak{co}(TM)\oplus TM$ between the Lie algebra bundles of section 2. (Note however, that different Weyl structures give rise to different isomorphisms!)

These decompositions (a reduction of the structure group of V to $CO(n)$) induce a decomposition of the connection $\mathfrak{D}$ into a T^*M-valued 1-form $r^{D,\mathfrak{D}}$, a $CO(n)$-connection $D^{\mathfrak{D}}$ (restricting to the Weyl derivative D on L and L^{-1} and a metric connection on $T^*M\,L$), and a TM-valued 1-form. This last is minus the soldering form which, with our identifications, is minus the identity. To summarize, if we write (σ,θ,ℓ) for the components of v, we have

$$\mathfrak{D}_X v=r^{D,\mathfrak{D}}_X\cdot v+D^{\mathfrak{D}}_X v-X\cdot v=\begin{bmatrix}D_X\sigma+r^{D,\mathfrak{D}}_X(\theta)\\ D^{\mathfrak{D}}_X\theta+r^{D,\mathfrak{D}}_X\ell-\sigma X\\ D_X\ell-\theta(X)\end{bmatrix}. \tag{3.1}$$

It is natural to ask how these components transform under a change $\hat\varLambda\mapsto\exp(-\gamma)\hat\varLambda$ of Weyl structure. We first examine how $\mathfrak{D}$ changes under gauge transformation by $\exp(\gamma)$: $(\exp\gamma\cdot\mathfrak{D})_X v=\exp(\gamma)\mathfrak{D}_X(\exp(-\gamma)v)$ differs from $\mathfrak{D}_X v$ by the right logarithmic derivative of the exponential map at γ, in the direction $\mathfrak{D}_X\gamma$, applied to v. Using the standard formula

$$d(R_{\exp(-\gamma)})_{\exp\gamma}\circ d\exp_\gamma(\chi)=F(ad\,\gamma)(\chi),\qquad F(t)=\tfrac{1}{t}(e^t-1)=1+\tfrac12 t+\cdots$$

for this logarithmic derivative (see [58]), we then obtain

$$\exp\gamma\cdot\mathfrak{D}=\mathfrak{D}-\mathfrak{D}\gamma-\tfrac12[\![\gamma,\pi\mathfrak{D}\gamma]\!], \tag{3.2}$$

since $\mathfrak{D}\gamma$ is in $\mathfrak{stab}(\varLambda)$, on which $ad\,\gamma$ is 2-step nilpotent, and hence $F(ad\,\gamma)(\mathfrak{D}_X\gamma)=\mathfrak{D}_X\gamma+\frac12[\gamma,\mathfrak{D}_X\gamma]$. We can expand $[\![\gamma,\pi\mathfrak{D}_X\gamma]\!]=-2\gamma(X)\gamma+\langle\gamma,\gamma\rangle X^\flat$ explicitly as needed.

This computation provides the required transformation formulae by switching from the active to passive view of gauge transformations: in the former, the connection is transformed but the ‘gauge’ (here a Weyl structure $\hat\varLambda\subset V$) is fixed; in the latter, the opposite process is performed. Since $\exp(\gamma)$ acts trivially on $\mathfrak{so}(V)_{\mathrm{gr}}$, we deduce that the components $r^{D,\mathfrak{D}}$ and $D^{\mathfrak{D}}$ transform as follows:

$$(D+\gamma)^{\mathfrak{D}}=D^{\exp\gamma\cdot\mathfrak{D}}=D^{\mathfrak{D}}-[\![\gamma,\cdot]\!] \tag{3.3}$$

$$r^{D+\gamma,\mathfrak{D}}=r^{D,\exp\gamma\cdot\mathfrak{D}}=r^{D,\mathfrak{D}}-D\gamma+\gamma\otimes\gamma-\tfrac12\langle\gamma,\gamma\rangle\mathsf{c}. \tag{3.4}$$

Curvature. We now compute the curvature of a conformal Cartan connection $\mathfrak{D}$ with respect to a Weyl structure $\hat\varLambda\subset V$. Using (3.1), *i.e.*, $\mathfrak{D}=r^{D,\mathfrak{D}}+D^{\mathfrak{D}}-id$, we have

$$R^{\mathfrak{D}}=d^{D^{\mathfrak{D}}}r^{D,\mathfrak{D}}+\;\;R^{D^{\mathfrak{D}}}-[\![id\wedge r^{D,\mathfrak{D}}]\!]\;\;-d^{D^{\mathfrak{D}}}id. \tag{3.5}$$

The last component, $-d^{D^{\mathfrak{D}}}id$, *i.e.*, minus the torsion of $D^{\mathfrak{D}}$, is equal to $p(R^{\mathfrak{D}})$, hence independent of the Weyl structure, so we call it the *torsion* of $\mathfrak{D}$. We refer to $W^{\mathfrak{D}}:=R^{D^{\mathfrak{D}}}-[\![id\wedge r^{D,\mathfrak{D}}]\!]$ as the *Weyl curvature* of $\mathfrak{D}$ (with respect to D or $\hat\varLambda$); if the torsion of $\mathfrak{D}$ is zero, $W^{\mathfrak{D}}=\pi(R^{\mathfrak{D}})$, independently of $\hat\varLambda$. Finally $C^{\mathfrak{D},D}:=d^{D^{\mathfrak{D}}}r^{D,\mathfrak{D}}$ is called the

Cotton–York curvature of $\mathfrak{D}$ (with respect to $\hat{\Lambda}$); it is the full curvature (independent of $\hat{\Lambda}$) if both the torsion and Weyl curvature vanish.

We shall be interested in the following natural conditions on the curvature of $\mathfrak{D}$.

Definition 3.4. Let $(V, \Lambda, \mathfrak{D})$ be a conformal Cartan connection. Then

- $\mathfrak{D}$ is *torsion-free* if $p(R^{\mathfrak{D}}) = 0$, *i.e.*, $R^{\mathfrak{D}} \in \Omega^2(M, \mathfrak{stab}(\Lambda))$, or equivalently $R^{\mathfrak{D}}\Lambda \subseteq \Lambda$;
- $\mathfrak{D}$ is *strongly torsion-free* if $R^{\mathfrak{D}}|_{\Lambda} = 0$;
- $\mathfrak{D}$ is *normal* if $\partial R^{\mathfrak{D}} = 0$.

In fact, a normal conformal Cartan connection is strongly torsion-free. For this, note that $\Gamma \mapsto [\![id \wedge \Gamma]\!]$ defines an isomorphism $T^*M \otimes \mathfrak{so}(TM) \to \wedge^2 T^*M \otimes TM$ (this amounts to the familiar algebra that determines a metric connection from its torsion). Thus $[\![id \wedge (\cdot)]\!]$ surjects so that $\partial \colon \wedge^2 T^*M \otimes TM \to T^*M \otimes \mathfrak{co}(TM)$ injects. Furthermore, it is easy to see that also $\partial \colon \wedge^2 T^*M \otimes \mathfrak{co}(T\Sigma) \to T^*M \otimes T^*M$ injects on $\wedge^2 T^*M \otimes \langle id_{TM} \rangle$.

If $\mathfrak{D}$ is torsion-free, $\pi(R^{\mathfrak{D}}) = W^{\mathfrak{D}}$. Thus $\mathfrak{D}$ is normal iff it is torsion-free and $\partial W^{\mathfrak{D}} = 0$.

We can now single out the conformal Cartan connections that will be of interest to us.

Definition 3.5. A *conformal Cartan geometry* is a conformal Cartan connection $(V, \Lambda, \mathfrak{D})$ with $\mathfrak{D}$ strongly torsion-free. It is said to be *normal* if $\mathfrak{D}$ is normal.

3B. Weyl connections.

Definition 3.6. Let c be a conformal metric on M. Then a *Weyl connection* is a torsion-free connection D on TM which is *conformal* in the sense that $D\mathsf{c} = 0$.

If ℓ is a length scale, then the Levi-Civita connection D^g of the metric $g = \ell^{-2}\mathsf{c}$ is a Weyl connection: since g is parallel and ℓ is parallel (with respect to the induced connection on L), so is $\mathsf{c} = \ell^2 g$. Any Weyl connection induces a Weyl derivative and Levi-Civita connections induce exact Weyl derivatives. The existence and uniqueness of the Levi-Civita connection associated to a length scale admits the following generalization [40, 48, 80].

Proposition 3.7. *The affine map sending a connection on TM to the induced covariant derivative on L induces a bijection between Weyl connections and Weyl derivatives.*

Indeed, if D is a given Weyl connection (for example, the Levi-Civita of a compatible metric), then any other such is of the form $D + \Gamma$ where Γ is a $\mathfrak{co}(TM)$-valued 1-form with $[\![id \wedge \Gamma]\!] = 0$. A standard argument shows that all such Γ are of the form $\Gamma = [\![id, \gamma]\!]$ for a section γ of T^*M and then the corresponding Weyl derivative on L is $D + \frac{1}{n}\, tr\, \Gamma = D + \gamma$.

One of the reasons why Weyl derivatives can be more convenient than length scales is that they form an affine space, modelled on $\Omega^1(M, \mathbb{R})$. When we construct a geometric object using a Weyl derivative D, we often want to know how it depends upon this choice. We view such an object as a function $F(D)$ and say that it is conformally-invariant if and only if this function is constant (*i.e.*, independent of D). For this, it is often helpful to use the fundamental theorem of calculus: $F(D)$ is constant iff its derivative with respect to D is zero. This amounts to checking that $\partial_\gamma F(D) = 0$ for all Weyl derivatives D and all 1-forms γ, *i.e.*, that for any one parameter family $D(t)$ with

$$D(0) = D \quad \text{and} \quad \frac{d}{dt} D(t)\Big|_{t=0} = \gamma$$

we have

$$\partial_\gamma F(D) := \frac{d}{dt} F(D(t))\Big|_{t=0} = 0.$$

More generally, Taylor's theorem implies that if $\partial^{k+1}_{\gamma,\ldots\gamma} F(D) = 0$ for all D and γ then

$$F(D + \gamma) = F(D) + \partial_\gamma F(D) + \tfrac{1}{2}\partial^2_{\gamma,\gamma} F(D) + \cdots + \tfrac{1}{k!}\partial^k_{\gamma,\ldots\gamma} F(D).$$

This approach often simplifies the calculation of nonlinear terms.

For example, for any section ℓ of L, $\partial_\gamma D_X \ell = \gamma(X)\ell$ and for any vector field Y, $\partial_\gamma D_X Y = [\![X,\gamma]\!]\cdot Y$. Similarly, if s is a section of a bundle associated to the conformal frame bundle, then $\partial_\gamma D_X s = [\![X,\gamma]\!]\cdot s$ where the dot denotes the natural action of $\mathfrak{co}(TM)$. This expresses the obvious fact that all first covariant derivatives depend affinely on D.

For the second derivative, we obtain

$$\begin{aligned}\partial_\gamma D^2_{X,Y}s &= [\![X,\gamma]\!]\cdot D_Y s - D_{[\![X,\gamma]\!]\cdot Y}s + D_X\big([\![Y,\gamma]\!]\cdot s\big) - [\![D_X Y,\gamma]\!]\cdot s\\ &= [\![Y, D_X\gamma]\!]\cdot s + [\![X,\gamma]\!]\cdot D_Y s + [\![Y,\gamma]\!]\cdot D_X s - D_{[\![X,\gamma]\!]\cdot Y}s\end{aligned}$$

In particular, for any section ℓ of L we have

$$\partial_\gamma D^2_{X,Y}\ell = (D_X\gamma)(Y)\ell + \langle X,Y\rangle\langle\gamma, D\ell\rangle. \tag{3.6}$$

Curvature. We now apply the transformation formula for the second derivative to the curvature R^D of the Weyl connection D, which is the $\mathfrak{co}(TM)$-valued 2-form such that $D^2_{X,Y}s - D^2_{Y,X}s = R^D_{X,Y}\cdot s$ for any section s of any bundle associated to the conformal frame bundle. We deduce that

$$\partial_\gamma R^D_{X,Y} = -[\![X, D_Y\gamma]\!] + [\![Y, D_X\gamma]\!] = -[\![id\wedge D\gamma]\!]_{X,Y}, \tag{3.7}$$

i.e., $\partial_\gamma R^D = -[\![id\wedge D\gamma]\!]$. Hence the part of R^D orthogonal to the image of the Ricci map is conformally-invariant: this is the *Weyl curvature* W of the conformal metric, and it has vanishing Ricci contraction: W itself vanishes for $n<4$ since the Ricci map is then surjective. On the other hand, for $n \geqslant 3$, the Ricci map is injective, and so we can write

$$R^D = W + [\![id\wedge r^D]\!], \tag{3.8}$$

with r^D uniquely determined by D and, in view of (3.7),

$$\partial_\gamma r^D = -D\gamma. \tag{3.9}$$

Let ric^D be the Ricci contraction ∂R^D of R^D, so that, by (2.13), we have

$$ric^D = (n-2)\,sym_0\, r^D + 2(n-1)\big(\tfrac{1}{n}\,tr_{\mathsf{c}}\, r^D\big)\mathsf{c} + \tfrac{n}{2}\,alt\, r^D. \tag{3.10}$$

A simple application of the Bianchi identity $[\![id\wedge R^D]\!]=0$ gives $alt\, ric^D = -\,tr\, R^D = -nF^D$ and hence we deduce r^D is the *normalized Ricci curvature*, defined by

$$r^D = r^D_0 + \tfrac{1}{n}s^D\mathsf{c} - \tfrac{1}{2}F^D,$$

where we set

$$r^D_0 = \frac{1}{n-2}\,sym_0\, ric^D, \qquad s^D = \frac{1}{2(n-1)}\,tr_{\mathsf{c}}\, ric^D.$$

When $n=3$, $d^D r^D$ is independent of D, and known as the *Cotton–York curvature* of c.

When $n=2$, s^D is still defined and the Ricci map is still injective on L^{-2} so that

$$\partial_\gamma s^D = -\,tr_{\mathsf{c}}\, D\gamma. \tag{3.11}$$

We wish to bring the cases $n=1$ and $n=2$ in to line with the higher dimensional case. One way to do this uses a pair of differential operators, which we call a *Möbius structure.*

4. Möbius manifolds

4A. The differential lift and Möbius operators. A conformal Cartan connection $(V,\Lambda,\mathfrak{D})$ induces more structure on M than just a conformal metric. This additional structure arises from the following differential operator, cf. [25, 35].

Proposition 4.1. *There is a unique linear map $j^{\mathfrak{D}}\colon C^\infty(M,L)\to C^\infty(M,V)$ such that*

- *$p(j^{\mathfrak{D}}\ell)=\ell$, i.e., $j^{\mathfrak{D}}\ell$ is a lift of ℓ with respect to the projection $p\colon V\to L$;*
- *$\partial\mathfrak{D}(j^{\mathfrak{D}}\ell) := \sum_i \varepsilon_i\cdot\mathfrak{D}_{e_i}j^{\mathfrak{D}}\ell = 0$, i.e., $\mathfrak{D}j^{\mathfrak{D}}\ell$ is a section of $Z_1(T^*M,V)$.*

The components of $j^{\mathfrak{D}}\ell$, relative to a Weyl structure with Weyl derivative D, are

$$(\tfrac{1}{n}\operatorname{tr}_{\mathsf{c}}\psi, D\ell, \ell) \quad \textit{where} \quad \psi = D^{\mathfrak{D}}D\ell + \mathrm{r}^{D,\mathfrak{D}}\ell. \tag{4.1}$$

In particular $j^{\mathfrak{D}}$ is a second order linear differential operator. Furthermore $\pi\mathfrak{D}j^{\mathfrak{D}}\ell = \psi - \frac{1}{n}(tr_{\mathsf{c}}\psi)\mathsf{c} \in T^*M \otimes T^*M\,L$.

Proof. With respect to any Weyl structure, a lift of ℓ has components (σ, θ, ℓ) and, using (2.12) and (3.1), we readily compute

$$\sum_i \varepsilon_i \cdot \mathfrak{D}_{e_i}\begin{bmatrix}\sigma\\ \theta\\ \ell\end{bmatrix} = \begin{bmatrix}-n\sigma + \operatorname{tr}_{\mathsf{c}}(D^{\mathfrak{D}}\theta + \mathrm{r}^{D,\mathfrak{D}}\ell)\\ D\ell - \theta\\ 0\end{bmatrix}$$

which vanishes precisely when $\theta = D\ell$ and $\sigma = \frac{1}{n}\operatorname{tr}_{\mathsf{c}}\psi$. The last part follows from (3.1). □

We refer to $j^{\mathfrak{D}}\ell$ as the *differential lift* of ℓ. Using this, we now define two second order differential operators on L, one linear, one quadratic.

Definition 4.2. Let $(V, \Lambda, \mathfrak{D})$ be a conformal Cartan connection. Define $\mathcal{H}^{\mathfrak{D}}\colon \mathrm{C}^\infty(M, L) \to \mathrm{C}^\infty(M, S^2_0T^*M\,L)$ and $\mathcal{S}^{\mathfrak{D}}\colon \mathrm{C}^\infty(M, L) \to \mathrm{C}^\infty(M, \mathbb{R})$ by

$$\mathcal{H}^{\mathfrak{D}}\ell = sym\,\pi\mathfrak{D}j^{\mathfrak{D}}\ell, \qquad \mathcal{S}^{\mathfrak{D}}\ell = (j^{\mathfrak{D}}\ell, j^{\mathfrak{D}}\ell).$$

Using (4.1) and (3.1), we obtain the explicit formulae

$$\begin{aligned}\mathcal{H}^{\mathfrak{D}}\ell &= sym_0\big(D^{\mathfrak{D}}D\ell + \mathrm{r}^{D,\mathfrak{D}}\ell\big)\\ \mathcal{S}^{\mathfrak{D}}\ell &= \langle D\ell, D\ell\rangle - \tfrac{2}{n}\ell\operatorname{tr}_{\mathsf{c}}\big(D^{\mathfrak{D}}D\ell + \mathrm{r}^{D,\mathfrak{D}}\ell\big)\end{aligned} \tag{4.2}$$

relative to a Weyl structure with Weyl derivative D. When $\mathfrak{D}$ is torsion-free, $D^{\mathfrak{D}}$ is the Weyl connection on TM induced by D. (Also in this case, the skew part of $\pi\mathfrak{D}j^{\mathfrak{D}}\ell$ is $F^D + alt\,\mathrm{r}^{D,\mathfrak{D}} = \frac{1}{n}\,tr\,W^{\mathfrak{D}}$ so that $\mathcal{H}^{\mathfrak{D}}\ell = \pi\mathfrak{D}j^{\mathfrak{D}}\ell$ precisely when $\mathfrak{D}$ is strongly torsion-free.)

Our goal, achieved in the next section, is to recover the conformal Cartan connection (up to isomorphism) from the underlying conformal metric and these two operators.

The differential lift in general. For later use we remark that the above is a special case of a more general construction [25, 35], which, for any representation W of $\mathfrak{so}(V)$, provides a differential lift of any section of a Lie algebra homology bundle $H_k(T^*M, W)$ to give a representative section of $Z_k(T^*M, W)$. The key ingredient in this construction is the first order quabla operator $\square_{\mathfrak{D}} := \partial \circ d^{\mathfrak{D}} + d^{\mathfrak{D}} \circ \partial$ on $C_k(T^*M, W)$ and the following observation.

Proposition 4.3. [25] *$\square_{\mathfrak{D}}$ is invertible on the image of ∂.*

This result is not difficult: using a Weyl structure, one shows that $\square_{\mathfrak{D}}$ differs from an invertible algebraic operator (Kostant's $\square$ [59]) by a nilpotent first order operator; the inverse is then given by a geometric series, and is a differential operator of finite order.

We then define $\Pi = id - \square_{\mathfrak{D}}^{-1} \circ \partial \circ d^{\mathfrak{D}} - d^{\mathfrak{D}} \circ \square_{\mathfrak{D}}^{-1} \circ \partial$, and note the following properties:

- Π vanishes on the image of ∂;
- Π maps into the kernel of ∂;
- $\Pi|_{ker\,\partial} \cong id \bmod im\,\partial$.

It follows that there is a canonical differential lift $j^{\mathfrak{D}}\colon [\alpha] \mapsto \Pi\alpha$ from Lie algebra homology to cycles [35]. According to [25], $j^{\mathfrak{D}}[\alpha]$ is characterized by the following properties: $\partial j^{\mathfrak{D}}[\alpha] = 0$, $\big[j^{\mathfrak{D}}[\alpha]\big] = [\alpha]$ and $\partial d^{\mathfrak{D}} j^{\mathfrak{D}}[\alpha] = 0$. This generalizes the characterization of the differential lift in Proposition 4.1.

The differential operators $\mathcal{H}^{\mathfrak{D}}, \mathcal{S}^{\mathfrak{D}}$ also have a homological interpretation and generalization. For $n \geqslant 2$, $\mathcal{H}^{\mathfrak{D}}$ is the Bernstein–Gelfand–Gelfand operator $[\alpha] \mapsto [\mathfrak{D}\Pi\alpha]$ from $\mathrm{C}^\infty(M, H_0(T^*M, V))$ to $\mathrm{C}^\infty(M, H_1(T^*M, V))$, which is part of the general theory of Čap–Slovák–Souček [35], whereas (for all n) $\mathcal{S}^{\mathfrak{D}}(\ell) = \ell \sqcup \ell$, $\sqcup\colon \mathrm{C}^\infty(M, H_0(T^*M, V)) \times \mathrm{C}^\infty(M, H_0(T^*M, V)) \to \mathrm{C}^\infty(M, \mathbb{R})$ being the bilinear operator of Calderbank–Diemer [25] induced by the metric pairing $V \times V \to \mathbb{R}$. We shall discuss this further in section 10.

4B. Möbius structures. We are now going to define a class of linear and quadratic differential operators on *any* manifold with a conformal metric. First, for ℓ a section of L and D a Weyl connection, define:

$$\begin{aligned} \mathcal{H}^D\ell &= \mathrm{sym}_0\, D^2\ell && \in C^\infty(M, S^2_0T^*M\, L),\\ \mathcal{S}^D\ell &= \langle D\ell, D\ell\rangle - \tfrac{2}{n}\ell\,\Delta^D\ell && \in C^\infty(M,\mathbb{R}), \end{aligned} \tag{4.3}$$

where $\Delta^D\ell = tr_{\mathsf{c}}\, D^2\ell$, a section of L^{-1}. A fundamental feature of $\mathcal{H}^D$ and $\mathcal{S}^D$ is that replacing D by $D+\gamma$ only alters them by zero-order terms. Indeed from (3.6), we have

$$\begin{aligned} &\partial_\gamma\mathcal{H}^D(\ell) = (\mathrm{sym}_0\, D\gamma)\ell,\\ &\partial_\gamma\mathcal{S}^D(\ell) = 2\langle\gamma\ell, D\ell\rangle - \tfrac{2}{n}n\ell\langle\gamma, D\ell\rangle - \tfrac{2}{n}\ell(tr_{\mathsf{c}}\, D\gamma)\ell = -\tfrac{2}{n}(tr_{\mathsf{c}}\, D\gamma)\ell^2. \end{aligned} \tag{4.4}$$

This observation will allow us to make a key definition. Before this, though, we compare these variations with (3.9) and (3.11) to deduce that the formulae

$$\begin{aligned} \mathcal{H}^{\mathsf{c}}\ell &= \mathcal{H}^D\ell + \mathrm{r}_0^D\ell && (n \geqslant 3)\\ \mathcal{S}^{\mathsf{c}}\ell &= \mathcal{S}^D\ell - \tfrac{2}{n}s^D\ell^2 && (n \geqslant 2) \end{aligned}$$

define conformally-invariant second order differential operators, one linear, one quadratic.

Remark 4.4. These operators, when they exist, may be defined in a manifestly invariant way on positive sections of L by associating to such a length scale ℓ the normalized Ricci curvature r^D of the unique Weyl connection D with $D\ell = 0$ (this is the Levi-Civita connection of the metric $g = \ell^{-2}\mathsf{c}$, and r^D is the normalized Ricci curvature of g). Then, by definition, since $D\ell = 0$ we have $\mathcal{H}^{\mathsf{c}}\ell = \mathrm{r}_0^D\ell$ and $\mathcal{S}^{\mathsf{c}}\ell = -\frac{2}{n}s^D\ell^2$, cf. [48]. Hence positive solutions of $\mathcal{H}^{\mathsf{c}}$ are sometimes called *Einstein gauges* and $\mathcal{S}^{\mathsf{c}}$ is sometimes called the *scalar curvature metric*. $\mathcal{H}^{\mathsf{c}}$ itself is often called the *conformal* (*tracefree*) *Hessian*.

Remark 4.5. We also note that differentiating (4.4) again, using $\partial_\gamma D\gamma = -2\gamma\otimes_0\gamma + \frac{n-2}{n}\langle\gamma,\gamma\rangle\mathsf{c}$, gives $\partial^2_{\gamma,\gamma}\mathcal{H}^D(\ell) = -2\gamma\otimes_0\gamma\,\ell$ and $\partial^2_{\gamma,\gamma}\mathcal{S}^D(\ell) = -\frac{2}{n}(n-2)\langle\gamma,\gamma\rangle\ell^2$, which are both independent of D so that $\mathcal{H}^D$ and $\mathcal{S}^D$ depend quadratically on D:

$$\begin{aligned} \mathcal{H}^{D+\gamma}\ell &= \mathcal{H}^D\ell + (\mathrm{sym}_0\, D\gamma - \gamma\otimes_0\gamma)\ell,\\ \mathcal{S}^{D+\gamma}\ell &= \mathcal{S}^D\ell - \tfrac{1}{n}\big(2\,\mathrm{tr}_{\mathsf{c}}\, D\gamma + (n-2)\langle\gamma,\gamma\rangle\big)\ell^2. \end{aligned}$$

We are now motivated to make our definition, cf. [24, 66].

Definition 4.6. A *Möbius structure* on a conformal manifold (M,c) is a pair $\mathcal{M} = (\mathcal{H},\mathcal{S})$ of differential operators such that

- $\mathcal{H}\colon \mathrm{C}^\infty(M,L) \to \mathrm{C}^\infty(M, S^2_0T^*M\, L)$ is a linear differential operator with $\mathcal{H}\,\ell - \mathcal{H}^D\ell$ zero order in ℓ for some (hence any) Weyl connection D.
- $\mathcal{S}\colon \mathrm{C}^\infty(M,L) \to \mathrm{C}^\infty(M,\mathbb{R})$ is a quadratic differential operator with $\mathcal{S}\,\ell - \mathcal{S}^D\ell$ zero order in ℓ for some (hence any) Weyl connection D.

Let us summarize two basic motivations leading to this definition.

Proposition 4.7. *First, if c is a conformal metric on a manifold M of dimension $n \geqslant 3$, then there is a canonical Möbius structure $\mathcal{M}^{\mathsf{c}} := (\mathcal{H}^{\mathsf{c}},\mathcal{S}^{\mathsf{c}})$. on (M,c).*

Second, if $(V,\varLambda,\mathfrak{D})$ is a torsion-free conformal Cartan connection over a manifold M of any dimension, then $\mathcal{M}^{\mathfrak{D}} := (\mathcal{H}^{\mathfrak{D}},\mathcal{S}^{\mathfrak{D}})$ is a Möbius structure on M.

Möbius structures form an affine space modelled on $C^\infty(M, S^2T^*M)$. Indeed, for Q a section of S^2T^*M, let Q_0 be its tracefree part, $tr_{\mathsf{c}}\, Q$ its trace, and define, for $\mathcal{M} = (\mathcal{H}, \mathcal{S})$,

$$\mathcal{M} + Q := \big(\ell \mapsto \mathcal{H}\,\ell + Q_0\ell, \ell \mapsto \mathcal{S}\,\ell - \tfrac{2}{n}(tr_{\mathsf{c}}\, Q)\ell^2\big). \tag{4.5}$$

In particular if we define $\mathcal{M}^D := (\mathcal{H}^D, \mathcal{S}^D)$, then for any Möbius structure $\mathcal{M}$, $\mathcal{M} - \mathcal{M}^D$ satisfies (for fixed $\mathcal{M}$):

$$\partial_\gamma(\mathcal{M} - \mathcal{M}^D) = -\, sym\, D\gamma, \tag{4.6}$$

which is the origin of the choice of coefficient of $tr_{\mathsf{c}}\, Q$ in our definition of $\mathcal{M} + Q$. By adding $-\frac{1}{2}F^D$ to this, we obtain a generalized notion of normalized Ricci curvature and an associated decomposition of R^D.

Definition 4.8. Let $\mathsf{c}, \mathcal{M}$ be a Möbius structure. Then the *Möbius Ricci curvature* of a Weyl derivative D with respect to $\mathcal{M}$ is defined by $r^{D,\mathcal{M}} := \mathcal{M} - \mathcal{M}^D - \frac{1}{2}F^D$. The *Weyl curvature* $W^{\mathcal{M}}$ of $\mathcal{M}$ is given by $W^{\mathcal{M}} := R^D - [\![id \wedge r^{D,\mathcal{M}}]\!]$ for any Weyl derivative D. The *Cotton–York curvature* $C^{\mathcal{M},D}$ of $\mathcal{M}$ with respect to D is given by $C^{\mathcal{M},D} := d^D r^{D,\mathcal{M}}$.

Note that $W^{\mathcal{M}}$ is independent of the choice of D and thus is an invariant of the Möbius structure: indeed, using (4.6) and $\partial_\gamma F^D = d\gamma$, we have $\partial_\gamma r^{D,\mathcal{M}} = -D\gamma$, which cancels with (3.7). (We remark that we can differentiate once more to find

$$r^{D+\gamma,\mathcal{M}} = r^{D,\mathcal{M}} - D\gamma + \gamma \otimes \gamma - \tfrac{1}{2}\langle \gamma, \gamma \rangle \mathsf{c}, \tag{4.7}$$

so that $r^{D,\mathcal{M}}$ has the same transformation law (3.4) as $r^{D,\mathfrak{D}}$.)

We now fix a distinguished class of Möbius structures by requiring $W^{\mathcal{M}}$ to coincide with the Weyl curvature W of the underlying conformal metric. In general, we have $R^D = W^{\mathcal{M}} + [\![id \wedge r^{D,\mathcal{M}}]\!]$, with the second term in the image of the Ricci map, so W is the component of $W^{\mathcal{M}}$ orthogonal to this image, *i.e.*, in $\ker \partial$. Hence

$$W^{\mathcal{M}} = W \quad \Leftrightarrow \quad W^{\mathcal{M}} \text{ is in the kernel of the Ricci contraction, } i.e., \partial W^{\mathcal{M}} = 0. \tag{4.8}$$

When $n \geqslant 3$, this amounts to requiring that the Möbius Ricci curvature with respect to $\mathcal{M}$ of each Weyl connection D coincides with the normalized Ricci curvature: $r^{D,\mathcal{M}} = r^D$. More generally, by (2.13) and (3.10), we see that this is equivalent to requiring that $sym_0\, r^{D,\mathcal{M}} = sym_0\, r^D$ for $n \geqslant 3$ and that $\frac{1}{n}\, tr_{\mathsf{c}}\, r^{D,\mathcal{M}} = s^D$ for $n \geqslant 2$ (note that $alt\, r^{D,\mathcal{M}} = alt\, r^D$ by definition). In other words, when $n \geqslant 3$, a conformal metric alone is sufficient to fix a canonical choice of Möbius structure; when $n = 2$, we still have a canonical choice $\mathcal{S}^{\mathsf{c}}$ of quadratic operator but lack the linear operator; when $n = 1$, the linear operator necessarily vanishes but we lack the quadratic operator. To obtain a uniform approach, we impose the missing data as extra structure as follows.

Definition 4.9. A *conformal Möbius structure* on a 1-manifold M is a second order quadratic differential operator $\mathcal{S}^{\mathsf{c}}$ from L to $\mathbb{R}$ such that $\mathcal{S}^{\mathsf{c}} - \mathcal{S}^D$ is zero order (for any D); c is the canonical section of $S^2T^*M\, L^2$ and $\mathcal{H}^{\mathsf{c}} = 0$.

A *conformal Möbius structure* on a 2-manifold M is a conformal metric, together with a second order linear differential operator $\mathcal{H}^{\mathsf{c}}$ from L to $S^2_0T^*M\, L$ such that $\mathcal{H}^{\mathsf{c}} - \mathcal{H}^D$ is zero order (for any D); $\mathcal{S}^{\mathsf{c}}$ is the canonical Möbius operator of c.

In higher dimensions a *conformal Möbius structure* is just a conformal metric; $\mathcal{H}^{\mathsf{c}}, \mathcal{S}^{\mathsf{c}}$ are the canonical Möbius operators of c.

A *Möbius manifold* is a manifold equipped with a conformal Möbius structure $(\mathsf{c}, \mathcal{H}^{\mathsf{c}}, \mathcal{S}^{\mathsf{c}})$.

We now summarize the situation.

Proposition 4.10. $(\mathsf{c}, \mathcal{M})$ *is a conformal Möbius structure if and only if* $\partial W^{\mathcal{M}} = 0$.

Remark 4.11. In the presence of a conformal Möbius structure, we define the normalized Ricci curvature of a Weyl derivative D to be $r^{D,\mathcal{M}^c}$. When $n=2$, this provides a hitherto unavailable tracefree part of the normalized Ricci curvature. This idea was exploited in [24] to do Einstein–Weyl geometry on Möbius 2-manifolds.

5. The conformal equivalence problem

It is our contention that conformal Cartan geometries are the same as Möbius structures and that normal conformal Cartan geometries are the same as conformal Möbius structures. For this, a common setting is provided by the bundle J^2L of 2-jets of sections of L.

A Möbius structure $\mathcal{M} = (\mathcal{H}, \mathcal{S})$ is defined by a linear bundle map $\mathcal{H}\colon J^2L \to S^2_0T^*M\, L$ and a quadratic form $\mathcal{S}\colon J^2L \to \mathbb{R}$. The kernel of $\mathcal{H}$ is a rank $n+2$ subbundle $\mathcal{V}$ of J^2L, on which we have a quadratic form by restricting $\mathcal{S}$. We easily see that this polarizes to give a nondegenerate inner product of signature $(n+1,1)$ on $\mathcal{V}$ and that the line L^{-1} of pure trace elements of $S^2T^*M\, L$ is a null line subbundle. We shall exploit the jet structure of J^2L to find a canonical choice of strongly torsion-free conformal Cartan connection on $\mathcal{V}$.

Conversely, a conformal Cartan geometry $(V, \Lambda, \mathfrak{D})$ gives rise to a Möbius structure $\mathcal{M}^{\mathfrak{D}} = (\mathcal{H}^{\mathfrak{D}}, \mathcal{S}^{\mathfrak{D}})$ and so we obtain a subbundle $\mathcal{V}$ of J^2L with a conformal Cartan connection. We shall see that the differential lift, viewed as a bundle map $J^2L \to V$, restricts to give an isomorphism from $\mathcal{V}$ to V intertwining the conformal Cartan connections.

5.1. The jet derivative and Möbius bundles. Recall (see [58] for further details) that for any vector bundle E over M, the *k-jet bundle* J^kE is the bundle whose fibre at x is $J^k_xE := \mathrm{C}^\infty(M,E)/\mathcal{I}^k_x$ where $\mathcal{I}^k_x$ is the subspace of sections which vanish to order k at x. Thus J^k_xE may be regarded as the space of kth order Taylor series, at x, of sections of E, and there is a kth order differential operator $j^k\colon \mathrm{C}^\infty(M,E) \to \mathrm{C}^\infty(M,J^kE)$ sending a section s to its equivalence class j^k_xs (the *k-jet* or kth order Taylor series) at each $x \in M$.

Since $\mathcal{I}^k \subset \mathcal{I}^{k-1}$, there is, for each k, a natural surjective bundle map $\pi_k = \pi^E_k\colon J^kE \to J^{k-1}E$, whose kernel is the bundle of kth derivatives $S^kT^*M \otimes E$. Note that $\pi_k \circ j^k = j^{k-1}$ and j^0 identifies E with J^0E.

Definition 5.1. The (*semiholonomic*) *jet derivative* on J^kE is the first order differential operator $d_k\colon \mathrm{C}^\infty(M,J^kE) \to \Omega^1(M,J^{k-1}E)$ defined by $d_ks = j^1(\pi_ks) - s$.

This requires a brief explanation: $j^1(\pi_ks)$ is a section of $J^1(J^{k-1}E)$. On the other hand, J^kE is a subbundle of $J^1(J^{k-1}E)$: the inclusion sends j^k_xs to $j^1_xj^{k-1}s$. The difference $j^1(\pi_ks) - s$ lies in the kernel of $\pi_1^{J^{k-1}E}$ and hence defines a section of $T^*M \otimes J^{k-1}E$.

The jet derivative is thus given by the difference between formal and actual differentiation: it measures the failure of a kth order Taylor series to be the first derivative of the $(k-1)$st order Taylor series obtained by truncation; note $d_k \circ j^k = 0$. Our main use for it is to distinguish a preferred class of connections on subbundles of jet bundles.

Definition 5.2. Let F be a subbundle of J^kE, let d and π be the restrictions to F of the jet derivative and the projection $J^kE \to J^{k-1}E$. Following [48], a connection ∇ on F is said to be *symmetric* if $\pi \circ \nabla = d$. The space of symmetric connections, if it is nonempty, is an affine space modelled on $\Omega^1(M, \mathit{Hom}(F, \ker\pi))$.

We shall only need this formalism on J^2L, where the jet derivative is a differential operator $\mathrm{C}^\infty(M,J^2L) \to \Omega^1(M,J^1L)$. We next define the subbundles we shall use.

Definition 5.3. A *Möbius bundle* on a manifold M is a subbundle $\mathcal{V}$ of J^2L with a nondegenerate metric such that $\ker\pi \cap V \subset S^2T^*M \otimes L$ is a null line subbundle.

The following result explains the canonical nature of our forthcoming constructions.

Proposition 5.4. [48] *Let $\mathcal{V}$ be a Möbius bundle on M. Then there is at most one symmetric metric connection on $\mathcal{V}$.*

Proof. We shall show that $Hom(\mathcal{V}, ker\,\pi)\cap\mathfrak{so}(\mathcal{V}) = 0$. Indeed a skew map $\mathcal{V} \to ker\,\pi$ must factor through the projection $\mathcal{V} \to \mathcal{V}/\,ker\,\pi^{\perp}$. However, $ker\,\pi$ and $\mathcal{V}/\,ker\,\pi^{\perp}$ are one dimensional, and the maps $\mathcal{V} \to \mathcal{V}/\,ker\,\pi^{\perp} \to ker\,\pi \to \mathcal{V}$ are symmetric, not skew. □

5.2. Möbius structures induce conformal Cartan geometries. Let $\mathcal{M} = (\mathcal{H}, \mathcal{S})$ be a Möbius structure and define $\mathcal{V} = ker\,\mathcal{H} \subset J^2L$. We wish to show that $\mathcal{V}$ is a Möbius bundle and that there is a symmetric metric connection on $\mathcal{V}$.

In order to do this, we use a Weyl derivative D to decompose J^2L compatibly with $\mathcal{M}$: given $\mathcal{M}, D$, identify J^2L with $S^2T^*M\,L \oplus T^*M\,L \oplus L$ by sending $j^2\ell$ to (ψ, θ, ℓ) where

$$\psi = D^2\ell + r^{D,\mathcal{M}}\ell, \qquad \theta = D\ell.$$

This induces an identification of J^1L with $T^*M\,L \oplus L$ sending $j^1\ell$ to $(D\ell, \ell)$. Observe that ψ *is* symmetric, since the skew part of $r^{D,\mathcal{M}}$ is $-\frac{1}{2}F^D$ so that $\psi = sym(D^2\ell + r^{D,\mathcal{M}}\ell)$.

We have $\mathcal{H}(\psi, \theta, \ell) = \psi_0$ (the tracefree part of ψ) so that $\mathcal{V}$ is identified with

$$\{(\psi, \theta, \ell) : \psi = \sigma\mathsf{c} \text{ for } \sigma \in L^{-1}\}.$$

We thus have an isomorphism $L^{-1} \oplus T^*M\,L \oplus L \cong \mathcal{V}$ sending (σ, θ, ℓ) to $(\sigma\mathsf{c}, \theta, \ell) \in J^2L$.

Now $\mathcal{S}(\psi, \theta, \ell) = \langle\theta, \theta\rangle - \frac{2}{n}(tr_{\mathsf{c}}\,\psi)\ell$ so that on $\mathcal{V}$ we have

$$\mathcal{S}(\sigma, \theta, \ell) = \langle\theta, \theta\rangle - 2\sigma\ell$$

which is a metric of signature $(n + 1, 1)$ for which L^{-1} is null. Hence $\mathcal{V}$ is a Möbius bundle. Furthermore, we have identified $\mathcal{V}$ with $L^{-1} \oplus T^*M\,L \oplus L$ and so we are in the algebraic setting of §2B. In particular, the Lie bracket on $\mathfrak{so}(\mathcal{V})$ is identified, under this decomposition, with the bracket $[\![\cdot\,, \cdot]\!]$ defined there.

The jet derivative on J^2L reads in our components

$$d_2(\psi, \theta, \ell) = (D\theta + r^{D,\mathcal{M}}\ell - \psi, D\ell - \theta) \tag{5.1}$$

so that on $\mathcal{V}$ we have

$$d(\sigma, \theta, \ell) = (D\theta + r^{D,\mathcal{M}}\ell - \sigma\mathsf{c}, D\ell - \theta).$$

Now define a connection ∇ on $\mathcal{V}$ by

$$\nabla_X \begin{bmatrix}\sigma\\ \theta\\ \ell\end{bmatrix} = (r^{D,\mathcal{M}}_X + D_X - X)\begin{bmatrix}\sigma\\ \theta\\ \ell\end{bmatrix} = \begin{bmatrix}D_X\sigma + r^{D,\mathcal{M}}_X(\theta)\\ D_X\theta + r^{D,\mathcal{M}}_X\ell - \sigma X\\ D_X\ell - \theta(X)\end{bmatrix} \tag{5.2}$$

where $r^{D,\mathcal{M}}_X \in T^*M$, $X \in TM$ are viewed as elements of $\mathfrak{so}(L^{-1} \oplus T^*M \oplus L)$ via (2.12). It is clear that ∇ is symmetric. Moreover, ∇ is metric since D is metric and differs from ∇ by an $\mathfrak{so}(\mathcal{V})$-valued 1-form. Further, with $\Lambda = L^{-1}$, ∇ is a conformal Cartan connection whose soldering form is precisely the 'musical' isomorphism $TM\,L^{-1} \to T^*M\,L \cong \Lambda^{\perp}/\Lambda$ (which is the origin of our choice of sign in the definition of the soldering form).

We have $\nabla = r^{D,\mathcal{M}} + D - id$ so that

$$R^{\nabla} = d^D r^{D,\mathcal{M}} + \; R^D - [\![id \wedge r^{D,\mathcal{M}}]\!] \; - d^D id = C^{\mathcal{M},D} + W^{\mathcal{M}}$$

since D is torsion-free. Furthermore,

$$tr[\![id \wedge r^{D,\mathcal{M}}]\!] = nF^D = tr\,R^D$$

so that $W^{\mathcal{M}}$ has vanishing trace, from which we conclude that $R^{\nabla}|_{L^{-1}} = 0$, *i.e.*, ∇ is strongly torsion-free. We therefore have the following result, cf. [37, 48, 75] when $n \geqslant 3$.

Proposition 5.5. *Let $\mathcal{M} = (\mathcal{H}, \mathcal{S})$ be a Möbius structure. Then the Möbius bundle $\mathcal{V} = \ker \mathcal{H}$ has a unique symmetric metric connection ∇ and $(\mathcal{V}, L^{-1}, \nabla)$ is a conformal Cartan geometry with curvature given (in components) by*

$$R^\nabla = C^{\mathcal{M},D} + W^{\mathcal{M}}.$$

When is ∇ normal? The answer is immediate: since T^*M is abelian, $\partial R^\nabla = \partial W^{\mathcal{M}}$ and so, in view of Proposition 4.10, we have the following conclusion.

Proposition 5.6. *$(\mathcal{V}, L^{-1}, \nabla)$ is a normal conformal Cartan geometry if and only if $\mathcal{M}$ is a conformal Möbius structure.*

5.3. Conformal Cartan geometries are Möbius structures. We have seen already that any conformal Cartan geometry $(V, \Lambda, \mathfrak{D})$ on M determines a Möbius structure in a canonical way: we saw already in §1A that a conformal Cartan connection determines a conformal metric c on M; then, in §4A we defined differential operators $\mathcal{H}^{\mathfrak{D}}\ell = sym\, \pi \mathfrak{D} j^{\mathfrak{D}}\ell$ and $\mathcal{S}^{\mathfrak{D}}\ell = (j^{\mathfrak{D}}\ell, j^{\mathfrak{D}}\ell)$ on L, using the differential lift $\ell \mapsto j^{\mathfrak{D}}\ell$, which define a Möbius structure $\mathcal{M}^{\mathfrak{D}} = (\mathcal{H}^{\mathfrak{D}}, \mathcal{S}^{\mathfrak{D}})$ if $\mathfrak{D}$ is torsion-free. It remains to show that these two constructions are mutually inverse up to natural isomorphism.

In one direction, this is straightforward: the Möbius structure associated to the conformal Cartan connection $(\mathcal{V}, L^{-1}, \nabla)$ in this way is $(\mathsf{c}, \mathcal{M})$ itself. This is clear for the conformal structure. For the rest, we recall that for any conformal Cartan connection $(V, \Lambda, \mathfrak{D})$, a Weyl structure determines an isomorphism between V and $L^{-1} \oplus T^*M\, L \oplus L$, with respect to which the Cartan connection may be written:

$$\mathfrak{D}_X \begin{bmatrix} \sigma \\ \theta \\ \ell \end{bmatrix} = (r^{D,\mathfrak{D}}_X + D_X - X) \begin{bmatrix} \sigma \\ \theta \\ \ell \end{bmatrix} = \begin{bmatrix} D_X\sigma + r^{D,\mathfrak{D}}_X(\theta) \\ D_X\theta + r^{D,\mathfrak{D}}_X \ell - \sigma X \\ D_X\ell - \theta(X) \end{bmatrix}. \tag{5.3}$$

In the case that $(V, \Lambda, \mathfrak{D}) = (\mathcal{V}, L^{-1}, \nabla)$, this decomposition coincides with that of §5.2, and we have $r^{D,\mathfrak{D}} = r^{D,\mathcal{M}}$, so that $\mathcal{M}^{\mathfrak{D}} = \mathcal{M}$.

It remains to prove that the constructions are inverse the other way around, *i.e.*, that we recover $(V, \Lambda, \mathfrak{D})$ as the canonical Cartan connection of $(\mathsf{c}, \mathcal{M}^{\mathfrak{D}})$. We can only expect to do this up to isomorphism, since V is an abstract bundle. Furthermore, since the canonical Cartan connection of $(\mathsf{c}, \mathcal{M}^{\mathfrak{D}})$ is *strongly* torsion-free, we must suppose that $\mathfrak{D}$ is strongly torsion-free also. In this case $alt\, r^{D,\mathfrak{D}} = -F^D$, so that $r^{D,\mathfrak{D}}$ is the Möbius Ricci curvature $r^{D,\mathcal{M}^{\mathfrak{D}}}$ of D with respect to $\mathcal{M}^{\mathfrak{D}}$.

Proposition 5.7. *Let $(V, \Lambda, \mathfrak{D})$ be a conformal Cartan geometry and let $\mathcal{V} \subset J^2L$ be the Möbius bundle of the induced Möbius structure $(\mathsf{c}, \mathcal{M}^{\mathfrak{D}})$. Then the differential lift $j^{\mathfrak{D}}$, as a bundle map $J^2L \to V$, restricts to an isomorphism from $(\mathcal{V}, L^{-1}, \nabla) \to (V, \Lambda, \mathfrak{D})$.*

Proof. We choose a Weyl derivative D and compute in components. Since $r^{D,\mathfrak{D}} = r^{D,\mathcal{M}^{\mathfrak{D}}}$, the bundle map $j^{\mathfrak{D}} \colon J^2L \to V$ is then given by $(\psi, \theta, \ell) \mapsto (\frac{1}{n} \operatorname{tr}_{\mathsf{c}} \psi, \theta, \ell)$. Hence it is an isometry when restricted to $\mathcal{V}$ and maps L^{-1} to Λ. Comparing (5.2) with (5.3) we see that this isomorphism identifies the connections ∇ and $\mathfrak{D}$. □

Let us combine Propositions 5.5, 5.6 and 5.7 to summarize our results so far.

Theorem 5.8. *Let c and $\mathcal{M} = (\mathcal{H}, \mathcal{S})$ define a Möbius structure on a manifold M. Then there is a unique symmetric metric connection on the Möbius bundle $\mathcal{V} = \ker \mathcal{H} \subset J^2L$ and this is a conformal Cartan geometry compatible with the same Möbius structure. Any conformal Cartan geometry arises in this way up to natural isomorphism, and is normal if and only if the Möbius structure is conformal.*

Remark 5.9. It is straightforward to extend this equivalence to general torsion-free conformal Cartan connections by extending the notion of a Möbius structure to include a 2-form $\mathcal{F}$. In order to make room for this 2-form in the jet bundle picture, it is necessary to work with semiholonomic jets—or equivalently, to extend the jet derivative to $J^2L \oplus (\wedge^2 T^*M \otimes L)$. By this device it is even possible to consider arbitrary conformal Cartan connections, but the notion of a Möbius structure must be modified further in the presence of torsion. Since we do not see any advantage in this generality, and it would have clouded the exposition considerably, we have restricted attention to the strongly torsion-free case.

5.4. Normalization of conformal Cartan geometries. Möbius structures compatible with a fixed conformal metric on a manifold M form an affine space modelled on sections of S^2T^*M. We now exploit Theorem 5.8 to lift this affine structure to conformal Cartan geometries. Let $(V, \varLambda, \mathfrak{D})$ be a conformal Cartan geometry and $Q \in \mathrm{C}^\infty(M, S^2T^*M)$. We view Q as an $\mathfrak{so}(V)$-valued 1-form via (2.3) so that

$$Q_X\sigma = 0, \qquad Q_X\mathfrak{D}_Y\sigma = -Q_X(Y)\sigma, \qquad Q_X(V) \subseteq \varLambda^\perp. \tag{5.4}$$

In particular, $\mathfrak{D} + Q$ is a conformal Cartan connection with the same soldering form as $\mathfrak{D}$ and so induces the same conformal metric on M. In fact, more can be said.

Lemma 5.10. *If $\mathfrak{D}$ is strongly torsion-free, so is $\mathfrak{D} + Q$ and $\mathcal{M}^{\mathfrak{D}+Q} = \mathcal{M}^{\mathfrak{D}} + Q$.*

Proof. Since $T^*M \subset \mathfrak{so}(V)$ is abelian, we have

$$R^{\mathfrak{D}+Q} = R^{\mathfrak{D}} + d^{\mathfrak{D}}Q + \tfrac{1}{2}[Q \wedge Q] = R^{\mathfrak{D}} + d^{\mathfrak{D}}Q,$$

so $\mathfrak{D} + Q$ is strongly torsion-free iff $d^{\mathfrak{D}}Q|_{\varLambda} = 0$. Now, using (5.4), we have

$$(d^{\mathfrak{D}}Q)_{X,Y}\sigma = (\mathfrak{D}_X Q)_Y\sigma - (\mathfrak{D}_Y Q)_X\sigma = -Q_Y\mathfrak{D}_X\sigma + Q_X\mathfrak{D}_Y\sigma = (Q_Y X - Q_X Y)\sigma = 0.$$

For the second assertion, we must compute $j^{\mathfrak{D}+Q}\ell$: using (2.7) and (2.6), we see that this differs from $j^{\mathfrak{D}}\ell$ by a section of $\varLambda$ and then the definition of the differential lift, along with (2.9), shows that $j^{\mathfrak{D}+Q}\ell = j^{\mathfrak{D}}\ell + \frac{1}{n}(tr_{\mathsf{c}}\, Q)\ell$. Thus $(j^{\mathfrak{D}+Q}\ell, j^{\mathfrak{D}+Q}\ell) = (j^{\mathfrak{D}}\ell, j^{\mathfrak{D}}\ell) - \frac{2}{n}(tr_{\mathsf{c}}\, Q)\ell^2$. Finally, for any section σ of $\varLambda$,

$$\pi(\mathfrak{D} + Q)_X(v + \sigma) = \pi\mathfrak{D}_X v + Q_X \otimes p(v) - X \otimes \sigma$$

from which we deduce that $\pi(\mathfrak{D} + Q)j^{\mathfrak{D}+Q}\ell = \pi\mathfrak{D}j^{\mathfrak{D}}\ell + (Q - \frac{1}{n}(tr_{\mathsf{c}}\, Q)\mathsf{c})\ell$ and the result immediately follows. □

This procedure allows us to 'normalize' a conformal Cartan geometry, which will be important later. For this we use the fact, cf. Proposition 5.6 and Proposition 4.10, that a conformal Cartan geometry is normal iff the Weyl curvature of the corresponding Möbius structure has zero Ricci contraction. Now for any Möbius structure $\mathcal{M}$, we have $W^{\mathcal{M}+Q} = W^{\mathcal{M}} - [\![id \wedge Q]\!]$ and so

$$\partial W^{\mathcal{M}+Q} = \partial W^{\mathcal{M}} - (n-2)Q_0 - 2(n-1)(\tfrac{1}{n}\, tr_{\mathsf{c}}\, Q)\mathsf{c}. \tag{5.5}$$

Since $\partial W^{\mathcal{M}}$ is symmetric (and is tracelike for $n = 2$ and vanishes for $n = 1$) we have:

Proposition 5.11. *Let $(V, \varLambda, \mathfrak{D})$ be a conformal Cartan geometry. Then there is a unique section Q of S^2T^*M such that*

(i) *$\mathfrak{D} - Q$ is normal;*

(ii) *$Q_0 = 0$ if $n = 2$ and $Q = 0$ if $n = 1$, i.e., Q is in the image of the Ricci contraction ∂.*

6. Möbius structures in low dimensions

The low dimensional cases $\dim M = 1$ or $\dim M = 2$ merit special attention for two related reasons. First, any such M admits conformal coordinates so that any conformal metric is flat. On the other hand, our theory introduces a new ingredient in addition to the conformal metric for, in this case, M supports many conformal Möbius structures compatible with a given conformal metric. We now discuss the geometrical meaning of these data and explain how to compute them from a normal conformal Cartan connection.

In all this we shall make use of a further feature of low dimensional geometry. As we noted in Remark 1.3, our conformal Cartan connections are a linear representation of the usual notion. We chose the standard representation for simplicity, but other choices are possible. If we take $G = Spin_0(n+1,1)$ (which is a double cover of the identity component of $Möb(n)$) we can use the spin representation. Such a choice is particularly effective for $n \leqslant 4$, when special isomorphisms of G with more familiar Lie groups give concrete realizations of spinors.

6.1. Computing the Möbius structure. Let us consider in general how to compute the Möbius structure $\mathcal{M}^{\mathfrak{D}}$ associated to a conformal Cartan geometry $(V, \Lambda, \mathfrak{D})$ via the differential lift: §4A provides formulae for this, but they can be difficult to use since one must decompose $\mathfrak{D}$ with respect to a Weyl structure. Instead, we shall write $\mathcal{M}^{\mathfrak{D}} = \mathcal{M}^D + sym\, r^{D,\mathfrak{D}}$ relative to a Weyl derivative D, and compute $sym\, r^{D,\mathfrak{D}}$ directly.

Recall that if $i_\Lambda \colon \Lambda \to V$ is the inclusion, we can define $\mathfrak{D}^D i_\Lambda \colon TM \otimes \Lambda \to \Lambda^\perp \subset V$. Differentiating again using (3.1), we find that

$$\big((\mathfrak{D}^D)^2 i_\Lambda\big)_{X,Y}\sigma + r^{D,\mathfrak{D}}_{X,Y}\sigma = X \cdot Y \cdot \sigma, \tag{6.1}$$

for vector fields X, Y and any section σ of Λ. In other words, $(\mathfrak{D}^D)^2 i_\Lambda + r^{D,\mathfrak{D}} i_\Lambda = \mathsf{c} \otimes i_{\hat\Lambda}$, where $i_{\hat\Lambda} \colon \hat\Lambda \to V$ is the inclusion of the Weyl structure corresponding to D. The tracefree part of this formula then yields:

$$sym_0\big((\mathfrak{D}^D)^2 i_\Lambda + r^{D,\mathfrak{D}} i_\Lambda\big) = 0. \tag{6.2}$$

The trace part is still difficult to use because of the appearance of $i_{\hat\Lambda}$. To get around this, we note that if $e_i \otimes \sigma$ is an orthonormal frame of $TM\,\Lambda$, and $\hat\sigma \in C^\infty(M, \hat\Lambda)$ with $(\sigma, \hat\sigma) = -1$, then σ, $(\mathfrak{D}^D_{e_i} i_\Lambda)\sigma$ and $\hat\sigma$ form a frame of V relative to which we have, for any $v \in V$,

$$(v, v) = \sum_{i=1}^n (v, (\mathfrak{D}^D_{e_i} i_\Lambda)\sigma) - 2(v, \sigma)(v, \hat\sigma).$$

Combining this with the trace of (6.1), using $X \cdot Y \cdot \sigma = \langle X, Y\rangle \hat\sigma$, we then have

$$\sum_{i=1}^n \Big\{ \big(v, (\mathfrak{D}^D_{e_i} i_\Lambda)\sigma\big)^2 - \tfrac{2}{n}(v,\sigma)\big(v, ((\mathfrak{D}^D)^2_{e_i,e_i} i_\Lambda)\sigma\big)\Big\} - \tfrac{2}{n} s^{D,\mathfrak{D}}(v, i_\Lambda)^2 = (v, v). \tag{6.3}$$

If we take $v = \hat\sigma$, this formula simplifies, because v is then null and orthogonal to $(\mathfrak{D}^D_{e_i} i_\Lambda)\sigma$. However, the computational advantage of (6.3) is the explicitness of the dependence on D.

Remark 6.1. The formulae (6.2)–(6.3) have a conceptual explanation: viewing i_Λ as a section of $L \otimes V$, the $\mathfrak{D}$-coupled differential lift of i_Λ is minus the metric in $V \otimes V$. This last is parallel, so that the $\mathfrak{D}$-coupled Möbius operator $\mathcal{H}^{\mathfrak{D},\mathfrak{D}}$ gives zero when applied to i_Λ. On the other hand $\mathcal{S}^{\mathfrak{D},\mathfrak{D}}(i_\Lambda)$ is a partial contraction of this differential lift with itself, which reproduces the metric.

Now specialize to the case where D is exact with gauge σ (*i.e.*, $D\sigma = 0$), so that $F^D = 0$. Assume also that v is null. Then (6.2)–(6.3) read:

$$sym_0\,\mathfrak{D}^D\mathfrak{D}\sigma + r_0^{D,\mathfrak{D}}\sigma = 0; \tag{6.4}$$

$$\tfrac{1}{n}s^{D,\mathfrak{D}}(v,\sigma)^2 = \tfrac{1}{2}|(v,\mathfrak{D}\sigma)|^2 - (v,\sigma)\big(v,\tfrac{1}{n}\,tr_{\mathsf{c}}(\mathfrak{D}^D\mathfrak{D}\sigma)\big). \tag{6.5}$$

6.2. Möbius 1-manifolds and projective structures. A conformal Cartan connection $(V, \Lambda, \mathfrak{D})$ on a 1-manifold M is automatically flat, hence a normal conformal Cartan geometry. Further, by Proposition 1.2, there is a distinguished family of local immersions of M into $P(\mathcal{L})$ (here a circle), which cover M and are related on overlaps by elements of $M\ddot{o}b(1) = O_+(2,1)$.

Now $Spin_0(2,1) \cong SL(2,\mathbb{R})$, corresponding to the realization of S^1 as the real projective line $\mathbb{R}P^1$. Thus $\mathbb{R}^{2,1} = S^2\mathbb{R}^2$, the set of symmetric matrices on $\mathbb{R}^2$, where the quadratic form is minus the determinant. More precisely, fix a symplectic form $\omega_{\mathbb{R}} \in \wedge^2\mathbb{R}^{2*}$ and consider $\mathbb{R}^2 \otimes \mathbb{R}^2$ with the metric

$$(v_1 \otimes v_2, w_1 \otimes w_2) = -\omega_{\mathbb{R}}(v_1, w_2)\omega_{\mathbb{R}}(v_2, w_1)$$

The symmetric tensors form a subspace on which the metric has signature $(2,1)$. The quadratic map $x \mapsto x \otimes x \colon \mathbb{R}^2 \to \mathbb{R}^{2,1}$ then induces a diffeomorphism $\mathbb{R}P^1 \to P(\mathcal{L})$.

The trivial $\mathbb{R}^2$ bundle and tautological line subbundle $\mathcal{O}(-1)_{\mathbb{R}} \subset \mathbb{R}P^1 \times \mathbb{R}^2$ may be abstracted as follows: a 'spin Cartan connection' is a real rank 2 symplectic vector bundle $(V_{\mathbb{R}}, \omega_{\mathbb{R}}) \to M$ with symplectic connection ∇ and a line subbundle $\Lambda_{\mathbb{R}}$ such that $\nabla|_{\Lambda_{\mathbb{R}}}$ *mod* $\Lambda_{\mathbb{R}}$ defines an isomorphism $TM \otimes \Lambda_{\mathbb{R}} \to V_{\mathbb{R}}/\Lambda_{\mathbb{R}}$. This is a one dimensional projective structure and is always flat. Given such a spin Cartan connection, we obtain a conformal Cartan connection on $V = S^2V_{\mathbb{R}}$ with $\Lambda = \Lambda_{\mathbb{R}}^2$: the induced connection preserves the bundle metric defined pointwise as above.

It follows that a conformal Cartan connection carries the information of a projective structure on M, and it is natural to enquire how this arises in explicit geometric terms. We do this now by showing how projective coordinates, Hill operators and schwarzian derivatives arise from $\mathfrak{D}$.

First, let x be a coordinate on M and D^x the unique connection on TM with $D^x\partial_x = 0$, where ∂_x is the vector field with $dx(\partial_x) = 1$. (Any connection on a 1-manifold is a Levi-Civita connection, hence a Weyl connection: D^x is the Levi-Civita connection of the metric dx^2 induced by x.) If $\sigma_x = |dx|$ and v is a null section of V with $(v, \sigma_x) \neq 0$, then (6.5) yields a formula for the only nontrivial datum of the Möbius structure:

$$s^{D^x,\mathfrak{D}}|\partial_x|^2 = \frac{(v, \mathfrak{D}_{\partial_x}\sigma_x)^2}{2(v,\sigma_x)^2} - \frac{\big(v, \mathfrak{D}_{\partial_x}\mathfrak{D}_{\partial_x}\sigma_x\big)}{(v,\sigma_x)}$$

Suppose now that w is another coordinate with $dw = w'dx$ and assume for simplicity that $w' > 0$ so that $\sigma_w := |dw| = w'\sigma_x$. Then $\mathfrak{D}_{\partial_w}\sigma_w = \mathfrak{D}_{\partial_x}\sigma_x + w''/w'$ and hence

$$s^{D^w,\mathfrak{D}}|\partial_w|^2 = s^{D^x,\mathfrak{D}}|\partial_x|^2 - \Big(\frac{w''}{w'}\Big)' + \frac{1}{2}\Big(\frac{w''}{w'}\Big)^2. \tag{6.6}$$

We define a projective coordinate to be a coordinate x with $s^{D^x,\mathfrak{D}} = 0$, so that $\mathcal{S}^{\mathfrak{D}} = \mathcal{S}^{D^x}$. Such coordinates can be found by solving a nonlinear ODE.

Proposition 6.2. *Let $(M, \mathcal{M}^{\mathfrak{D}})$ be a Möbius 1-manifold. Then there are (local) projective coordinates on M, and if x is such a coordinate, then another coordinate $w(x)$ is projective if and only if the schwarzian $S_x(w) := (w''/w')' - \frac{1}{2}(w''/w')^2$ of w with respect to x is zero.*

Another approach to one dimensional projective structures is via Hill (or Hill's) operators Δ, *i.e.*, second order differential operators on $\mathrm{C}^\infty(M, L^{1/2})$ which differ from a second derivative D^2 by a zero order term. For this we view $\mathfrak{D}$ as a spin Cartan connection on $V_{\mathbb{R}}$

(with $S^2 V_{\mathbb{R}} = V$) preserving a symplectic form $\omega_{\mathbb{R}}$ and such that $X \otimes \lambda \mapsto -\mathfrak{D}_X \lambda \mod \Lambda_{\mathbb{R}}$ defines an isomorphism from $TM \otimes \Lambda_{\mathbb{R}}$ to $V_{\mathbb{R}}/\Lambda_{\mathbb{R}} \cong_{\omega_{\mathbb{R}}} \Lambda_{\mathbb{R}}^* = L^{1/2}$.

As in Proposition 4.1 any section f of $L^{1/2}$ has a differential lift $j^{\mathfrak{D}} f$ which is the unique section of $V_{\mathbb{R}}$ that projects onto f and satisfies $\mathfrak{D} j^{\mathfrak{D}} f \in \ker \partial \cap \Omega^1(M, V_{\mathbb{R}}) = \Omega^1(M, \Lambda_{\mathbb{R}})$. If we choose a Weyl connection D, then the derivative $\mathfrak{D}^D i_{\Lambda_{\mathbb{R}}}$ of the inclusion $i_{\Lambda_{\mathbb{R}}} : \Lambda_{\mathbb{R}} \to V_{\mathbb{R}}$ determines a splitting $V_{\mathbb{R}} = \Lambda_{\mathbb{R}} \oplus \Lambda_{\mathbb{R}}^*$, and using this decomposition we find

$$\mathfrak{D}(\lambda, f) = (D\lambda + \tfrac{1}{2} s^{D,\mathfrak{D}} f, Df - \lambda).$$

It follows that $j^{\mathfrak{D}} f$ is essentially the 1-jet of f and $f \mapsto \Delta^{\mathfrak{D}} f := \mathfrak{D} j^{\mathfrak{D}} f$ is a Hill operator: with respect to any Weyl derivative D, we have $j^{\mathfrak{D}} f = (Df, f)$, and $\Delta^{\mathfrak{D}} f = D^2 f + \frac{1}{2} s^{D,\mathfrak{D}} f$. This provides another way of computing the Möbius structure, and it is well known that the zero order term of a Hill operator transforms by schwarzian derivative under changes of coordinate.

The two approaches are related in a simple way.

Proposition 6.3. *The quadratic operator $\mathcal{S}^{\mathfrak{D}}(\ell)$ is related to the Hill operator $\Delta^{\mathfrak{D}} f$, when $\ell = f^2$, by $\mathcal{S}^{\mathfrak{D}}(\ell) = -4 f^3 \Delta^{\mathfrak{D}} f$.*

Proof. This is straightforward to prove using a Weyl derivative. Alternatively, observe that $j^{\mathfrak{D}} \ell = (j^{\mathfrak{D}} f) \otimes (j^{\mathfrak{D}} f) + \sigma$, where $\sigma = \partial \mathfrak{D}((j^{\mathfrak{D}} f) \otimes (j^{\mathfrak{D}} f))$. (This follows because $\partial \mathfrak{D} \sigma = -\sigma$.) We then compute, using dual frames e, ε (*i.e.*, $\varepsilon(e) = 1$), that $\mathcal{S}^{\mathfrak{D}}(\ell) = 2(j^{\mathfrak{D}} f \otimes j^{\mathfrak{D}} f, \sigma) = 4\omega_{\mathbb{R}}(j^{\mathfrak{D}} f, \mathfrak{D}_e j^{\mathfrak{D}} f) \omega_{\mathbb{R}}(j^{\mathfrak{D}} f, \varepsilon \cdot j^{\mathfrak{D}} f) = -4 f^3 \mathfrak{D} j^{\mathfrak{D}} f$. □

We have already noted that $V_{\mathbb{R}}$ can be identified the 1-jet bundle of $L^{1/2}$; its parallel sections are 'affine with respect to $\mathfrak{D}$', *i.e.*, 1-jets of solutions of $\Delta^{\mathfrak{D}} f = 0$. Similarly, $V = S^2 \Lambda_{\mathbb{R}}$ is the 2-jet bundle of L, and L is the tangent bundle if M is oriented; the parallel sections are 2-jets of projective vector fields, *i.e.*, vector fields X with $\mathcal{L}_X \Delta^{\mathfrak{D}} = 0$.

Remark 6.4. The interpretation of s^D as a one dimensional analogue of scalar curvature suggests the following Yamabe-like problem: given a projective structure Δ on $M = \mathbb{R}/\mathbb{Z}$, is there a gauge of constant scalar curvature? This amounts to solving $\Delta\mu = c\mu^{-3}$ for a constant c and a positive section μ of $L^{1/2}$ which we normalize via $\int_M \mu^{-2} = 1$: if $D\mu = 0$, we then have $s^D = c\mu^{-2}$. Solutions are critical points of the Yamabe-like functional $Y(\mu) = (\int_M \mu^{-2})(\int_M \mu\, \Delta\mu)$. Given a solution μ, we have $\Delta(f\mu) = (f'' + cf)\mu^{-3}$ for any $f \in C^\infty(M, \mathbb{R})$, from which it is straightforward to determine the developing map δ from $\tilde{M} = \mathbb{R}$ to $\mathbb{R}P^1 \cong \mathbb{R} \cup \{\infty\}$: $\delta(x) = \frac{1}{\sqrt{c}} \tan(\sqrt{c}x)$ for $c > 0$, $\delta(x) = x$ for $c = 0$ and $\delta(x) = \frac{1}{\sqrt{-c}} \tanh(\sqrt{-c}x)$ for $c < 0$. The constant c determines the conjugacy class of the holonomy and, for $c > 0$ (when the developing map surjects), the winding number of the fundamental domain $[0, 1)$: the fundamental domain injects for $c \leqslant 4\pi^2$. Any such data arise for some (unique) $c \in \mathbb{R}$, and so any projective structure admits a constant scalar curvature gauge. Without the geometric interpretation, the existence of critical points for Y is not obvious, since Y is unbounded: if $\mu_\varepsilon(x) = (1 + \varepsilon + \cos(2\pi x)) dx^{-1/2}$, where $x : M \xrightarrow{\cong} \mathbb{R}/\mathbb{Z}$, then $\int_M \mu_\varepsilon^{-2}$ can be arbitrarily large for $\varepsilon > 0$.

6.3. Möbius 2-manifolds and schwarzian derivatives. When $n = 2$, $M\ddot{o}b(2) = O_+(3,1)$, and $Spin_0(3,1) \cong SL(2, \mathbb{C})$ corresponding to the realization of S^2 as the complex projective line $\mathbb{C}P^1$. Thus $\mathbb{R}^{3,1} = (\mathbb{C}^2 \otimes \bar{\mathbb{C}}^2)_{\mathbb{R}}$, the set of hermitian matrices on $\mathbb{C}^2$, where the quadratic form is minus the determinant. More precisely, fix a symplectic form $\omega_{\mathbb{C}} \in \wedge^2 \mathbb{C}^{2*}$ and consider $\mathbb{C}^2 \otimes_{\mathbb{C}} \bar{\mathbb{C}}^2$ with the metric

$$(v_1 \otimes \bar{v}_2, w_1 \otimes \bar{w}_2) = -\omega_{\mathbb{C}}(v_1, w_1) \overline{\omega_{\mathbb{C}}(v_2, w_2)}.$$

The conjugation $x \otimes \bar{y} \mapsto y \otimes \bar{x}$ fixes a real form $\mathbb{R}^{3,1}$ (the hermitian matrices) on which the metric is real of signature $(3,1)$. The quadratic map $x \mapsto x \otimes \bar{x} \colon \mathbb{C}^2 \to \mathbb{R}^{3,1}$ then induces a conformal diffeomorphism $\mathbb{C}P^1 \to P(\mathcal{L})$.

The trivial $\mathbb{C}^2$-bundle and the tautological complex line subbundle $\mathcal{O}(-1)_{\mathbb{C}} \subset \mathbb{C}P^1 \times \mathbb{C}^2$ may be abstracted as follows: a 'spin Cartan connection' as a complex rank 2 symplectic vector bundle $(V_{\mathbb{C}}, \omega_{\mathbb{C}}) \to M$ with complex symplectic connection ∇ and a complex line subbundle $\Lambda_{\mathbb{C}}$ such that the linear bundle map $\nabla|_{\Lambda_{\mathbb{C}}}$ *mod* $\Lambda_{\mathbb{C}} \colon TM \otimes \Lambda_{\mathbb{C}} \to V_{\mathbb{C}}/\Lambda_{\mathbb{C}}$ is an isomorphism. A spin Cartan connection induces a conformal Cartan connection via $V = (V_{\mathbb{C}} \otimes \bar{V}_{\mathbb{C}})_{\mathbb{R}}$ and $\Lambda = (\Lambda_{\mathbb{C}} \otimes \bar{\Lambda}_{\mathbb{C}})_{\mathbb{R}}$ (the fixed points of the conjugation involution): the induced connection preserves the bundle metric defined pointwise as above.

The conformal Möbius structures compatible with a given conformal metric form an affine space modelled on $C^\infty(M, S^2_0 T^*M)$. To compute this extra datum, we compare a given Möbius structure with that provided by a holomorphic coordinate z: given this, $dz\, d\bar{z}$ is a flat metric compatible with c and so arises from a length scale ℓ which is parallel for the (flat) Levi-Civita connection D^z of this metric. Contemplate the Möbius structure $\mathcal{M}^z := \mathcal{M}^{D^z}$: since F^{D^z} vanishes, $r^{D^z,\mathcal{M}^z} = 0$ so that, first, $C^{D^z,\mathcal{M}^z} = 0$ and, second, $W^{\mathcal{M}^z} = R^{D^z} = 0$, whence the corresponding Cartan connection $\mathfrak{D}^z$ is flat by Proposition 5.5.

Since $S^2_0 T^*M \otimes \mathbb{C} \cong \Omega^{2,0}M \oplus \Omega^{0,2}M$, where $\Omega^{2,0}M = (T^{1,0}M)^{-2}$, $\Omega^{0,2}M = (T^{0,1}M)^{-2}$, and $TM \otimes \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$ is the $\pm\boldsymbol{i}$-eigenspace decomposition of the complex structure $J \colon TM \to TM$, we can identify $C^\infty(M, S^2_0 T^*M)$ with the space of smooth quadratic differentials and write $r_0^{D^z,\mathfrak{D}} = q\, dz^2 + \bar{q}\, d\bar{z}^2$. We now take $D = D^z$ in equation (6.4) and note that $D^z_{\partial_z} \partial_z = 0$ to conclude:

$$\mathfrak{D}_{\partial_z} \mathfrak{D}_{\partial_z} \sigma + q\sigma = 0.$$

This gives an effective method to compute $r_0^{D^z,\mathfrak{D}}$.

Proposition 6.5. *Let $(V, \Lambda, \mathfrak{D})$ a conformal Cartan geometry on a 2-manifold M with a holomorphic coordinate z for the induced conformal metric. Let $\mathcal{M} = \mathcal{M}^z + r^{D^z,\mathfrak{D}}$ be the induced Möbius structure and write $r_0^{D^z,\mathfrak{D}} = q\, dz^2 + \bar{q}\, d\bar{z}^2$. Then*

$$\mathfrak{D}_{\partial_z} \mathfrak{D}_{\partial_z} \sigma + q\sigma = 0 \tag{6.7}$$

where σ is the (unique) positive section of Λ with $(\mathfrak{D}\sigma, \mathfrak{D}\sigma) = dz\, d\bar{z}$.

It is instructive to apply this analysis to the Möbius structure $\mathcal{M}^w$ where w is another holomorphic coordinate. Let $\mathfrak{D}^w$ be the corresponding Cartan connection and σ_w the section of Λ with $(\mathfrak{D}^w \sigma_w, \mathfrak{D}^w \sigma_w) = dw\, d\bar{w} = |w'|^2 dz\, d\bar{z}$, where $'$ denotes differentiation with respect to z. Then $\sigma = \sigma_w / |w'|$ is D^z-parallel, and we have from Proposition 6.5 that $0 = \mathfrak{D}^w_{\partial_w} \mathfrak{D}^w_{\partial_w} \sigma_w = (w')^{-1} \mathfrak{D}^w_{\partial_z} \big((w')^{-1} \mathfrak{D}^w_{\partial_z} (|w'|\sigma) \big)$, from which we compute that

$$\mathfrak{D}^w_{\partial_z} \mathfrak{D}^w_{\partial_z} \sigma + \tfrac{1}{2} S_z(w) \sigma = 0$$

where

$$S_z(w) = \Big(\frac{w''}{w'}\Big)' - \frac{1}{2}\Big(\frac{w''}{w'}\Big)^2$$

is the classical schwarzian derivative of w. We thus have the following transformation law.

Proposition 6.6. $\mathcal{M}^w = \mathcal{M}^z + \mathrm{Re}(S_z(w) dz^2)$.

This prompts us, with [24], to define the *schwarzian derivative $S_z(\mathcal{M})$ of a Möbius structure $\mathcal{M}$ with respect to z* by

$$(\mathcal{M} - \mathcal{M}^z)^{2,0} = \tfrac{1}{2} S_z(\mathcal{M}) dz^2.$$

For coordinate Möbius structures, we have seen that $S_z(\mathcal{M}^w) = S_z(w)$, and in all cases, it has the classical transformation law: $\mathcal{M} - \mathcal{M}^w = (\mathcal{M} - \mathcal{M}^z) + (\mathcal{M}^z - \mathcal{M}^w)$ giving

$$S_w(\mathcal{M}) = S_z(\mathcal{M})(dz/dw)^2 + S_w(z).$$

Note that the holomorphicity of $S_z(\mathcal{M})$ is independent of the coordinate z and amounts to the flatness of $\mathcal{M}$: if $\mathcal{M}$ is flat then, since D^z is flat and $\mathcal{M}$ is canonical, $r^{D^z,\mathcal{M}} = Q_0^D$; hence $d^{D^z} Q_0^D = 0$, *i.e.*, $S_z(\mathcal{M})$ is holomorphic. Conversely, if $S_z(\mathcal{M})$ is holomorphic, solving a holomorphic ODE yields a coordinate w with $S_w(\mathcal{M}) = 0$, *i.e.*, $\mathcal{M} = \mathcal{M}^w$, which is flat. In particular, any flat $\mathcal{M}$ is locally of the form $\mathcal{M}^w$ for some holomorphic coordinate w.

(Alternatively, note that since D^z is flat, a canonical $\mathcal{M}$ has $r^{D^z,\mathcal{M}} = Q_0^D$ while, since $\dim M = 2$, $W^{\mathcal{M}} = 0$ whence $\mathcal{M}$ is flat if and only if $d^{D^z} Q_0^D = 0$ which last amounts to the holomorphicity of $S_z(\mathcal{M})$.)

As in the one dimensional case, we relate the schwarzian derivative appearing here to Hill operators and projective structures using the spin representation $\varLambda_{\mathbb{C}} \subset V_{\mathbb{C}}$ of $\mathfrak{D}$ above.

A Weyl derivative D provides a decomposition $V_{\mathbb{C}} = \varLambda_{\mathbb{C}} \oplus \varLambda_{\mathbb{C}}^*$ with respect to which the differential lift of $f \in \varLambda_{\mathbb{C}}^*$ may be written $j^{\mathfrak{D}} f = (\partial^D f, f)$, where $\partial^D f$ is the complex linear part of Df, *i.e.*, $\partial^D f = Df - \bar\partial f$. (For $D = D^z$, this was denoted by a prime above.) It follows that $[\mathfrak{D} j^{\mathfrak{D}} f] = ((\partial^D)^2 f + q^{D,\mathfrak{D}} f, \bar\partial f)$ and the first component is a complex Hill operator, where $q^{D,\mathfrak{D}} = (r_0^{D,\mathfrak{D}})^{2,0}$: if $D = D^z$, this may be written $f'' + qf$, explaining the appearance of the complex schwarzian derivative above.

It is easy to relate the two pictures: the 'real' Möbius operator $\mathcal{H}\ell = sym_0\, D^2\ell + r_0^{D,\mathfrak{D}}\ell$ is obtained from the complex Hill operator $(\partial^D)^2 f + q^D f$, when $\ell = f \otimes \bar f$, by coupling the latter to the antiholomorphic structure ∂ on $\bar\varLambda_{\mathbb{C}}$ and taking the real part.

6.4. Quaternionic geometry of Möbius 3- and 4-manifolds. The spin formalism can also be applied to conformal 3- and 4-manifolds. When $n = 4$, $M\ddot{o}b(4) = Spin_0(5,1)$, and $Spin_0(5,1) \cong SL(2,\mathbb{H})$ corresponding to the realization of S^4 as the quaternionic projective line $\mathbb{H}P^1$. Thus $\mathbb{R}^{5,1}$ is the set of quaternionic hermitian matrices on $\mathbb{H}^2$.

The trivial $\mathbb{H}^2$-bundle and the tautological quaternionic line subbundle $\mathcal{O}(-1)_{\mathbb{H}} \subset \mathbb{H}P^1 \times \mathbb{H}^2$ may be abstracted as follows: a 'spin Cartan connection' as a quaternionic rank 2 vector bundle $V_{\mathbb{H}} \to M$ with unimodular quaternionic connection ∇ and a quaternionic line subbundle $\varLambda_{\mathbb{H}}$ such that the linear bundle map $\nabla|_{\varLambda_{\mathbb{H}}}\ mod\ \varLambda_{\mathbb{H}} \colon TM \otimes \varLambda_{\mathbb{H}} \to V_{\mathbb{H}}/\varLambda_{\mathbb{H}}$ defines an isomorphism $TM \cong Hom_{\mathbb{H}}(\varLambda_{\mathbb{H}}, V_{\mathbb{H}}/\varLambda_{\mathbb{H}})$: then $End_{\mathbb{H}}(\varLambda_{\mathbb{H}})$ and $End_{\mathbb{H}}(V_{\mathbb{H}}/\varLambda_{\mathbb{H}})$ are bundles of quaternions acting on TM by dilations and $\pm$selfdual rotations with respect to the induced conformal metric. The quaternionic realization of S^4 has proved to be very effective in the conformal geometry of surfaces in S^3 and S^4 [18, 56], and we shall discuss this briefly later. The conformal Cartan connection may be recovered using the real structure $j \wedge j$ on $\wedge^2_{\mathbb{C}} V_{\mathbb{H}}$.

The case $n = 3$ is similar: $Spin_0(4,1) \cong Sp(1,1)$ corresponds to the realization of S^3 as a real quadric in $\mathbb{H}P^1$. The 'spin Cartan connections' are therefore similar to the four dimensional case, except that $V_{\mathbb{H}}$ is equipped with a signature $(1,1)$ quaternion-hermitian metric with respect to which $\varLambda_{\mathbb{H}}$ is null. We leave the details to the interested Reader.

In three and four dimensions, a normal Cartan connection is determined by the conformal metric, so there is no additional data to compute. Nevertheless, we have a differential lift and quaternionic Möbius operators on sections of $V_{\mathbb{H}}/\varLambda_{\mathbb{H}}$, which we momentarily discuss in the four dimensional case, purely for general interest. Using a Weyl structure D, $V_{\mathbb{H}} \cong \varLambda_{\mathbb{H}} \oplus V_{\mathbb{H}}/\varLambda_{\mathbb{H}}$ and $\mathfrak{D}_X(\lambda, f) = (D_X\lambda + r_X^{D,\mathfrak{D}}(f), D_X f - X(\lambda))$, where we identify TM with $Hom(\varLambda_{\mathbb{H}}, V_{\mathbb{H}}/\varLambda_{\mathbb{H}})$. In these terms $j^{\mathfrak{D}} f = (\partial_{\mathbb{H}}^D f, f)$, where the first term is a Dirac operator, and $[\mathfrak{D} j^{\mathfrak{D}} f] = Df - \partial_{\mathbb{H}}^D f$ is the twistor operator.

7. Conformal geometry revisited

7.1. Gauge theory and moduli of conformal Cartan geometries. Consider an n-manifold M equipped with a Cartan vector bundle V with null line subbundle $\varLambda$, and let $\mathcal{A}$ denote the set of conformal Cartan connections on $(V, \varLambda)$. The *gauge group* $\mathcal{G} := \{g \in$

$C^\infty(M, O(V))\colon g\Lambda^+ \subset \Lambda^+\}$ is the space of sections of a bundle of parabolic subgroups of $O(V)$ whose Lie algebra bundle is $\mathfrak{stab}(\Lambda) \subset \mathfrak{so}(V)$. It acts on $\mathcal{A}$ is the usual way: for $g \in \mathcal{G}$ we have

$$g \cdot \mathfrak{D} := g \circ \mathfrak{D} \circ g^{-1} = \mathfrak{D} - (\mathfrak{D}g)g^{-1}. \tag{7.1}$$

Since g acts orthogonally on V and preserves Λ^+, $g \cdot \mathfrak{D}$ and $\mathfrak{D}$ induce the same conformal metric on M. Hence if $\mathcal{A}^{\mathsf{c}} \subset \mathcal{A}$ denotes the subset of connections inducing a fixed conformal metric c, $\mathcal{A}^{\mathsf{c}}$ is preserved by $\mathcal{G}$. We next ask when gauge equivalent conformal Cartan connections induce the same soldering form.

Proposition 7.1. *Let $\mathfrak{D}$ be a conformal Cartan connection and g a gauge transformation. Then $\mathfrak{D}$ and $g \cdot \mathfrak{D}$ have the same soldering form iff $g = \exp\gamma$ with $\gamma \in T^*M \subset \mathfrak{stab}(\Lambda)$.*

Proof. $g \cdot \mathfrak{D}$ has the same soldering form as $\mathfrak{D}$ if and only if g commutes with $\beta^{\mathfrak{D}}$, *i.e.*, $g\beta^{\mathfrak{D}}\sigma = \beta^{\mathfrak{D}}g\sigma$. However, $g\Lambda^+ \subset \Lambda^+$ so that $g\sigma = e^u\sigma$ for some function u and now $g\beta^{\mathfrak{D}}\sigma = e^u\beta^{\mathfrak{D}}\sigma$ implies, by the Cartan condition, that

$$g|_{\Lambda^\perp} = e^u id + \gamma \circ \pi$$

with $\gamma\colon \Lambda^\perp/\Lambda \to \Lambda$. Now g acts orthogonally so, for $v \in \Lambda^\perp$,

$$(v, v) = (gv, gv) = e^{2u}(v, v)$$

whence $e^u = 1$. In particular, $g|_{\Lambda^\perp} = id + \gamma \circ \pi$. However, it is not difficult to see that any orthogonal g is determined by its values on $\Lambda^\perp$ and so we conclude that $g = \exp\gamma$ under the usual soldering identification of $Hom(\Lambda^\perp/\Lambda, \Lambda)$ with T^*M. Certainly any g of this form preserves $\beta^{\mathfrak{D}}$. □

We now combine this gauge theory with the normalization of conformal Cartan connections discussed in §5.4. Let $\mathfrak{D}$ be a fixed strongly torsion-free connection inducing the conformal metric c. Then Lemma 5.10 assures us that any other strongly torsion-free connection induces the same Möbius structure as one of the form $\mathfrak{D} + Q$, and is therefore gauge equivalent to it by Proposition 5.7. In fact the connections of the form $\mathfrak{D} + Q$ provide a slice to the gauge action: it just remains to show that $\mathfrak{D}$ and $\mathfrak{D} + Q$ are only gauge equivalent if $Q = 0$. However, since they have the same soldering form, Proposition 7.1 and equation (3.2) show that $Q = -\mathfrak{D}\gamma - \frac{1}{2}[\![\gamma, \pi\mathfrak{D}\gamma]\!]$. Therefore $0 = -Q_X\sigma = (\mathfrak{D}_X\gamma)\sigma = -\gamma \cdot \mathfrak{D}_X\sigma = \gamma(X)\sigma$ for any vector field X. Thus $\gamma = 0$ and $Q = 0$.

Theorem 7.2. *Let $\mathcal{A}^{\mathsf{c}}_{\mathrm{stf}}$ be the set of strongly torsion-free conformal Cartan connections inducing the conformal metric c. Then $\mathcal{G}$ acts freely on $\mathcal{A}^{\mathsf{c}}_{\mathrm{stf}}$ and, if $\mathfrak{D}$ is a fixed element of $\mathcal{A}^{\mathsf{c}}_{\mathrm{stf}}$, there is a unique connection of the form $\mathfrak{D} + Q$ in each $\mathcal{G}$-orbit.*

7.2. Constant vectors, spaceform geometries and stereoprojection. If $(V, \Lambda, \mathfrak{D})$ is a conformal Cartan geometry on M, then we have seen that the compatible riemannian metrics correspond to sections σ of $\mathcal{L}^+ \subset \Lambda$ via $g_\sigma(X, Y) = (\mathfrak{D}_X\sigma, \mathfrak{D}_Y\sigma)$. This induces a Weyl structure $\hat{\Lambda} \subset V$ spanned by $\hat{\sigma}$ with $(\hat{\sigma}, \hat{\sigma}) = 0$ and $(\hat{\sigma}, \mathfrak{D}_X\sigma) = 0$.

When $\mathfrak{D}$ is flat, a particularly important class of sections of $\mathcal{L}^+$ are the *conic sections*: these sections are given by the intersection of $\mathcal{L}^+$ with an affine hyperplane, *i.e.*, $\{\sigma \in \mathcal{L}^+ : (v_\infty, \sigma) = -1\}$, where v_∞ is a parallel section of V and we assume that $\Lambda \not\subseteq v_\infty^\perp$ on M. Since $(\mathfrak{D}_X\sigma, v_\infty) = 0$ and $(\mathfrak{D}_X\sigma, \sigma) = 0$, the corresponding Weyl structure is spanned by

$$\hat{\sigma} = v_\infty + \tfrac{1}{2}(v_\infty, v_\infty)\sigma. \tag{7.2}$$

The length scale ℓ dual to σ (with $\langle\sigma, \ell\rangle = 1$) is the homology class $v_\infty \mathit{ mod } \Lambda^\perp$, and v_∞ is the differential lift $j^{\mathfrak{D}}\ell$ of Proposition 4.1. In this case the corresponding metric g is an Einstein metric (since $\mathcal{H}\ell = 0$) of constant curvature $\frac{2}{n}s^g\ell^2 = -\mathcal{S}\,\ell = -(v_\infty, v_\infty)$, and is the metric of a spherical, euclidean or hyperbolic spaceform.

To see this explicitly, suppose that M is simply connected, so we may assume $V = M\times\mathbb{R}^{n+1,1}$ with $\mathfrak{D}=d$ and v_∞ constant. Then $x\mapsto \Lambda_x$ defines a local diffeomorphism from M to $P(\mathcal{L})\smallsetminus P(v_\infty^\perp\cap\mathcal{L})$—which is $P(\mathcal{L})$, $P(\mathcal{L})\smallsetminus\{<v_\infty>\}$ or $P(\mathcal{L})$ with a hypersphere removed, according to whether v_∞ is timelike, null, or spacelike, cf. §1A.

- If v_∞ is non-null, then define $v=\sigma+v_\infty/(v_\infty,v_\infty)$. Then $(v,v_\infty)=0$ and $(v,v)=-1/(v_\infty,v_\infty)$, so that $x\mapsto v(x)$ is a local diffeomorphism from M to $\{v\in v_\infty^\perp : (v,v)=\pm r^2\}$, where $1/r^2=|(v_\infty,v_\infty)|$, which is a hyperboloid or a sphere of radius r according to whether (v_∞,v_∞) is positive or negative. Since $d_Xv=d_X\sigma$, this local diffeomorphism is an isometry.
- If v_∞ is null, let v_0 be another constant null vector with $(v_0,v_\infty)=-1$ and identify $\{v_0,v_\infty\}^\perp$ with euclidean $\mathbb{R}^n$. Then the *stereographic projection* or *stereoprojection*

$$\sigma\mapsto v = pr_{\mathbb{R}^n}\sigma=\sigma-v_0+(\sigma,v_0)v_\infty \tag{7.3}$$

is a local diffeomorphism from M to $\mathbb{R}^n$ with inverse

$$v\mapsto\sigma=\exp(v\wedge v_\infty)\cdot v_0 = v+v_0+\tfrac12(v,v)v_\infty. \tag{7.4}$$

Since $(d_X\sigma,v_\infty)=0$, we have $(d_Xv,d_Yv)=(d_X\sigma,d_Y\sigma)=g(X,Y)$ and so stereoprojection is also an isometry for the metric g induced by σ. This proves that flat conformal Cartan geometries are conformally flat in the usual sense, the charts given by stereoprojection being conformal. Furthermore, these charts are not just conformal, but Möbius (which is only an issue for curves and surfaces): since $\mathcal{H}^c\ell=0=\mathcal{S}^c\ell$ (*i.e.*, $r^g=0$), the flat Möbius structure given by the flat metric on $\mathbb{R}^n$ pulls back to the flat Möbius structure on S^n.

7.3. Symmetry breaking. The introduction of a constant vector $v_\infty\in\mathbb{R}^{n+1,1}$ breaks symmetry from the Möbius group of S^n to the isometry group $Stab(v_\infty)$ of a spaceform (or the homothety group in the flat case). Such constant vectors sometimes arise naturally in conformal submanifold geometry. Let us study, more generally, a constant $(k+1)$-plane W (or equivalently a constant decomposable $(k+1)$-vector ω up to scale) for $0\leqslant k\leqslant n$.

- If the induced metric on W is nondegenerate, then replacing W by $W^\perp$ and k by $n-k$ if necessary, we may assume that W has signature $(k,1)$ and $W^\perp$ has signature $(n-k+1,0)$. Now any null vector σ not in W can be scaled so that $\sigma=w+x$ where w is a positively oriented timelike unit vector in W and x is spacelike unit vector in $W^\perp$, both uniquely determined. This defines an isometry from $(\mathcal{H}^k+W^\perp)\cap\mathcal{L}$ to $\mathcal{H}^k\times\mathcal{S}^{n-k}$, where $\mathcal{H}^k$ is (the positive sheet of) the hyperboloid in W, and $\mathcal{H}^k\times\mathcal{S}^{n-k}$ is equipped with a product of constant curvature metrics. Hence we obtain a conformal diffeomorphism from $P(\mathcal{L})\smallsetminus P(W\cap\mathcal{L})$, *i.e.*, $S^n\smallsetminus S^{k-1}$, to $\mathcal{H}^k\times\mathcal{S}^{n-k}$.
- If the induced metric on W is degenerate, *i.e.*, W has signature $(k,0)$ and $W^\perp$ has signature $(n-k,0)$, then $W\cap W^\perp=W\cap\mathcal{L}=W^\perp\cap\mathcal{L}$ is a null line spanned by a null vector $\hat w$. If w_0 is another null vector with $(\hat w,w_0)=-1$ then any null vector σ not in the span of W is a constant multiple of $w_0+\lambda\hat w+x+y$ for unique $x\in W\cap\hat w_0^\perp$ and $y\in W^\perp\cap\hat w_0^\perp$, identifying $(w_0+W+W^\perp)\cap\mathcal{L}$ isometrically with a product $\mathbb{R}^k\times\mathbb{R}^{n-k}$ of euclidean spaces. Up to translation and scale this isometry is independent of the choice of w_0 and $\hat w$, we have a conformal diffeomorphism from $S^n\smallsetminus\{pt\}$ to $\mathbb{R}^k\times\mathbb{R}^{n-k}$.

As in the case of vectors, the choice W induces a Weyl structure $\hat\Lambda$, hence a compatible metric (up to homothety), on the open subset of S^n where $\Lambda\not\subseteq W,W^\perp$ such that $\Lambda\oplus\hat\Lambda$ is the $(1,1)$-plane $(\Sigma\times W\oplus\Lambda)\cap(\Sigma\times W^\perp\oplus\Lambda)$ on this open subset: in the nondegenerate case, $\hat\Lambda=\rho_W\Lambda$ where $\rho_W=id_W-id_{W^\perp}$; in the degenerate case $\hat\Lambda=W\cap W^\perp=<\hat w>$ is constant. Since $\Sigma\times W^\perp\cap(\Lambda\oplus\hat\Lambda)$ is a constant line inside $\Sigma\times W\oplus\hat\Lambda$ (and similarly for $\Sigma\times W\cap(\Lambda\oplus\hat\Lambda)$ inside $\Sigma\times W^\perp\oplus\hat\Lambda$), $\hat\Lambda$ is the Weyl structure of the product of constant curvature metrics.

If we view the representative $\omega \in \wedge^{k+1}W$ as a constant $(k+1)$-vector on S^n, then its Lie algebra homology class $[\omega]$ defines a section of $H_0(T^*S^n, S^n \times \wedge^{k+1}\mathbb{R}^{n+1,1})$, which turns out to be the bundle $\wedge^k(TS^n\,\Lambda)\,L \cong \wedge^k T^*S^n\,L^{k+1}$ and $[\omega]$ is a decomposable section tangent to one of the factors of the product structure. The equation $d\omega = 0$, viewed as an equation on $[\omega]$ implies that the latter is a *conformal Killing form* in the sense of [69].

This symmetry breaking is particularly straightforward when $k = 1$: a decomposable 2-form ω identifies $S^n \smallsetminus S^{n-2}$, $S^n \smallsetminus \{\mathrm{pt}\}$ or $S^n \smallsetminus S^0$ with $\mathcal{S}^1 \times \mathcal{H}^{n-1}$, $\mathbb{R} \times \mathbb{R}^{n-1}$ or $\mathcal{H}^1 \times \mathcal{S}^{n-1}$ respectively, according to whether ω is spacelike, null, or timelike, respectively (*i.e.*, whether the corresponding 2-plane is euclidean, degenerate, or lorentzian respectively). Note that ω may also be viewed as an element of $\mathfrak{so}(n+1, 1)$, *i.e.*, an infinitesimal Möbius transformation of S^n, its Lie algebra homology class $[\omega]$ being the vector field generating this action. Since ω is a wedge product of two vectors in W, it follows that $[\omega]$ generates of the obvious S^1 or $\mathbb{R}$ action on $\mathcal{S}^1 \times \mathcal{H}^{n-1}$, $\mathbb{R} \times \mathbb{R}^{n-1}$ or $\mathcal{H}^1 \times \mathcal{S}^{n-1}$.

Part II. Submanifolds and The Conformal Bonnet Theorem

Now we come to the heart of the matter: an immersed submanifold of a Möbius manifold inherits a conformal Möbius structure in a canonical way. In particular, a curve or surface immersed in a conformal n-manifold with $n \geqslant 3$ acquires more intrinsic geometry from the ambient space than merely a conformal metric.

The key ingredient of our theory is the notion of a Möbius reduction, introduced in §8.2 as a reduction of structure group along a submanifold Σ immersed in a Möbius manifold M. In §8.3, we show that this generalizes the classical notion of an enveloped sphere congruence. In §9.1 we compute the data induced on Σ by a Möbius reduction, and then in §9.2, using a gauge theoretic interpretation of Möbius reductions, we show how these data depend on the choice of Möbius reduction.

One of these data is an analogue of the second fundamental form (or shape operator) in riemannian geometry (we make the analogy precise in §11.3) and this datum only depends on the Möbius reduction through its trace, a generalized mean curvature conormal vector. This allows us to achieve our first goal in §9.3, where we show that there is a unique Möbius reduction whose second fundamental form is tracefree. By normalizing the induced Cartan connection, we thus obtain a canonical conformal Cartan geometry on Σ.

For submanifolds of the conformal n-sphere, the canonical Möbius reduction is the central sphere congruence [8]. In this case, we show in §10.2 that the primitive data induced on Σ by the reduction suffice to recover the ambient flat conformal Cartan connection. As we show in §10.3, this leads at once to a conformal Bonnet theorem, and we present two approaches to the Gauß–Codazzi–Ricci equations characterizing conformal immersions. The first, introduced in §10.4, is an unashamedly abstract treatment using Lie algebra homology and the Bernstein–Gelfand–Gelfand machinery of [25]. The second, presented in §11.5, uses the decomposition with respect to a Weyl structure (described and studied in in §§11.1–11.4) to give a more explicit formulation.

In §12.1, we specialize our theory to curves and study their Möbius invariants. We turn to surfaces in §12.2, where, with an eye on the applications in Part III, we study quadratic differentials and the relation between the differential lift and the Hodge star operator. We also relate our theory to the work of [18, 21].

8. Submanifolds and Möbius reductions

8.1. Submanifolds of Möbius manifolds. Let $\phi\colon \Sigma \to M$ be an immersion of an m-manifold Σ into a Möbius n-manifold M, with $n = m + k$ and $m, k > 0$. Thus, for $n \geqslant 3$, our assumption is no more than that M be a conformal manifold: it is only for the case of curves immersed in surfaces that we are imposing extra structure on the ambient space. In any case, M is equipped with a normal conformal Cartan geometry $(V_M, \Lambda, \mathfrak{D}^M)$.

The conformal metric on M restricts to give a conformal metric on Σ. At first we obtain a section of $S^2T^*\Sigma\,(L_M|_\Sigma)^2$, but then Proposition 1.4 identifies $L_M|_\Sigma$ with the density bundle L_Σ of Σ. (To generalize our theory to indefinite signature metrics we need to assume that the induced conformal metric on Σ is nondegenerate.) Henceforth, we write L and $\Lambda = L^{-1}$ for both the intrinsic and ambient line bundles, and often omit restriction maps (pullbacks to Σ). Thus $(V_M, \mathfrak{D}^M)$ will be viewed as a vector bundle with connection over Σ. Along Σ, we have an orthogonal decomposition

$$TM = T\Sigma \oplus N\Sigma,$$

where $N\Sigma$ is the normal bundle to Σ in M. Further, the operators π and p, defined by the soldering isomorphism of M, restrict to operators along Σ, so that in particular, we have $\pi\colon \Lambda^\perp \to TM\,\Lambda (\cong T^*M\,L)$ over Σ. We thus obtain two subbundles $\pi^{-1}(N\Sigma\,\Lambda)$ and $\pi^{-1}(T\Sigma\,\Lambda)$ of V_M with sum $\Lambda^\perp$ and intersection Λ. We denote the latter subbundle by $\Lambda^{(1)}$: it is the smallest subbundle of $\Lambda^\perp$ containing Λ and its tangential first derivatives, *i.e.*, $\mathfrak{D}^M_X\sigma \in \Lambda^{(1)}$ for all sections σ of Λ and $X \in T\Sigma$.

Our approach to Möbius submanifold geometry is to forget about M and study only $(V_M, \Lambda, \mathfrak{D}^M)$ along Σ. This will be particularly effective in the case $M = S^n$, since then V_M (restricted to Σ) is $\Sigma \times \mathbb{R}^{n+1,1}$ with $\mathfrak{D}^M$ just the flat derivative d, and the immersion of Σ into $S^n = P(\mathcal{L})$ can be recovered from the subbundle $\Lambda \subset \Sigma \times \mathbb{R}^{n+1,1}$ (whose fibres are null lines in $\mathbb{R}^{n+1,1}$, hence points in $P(\mathcal{L})$).

8.2. Möbius reductions. We regard $(V_M, \Lambda, \mathfrak{D}^M)$ as a generalized conformal Cartan geometry on Σ. It is not actually a conformal Cartan geometry, since V_M has rank $n+2 > m+2$ so a reduction of structure group is required. This prompts the following definition.

Definition 8.1. A signature $(m+1,1)$ subbundle $V \subset V_M$ (over Σ) is called a *Möbius reduction* iff V contains the rank $m+1$ bundle $\Lambda^{(1)}$ (so that $\Lambda^{(1)} = \Lambda^\perp \cap V$).

Otherwise said, $V^\perp$ is a complement to Λ in $\pi^{-1}(N\Sigma\,\Lambda) \subset \Lambda^\perp$, or equivalently, an orthogonal complement to $\Lambda^{(1)}$ in $\Lambda^\perp$. From this, it is clear that Möbius reductions exist and that $V^\perp$ is a rank k bundle on which the metric is definite. Also $\pi\colon \Lambda^\perp \to TM\,\Lambda$ restricts to a metric isomorphism from $V^\perp$ onto the weightless normal bundle $N\Sigma\,\Lambda$.

A Möbius reduction V determines and is determined by a bundle map $\mathfrak{I}_V\colon N\Sigma\,\Lambda \to \pi^{-1}(N\Sigma\,\Lambda) \subset \Lambda^\perp$ with $\pi \circ \mathfrak{I}_V = \mathit{id}_{N\Sigma\,\Lambda}$ via $V^\perp = \mathit{im}\,\mathfrak{I}_V$. It follows that Möbius reductions form an affine space modelled on the sections of $\mathit{Hom}(N\Sigma\,\Lambda, \Lambda) \cong N^*\Sigma$. We write $V \mapsto V + \nu$ for this affine structure, where $\nu \in \mathrm{C}^\infty(\Sigma, N^*\Sigma)$: explicitly, $\mathfrak{I}_{V+\nu} = \mathfrak{I}_V - \nu \otimes \mathit{id}_\Lambda$. (The sign here is for consistency with our sign convention for Weyl structures.)

8.3. Sphere congruences. When $M = S^n$, Möbius reductions admit a classical interpretation. In this case, $V_M \to \Sigma$ is just the trivial $\mathbb{R}^{n+1,1}$ bundle, $\mathfrak{D}^M$ is flat differentiation d, and the line bundle $\Lambda \to \Sigma$ is the pullback of the tautological bundle by the immersion $\phi\colon \Sigma \to S^n$. Positive sections of Λ are precisely the *lifts* $f\colon \Sigma \to \mathcal{L}^+$ of ϕ with $q \circ f = \phi$.

Recall from §1A that ℓ-spheres in the conformal sphere $S^n = P(\mathcal{L})$ correspond bijectively to signature $(\ell+1,1)$ subspaces of $\mathbb{R}^{n+1,1}$ (or equivalently to the orthogonal spacelike $(n-\ell)$-planes) via $W \mapsto P(W \cap \mathcal{L})$. Hence the space of ℓ-spheres in S^n is identified with the grassmannian of signature $(\ell+1,1)$ subspaces of $\mathbb{R}^{n+1,1}$, and a signature $(\ell+1,1)$ subbundle of $\Sigma \times \mathbb{R}^{n+1,1}$ is the same as a map of Σ into this space.

Definition 8.2. A *ℓ-sphere congruence* parameterized by an m-manifold Σ is a signature $(\ell+1,1)$ subbundle V of $\Sigma \times \mathbb{R}^{n+1,1}$. It is said to be *enveloped* by an immersion ϕ, *i.e.*, a null line subbundle Λ satisfying the Cartan condition, iff $\Lambda^{(1)} \subset V$, *i.e.*, V contains the image of every lift f and its first derivatives $d_X f$, $X \in T\Sigma$.

Geometrically, the enveloping condition means that each ℓ-sphere $P(V_x \cap \mathcal{L})$ has first order contact with $\phi(\Sigma)$ at $\phi(x)$. When $\ell = m$, this is exactly the condition that V be a Möbius reduction for Λ. To summarize:

Möbius reductions are the same as m-sphere congruences enveloped by Σ.

9. Geometry of Möbius reductions

9.1. Splitting of the ambient connection. Let V a Möbius reduction of V_M along Σ. The orthogonal decomposition $V_M = V \oplus V^\perp$ induces a decomposition of $\mathfrak{D}^M$: we have metric connections $\mathfrak{D}^V$, ∇^V on V, $V^\perp$, respectively, given by composing the restriction of $\mathfrak{D}^M$ with orthoprojection. The remaining, off-diagonal, part of $\mathfrak{D}^M$ is the following.

Definition 9.1. The *Möbius differential* $\mathcal{N}^V : T\Sigma \to \mathit{Hom}(V, V^\perp) \oplus \mathit{Hom}(V^\perp, V)$ of a Möbius reduction V is defined by

$$\mathcal{N}^V_X \xi = (\mathfrak{D}^M_X \xi)^\top, \qquad \mathcal{N}^V_X v = (\mathfrak{D}^M_X v)^\perp,$$

for $v \in V$, $\xi \in V^\perp$, where $^\top, ^\perp$ denote the orthoprojections onto $V, V^\perp$ respectively.

Note that $\mathcal{N}^V$ is zero order in v, ξ, and $(\mathcal{N}^V_X \xi, v) = -(\xi, \mathcal{N}^V_X v)$ since $\mathfrak{D}^M$ is a metric connection. In Lie-theoretic terms, we have a fibrewise symmetric decomposition

$$\mathfrak{g} := \mathfrak{so}(V_M) = \mathfrak{h}_V \oplus \mathfrak{m}_V, \tag{9.1}$$

where $\qquad \mathfrak{h}_V = \mathfrak{so}(V) \oplus \mathfrak{so}(V^\perp), \qquad \mathfrak{m}_V = \mathfrak{g} \cap \big(\mathit{Hom}(V, V^\perp) \oplus \mathit{Hom}(V^\perp, V)\big).$

Accordingly $\mathfrak{D}^M$ decomposes as

$$\mathfrak{D}^M = \mathfrak{D}^{\nabla,V} + \mathcal{N}^V, \tag{9.2}$$

where $\mathfrak{D}^{\nabla,V} = \mathfrak{D}^V + \nabla^V$ is the direct sum connection and $\mathcal{N}^V$ is a $\mathfrak{m}_V$-valued 1-form.

The fundamental tools in submanifold geometry are the equations which give the curvature of $\mathfrak{D}^M$ in terms of this decomposition:

$$(R^{\mathfrak{D}^M})_{\mathfrak{h}} = R^{\mathfrak{D}^{\nabla,V}} + \tfrac{1}{2}[\mathcal{N}^V \wedge \mathcal{N}^V]; \tag{9.3a}$$

$$(R^{\mathfrak{D}^M})_{\mathfrak{m}} = d^{\mathfrak{D}^{\nabla,V}} \mathcal{N}^V. \tag{9.3b}$$

We shall see that the $\mathfrak{so}(V)$ and $\mathfrak{so}(V^\perp)$ components of (9.3a) are generalized versions of the Gauß and Ricci equations respectively, while (9.3b) is a Codazzi equation. As they stand, these equations are conceptually simple and general: V_M could be any vector bundle with $\mathfrak{g}$ connection and V any nondegenerate subbundle. However, they depend on the choice of Möbius reduction and we need to understand this dependence. First, though, there is a lot of information hidden in the interaction with the tautological line bundle Λ, which we now begin to unravel.

Let V a Möbius reduction of $(V_M, \Lambda, \mathfrak{D}^M)$ along Σ. Then $\Lambda^{(1)} \subset V$, so that $\mathfrak{D}^M_X \sigma \in V$ and thus

$$\mathfrak{D}^V_X \sigma = \mathfrak{D}^M_X \sigma, \qquad \mathcal{N}^V_X \sigma = 0,$$

for all sections σ of Λ and $X \in T\Sigma$. This has several consequences for the components $\mathfrak{D}^V$, ∇^V and $\mathcal{N}^V$ of $\mathfrak{D}^M$. First, $\mathfrak{D}^V$ is a conformal Cartan connection for the conformal structure on Σ inherited from M: the soldering form of $\mathfrak{D}^V$ is the pullback of that of $\mathfrak{D}^M$. In particular, $\pi|_{\Lambda^{(1)}}$ is the operator $\pi_\Sigma : \Lambda^{(1)} \to T\Sigma\,\Lambda$ defined by the soldering form of $\mathfrak{D}^V$, and we shall drop the subscript henceforth.

Moreover, $\mathfrak{D}^V_X \sigma = \mathfrak{D}^M_X \sigma$ implies that $R^{\mathfrak{D}^V}_{X,Y} \sigma$ is the projection onto V of $R^{\mathfrak{D}^M}_{X,Y} \sigma$ which vanishes since $\mathfrak{D}^M$ is strongly torsion-free. We summarize this as follows.

Proposition 9.2. *$(V, \varLambda, \mathfrak{D}^V)$ is a conformal Cartan geometry on $\varSigma$ whose induced conformal metric is the restriction of the conformal metric on M.*

Thus in addition to a conformal metric, $\varSigma$ acquires a Möbius structure $\mathcal{M}^V := \mathcal{M}^{\mathfrak{D}^V}$. When $dim\, \varSigma \leqslant 2$, this packs a punch as we shall see.

Now contemplate $\mathcal{N}^V$: since $\mathcal{N}^V|_\varLambda$ vanishes, $\mathcal{N}^V$ takes values in $\mathfrak{stab}(\varLambda) \cap \mathfrak{m}_V$ so that $\pi\mathcal{N}^V$ is a tensorial object: a 1-form with values in $\mathfrak{co}(TM)$.

Definition 9.3. The *second fundamental form* and *shape operator* of a Möbius reduction V are the bundle maps $\mathrm{II}^V : T\varSigma \otimes T\varSigma \to N\varSigma$ and $\mathbb{S}^V : T\varSigma \otimes N\varSigma \to T\varSigma$ defined in terms of the Möbius differential $\mathcal{N}^V$ by the following formulae:

$$\mathrm{II}^V_X Y\, \sigma = -\pi\big(\mathcal{N}^V_X(\mathfrak{D}^M_Y \sigma)\big), \qquad \langle \mathbb{S}^V_X U, Y\rangle \sigma^2 = (\mathcal{N}^V_X \mathcal{I}_V(U\otimes\sigma), \mathfrak{D}^M_Y\sigma);$$

thus $\mathbb{S}^V \in \Omega^1(\varSigma, Hom(N\varSigma, T\varSigma))$ is the transpose of $\mathrm{II}^V \in \Omega^1(\varSigma, Hom(T\varSigma, N\varSigma))$ and $\pi\mathcal{N}^V = \mathrm{II}^V - \mathbb{S}^V$. The *mean curvature covector* $H^V \in \mathrm{C}^\infty(\varSigma, N^*\varSigma)$ of V is given by $H^V = \frac{1}{m}\, tr\, \mathbb{S}^V = \frac{1}{m}\langle tr_{\mathsf{c}}\, \mathrm{II}^V, \cdot\rangle$.

Proposition 9.4. *Let V be a Möbius reduction of V_M on $\varSigma$. Then II^V is symmetric, i.e., is a section of $S^2T^*\varSigma \otimes N\varSigma$. (Equivalently $\mathbb{S}^V$ is a section of $N^*\varSigma \otimes Sym_{\mathsf{c}}(T\varSigma)$.)*

Proof. We have

$$(\mathrm{II}^V_Y X - \mathrm{II}^V_X Y)\sigma = \pi(\mathcal{N}^V_X(\mathfrak{D}^M_Y\sigma) - \mathcal{N}^V_Y(\mathfrak{D}^M_X\sigma)) = \pi((R^{\mathfrak{D}^M}_{X,Y}\sigma)^\perp) = 0$$

since $\mathfrak{D}^M$ is torsion-free. □

Finally ∇^V induces a metric connection $\nabla = \pi \circ \nabla^V \circ \mathcal{I}_V$ on $N\varSigma\, \varLambda$.

We now see how all these data depend on the choice of reduction V which will require a closer look at the space of Möbius reductions.

9.2. Gauge theory of Möbius reductions. The affine structure on the space of Möbius reductions may be usefully understood in terms of gauge transformations of V_M: we have $V + \nu = \exp(-\nu)V$. Indeed, for $v \in V_M$, $U \otimes \sigma \in N\varSigma\, \varLambda$, we compute

$$\exp(-\nu)\mathcal{I}_V(U\otimes\sigma) = \mathcal{I}_V(U\otimes\sigma) - \nu\cdot\mathcal{I}_V(U\otimes\sigma) = \mathcal{I}_V(U\otimes\sigma) - \langle\nu, U\rangle\sigma = \mathcal{I}_{V+\nu}(U\otimes\sigma)$$

so that $\exp(-\nu)\circ\mathcal{I}_V = \mathcal{I}_{V+\nu}$. Thus if $v \in V+\nu$, $\exp(\nu)v \in V$.

Proposition 9.5. *Let V be a Möbius reduction of V_M on $\varSigma$. Then, for $\nu \in \mathrm{C}^\infty(\varSigma, N^*\varSigma)$,*

$$\begin{aligned} \exp(\nu)\cdot\mathfrak{D}^{V+\nu} &= \mathfrak{D}^V + [\![\nu, \pi\mathcal{N}^V]\!] - \tfrac12 [\![\nu, \pi\mathfrak{D}^{\nabla,V}\nu]\!], \\ \exp(\nu)\cdot\mathcal{N}^{V+\nu} &= \mathcal{N}^V - \mathfrak{D}^{\nabla,V}\nu, \\ \exp(\nu)\cdot\nabla^{V+\nu} &= \nabla^V, \end{aligned} \tag{9.4}$$

and $[\mathcal{N}^V, \nu] - \frac12[\nu, \mathfrak{D}^{\nabla,V}\nu]$, $\mathfrak{D}^{\nabla,V}\nu$ take values in $\mathfrak{stab}(\varLambda)^\perp \cap \mathfrak{h}_V$, $\mathfrak{stab}(\varLambda)\cap\mathfrak{m}_V$ respectively.

Proof. From (3.2), we have $\exp(\nu)\cdot\mathfrak{D}^M = \mathfrak{D}^M - \mathfrak{D}^M\nu - \frac12[\![\nu, \pi\mathfrak{D}^M\nu]\!]$. Now $\mathfrak{D}^M\nu = \mathfrak{D}^{\nabla,V}\nu + [\mathcal{N}^V, \nu] = \mathfrak{D}^{\nabla,V}\nu - [\![\nu, \pi\mathcal{N}^V]\!]$, since $\mathcal{N}^V$ takes values in $\mathfrak{stab}(\varLambda)$. In particular, $\pi[\mathcal{N}^V, \nu] = 0$.

Since $V + \nu = \exp(-\nu)V$, $\exp(\nu)\cdot\mathfrak{D}^{\nabla,V}$ is the reduction of $\exp(\nu)\cdot\mathfrak{D}^M$ to $\mathfrak{h}_V$, whence

$$\begin{aligned} \exp(\nu)\cdot\mathfrak{D}^{\nabla,V+\nu} &= \big(\exp(\nu)\cdot\mathfrak{D}^M\big)_{\mathfrak{h}_V} = \mathfrak{D}^{\nabla,V} + [\![\nu, \pi\mathcal{N}^V]\!] - \tfrac12[\![\nu, \pi\mathfrak{D}^{\nabla,V}\nu]\!], \\ \exp(\nu)\cdot\mathcal{N}^{V+\nu} &= \big(\exp(\nu)\cdot\mathfrak{D}^M\big)_{\mathfrak{m}_V} = \mathcal{N}^V - \mathfrak{D}^{\nabla,V}\nu. \end{aligned}$$

(9.4) follows after restricting the first of these formulae to V and $V^\perp$. □

With this in hand, we can determine the dependence of our data on the Möbius reduction.

Proposition 9.6. *The conformal structure* c *on* Σ *and connection* ∇ *on* $N\Sigma\,\Lambda$ *are independent of* V*, whereas*

$$\mathcal{M}^{V+\nu} = \mathcal{M}^V + \nu(\mathrm{II}^V) - \tfrac{1}{2}\langle \nu, \nu\rangle \mathsf{c}, \tag{9.5}$$

$$\mathrm{II}^{V+\nu} = \mathrm{II}^V - \mathsf{c}\otimes\nu, \qquad \mathbb{S}^{V+\nu} = \mathbb{S}^V - \nu\otimes id. \tag{9.6}$$

Proof. The independence of the conformal structure is immediate from Proposition 9.2. Moreover, since $\pi\circ\exp(\nu)=\pi$, we have $\pi\circ\nabla^{V+\nu}\circ\mathcal{I}_{V+\nu} = \pi\circ\exp(\nu)\cdot\nabla^{V+\nu}\circ\mathcal{I}_V$ which is $\pi\circ\nabla^V\circ\mathcal{I}_V$ by Proposition 9.5. Thus ∇ is independent of V.

Similarly, $\pi\mathcal{N}^{V+\nu} = \pi(\exp(\nu)\cdot\mathcal{N}^{V+\nu}) = \pi\mathcal{N}^V - \pi\mathfrak{D}^{\nabla,V}\nu$ by Proposition 9.5. Moreover

$$\pi\mathfrak{D}^{\nabla,V}\nu = -[\![id,\nu]\!] = id\otimes\nu - \mathsf{c}\otimes\nu \tag{9.7}$$

and so (9.6) follows since $\mathrm{II}^V - \mathbb{S}^V = \pi\mathcal{N}^V$.

Finally, for (9.5), note that $\mathfrak{D}^{V+\nu}$ is gauge-equivalent to $\mathfrak{D}^V + [\![\nu,\pi\mathcal{N}^V]\!] - \frac{1}{2}[\![\nu,\pi\mathfrak{D}^{\nabla,V}\nu]\!]$ by Proposition 9.5, so that these connections induce the same Möbius structures, namely $\mathcal{M}^V + [\![\nu,\pi\mathcal{N}^V]\!] - \frac{1}{2}[\![\nu,\pi\mathfrak{D}^{\mathfrak{h}_V}\nu]\!]$. Moreover,

$$[\![\nu,\pi\mathcal{N}^V]\!] = [\![\nu,\mathrm{II}^V - \mathbb{S}^V]\!] = \nu(\mathrm{II}^V),$$
$$[\![\nu,\pi\mathfrak{D}^{\nabla,V}\nu]\!] = -[\![\nu,[\![id,\nu]\!]]\!] = \langle\nu,\nu\rangle\mathsf{c},$$

(using (9.7)), whence (9.5) follows. ∎

9.3. Normalization and the primitive data. The submanifold geometry developed thus far is not a pure theory of conformal submanifolds as it depends on the choice of a Möbius reduction. Furthermore, there is no reason for the induced conformal Cartan geometry to be normal. The key to rectifying this is equation (9.6), which shows that the tracefree part II^0 of the second fundamental form II^V (or equivalently the tracefree part $\mathbb{S}^0$ of the shape operator $\mathbb{S}^V$) is independent of the choice of V, whereas the mean curvature depends on V via $H^{V+\nu} = H^V - \nu$. Hence there is a unique Möbius reduction with tracefree second fundamental form—*i.e.*, vanishing mean curvature—given by $V + H^V = \exp(-H^V)V$ for any Möbius reduction V.

Definition 9.7. The unique Möbius reduction V_Σ with $H^{V_\Sigma}=0$ will be called the *canonical* Möbius reduction. We denote the Möbius differential of V_Σ by $\mathcal{N}$ and identify V_Σ with $N\Sigma\,\Lambda$ so that $\nabla^{V_\Sigma}=\nabla$. We also write $\mathfrak{h}$ for $\mathfrak{h}_{V_\Sigma}$ and $\mathfrak{m}$ for $\mathfrak{m}_{V_\Sigma}$ so that $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}$.

The canonical Möbius reduction is picked out by a homological condition: for any reduction V, $\mathcal{N}^V$ is a 1-chain in the complex defining the Lie algebra homology $H_\bullet(T^*\Sigma,\mathfrak{m}_V)$ (where the standard action on V and the trivial action on $V^\perp$ are used to define the $\mathfrak{so}(V)$-module structure on $\mathfrak{m}_V$). Applying the boundary operator to this 1-chain gives

$$\partial\mathcal{N}^V = \textstyle\sum_i[\varepsilon_i,\mathcal{N}^V_{e_i}] = \sum_i[\![\varepsilon_i,\pi\mathcal{N}^V_{e_i}]\!] = \sum_i[\![\varepsilon_i,\mathrm{II}^V_{e_i} - \mathbb{S}^V_{e_i}]\!] = -\,tr\,\mathbb{S}^V.$$

Thus V_Σ is the unique reduction whose Möbius differential is a 1-cycle. When $m=1$, $\mathcal{N}$ is a section of $T^*\Sigma\otimes N^*\Sigma$, which we shall call the *conformal acceleration* A of the curve.

We now have a canonical choice $(V_\Sigma,\Lambda,\mathfrak{D}^{V_\Sigma})$ of conformal Cartan geometry on Σ. The remaining problem is that the associated Möbius structure need not be a conformal Möbius structure, that is, $\mathfrak{D}^{V_\Sigma}$ need not be normal (when $m\geqslant 2$). But now Proposition 5.11 comes to the rescue and assures us of a unique section Q of $S^2T^*\Sigma$ such that

- $\mathfrak{D}^{V_\Sigma} - Q$ is a normal Cartan connection;
- $Q_0 = 0$ when $m=2$ and $Q=0$ when $m=1$ (*i.e.*, Q is in the image of ∂).

We denote the normal Cartan connection so defined by $\mathfrak{D}^\Sigma = \mathfrak{D}^{V_\Sigma} - Q$ and the induced conformal Möbius structure by $\mathcal{M}^\Sigma = \mathcal{M}^{V_\Sigma} - Q$.

Theorem 9.8. *Let $\phi\colon \Sigma \to M$ be an immersion of an m-manifold Σ into a Möbius n-manifold M. Then ϕ equips Σ canonically with the following primitive data*:

- *a conformal Möbius structure* $(\mathsf{c}, \mathcal{M}^{\Sigma})$;
- *a rank $n-m$ euclidean vector bundle $N\Sigma\,\Lambda$ with a metric connection ∇*;
- *a section $\mathrm{I\!I}^0$ of $S^2_0 T^*\Sigma \otimes N\Sigma$ for $m \geqslant 2$*;
- *a section A of $T^*\Sigma \otimes N^*\Sigma$ for $m = 1$.*

In addition there is an affine bijection $V \mapsto -H^V$ from Möbius reductions to sections of $N^*\Sigma$ such that the inverse image of zero is the canonical Möbius reduction.

It is not immediately apparent that these data suffice to recover the connection $\mathfrak{D}^M = \mathfrak{D}^{\Sigma} + \nabla + Q + \mathcal{N}$ on $V_M = V_\Sigma \oplus V_\Sigma^\perp$, but we shall see that they do when $\mathfrak{D}^M$ is flat.

The primitive data can be computed from an arbitrary Möbius reduction V by writing $V = V_\Sigma - H^V$ and applying Proposition 9.6 to see that V gives the same conformal metric and normal connection as V_Σ, whereas the remaining data are related by

$$\mathrm{I\!I}^0 = \mathrm{I\!I}^V - \mathsf{c} \otimes H^V, \qquad \mathbb{S}^0 = \mathbb{S}^V - id \otimes H^V \qquad (m \geqslant 2) \tag{9.8}$$

$$A = \mathcal{N}^V - \mathfrak{D}^{\nabla,V} H^V \qquad (m = 1) \tag{9.9}$$

$$\mathcal{M}^{\Sigma} = \mathcal{M}^{V_\Sigma} - Q = \mathcal{M}^V - Q^V, \qquad \text{where} \tag{9.10}$$

$$Q^V := Q - H^V(\mathrm{I\!I}^V) + \tfrac{1}{2}|H^V|^2 \mathsf{c} = Q - H^V(\mathrm{I\!I}^0) - \tfrac{1}{2}|H^V|^2 \mathsf{c}. \tag{9.11}$$

We can also decompose $\mathfrak{D}^M$ on $V \oplus V^\perp$ as $\mathfrak{D}^M = \mathfrak{D}^{\Sigma,V} + \nabla^V + Q^V + \mathcal{N}^V$, where $\mathfrak{D}^{\Sigma,V} = \mathfrak{D}^V - Q^V$ is the normal Cartan connection $\exp(H^V) \cdot \mathfrak{D}^{\Sigma}$ on V.

To summarize, given a submanifold $\phi\colon \Sigma \to M$ and a Möbius reduction V, we have a decomposition of the ambient Cartan connection

$$\mathfrak{D}^M = \mathfrak{D}^{\mathfrak{h}_V} + Q^V + \mathcal{N}^V \tag{9.12}$$

where $\mathfrak{D}^{\mathfrak{h}_V} := \mathfrak{D}^{\Sigma,V} + \nabla^V$ for a normal conformal Cartan connection $\mathfrak{D}^{\Sigma,V}$. By the formulae above, this decomposition induces the primitive data of Theorem 9.8. The curvature of $\mathfrak{D}^M$ with respect to this decomposition is given by:

$$(R^{\mathfrak{D}^M})_{\mathfrak{h}_V} = R^{\mathfrak{h}_V} + d^{\mathfrak{h}_V} Q^V + \tfrac{1}{2}[\mathcal{N}^V \wedge \mathcal{N}^V]; \tag{9.13a}$$

$$(R^{\mathfrak{D}^M})_{\mathfrak{m}_V} = d^{\mathfrak{h}_V} \mathcal{N}^V + [Q^V \wedge \mathcal{N}^V]. \tag{9.13b}$$

Here $d^{\mathfrak{h}_V}$ is the exterior derivative coupled to $\mathfrak{D}^{\mathfrak{h}_V}$, and $R^{\mathfrak{h}_V}$ is the curvature of $\mathfrak{D}^{\mathfrak{h}_V}$.

10. Lie algebra homology and the conformal Bonnet theorem

We now specialize to the case that the ambient manifold M is the conformal sphere S^n so that V_M is the trivial $\mathbb{R}^{n+1,1}$-bundle over Σ and $\mathfrak{D}^M = d$. In this setting, we will see that the immersion Λ of Σ is completely determined by the primitive data of Theorem 9.8. It follows that the equations of (9.13), with $V = V_\Sigma$, are equations on these data, the *Gauß–Codazzi–Ricci equations*. Furthermore, we shall see that one can recover (locally) an immersion of Σ into S^n from an arbitrary solution of these equations. In all this, we make substantial use of the machinery of Lie algebra homology and the associated theory of Bernstein–Gelfand–Gelfand operators. The Reader with no taste for such abstraction is directed to section 11 where Weyl structures are used to provide a more explicit formulation.

Before all this, however, we show that the canonical Möbius reduction is an object well-known to conformal submanifold geometers.

10.1. The central sphere congruence. For $M = S^n$, we identified Möbius reductions with the classical notion of enveloped sphere congruences in §8.3. The canonical Möbius reduction is also known classically: it is *central sphere congruence* or *conformal Gauß map* [10, 76] which may be defined in terms of a lift $f\colon \Sigma \to \mathcal{L}^+$ by

$$V_{\mathrm{cent}} = <f, df, tr\, Ddf>.$$

Here D is the Levi-Civita connection of (df, df), which is also used to compute the trace. Equivalently, if $e_i \otimes f$ is an orthonormal basis of $T\Sigma\,\Lambda$, $V_{\mathrm{cent}} = <f, df, \sum_i d_{e_i} d_{e_i} f>$. Now we compute the $V_\Sigma^\perp$ component of $\sum_i d_{e_i} d_{e_i} f$ to be $\sum_i \mathcal{N}_{e_i}(d_{e_i} f)$, π of which is $-\sum_i \mathrm{II}^0_{e_i} e_i\, f = 0$. Thus $V_{\mathrm{cent}} = V_\Sigma$.

More geometrically, $\Sigma_{(x)} := P((V_\Sigma)_x \cap \mathcal{L})$ is the *mean curvature sphere*, *i.e.*, the unique sphere tangent to Σ at x, which has the same mean curvature covector as Σ with respect to some (hence any) enveloped sphere congruences V of Σ and $V_{(x)}$ of $\Sigma_{(x)}$ which agree at x. Indeed, the central sphere congruence of $\Sigma_{(x)}$ is clearly constant, equal to $(V_\Sigma)_x$ (the Möbius differential is zero, which is certainly a cycle). Hence Σ and $\Sigma_{(x)}$ have central sphere congruences which agree at x. As we shall in §11.3, it follows that Σ and $\Sigma_{(x)}$ have the same mean curvature covector at x with respect to some (hence any) ambient metric.

10.2. Lifting the primitive data. Consider an immersion $\phi\colon \Sigma \to S^n$ with central sphere congruence (*i.e.*, canonical Möbius reduction) V_Σ. Then, using the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ induced by $\Sigma \times \mathbb{R}^{n+1,1} = V_\Sigma \oplus V_\Sigma^\perp$, (9.12) specializes to give

$$d = \mathfrak{D}^{\mathfrak{h}} + Q + \mathcal{N}, \tag{10.1}$$

where $\mathfrak{D}^{\mathfrak{h}}$ is the $\mathfrak{h}$-connection $\mathfrak{D}^\Sigma + \nabla$, $\mathcal{N} \in \Omega^1(\Sigma, \mathfrak{m})$, and Q is the unique section of $S^2T^*\Sigma \cap im\,\partial$ for which $\mathfrak{D}^\Sigma$ is normal.

Write $d^{\mathfrak{h}}$ for $d^{\mathfrak{D}^{\mathfrak{h}}}$ and $R^{\mathfrak{h}}$ for the curvature of $\mathfrak{D}^{\mathfrak{h}}$. Then (9.13) specializes as follows:

$$0 = R^{\mathfrak{h}} + d^{\mathfrak{h}} Q + \tfrac{1}{2}[\mathcal{N} \wedge \mathcal{N}]; \tag{10.2a}$$

$$0 = d^{\mathfrak{h}} \mathcal{N} + [Q \wedge \mathcal{N}]. \tag{10.2b}$$

Recall that our normalization conditions on the Möbius reduction V_Σ and the Möbius structure $\mathcal{M}^\Sigma$ amount to the requirement that $R^{\mathfrak{h}}$ and $\mathcal{N}$ are cycles in $Z_2(\Sigma, \mathfrak{h})$ and $Z_1(\Sigma, \mathfrak{m})$ respectively (the former condition is equivalent to $R^{\mathfrak{D}^\Sigma}$ being a cycle). Moreover, we see that the homology class of $\mathcal{N}$ yields primitive data:

$$[\mathcal{N}] = \begin{cases} \mathrm{II}^0 - \mathbb{S}^0 & \text{for } m \geqslant 2; \\ A & \text{for } m = 1. \end{cases}$$

$$H_1(T^*\Sigma, \mathfrak{m}) = \begin{cases} \big(S^2_0 T^*\Sigma \otimes N\Sigma \oplus Sym_c(T\Sigma) \otimes N^*\Sigma\big) \cap \Omega^1(\Sigma, \mathfrak{co}(TM)) & \text{for } m \geqslant 2; \\ T^*\Sigma \otimes N^*\Sigma \subset \Omega^1(\Sigma, T^*M) & \text{for } m = 1. \end{cases}$$

This is analogous to the primitive data given by the homology class $[R^{\mathfrak{D}^\Sigma}]$, which is the Weyl curvature of (Σ, c) when $m \geqslant 4$ and the Cotton–York curvature of (Σ, c) or $(\Sigma, \mathsf{c}, \mathcal{M}^\Sigma)$ when $m = 3$ or $m = 2$ respectively.

We now show that both $\mathcal{N}$ and Q can be recovered from these homology classes, from which it will follow that the Gauß–Codazzi–Ricci equations (10.2) are equations on these data. In §10.4, we shall realize these equations in terms of Bernstein–Gelfand–Gelfand (BGG) differential operators on Lie algebra homology bundles. A key ingredient is the differential lift [25, 35] associated to $\mathfrak{D}^\Sigma$ that we introduced in §4A. We let $\Box_{\mathfrak{h}}$ denote $\Box_{\mathfrak{D}^\Sigma}$ coupled to the connection ∇ on $V_\Sigma^\perp$, to obtain a generalized differential lift $j^{\mathfrak{h}}$ which can be applied to homology classes, such as $[\mathcal{N}]$, with a normal component.

Proposition 10.1. *Let $(V_\Sigma, \varLambda, \mathfrak{D}^\Sigma)$ be a normal conformal Cartan geometry on Σ, let $(V_\Sigma^\perp, \nabla)$ be a euclidean vector bundle with metric connection, let $\mathcal{N}$ be a $\mathfrak{m}$-valued 1-form in $\ker \partial$, and let Q be a $\mathfrak{h}$-valued 1-form in $\operatorname{im} \partial$.*

Suppose that these data satisfy the full Gauß–Codazzi–Ricci equations (10.2). *Then $Q = -\frac{1}{2}\Box_{\mathfrak{h}}^{-1}\partial[\![[\mathcal{N}] \wedge [\mathcal{N}]]\!]$ and $\mathcal{N} = j^{\mathfrak{h}}[\mathcal{N}]$. Furthermore, $\mathcal{N} \in \Omega^1(\Sigma, \mathfrak{m} \cap \mathfrak{stab}(\varLambda))$ and $\pi\mathcal{N}$ is symmetric and tracefree, whereas $Q \in \Omega^1(\Sigma, T^*\Sigma)$ and is symmetric.*

Proof. Since $\partial R^{\mathfrak{h}} = 0$, (10.2a) implies that $\Box_{\mathfrak{h}} Q = \partial d^{\mathfrak{h}} Q = -\frac{1}{2}\partial[\mathcal{N} \wedge \mathcal{N}]$, so that $Q = -\frac{1}{2}\Box_{\mathfrak{h}}^{-1}\partial[\mathcal{N} \wedge \mathcal{N}]$ since it is in the image of ∂. Now since $\partial\mathcal{N} = 0$, $\mathcal{N}$ takes values in $\mathfrak{stab}(\varLambda) \cap \mathfrak{m}$ and $\pi\mathcal{N}$ is tracefree. Thus Q takes values in $\mathfrak{stab}(\varLambda)^\perp \cap \mathfrak{h} = T^*\Sigma$, and hence $[Q \wedge \mathcal{N}]$ is a 2-form with values in $\mathfrak{stab}(\varLambda)^\perp \cap \mathfrak{m} \subset \mathit{Hom}(V_\Sigma/\varLambda^\perp, V_\Sigma^\perp) \oplus \mathit{Hom}(V_\Sigma^\perp, \varLambda)$. Since ∂ vanishes on $\varLambda$, $\partial[Q \wedge \mathcal{N}] = 0$, so that (10.2b) implies $\partial d^{\mathfrak{h}}\mathcal{N} = 0$ and hence $\mathcal{N} = j^{\mathfrak{h}}[\mathcal{N}]$. We also deduce $\partial[\![\mathit{id} \wedge \pi\mathcal{N}]\!] = 0$, and then $[\![\mathit{id} \wedge \pi\mathcal{N}]\!] = 0$, *i.e.*, $\pi\mathcal{N}$ is symmetric.

It remains to establish the symmetry of Q. Since $\Box_{\mathfrak{h}} Q = \partial[\![\mathit{id} \wedge Q]\!]$, $\Box_{\mathfrak{h}}$ is an algebraic isomorphism on $\mathit{im}\,\partial$, so it suffices to show that $\partial[\mathcal{N} \wedge \mathcal{N}]$ is symmetric. This equals ∂R where $R = [\![\pi\mathcal{N} \wedge \pi\mathcal{N}]\!]$, which is in $\Omega^2(\Sigma, \mathfrak{so}(T\Sigma))$. By the Jacobi identity and the symmetry of $\pi\mathcal{N}$, R satisfies the algebraic Bianchi identity $[\![\mathit{id} \wedge R]\!] = 0$. It follows that its associated Ricci contraction ∂R is symmetric. $\square$

We end by noting that since Q and $\mathcal{N}$ are determined by the primitive data, then, given H^V, so are $Q^V = Q - H^V(\mathrm{II}^0) - \frac{1}{2}|H^V|^2\mathfrak{c}$ and $\mathcal{N}^V = \exp(H^V)\cdot\mathcal{N} + \mathfrak{D}^{\mathfrak{h}_V} H^V$. This generality turns out to be convenient in applications, so we digress from the homological theory to give a general formulation of the conformal Bonnet theorem.

10.3. The conformal Bonnet theorem for enveloped sphere congruences. Let $\varLambda$ be an immersion of Σ, and V an enveloped sphere congruence with mean curvature covector H^V. Then (9.12) gives a decomposition of the flat connection on $\Sigma \times \mathbb{R}^{n+1,1} = V \oplus V^\perp$ as

$$\mathfrak{D}^{\mathfrak{h}_V} + Q^V + \mathcal{N}^V, \tag{10.3}$$

where $Q^V \in \Omega^1(\Sigma, T^*\Sigma)$ is the unique section of $S^2T^*\Sigma$ such that $\mathfrak{D}^{\mathfrak{h}_V}$ restricts to the normal conformal Cartan connection on V inducing the conformal Möbius structure $\mathcal{M}^\Sigma$ of the central sphere congruence $V_\Sigma = V + H^V$. It follows from §10.2 that the primitive data of Theorem 9.8, together with H^V, determine Q^V and $\mathcal{N}^V$. These data then satisfy Gauß–Codazzi–Ricci equations, which are *equivalent* to the flatness of the connection (10.3):

$$0 = R^{\mathfrak{h}_V} + d^{\mathfrak{h}_V} Q^V + \tfrac{1}{2}[\mathcal{N}^V \wedge \mathcal{N}^V]; \tag{10.4a}$$

$$0 = d^{\mathfrak{h}_V}\mathcal{N}^V + [Q^V \wedge \mathcal{N}^V]. \tag{10.4b}$$

A converse is therefore available: the conformal immersion and enveloped sphere congruence can be recovered from the data they induce (*i.e.*, the primitive data of Theorem 9.8 and the mean curvature covector H^V). Our goal now is to establish such a converse.

To this end, let Σ be an m-manifold with conformal Möbius structure $(\mathfrak{c}, \mathcal{M}^\Sigma)$. Suppose that $N\Sigma \to \Sigma$ is a rank $n - m$ vector bundle with a metric and metric connection ∇ on $N\Sigma\, L^{-1}$, and that II^0 is a section of $S_0^2 T^*\Sigma \otimes N\Sigma$ for $m \geqslant 2$ (with transpose $\mathbb{S}^0$) and A is a section of $T^*\Sigma \otimes N^*\Sigma$ for $m = 1$. Finally let H^V be a section of $N^*\Sigma$.

Denote by $(V, \varLambda, \mathfrak{D}^{\Sigma,V})$ the induced normal conformal Cartan geometry (which identifies $\varLambda$ with L^{-1}) and set $V^\perp = N\Sigma\,\varLambda$, $\nabla^V = \nabla$ and $\mathfrak{D}^{\mathfrak{h}_V} = \mathfrak{D}^{\Sigma,V} + \nabla^V$. Let $\mathcal{N}^V = j^{\mathfrak{h}_V}(\mathrm{II}^0 - \mathbb{S}^0) + \mathfrak{D}^{\mathfrak{h}_V} H^V$ for $m \geqslant 2$ and $\mathcal{N}^V = A + \mathfrak{D}^{\mathfrak{h}_V} H^V$ for $m = 1$, where $j^{\mathfrak{h}_V}$ is the differential lift operator defined by $\mathfrak{D}^{\Sigma,V}$ and ∇^V. Last of all, let Q^V be given by (9.11), where $Q = -\frac{1}{2}\Box_{\mathfrak{h}}^{-1}\partial[\![(\mathrm{II}^0 - \mathbb{S}^0) \wedge (\mathrm{II}^0 - \mathbb{S}^0)]\!]$.

Then the connection (10.3) given by these data is flat iff (10.4) hold.

Theorem 10.2. *Let Σ be an m-manifold with conformal Möbius structure $(\mathsf{c}, \mathcal{M}^\Sigma)$, let $(N\Sigma\, L^{-1}, \nabla)$ be a rank $n-m$ euclidean vector bundle on Σ with metric connection, and let $\mathcal{N}^V$, Q^V be determined as above by these data together with a section H^V of $N^*\Sigma$ and a section II^0 of $S^2_0 T^*\Sigma \otimes N\Sigma$ (for $m \geqslant 2$) or A of $T^*\Sigma \otimes N^*\Sigma$ (for $m = 1$).*

Then there is locally an immersion of Σ into S^n with enveloped sphere congruence V inducing these data if and only if the equations of (10.4) hold. If so, then the immersion and enveloped sphere congruence are unique up to Möbius transformations of S^n.

(Obviously, when $m = 1$, the equations of this theorem are vacuous, so the theorem then shows that a projective curve can be embedded into S^n with arbitrary conformal acceleration A, and that H^V parameterizes the possible enveloped circle congruences.)

Proof. It remains to recover the immersion, assuming (10.4) holds. The bundle $V \oplus V^\perp$ has a metric of signature $(n+1,1)$—given by the sum of the metrics on the summands—for which Λ is null, and the connection $\mathfrak{D}^{\mathfrak{h}_V} + Q^V + \mathcal{N}^V$ on $V \oplus V^\perp$ is a metric connection, which is flat by (10.4). Thus, locally, we have a parallel metric isomorphism $V \oplus V^\perp \cong \Sigma \times \mathbb{R}^{n+1,1}$, unique up to constant gauge transformations, and the inclusion $\Lambda \to V_\Sigma \oplus V_\Sigma^\perp$ induces a map $\phi\colon \Sigma \to P(\mathcal{L})$; $x \mapsto \Lambda_x \subset \mathbb{R}^{n+1,1}$. The Cartan condition on $\mathfrak{D}^\Sigma$ ensures that ϕ is an immersion and $\mathcal{N}^V|_\Lambda = 0$ ensures that (the image in $\Sigma \times \mathbb{R}^{n+1,1}$ of) V is an enveloped sphere congruence, with mean curvature $\frac{1}{m}\, tr\, \mathrm{II}^V = H^V$. □

The above theorem is very general, but depends on a choice of enveloped sphere congruence and so it is not truly conformal: as we shall see in §13.2, it includes, as a special case, the Bonnet theorem for submanifolds of spaceforms. However, if we set $H^V = 0$ so that $V = V_\Sigma$ is the central sphere congruence, then (10.4) reduces to (10.2) and this specialization of Theorem 10.2 is purely conformal. In the next paragraph, though, we see that such a formulation of the conformal Bonnet Theorem is not optimal.

10.4. The abstract homological conformal Bonnet theorem. The normalization conditions used to fix the Möbius reduction and Möbius structure on Σ have the homological interpretation that $\mathcal{N}$ and $R^{\mathfrak{D}^\Sigma}$ are cycles, in $Z_1(\Sigma, \mathfrak{m})$ and $Z_2(\Sigma, \mathfrak{so}(V_\Sigma))$ respectively. We now show that the Gauß–Codazzi–Ricci equations also have a homological description, in terms of Bernstein–Gelfand–Gelfand (BGG) operators associated to the representations $\mathfrak{h} = \mathfrak{so}(V_\Sigma) \oplus \mathfrak{so}(V_\Sigma^\perp)$ and $\mathfrak{m}$ of $SO(V_\Sigma)$, where $V_\Sigma^\perp$ carries the trivial representation.

BGG sequences of invariant linear differential operators were first introduced on curved geometries by Eastwood–Rice [41], for 4-dimensional conformal geometry, and by Baston [4] and Čap–Slovak–Souček [35] in more general contexts. These constructions were simplified and extended to invariant multilinear differential operators in [25]. We need this generality here, and in particular, we need the operators $\Box_{\mathfrak{h}} = \partial \circ d^{\mathfrak{h}} + d^{\mathfrak{h}} \circ \partial$ and $\Pi = id - \Box_{\mathfrak{h}}^{-1} \circ \partial \circ d^{\mathfrak{h}} - d^{\mathfrak{h}} \circ \Box_{\mathfrak{h}}^{-1} \circ \partial$ of §4A, but here coupled to the normal connection ∇ (if required).

Using this, we can define linear and bilinear BGG operators between sections of Lie algebra homology bundles, closely related to the differential lift $j^{\mathfrak{h}}$. We shall only need them on $H_1(T^*\Sigma, \mathfrak{m})$, where, for sections $[\alpha]$ and $[\beta]$, we have

$$d_{BGG}[\alpha] = [\Pi d^{\mathfrak{h}} \Pi \alpha], \qquad\qquad [\alpha] \sqcup [\beta] = [\Pi([\Pi\alpha \wedge \Pi\beta])],$$

which are sections of $H_2(T^*\Sigma, \mathfrak{m})$ and $H_2(T^*\Sigma, \mathfrak{h})$ respectively.

We now have all the ingredients to establish our homological formulation of the Gauß–Codazzi–Ricci equations. These equations are simpler than the full equations since they only involve the homological objects $[\mathcal{N}]$ and $[R^{\mathfrak{h}}] = [R^{\mathfrak{D}^\Sigma}] + R^\nabla$. The Bernstein–Gelfand–Gelfand operators provides a conceptually elegant formulation of these equations, and the associated homological machinery yields a remarkably efficient proof.

Theorem 10.3. *Let $(V_\Sigma, \Lambda, \mathfrak{D}^\Sigma)$ be a normal conformal Cartan geometry on Σ, let $(V_\Sigma^\perp, \nabla)$ be a euclidean vector bundle with metric connection, let $\mathcal{N}$ be a $\mathfrak{m}$-valued 1-form in $\ker\partial$, and let Q be a $T^*\Sigma$-valued 1-form in $\operatorname{im}\partial$.*

Then the data $(\mathfrak{D}^\mathfrak{h}, \mathcal{N}, Q)$ satisfy the full Gauß–Codazzi–Ricci equations (10.2) if and only if $\mathcal{N} = j^\mathfrak{h}[\mathcal{N}]$, $Q = -\frac{1}{2}\square_\mathfrak{h}^{-1}\partial[\mathcal{N}\wedge\mathcal{N}]$, and the following homological Gauß–Codazzi–Ricci equations hold:

$$0 = [R^\mathfrak{h}] + \tfrac{1}{2}[\mathcal{N}] \sqcup [\mathcal{N}] \tag{10.5a}$$

$$0 = d_{BGG}[\mathcal{N}]. \tag{10.5b}$$

Proof. We first observe that $\partial R^\mathfrak{h} = 0$ and $d^\mathfrak{h} R^\mathfrak{h} = 0$, so that $R^\mathfrak{h} = j^\mathfrak{h}[R^\mathfrak{h}]$. By Proposition 10.1, we also know that (10.2) imply that $\mathcal{N} = j^\mathfrak{h}[\mathcal{N}]$ and $Q = -\frac{1}{2}\square_\mathfrak{h}^{-1}\partial[\mathcal{N}\wedge\mathcal{N}]$.

It remains to show that under these conditions, the equations (10.2) are equivalent to (10.5). By the uniqueness of the canonical differential representative, the latter are equivalent to

$$\begin{aligned} 0 &= R^\mathfrak{h} + \tfrac{1}{2}\Pi^2[\mathcal{N}\wedge\mathcal{N}] = R^\mathfrak{h} + \tfrac{1}{2}\Pi[\mathcal{N}\wedge\mathcal{N}] \\ &= R^\mathfrak{h} + \tfrac{1}{2}[\mathcal{N}\wedge\mathcal{N}] + d^\mathfrak{h} Q - \tfrac{1}{2}\square_\mathfrak{h}^{-1}\partial d^\mathfrak{h}[\mathcal{N}\wedge\mathcal{N}] \\ 0 &= \Pi d^\mathfrak{h}\mathcal{N} = d^\mathfrak{h}\mathcal{N} - \square_\mathfrak{h}^{-1}\partial[R^\mathfrak{h}\wedge\mathcal{N}]. \end{aligned}$$

Here we have expanded the definition of Π: for the first equation we have used the fact that $\Pi d^\mathfrak{h} Q = d^\mathfrak{h} Q - d^\mathfrak{h}\square_\mathfrak{h}^{-1}\partial d^\mathfrak{h} Q - \square_\mathfrak{h}^{-1}\partial(d^\mathfrak{h})^2 Q = -\square_\mathfrak{h}^{-1}\partial[R^\mathfrak{h}\wedge Q] = 0$, since $[R^\mathfrak{h}\wedge Q]$ has values in $T^*\Sigma$, hence is in the kernel of ∂. Now the second equation implies that $d^\mathfrak{h}\mathcal{N}$ has values in $V_\Sigma^\perp\otimes\Lambda$ and hence $\square_\mathfrak{h}^{-1}\partial d^\mathfrak{h}[\mathcal{N}\wedge\mathcal{N}] = 0$. By substitution of the first equation into the second, the equations (10.5) are therefore equivalent to

$$\begin{aligned} 0 &= R^\mathfrak{h} + d^\mathfrak{h} Q + \tfrac{1}{2}[\mathcal{N}\wedge\mathcal{N}] \\ 0 &= d^\mathfrak{h}\mathcal{N} + \square_\mathfrak{h}^{-1}\partial[d^\mathfrak{h} Q\wedge\mathcal{N}], \end{aligned}$$

since $[\mathcal{N}\wedge[\mathcal{N}\wedge\mathcal{N}]] = 0$ by the Jacobi identity. Now note that $[Q\wedge d^\mathfrak{h}\mathcal{N}] = 0$ since the Lie bracket of $T^*\Sigma$ with $V_\Sigma^\perp\otimes\Lambda$ is trivial. Hence the last term is equal to $\square_\mathfrak{h}^{-1}\partial d^\mathfrak{h}[Q\wedge\mathcal{N}]$.

We noted in the proof of Proposition 10.1 that $[Q\wedge\mathcal{N}]$ is in the kernel of ∂. The equivalence of the full and homological Gauß–Codazzi–Ricci equations now rests on the fact that it is actually in the image of ∂. For $m \geqslant 3$ this follows easily from the fact that $[Q\wedge\mathcal{N}]$ is a 2-form with values in $V_\Sigma^\perp\Lambda$, since $\partial\colon \wedge^3 T^*\Sigma\otimes\Lambda^\perp \to \wedge^2 T^*\Sigma\otimes\Lambda$ is then surjective. The result is vacuous for $m = 1$, so it remains to prove that for $m = 2$, $[Q\wedge\mathcal{N}] = 0$. For this, it suffices to show that $(Q\wedge\mathcal{N})_{X,Y}\nu = 0$ for any normal ν and vector fields X, Y. Now Q vanishes on Λ, so this reduces immediately to $Q_X(\mathbb{S}^0_Y\nu) - Q_Y(\mathbb{S}^0_X\nu)$, which vanishes because (for $m = 2$) $Q \in \operatorname{im}\partial$ is tracelike and $\mathbb{S}^0$ is symmetric. □

Remark 10.4. The subtle point in the above proof was the fact that $[Q\wedge\mathcal{N}] \in \operatorname{im}\partial$. If this had not worked out, the homology class would be $\langle[\mathcal{N}], [\mathcal{N}], [\mathcal{N}]\rangle$, where $\langle\cdot,\cdot,\cdot\rangle$ is one of the trilinear differential operators defined in [25]. It is thus a homological fluke that the Gauß–Codazzi–Ricci equations are quadratic in $\mathrm{I\!I}^0$ and $\mathbb{S}^0$. In more general circumstances, one should anticipate the appearance of multilinear differential operators of higher degree.

Although elegant, this theorem is quite abstract at the present, and we have already suggested that the Reader who prefers direct calculations to abstract machinery should turn to the next section, where we make the above equations and computations more explicit. For this reason, we postpone the statement of the homological conformal Bonnet theorem until we have understood the homological Gauß–Codazzi–Ricci equations more explicitly. However, for the Reader with no taste for the nitty-gritty, we note that the homological conformal Bonnet theorem which we will state as Theorem 11.4 is essentially the obvious corollary of Theorem 10.2, with $V = V_\Sigma$, and Theorem 10.3 above.

11. Weyl structures and the conformal Bonnet theorem

We have seen that Weyl derivatives and the associated apparatus of Weyl structures and connections provide an efficient computational tool in conformal geometry. The same is true in conformal submanifold geometry.

11.1. Decomposition with respect to a Weyl structure. Let Σ be an immersed submanifold of M. A Möbius reduction V provides a decomposition $\mathfrak{so}(V_M) = \mathfrak{h}_V \oplus \mathfrak{m}_V$ over Σ (9.1). On the other hand, a choice of Weyl structure on M may be restricted (pulled back) to Σ to give a decomposition $V_M \cong \Lambda \oplus U_M \oplus \hat{\Lambda}$, and hence $\mathfrak{so}(V_M) \cong T^*M \oplus \mathfrak{co}(TM) \oplus TM$ (§3A). (Here, as before, we omit pullbacks to Σ.)

If these data are compatible (*i.e.*, $\hat{\Lambda} \subset V$) then $V \cong \Lambda \oplus U \oplus \hat{\Lambda}$ with $U_M = U \oplus V^\perp$, so that

$$\mathfrak{so}(V_M) = \begin{array}{c}\mathfrak{h}_V\\ \oplus\\ \mathfrak{m}_V\end{array} \cong \begin{array}{c}T^*\Sigma \oplus \mathfrak{h}_0 \oplus T\Sigma\\ \oplus\\ N^*\Sigma \oplus \mathfrak{m}_0 \oplus N\Sigma\end{array} \cong T^*M \oplus \mathfrak{co}(TM) \oplus TM, \tag{11.1}$$

where $\mathfrak{h}_0 = \mathfrak{so}(T\Sigma) \oplus \mathfrak{so}(N\Sigma) \oplus \mathbb{R}\, id_{TM} \subset \mathfrak{co}(TM)$ and $\mathfrak{m}_0 = \mathfrak{co}(TM) \cap \big(Hom(T\Sigma, N\Sigma) \oplus Hom(N\Sigma, T\Sigma)\big)$. In the top line, $\mathfrak{h}_V = \mathfrak{so}(V) \oplus \mathfrak{so}(V^\perp)$, and the isomorphism identifies $\mathfrak{h}_0$ with $\mathfrak{co}(T\Sigma) \oplus \mathfrak{so}(V^\perp)$, and $\mathfrak{so}(V)$ with $T^*\Sigma \oplus \mathfrak{co}(T\Sigma) \oplus T\Sigma$. In the bottom line, $N^*\Sigma$ ($\cong V^\perp \Lambda$), $\mathfrak{m}_0$ and $N\Sigma$ are identified with the intersections of $\mathfrak{m}_V$ with $Hom(\Lambda, V^\perp) \oplus Hom(V^\perp, \hat{\Lambda})$, $Hom(U, V^\perp) \oplus Hom(V^\perp, U)$ and $Hom(\hat{\Lambda}, V^\perp) \oplus Hom(V^\perp, \Lambda)$ respectively.

For the Möbius reduction V, (9.12) gives

$$\mathfrak{D}^M = \mathfrak{D}^{\mathfrak{h}_V} + Q^V + \mathcal{N}^V,$$

where $\mathfrak{D}^{\mathfrak{h}_V} = \mathfrak{D}^{\Sigma,V} + \nabla^V$ and $\mathfrak{D}^{\Sigma,V}$ is a normal conformal Cartan connection on V. We now apply (11.1) in two ways.

First, $\hat{\Lambda} \subset V_M$ is the restriction (pullback) to Σ of a Weyl structure on M, and so $\mathfrak{D}^M = r^{D^M} + D^M - id$, where the normalized Ricci tensor r^{D^M} and the identity map id are viewed, by restriction, as 1-forms on Σ with values in T^*M and TM respectively.

Second, $\hat{\Lambda} \subset V$ is a Weyl structure on Σ, so $\mathfrak{D}^{\Sigma,V} = r^{D,\Sigma} + D^V - id$, where D is the corresponding Weyl derivative, $r^{D,\Sigma}$ is the normalized Ricci curvature of the induced conformal Möbius structure, D^V is the induced conformal connection, and $id = id_{T\Sigma}$.

Putting these together using (11.1), we have

$$\left\{\begin{array}{ccccccccc} \mathfrak{D}^M & = & r^{D^M} & + & D^M & - & id & = & (\mathfrak{D}^{\mathfrak{h}_V} + Q^V) + \mathcal{N}^V\\ \mathfrak{D}^{\mathfrak{h}_V} + Q^V & = & r^{D,\Sigma} + Q^V & + & D^V + \nabla^V & - & id & & \\ \mathcal{N}^V & = & A^{D,V} & + & \mathrm{II}^V - \mathbb{S}^V, & & & & \end{array}\right. \tag{11.2}$$

where we note that the $\mathfrak{m}_0$-component of $\mathcal{N}^V$ is given by the second fundamental form and shape operator of V. Comparing coefficients, we deduce that $r^{D^M} = (r^{D,\Sigma} + Q^V) + A^{D,V}$ and $D^M = (D^V + \nabla^V) + (\mathrm{II}^V - \mathbb{S}^V)$. The second equation is the decomposition of D^M (along Σ) into a direct sum connection on $U \oplus V^\perp \cong (T\Sigma \oplus N\Sigma)\,\Lambda$ and 1-forms valued in $Hom(T\Sigma, N\Sigma)$ and $Hom(N\Sigma, T\Sigma)$.

11.2. The Gauß–Codazzi–Ricci equations. The decomposition (11.1)–(11.2) may be used to expand (9.13a) as

$$(R^{\mathfrak{D}^M})_{T^*\Sigma} = d^D r^{D,\Sigma} + d^D Q^V + [\![(\mathrm{I\!I}^V - \mathbb{S}^V) \wedge A^{D,V}]\!] \tag{11.3a}$$

$$(R^{\mathfrak{D}^M})_{\mathfrak{h}_0} = R^D + R^\nabla - [\![id \wedge (r^{D,\Sigma} + Q^V)]\!] + \tfrac{1}{2}[\![(\mathrm{I\!I}^V - \mathbb{S}^V) \wedge (\mathrm{I\!I}^V - \mathbb{S}^V)]\!] \tag{11.3b}$$

$$0 = d^D id, \tag{11.3c}$$

while (9.13b) reads

$$(R^{\mathfrak{D}^M})_{N^*\Sigma} = d^{\nabla,D} A^{D,V} + [\![(\mathrm{I\!I}^V - \mathbb{S}^V) \wedge (r^{D,\Sigma} + Q^V)]\!] \tag{11.3d}$$

$$(R^{\mathfrak{D}^M})_{\mathfrak{m}_0} = d^{\nabla,D}\mathrm{I\!I}^V - d^{\nabla,D}\mathbb{S}^V - [\![id \wedge A^{D,V}]\!] \tag{11.3e}$$

$$0 = [\![id \wedge (\mathrm{I\!I}^V - \mathbb{S}^V)]\!], \tag{11.3f}$$

where $[\![\cdot]\!]$ denotes the algebraic bracket on $T^*M \oplus \mathfrak{co}(TM) \oplus TM$ and D^V is denoted by D for simplicity.

Since $\mathfrak{D}^M$ is the pullback of a normal Cartan connection, $R^{\mathfrak{D}^M} = C^{M,D^M} + W^M$ with C^{M,D^M} the pullback of the Cotton–York curvature of M (defined using any extension of D^M to M) and W^M the Weyl curvature of M. On the other hand, $R^D - [\![id \wedge r^{D,\Sigma}]\!] = W^\Sigma$, the Weyl curvature of $\mathcal{M}^\Sigma$, and $d^D r^{D,\Sigma} = C^{\Sigma,D} := C^{\mathcal{M}^\Sigma,D}$, the Cotton–York curvature of $\mathcal{M}^\Sigma$ with respect to D.

Using this, and expanding the algebraic brackets, we can rewrite (11.3) as follows. First, in (11.3a) and (11.3d), the algebraic brackets are simply contractions. Second, in (11.3b), the algebraic bracket is just the commutator, while in (11.3e) we have

$$[\![id \wedge A^{D,V}]\!] \cdot Z = (A^{D,V})^\sharp \wedge \mathsf{c}(Z, \cdot), \qquad [\![id \wedge A^{D,V}]\!] \cdot U = id \wedge A^{D,V}(U)$$

for tangent vectors Z and normal vectors U. Finally, (11.3c) and (11.3f) are identities since D is torsion-free and $\mathrm{I\!I}^V$ is symmetric.

To summarize, after restricting $\mathfrak{h}_0$ components to $T\Sigma$ and $N\Sigma$, the system (11.3), which is equivalent to (9.13), yields the following Gauß–Codazzi–Ricci equations:

The Gauß equations.

$$(W^M|_{T\Sigma})^\top = W^\Sigma - [\![id \wedge Q^V]\!] - \mathbb{S}^V \wedge \mathrm{I\!I}^V; \tag{11.4}$$

$$(C^{M,D^M})^\top = C^{\Sigma,D} + d^D Q^V + A^{D,V} \wedge \mathrm{I\!I}^V; \tag{11.5}$$

The Codazzi equations.

$$\begin{aligned} (W^M|_{T\Sigma})^\perp &= d^{\nabla,D}\mathrm{I\!I}^V - (A^{D,V})^\sharp \wedge \mathsf{c}, \\ (W^M|_{N\Sigma})^\top &= -d^{\nabla,D}\mathbb{S}^V - id \wedge A^{D,V}; \end{aligned} \tag{11.6}$$

$$(C^{M,D^M})^\perp = d^{\nabla,D} A^{D,V} - (r^{D,\Sigma} + Q^V) \wedge \mathbb{S}^V; \tag{11.7}$$

The Ricci equation.

$$(W^M|_{N\Sigma})^\perp = R^\nabla - \mathrm{I\!I}^V \wedge \mathbb{S}^V. \tag{11.8}$$

Note that the second equation of (11.6) is minus the transpose of the first.

11.3. Möbius reductions, ambient Weyl structures and riemannian metrics. We now discuss the relationship between Möbius reductions and Weyl structures on M, and hence describe the conformal geometry of submanifolds in the more familiar context of submanifolds in riemannian geometry.

For this, recall from section 3 that a Weyl derivative on M is a covariant derivative on L or $\Lambda = L^{-1}$; using $\mathfrak{D}^M$ and the induced conformal metric, this datum is equivalently a torsion-free conformal connection on TM or a null line subbundle $\hat{\Lambda} \subset V_M$ complementary to $\Lambda^\perp$. This last definition makes sense along a submanifold Σ, *i.e.*, we define an *ambient Weyl structure along* Σ to be such a complement $\hat{\Lambda}$ in $V_M|_\Sigma$. Thus any ambient Weyl structure is the restriction to Σ of some Weyl structure on M.

An ambient Weyl structure $\hat{\Lambda}$ along Σ determines a Möbius reduction $V = \Lambda^{(1)} \oplus \hat{\Lambda}$, together with a Weyl structure $\hat{\Lambda} \subset V$ for the conformal Cartan geometry $(V, \Lambda, \mathfrak{D}^V)$, or equivalently, a Weyl derivative D on Σ. The map sending $\hat{\Lambda}$ to the pair (V, D) is an affine bijection. Indeed, since ambient Weyl structures are equivalently complements $(\Lambda \oplus \hat{\Lambda})^\perp \cap V$ to Λ in $\Lambda^\perp$ (along Σ), they form an affine space modelled on $\mathrm{C}^\infty(\Sigma, \mathit{Hom}(\Lambda^\perp/\Lambda, \Lambda)) \cong \mathrm{C}^\infty(\Sigma, T^*M)$: $\hat{\Lambda} + \gamma = \exp(-\gamma) \cdot \hat{\Lambda}$ and one readily checks that this affine structure is induced by the natural affine structure $(V, D) + \gamma = (V + \gamma^\perp, D + \gamma^\top)$.

Remark 11.1. An ambient Weyl structure can be regarded as an operator

$$V_M \to T^*M\, L = T^*\Sigma\, L \oplus N^*\Sigma\, L$$

vanishing on Λ, whose restriction to $\Lambda^\perp/\Lambda$ is the soldering isomorphism. Identifying $V_M\, mod$ Λ with the restriction to Σ of $J^1 L$ on M, such operators arise as restrictions to Σ of the jet bundle map induced by a Weyl derivative on M. The $T^*\Sigma\, L$ component of this operator gives the jet bundle map $J^1 L \to T^*\Sigma\, L$ induced by the induced Weyl derivative on Σ, while the $N^*\Sigma L$ component is the operator $\mathcal{P}_V\colon V_M \to N^*\Sigma L$ with $\mathcal{P}_V(v) = \pi(v^\perp)$ induced by the Möbius reduction V. This provides another way to see the affine bijection between ambient Weyl structures and pairs (D, V)—furthermore, since D^V is the pullback to Σ of a Weyl connection on M, it transforms as a Weyl connection should: $(D + \gamma^\top)^{V+\gamma^\perp} = D^V + [\![\cdot, \gamma]\!]$.

As a special case of this, the canonical Möbius reduction may be described by a natural differential operator, just like the canonical Möbius structure $(\mathcal{H}^{\mathsf{c}}, \mathcal{S}^{\mathsf{c}})$, cf. Remark 4.4. Indeed, a positive section ℓ of L_M on a neighbourhood of Σ in M provides a metric with respect to which we can compute the mean curvature covector H^ℓ of Σ in M. Now if D is the Weyl derivative and Levi-Civita connection of the metric $\ell^2\mathsf{c}$, then $\mathcal{P}_{V_\Sigma} = \mathcal{P}_D + H^D p$, where $\mathcal{P}_D$ defines the Möbius reduction associated to D. Since $(\mathcal{P}_D\ mod\ \Lambda)(j^1\ell) = (D\ell)|_\Sigma^\perp = 0$ and $H^D = H^\ell$, we have $(\mathcal{P}_{V_\Sigma}\ mod\ \Lambda)(j^1\ell) = H^\ell \ell$.

If D^M is a Weyl connection on (a neighbourhood of Σ in) M, it induces an ambient Weyl structure along Σ, hence a Möbius reduction V and a Weyl structure on Σ, and any Möbius reduction and Weyl structure arise in this way. Further, the second fundamental form is given by the familiar expression $\mathrm{II}^V_X Y = (D^M_X Y)^\perp \in N\Sigma$.

In particular, let g be a compatible riemannian metric on (a neighbourhood of Σ) in M. Then, using the Levi-Civita connection of g, we obtain:

- a Möbius reduction V_g along Σ with $H^{V_g} = H^g$, *i.e.*, $\mathrm{II}^{V_g} = \mathrm{II}^g$, where II^g and H^g are the usual riemannian second fundamental form and mean curvature covector respectively;
- a Weyl structure $\hat{\Lambda}_g \subset V_g$ on Σ corresponding to the Levi-Civita connection D of the induced metric on Σ.

The other quantities associated with this reduction and Weyl structure are:

$$r^{D,V_g} = (r^g)^\top, \qquad A^{D,V_g} = (r^g)^\perp, \tag{11.9}$$

i.e., the components of the ambient normalized Ricci curvature r^g after the latter is pulled back to give a T^*M-valued 1-form on Σ. (See §11.1.)

According to §9.3, we can then compare V_g with the canonical Möbius reduction V_Σ by writing $V_\Sigma = V_g + H^g$, and hence relate the riemannian and conformal quantities. We have

$$\mathrm{I\!I}^0 = \mathrm{I\!I}^g - (H^g)^\sharp \otimes g, \qquad \mathbb{S}^0 = \mathbb{S}^g - id \otimes H^g \tag{11.10}$$

and the pullback D^g of the Levi-Civita connection of M to Σ is related to the intrinsic Levi-Civita connection D by

$$D^g = D + (\mathrm{I\!I}^g - \mathbb{S}^g) + \nabla^D = D^{V_\Sigma} + [\![\cdot, H^g]\!], \tag{11.11}$$

where ∇^D is the induced connection on $N\Sigma$ and $D^{V_\Sigma} = D + (\mathrm{I\!I}^0 - \mathbb{S}^0) + \nabla^D$ is the connection on $TM|_\Sigma$ induced by D and the canonical Möbius reduction V_Σ. We also have

$$r^{D,\Sigma} = (r^g)^\top - Q^g \tag{11.12}$$

$$A^D := A^{D,V_\Sigma} = (r^g)^\perp - \nabla^D H^g, \tag{11.13}$$

where $Q^g = Q - H^g(\mathrm{I\!I}^0) - \frac{1}{2}\langle H^g, H^g\rangle \mathsf{c}$. Substituting into (11.4)–(11.8), with $V = V_\Sigma$, the conformally invariant equations (11.4), (11.6) and (11.8) can be viewed as trace-free parts of the riemannian Gauß–Codazzi–Ricci equations.

When g is an Einstein metric, $(r^g)^\top$ is a multiple of c and $(r^g)^\perp = 0$. As we have seen in §7.2, such metrics arise when $M = S^n$, and we shall discuss this further in §13.2 and §13.3.

We emphasise that Möbius reductions (or Weyl structures) do *not* all arise from from ambient metrics.

11.4. Reduction to homological data. We next apply ∂ to the Gauß and Codazzi equations to get formulae for $A^{D,V}$ and Q^V.

Proposition 11.2. *Let $\phi\colon \Sigma \to M$ be a conformal immersion. Then*

$$(m-1)A^{D,V}_X = -(div^{\nabla,D}\,\mathrm{I\!I}^V)(X) + m\nabla^D_X H^V + \big(\textstyle\sum_i W^M_{e_i,X}\varepsilon_i\big)^\perp \tag{11.14}$$

$$\begin{aligned}(m-2)Q^V_0(X,Y) &+ 2(m-1)\big(\tfrac{1}{m}\, tr_{\mathsf{c}}\, Q^V\big)\langle X,Y\rangle \\ &= \langle \mathrm{I\!I}^V_X, \mathrm{I\!I}^V_Y\rangle - mH^V(\mathrm{I\!I}^V_X Y) - \textstyle\sum_i \varepsilon_i(W^M_{e_i,X}Y).\end{aligned} \tag{11.15}$$

Proof. Using an orthonormal frame e_i with dual frame ε_i, we compute that

$$\partial(id \wedge A^{D,V}(\xi))_X = \textstyle\sum_i \varepsilon_i \cdot (A^{D,V}_X(\xi)e_i - A^{D,V}_{e_i}(\xi)X) = (m-1)A^{D,V}_X(\xi),$$

while, since D is torsion-free, we have

$$\begin{aligned}\partial(d^{\nabla,D}\mathbb{S}^V)_X\xi &= \textstyle\sum_i \langle e_i, (\nabla\otimes D^V\,\mathbb{S}^V)_{e_i,X}\xi - (\nabla\otimes D^V\,\mathbb{S}^V)_{X,e_i}\xi\rangle \\ &= \textstyle\sum_i \langle -(\nabla\otimes D^V\,\mathrm{I\!I}^V)_{e_i,X}e_i + (\nabla\otimes D^V\,\mathrm{I\!I}^V)_{X,e_i}e_i, \xi\rangle \\ &= -\langle (div^{\nabla,D}\,\mathrm{I\!I}^V)(X), \xi\rangle + m\langle \nabla^D_X H^V, \xi\rangle.\end{aligned}$$

Applying ∂ to (11.6) therefore yields (11.14).

For the other equation, note that

$$\partial[\![id \wedge Q^V]\!] = (m-2)Q^V_0 + 2(m-1)\big(\tfrac{1}{m}\, tr_{\mathsf{c}}\, Q^V\big)\mathsf{c},$$

where we use (2.13), cf. (5.5). Next, using the symmetry properties of $\mathrm{I\!I}^V$, we have

$$\partial(\mathbb{S}^V \wedge \mathrm{I\!I}^V)_X Y = \textstyle\sum_i \big(\langle e_i, \mathbb{S}^V_{e_i}\mathrm{I\!I}^V_X Y\rangle - \langle e_i, \mathbb{S}^V_X \mathrm{I\!I}^V_{e_i} Y\rangle\big) = mH^V(\mathrm{I\!I}^V_X Y) - \langle \mathrm{I\!I}^V_X, \mathrm{I\!I}^V_Y\rangle.$$

Substituting these into the Ricci contraction of (11.4) yields (11.15). ∎

In particular, observe that (11.14) with $V = V_\Sigma$ and $A^D := A^{D,V_\Sigma}$, *i.e.*,

$$(m-1)A^D_X = -(div^{\nabla,D}\,\mathrm{II}^0)(X) + (\textstyle\sum_i W^M_{e_i,X}\varepsilon_i)^\perp \tag{11.16}$$

determines A^D and hence $\mathcal{N}$ entirely from the conformal metric, II^0 and the normal connection ∇, provided that $(\sum_i W^M_{e_i,X}\varepsilon_i)^\perp = 0$.

Remark 11.3. With a little more work, one can deduce a manifestly conformally invariant formula for $\mathcal{N}$: first note that div^D is independent of the Weyl derivative D when applied to symmetric trace-free 2-tensors of weight $-m$. If ℓ is a length scale, $\ell^{1-m}\mathrm{II}^0$ is such a tensor with values in $N\Sigma\Lambda$ so that $div^\nabla(\ell^{1-m}\mathrm{II}^0)$ is invariantly defined. Now a calculation using (2.12) yields

$$(m-1)\mathcal{N}_X(j^{\mathfrak{D}^\Sigma}\ell) = -\ell^m\, div^\nabla\left(\ell^{1-m}\mathrm{II}^0\right)(X) + \textstyle\sum_i (W^M_{e_i,X}\varepsilon_i)^\perp \otimes \ell. \tag{11.17}$$

As remarked in the introduction, this corrects an error in [70, Proposition **7**.4.9 (a)].

For submanifolds of S^n, equation (11.17) amounts to the assertion that $\mathcal{N} = j^{\mathfrak{h}}[\mathcal{N}]$, which we have already seen in Theorem 10.3.

Similarly (11.15), with $V = V_\Sigma$, determines Q completely from II^0 and the conformal metric (recall that Q_0 vanishes when $m = 2$ and $Q = 0$ when $m = 1$) provided $\sum_i \varepsilon_i(W^M_{e_i,X}Y) = 0$ (which obviously holds when M is conformally flat, in which case (11.15) is an explicit form of $\Box_{\mathfrak{h}} Q = -\frac{1}{2}\partial[\mathcal{N}\wedge\mathcal{N}]$ as in Theorem 10.3):

$$Q_X(Y) = \begin{cases} 0 & \text{for } m = 1;\\ \frac{1}{4}|\mathrm{II}^0|^2\langle X,Y\rangle & \text{for } m = 2;\\ \frac{1}{m-2}\big(\langle \mathrm{II}^0_X, \mathrm{II}^0_Y\rangle - \frac{1}{2(m-1)}|\mathrm{II}^0|^2\langle X,Y\rangle\big) & \text{for } m \geqslant 3.\end{cases} \tag{11.18}$$

When $m = 2$ and $\sum_i \varepsilon_i(W^M_{e_i,X}Y) = 0$, $Q = \frac{1}{4}|\mathrm{II}^0|^2\mathsf{c}$ is essentially the Willmore integrand [81] and from (9.10), we obtain

$$\mathcal{M}^\Sigma = \mathcal{M}^V - Q^V \qquad \text{where} \qquad Q^V = -H^V(\mathrm{II}^0) - \tfrac{1}{2}K^V\mathsf{c}, \tag{11.19}$$

and $K^V = |H^V|^2 - \frac{1}{2}|\mathrm{II}^0|^2 = \langle det\,\mathrm{II}^V\rangle$, the determinant being evaluated using the metric on $V^\perp$. We refer to K^V as the *gaussian curvature* of the surface with respect to V. With respect to a Weyl derivative D, we then have the following formula:

$$r^{D,\Sigma} = r^{D,V} + H^V(\mathrm{II}^0) + \tfrac{1}{2}K^V\mathsf{c}. \tag{11.20}$$

11.5. The concrete homological conformal Bonnet theorem. We now specialize once more to immersions of Σ into the conformal n-sphere $M = S^n$, when $V_M = \Sigma\times\mathbb{R}^{n+1,1}$ and $\mathfrak{D}^M$ is flat differentiation d. Our goal is to illuminate the abstract aspect of the homological conformal Bonnet theorem by describing the homological Gauß–Codazzi–Ricci equations in a more explicit way. We shall see that there really are fewer homological than full Gauß–Codazzi–Ricci equations: although the Ricci equation is unaffected, the Gauß and Codazzi equations simplify. We also indicate how to show this by direct computation.

To this end, we let $V = V_\Sigma$ be the central sphere congruence and introduce an arbitrary compatible Weyl structure $V_\Sigma = \Lambda\oplus\Lambda T\Sigma\oplus\hat\Lambda$ (with induced Weyl derivative D)

The Gauß–Codazzi–Ricci equations (10.4), with $\mathfrak{D}^M = d$ and $V = V_\Sigma$ (hence $H^V = 0$) specialize to the following:

$$0 = W^\Sigma - [\![id \wedge Q]\!] - \mathbb{S}^0 \wedge \mathrm{II}^0 \tag{11.21a}$$

$$0 = C^{\Sigma,D} + d^D Q + A^D \wedge \mathrm{II}^0 \tag{11.21b}$$

$$0 = d^{\nabla,D}\mathrm{II}^0 - (A^D)^\sharp \wedge \mathsf{c}, \qquad 0 = d^{\nabla,D}\mathbb{S}^0 + id \wedge A^D; \tag{11.21c}$$

$$0 = d^{\nabla^D} A^D - (r^{D,\Sigma} + Q) \wedge \mathbb{S}^0 \tag{11.21d}$$

$$0 = R^\nabla - \mathrm{II}^0 \wedge \mathbb{S}^0. \tag{11.21e}$$

Now the homological formulation shows that of the first four equations, only two are relevant in each dimension.

- If $m = 2$, there are no Weyl tensors or Codazzi tensors on Σ: (11.21a) and (11.21c) are trivial, while (11.21b) and (11.21d) are independent of D.
- If $m = 3$, there are still no Weyl tensors on Σ: (11.21a) is trivial, while (11.21d) is an automatic consequence of the others, which are independent of D,
- If $m \geqslant 4$, (11.21b) and (11.21d) follow from the others, which are independent of D.

It is possible to check these facts directly, without recourse to the general theory of [25] used above: it is straightforward that (11.21a) and (11.21c) are trivial in low dimensions, but rather more difficult to check (by differentiating) that (11.21b) and (11.21d) are consequences of these in higher dimensions. In fact these relations follow easily from the theory of the differential lift [25]: one must check that the right hand sides of the above equations satisfy $d^D(11.21a) = [\![id \wedge (11.21b)]\!]$ and $d^{D,\nabla}(11.21c) = [\![id \wedge (11.21d)]\!]$.

Instead of doing this, we shall instead concentrate on the fact that the relevant equations in each dimension are Möbius-invariant, *i.e.*, independent of D. This is clear for the Ricci equation (11.21e) and for (11.21a) (which is only relevant when $m \geqslant 4$). We now look at the other equations, and describe explicitly the Möbius-invariant operators involved.

We first note that (11.21c) (which is only relevant when $m \geqslant 3$) may be rewritten

$$\mathit{CCoda}^\nabla\, \mathbb{S}^0 = 0,$$

where *CCoda* is the conformal Codazzi operator

$$\mathit{CCoda}\, S = d^D S + \frac{1}{m-1} id \wedge (\mathit{div}^D\, S)^\flat \tag{11.22}$$

on symmetric tracefree endomorphisms S of weight -1. It is easy to check that *CCoda* is a conformally-invariant first order differential operator (see [44] for the general theory), which is identically zero for $m = 2$. Coupling to the connection ∇ on $LN^*\Sigma$ gives CCoda^∇.

We next consider (11.21d) when $m = 2$. Applying the Hodge star operator, we have

$$\mathit{div}^{D,\nabla}\, \mathit{div}^{D,\nabla}\, J\mathrm{II}^0 + \langle J\mathrm{II}^0, r_0^{D,\Sigma} \rangle = 0,$$

where we have used the fact that only the symmetric tracefree part $r_0^{D,\Sigma}$ of $r^{D,\Sigma} + Q$ can contribute to the second term, and the definition $\langle (J\mathrm{II}^0)_X, \xi \rangle = *\langle \mathrm{II}^0_X, \xi \rangle$ of the complex structure on $S^2_0 T^*\Sigma \otimes N\Sigma$. We can rewrite this as

$$(\mathcal{H}^\nabla)^*(J\mathrm{II}^0) = 0,$$

where $(\mathcal{H}^\nabla)^*$ is the formal adjoint of the Möbius structure coupled to ∇, *i.e.*, $\mathcal{H}^\nabla \nu = \mathit{sym}_0(D^\nabla)^2\nu + r_0^{D,\Sigma}\nu$. This is a Möbius-invariant second order differential operator.

Finally we must consider (11.21b) when $m = 2$ or 3. In both cases $Q_X(Y) = \langle \mathrm{II}^0_X, \mathrm{II}^0_Y \rangle - \frac{1}{4}|\mathrm{II}^0|^2 \langle X, Y \rangle$ (which equals $\frac{1}{4}|\mathrm{II}^0|^2\langle X, Y\rangle$ for $m = 2$) and $\mathit{CCoda}^\nabla\, \mathrm{II}^0 = 0$ so that $A^D \wedge \mathrm{II}^0 = \langle \mathrm{II}^0, d^{\nabla,D}\mathrm{II}^0 \rangle$. Hence (11.21b) is equivalent to the equation

$$C^\Sigma + \langle \mathcal{B}^\nabla(\mathrm{II}^0) \rangle = 0,$$

where

$$\mathcal{B}(S) = d^D(S^2 - \tfrac{1}{4}|S|^2 id) + S(d^D S) \tag{11.23}$$

is a quadratic first order differential operator on symmetric tracefree endomorphisms S of weight -1 (viewed also as $L^{-1}\,T\Sigma$-valued 1-forms), which is applied to II^0 by coupling to the connection ∇ and contracting by the metric $\langle\cdot\,,\cdot\rangle$ on the weightless normal bundle. Thus $\langle \mathcal{B}^\nabla(\mathrm{II}^0)\rangle = d^D Q + \langle \mathrm{II}^0, d^{\nabla,D}\mathrm{II}^0\rangle$. Again one can check (by differentiating with respect to D) that $\mathcal{B}$ is conformally-invariant for $m=2,3$. In the case $m=3$, the tensor Q was introduced by Cartan [36] in his study of conformally-flat hypersurfaces in S^4. We shall see its use in the study of conformally-flat submanifolds in Theorem 16.9.

Apart from $(\mathcal{H}^\nabla)^*$, the operators we have introduced are at most first order, and so depend only on c, not on the Möbius structure $\mathcal{M}$. The dependence of $(\mathcal{H}^\nabla)^*(J\mathrm{II}^0)$ is straightforward: if q is a quadratic differential then

$$((\mathcal{H}+q)^\nabla)^*(J\mathrm{II}^0) = (\mathcal{H}^\nabla)^*(J\mathrm{II}^0) + \langle q, J\mathrm{II}^0\rangle. \tag{11.24}$$

We summarize this discussion by stating our homological conformal Bonnet theorem.

Theorem 11.4. *Σ can be locally immersed in S^n with induced conformal Möbius structure $(\mathsf{c},\mathcal{M}^\Sigma)$, weightless normal bundle $(N\Sigma\Lambda,\nabla)$ and tracefree second fundamental form II^0 (or conformal acceleration A) if and only if (with $\mathbb{S}^0$ the transpose of II^0 and Q as in (11.18))*

$$0 = W^\Sigma - [\![id\wedge Q]\!] - \mathbb{S}^0\wedge \mathrm{II}^0 \qquad m\geqslant 4 \tag{11.25}$$

$$0 = C^\Sigma + \langle \mathcal{B}^\nabla(\mathrm{II}^0)\rangle \qquad m=2,3 \tag{11.26}$$

$$0 = CCoda^\nabla\,\mathrm{II}^0 \qquad m\geqslant 3 \tag{11.27}$$

$$0 = (\mathcal{H}^\nabla)^*(J\mathrm{II}^0) \qquad m=2 \tag{11.28}$$

$$0 = R^\nabla - \mathrm{II}^0\wedge \mathbb{S}^0. \tag{11.29}$$

Moreover, in this case, the immersion is unique up to Möbius transformations of S^n.

12. Submanifold geometry in the conformal sphere

12.1. Projective geometry of curves. In this section we briefly review the geometry of curves in $M=S^n$. In this case the primitive data on the curve is a projective structure $\Delta\colon J^2L^{1/2}\to L^{-3/2}$ (equivalently given by the bilinear operator $\mathcal{M}^\Sigma$), metric connection ∇ on a metric vector bundle $N\Sigma\Lambda$ of rank $n-1$, and the conformal acceleration A, a $N^*\Sigma$-valued 1-form. Any such data give rise to a local conformal immersion of Σ, unique up to Möbius transformation. The Möbius invariants of Σ can be computed in terms of an ambient Weyl connection D^M. This induces an intrinsic Weyl structure D with respect to which $\Delta = D^2 + \frac12 s^{D,\Sigma}$, where $s^{D,\Sigma} = r^{D^M}(T,T) + \frac12|H|^2$ for a weightless unit tangent vector T and (mean) curvature vector $H = D^M_T T$. The conformal acceleration A is given by $A = r^{D^M}(T)^\perp - \nabla^D_T H = r^{D^M}(T) - r^{D^M}(T,T)T - D^M_T H - |H|^2 T$.

By specializing to the case $r^{D^M}=0$, these invariants can be related to the standard euclidean theory of curves: here $H=\kappa N$ where N is the principal unit normal and κ is the curvature, and $D^M_T N = \tau B$ where B is the unit binormal, and τ is the torsion. Thus $s^{D,\Sigma} = \frac12\kappa^2$ and $A = -\dot\kappa N - \kappa\tau B$.

Vertices are points where $A=0$ and we can identify $\int_\Sigma |A|^{1/2}$ with the conformal arclength of Musso [63]: this is well defined because A is a section of $L^{-3}N\Sigma\cong L^{-2}V^\perp$, hence $|A|^{1/2}$ is a section of L^{-1} which can be integrated on a compact curve. Note if we now define $\Delta^D\mu = D^2\mu + ws^{D,\Sigma}\mu$ on sections μ of L^w, then $\partial_\gamma\Delta^D = (2w-1)\langle\gamma, D\mu\rangle$, which is algebraic in γ. This allows for the construction of further invariants such as $4\langle \Delta^{D,\nabla}A, A\rangle - 5\langle DA, DA\rangle$.

12.2. Surfaces, the conformal Bonnet theorem, and quaternions. Let Σ be a surface in S^n. Then the Hodge star operator on $\Omega^1(\Sigma, T^*\Sigma)$ restricts to give a complex structure on $S^2_0T^*\Sigma$. As we have noted already, this identifies $S^2_0T^*\Sigma$ with the bundle $\Omega^{2,0}\Sigma$ of quadratic differentials. More precisely the $\pm\boldsymbol{i}$ eigenspace decomposition of this complex structure may be written $S^2_0T^*\Sigma\otimes\mathbb{C}\cong\Omega^{2,0}\Sigma\oplus\Omega^{0,2}\Sigma$, where $\Omega^{2,0}\Sigma=(T^{1,0}\Sigma)^{-2}$, $\Omega^{0,2}\Sigma=(T^{0,1}\Sigma)^{-2}$, and $T\Sigma\otimes\mathbb{C}=T^{1,0}\Sigma\oplus T^{0,1}\Sigma$ is the $\pm\boldsymbol{i}$ eigenspace decomposition of the complex structure on the tangent bundle into the two null lines of the complexified conformal metric. The isomorphism $S^2_0T^*M\to\Omega^{2,0}\Sigma$ is the projection $q\mapsto q^{2,0}$ with respect to this decomposition.

We now note that the exterior derivative $d^D\colon\Omega^1(\Sigma,T^*\Sigma)\to\Omega^2(\Sigma,T^*\Sigma)$ coupled to a Weyl connection D restricts to give the holomorphic structure on $S^2_0T^*\Sigma$: $(d^Dq)^{2,1}=\overline{\partial}(q^{2,0})\in\Omega^{1,1}(\Sigma,(T^{1,0}\Sigma)^*)$. Since this is independent of the choice of Weyl structure, we denote d^Dq by dq. Similarly, $(dq)^{1,2}=\partial(q^{0,2})$. We therefore have

$$\begin{aligned} dq &= \overline{\partial}(q^{2,0})+\partial(q^{0,2})\\ \mathit{div}\,q &= *(\overline{\partial}(q^{2,0})-\partial(q^{0,2})), \end{aligned}$$

the second line being obtained from the first via $d{*q}={*\mathit{div}\,q}$, showing that the divergence is also a conformally invariant operator on quadratic differentials. Thus $dq=0$ iff $\mathit{div}\,q=0$ iff $\overline{\partial}(q^{2,0})=0$ iff $\partial(q^{0,2})=0$, in which case we say q is *holomorphic* quadratic differential.

Applying similar ideas to the Möbius differential $\mathcal{N}$ yields the following fact.

Proposition 12.1. *Let Λ be an immersion of a surface Σ in S^n whose central sphere congruence V_Σ has Möbius differential $\mathcal{N}$. Then $\partial{*\mathcal{N}}=0$ and $*\mathcal{N}$ is the differential representative of its homology class $J\mathrm{I\!I}^0-J\mathbb{S}^0$.*

Proof. The fact that $\partial{*\mathcal{N}}=0$ follows easily from the symmetry of the second fundamental form (or shape operator). To show $*\mathcal{N}$ is the differential representative, we need to show that $\partial d^{\mathfrak{h}}{*\mathcal{N}}=0$, *i.e.*, $d^{\mathfrak{h}}{*\mathcal{N}}|_{V_\Sigma^\perp}\in\Omega^2(\Sigma,\mathit{Hom}(V_\Sigma^\perp,\Lambda))$. For this we write $\Lambda^{(1)}\otimes\mathbb{C}=\Lambda^{(1,0)}+\Lambda^{(0,1)}$, where $\Lambda^{(1,0)}\cap\Lambda^{(0,1)}=\Lambda\otimes\mathbb{C}$, $\Lambda^{(1,0)}/(\Lambda\otimes\mathbb{C})\cong\Lambda T^{1,0}\Sigma$ and $\Lambda^{(0,1)}/(\Lambda\otimes\mathbb{C})\cong\Lambda T^{0,1}\Sigma$. Now $\mathcal{N}=\mathcal{N}^{1,0}+\mathcal{N}^{0,1}$ and $*\mathcal{N}=\boldsymbol{i}(\mathcal{N}^{1,0}-\mathcal{N}^{0,1})$, where

$$\mathcal{N}^{1,0}|_{V_\Sigma^\perp}\in\Omega^{1,0}(\Sigma,\mathit{Hom}(V_\Sigma^\perp,\Lambda^{(1,0)})),\qquad \mathcal{N}^{0,1}|_{V_\Sigma^\perp}\in\Omega^{0,1}(\Sigma,\mathit{Hom}(V_\Sigma^\perp,\Lambda^{(0,1)})).$$

Since the $(1,0)$ and $(0,1)$ directions are null, $\mathfrak{D}^{1,0}\Lambda^{(1,0)}\subset\Lambda^{(1,0)}$ and $\mathfrak{D}^{0,1}\Lambda^{(0,1)}\subset\Lambda^{(0,1)}$. It follows that

$$d^{\mathfrak{h}}(\mathcal{N}^{1,0})|_{V_\Sigma^\perp}\in\Omega^{1,1}(\Sigma,\mathit{Hom}(V_\Sigma^\perp,\Lambda^{(1,0)})),\quad d^{\mathfrak{h}}(\mathcal{N}^{0,1})|_{V_\Sigma^\perp}\in\Omega^{1,1}(\Sigma,\mathit{Hom}(V_\Sigma^\perp,\Lambda^{(0,1)})).$$

Now $d^{\mathfrak{h}}\mathcal{N}|_{V_\Sigma^\perp}$ takes values in $\mathit{Hom}(V_\Sigma^\perp,\Lambda)$, hence so do $d^{\mathfrak{h}}(\mathcal{N}^{1,0})|_{V_\Sigma^\perp}$, $d^{\mathfrak{h}}(\mathcal{N}^{0,1})|_{V_\Sigma^\perp}$ and $d^{\mathfrak{h}}{*\mathcal{N}}|_{V_\Sigma^\perp}$. □

There is a conceptual explanation for the above result: in complexified conformal geometry (or in signature $(1,1)$), $\mathrm{I\!I}^0$ does not lie in an irreducible homology bundle and its differential lift is the sum of the differential lifts of the two irreducible components.

There is another fact we shall need: note that if $\eta\in\Omega^1(\Sigma,T^*M)$ then $\partial\eta=0$ and $[\eta]$ is a quadratic differential q given by the trace-free symmetric tangential part of η.

Proposition 12.2. *Suppose $[\eta]=q$ and and $\partial d^{\mathfrak{h}}\eta=0$. Then $\eta=q$ (i.e., η is trace-free, symmetric and tangential), $d^{\mathfrak{h}}\eta=0$ if and only if q is holomorphic (i.e., $dq=0$), and $[\mathcal{N}\wedge\eta]=0$ if and only if q commutes with the shape operators (i.e., $q\wedge\mathbb{S}^0=0$).*

Proof. If $0=\partial d^{\mathfrak{h}}\eta=\partial[\![id\wedge\eta]\!]$ then by adjointness $0=[\![id\wedge\eta]\!]$, from which it easily follows that η is tangential and trace-free symmetric. Then $d^{\mathfrak{h}}\eta=0$ if and only if $d^D\eta=0$ for some (hence any) Weyl derivative d, *i.e.*, q is holomorphic. The last part is immediate. □

In this statement we are, as usual, identifying (the pullback to Σ of) T^*M with the nilradical bundle $\mathfrak{stab}(\Lambda)^\perp$ in $\mathfrak{stab}(\Lambda) \subset \Sigma \times \mathfrak{so}(n+1,1)$. If we regard η in this way, then the statement $\eta = q$ means, more precisely:

$$\eta_X \cdot \sigma = 0, \qquad \eta_X \cdot \mathfrak{D}_Y \sigma = -q(X,Y)\sigma, \qquad \eta_X(V_\Sigma) \subset \Lambda^{(1)}, \qquad \eta_X|_{V_\Sigma^\perp} = 0. \tag{12.1}$$

The conformal Bonnet theorem in conformal coordinates. Surfaces are conformally flat, *i.e.*, locally, we can introduce a holomorphic coordinate $z = x + iy$, so that (x,y) are conformal (aka. isothermal) coordinates. Hence we can refer everything to the flat Möbius structure $\mathcal{M}^z$ determined by z and, in this setting, the Gauß–Codazzi–Ricci equations have a very down-to-earth flavour. Recall that the flat connection $d = \mathfrak{D}^{S^n}$ on $\Sigma \times \mathbb{R}^{n+1,1}$ may be written $d = \mathfrak{D}^\Sigma + Q + \nabla + \mathcal{N}$ with Q pure trace so that, for σ a section of Λ,

$$\partial_z^2 \sigma = (\mathfrak{D}^\Sigma + Q)_{\partial_z}(\mathfrak{D}^\Sigma + Q)_{\partial_z}\sigma + \mathcal{N}_{\partial_z}(\mathfrak{D}^\Sigma_{\partial_z}\sigma) = \mathfrak{D}^\Sigma_{\partial_z}\mathfrak{D}^\Sigma_{\partial_z}\sigma + \mathcal{N}_{\partial_z}(\mathfrak{D}^\Sigma_{\partial_z}\sigma).$$

Fix σ by demanding that $(d\sigma, d\sigma) = dz\, d\bar{z}$ so that σ is parallel for the (flat) Weyl derivative D^z. Then Proposition 6.5 gives

$$\partial_z^2 \sigma + q\sigma = -\mathrm{II}^0(\partial_z, \partial_z)\sigma,$$

where $r^{D^z} = q dz^2 + \bar{q} d\bar{z}^2$ (so that $2q$ is the schwarzian derivative of $\mathcal{M}^{\mathrm{c}}$ with respect to $\mathcal{M}^z$). Define κ, a section of $V_\Sigma^\perp$ by $\kappa = -\mathrm{II}^0(\partial_z, \partial_z)\sigma$ so that

$$\partial_z^2 \sigma + q\sigma = \kappa.$$

The remaining ingredients of the Gauß–Codazzi–Ricci equations are now readily expressed in terms of q, κ, ∇:

$$Q(\partial_z, \partial_z) = (\kappa, \bar{\kappa}) \qquad \text{and} \qquad A^{D^z} = 2(\nabla_{\partial_z}\kappa)\sigma.$$

An easy computation then gives the following form of the Gauß–Codazzi–Ricci equations:

$$\begin{gathered} \partial_{\bar{z}} q = 3(\nabla_{\partial_z}\bar{\kappa}, \kappa) + (\bar{\kappa}, \nabla_{\partial_z}\kappa) \\ \operatorname{Im}(\nabla_{\partial_{\bar{z}}}\nabla_{\partial_{\bar{z}}}\kappa + \bar{q}\kappa) = 0 \\ R^\nabla_{\partial_{\bar{z}}, \partial_z}\xi = 2(\kappa, \xi)\bar{\kappa} - 2(\bar{\kappa}, \xi)\kappa. \end{gathered}$$

This formulation of the Gauß–Codazzi–Ricci equations and the Bonnet theorem was developed in [21] using a bare-hands approach.

Surfaces in the 4-sphere and quaternions. For surfaces in S^4, §6.4 reveals an alternative approach to submanifold geometry, using the spin representation of $\mathit{Spin}(5,1) \cong \mathit{SL}(2,\mathbb{H})$ and quaternions. This is the setting for the book [18], in which quaternionic holomorphic structures are used to study the global theory of conformal immersions. We shall content ourselves here with a consideration of the local invariants from a quaternionic point of view.

Let $\Lambda_{\mathbb{H}} \subset H := \Sigma \times \mathbb{H}^2$ be an immersion of a surface Σ into $S^4 \cong \mathbb{H}P^1$. The differential of this immersion is $\beta \colon T\Sigma \to \mathit{Hom}_{\mathbb{H}}(\Lambda_{\mathbb{H}}, H/\Lambda_{\mathbb{H}}) \cong TS^4|_\Sigma$. It follows that there are unique quaternion linear complex structures on $\Lambda_{\mathbb{H}}$ and $H/\Lambda_{\mathbb{H}}$ up to sign which preserve, and agree on, the image of β (geometrically, they act by selfdual and antiselfdual rotations on TS^4). We denote these complex structures by J: in symbols $*\beta = J \circ \beta = \beta \circ J$, where $*$ is the Hodge star operator of the induced conformal metric on Σ. The normal bundle of Σ then consists of the elements of $\mathit{Hom}_{\mathbb{H}}(\Lambda_{\mathbb{H}}, H/\Lambda_{\mathbb{H}})$ which anticommute with J, and we equip it with the complex structure given by precomposition with J.

An enveloped sphere congruence in this language is a complex structure J_V on H which preserves $\Lambda_{\mathbb{H}}$ and induces J on $\Lambda_{\mathbb{H}}$ and $H/\Lambda_{\mathbb{H}}$. Flat differentiation then splits as $d = \mathfrak{D}^V + \mathcal{N}^V$, where $\mathfrak{D}^V J_V = 0$ and $\mathcal{N}^V$ anticommutes with J_V. The enveloping condition means equivalently that $\mathcal{N}^V$ preserves $\Lambda_{\mathbb{H}}$. The second fundamental form II^V arises from the induced endomorphisms of $\Lambda_{\mathbb{H}}$ and $H/\Lambda_{\mathbb{H}}$ by restricting pre- and post-composition to the image of β, yielding 1-forms with values in $Hom(T\Sigma, N\Sigma)$, one J-linear, the other J-antilinear. The central sphere congruence condition that II^V is tracefree may then be interpreted as saying that $(\mathrm{II}^V)^{1,0}$ is J-linear and $(\mathrm{II}^V)^{0,1}$ is J-antilinear (or vice versa). These are the objects are referred to as A and Q in [18].

Preview of Part IV

In this second part of our work, sphere congruences have been used as a tool to define homological invariants which characterize conformal immersions into S^n up to Möbius transformation. Part III is devoted to some applications of this homological machinery to integrable conformal submanifold geometries.

Sphere congruences in conformal geometry (i.e., maps from an m-manifold Σ into the grassmannian of k-spheres in S^n) may also be studied in their own right, and were of considerable interest classically [8, 38]. There is a rich interplay between sphere congruences and submanifolds: for example, sphere congruences may have enveloping submanifolds (which we have studied here for $k = m$) or orthogonal submanifolds (which are of particular interest for $k = n - m$).

In Part IV of our work, we develop an approach to sphere congruences using the bundle formalism, in which a sphere congruence is studied as a signature $(k+1, 1)$ subbundle V of the trivial $\mathbb{R}^{n+1,1}$ bundle over Σ. This theory is essentially self-contained, since it depends only on the straightforward idea to write $d = \mathfrak{D}^V + \mathcal{N}^V$, where $\mathfrak{D}^V$ is the induced direct sum connection on $\Sigma \times \mathbb{R}^{n+1,1} = V \oplus V^\perp$, and $\mathcal{N}^V$ is the remaining, off-diagonal, part of d. We have already seen, for example, that an enveloping submanifold is a null line subbundle Λ of V on which $\mathcal{N}^V$ vanishes (i.e., $\Lambda^{(1)} := d\Lambda \subseteq V$). Similarly, an orthogonal submanifold is $\mathfrak{D}^V$-parallel null $\Lambda \subseteq V$ (so $\Lambda^{(1)} \subseteq \Lambda \oplus V^\perp$).

We focus in particular on *Ribaucour* sphere congruences (with $k = m$) and *spherical systems* (with $k = n - m$—known as *cyclic systems* when $n - m = 1$). For Ribaucour sphere congruences, we establish the well-known Bianchi permutability of Ribaucour transformations. Although Lie sphere geometry provides a more general setting for these [19], extra information is available in the conformal approach, leading to additional applications.

For spherical systems, we devote our attention to flat spherical systems (for which $\mathfrak{D}^V$ is a flat connection—for a general spherical system, it is only assumed to be flat on $V^\perp$). These are examples of "curved flats" arising in conformal geometry [45, 74]. Darboux pairs of isothermic surfaces provide another example, in which a 2-dimensional 0-sphere congruence is a curved flat [20].

It is well-known that curved flats admit Bäcklund transformations which can be derived from a loop-group formalism, but we develop instead a direct approach. This has (at least) two advantages: first, we do not need to introduce frames; second, the *definition* of the Bäcklund transformations does not involve dressing (*i.e.*, Birkhoff factorization)—this is only needed (and then only implicitly and in a very simple form) for the *permutability* of Bäcklund transformations. We also introduce a theory of polynomial conserved quantities for such curved flats, which has several applications.

Part III. Applications and Examples

We now explore some applications of our conformal submanifold geometry theory. We begin in §13.1 with the simplest examples, the totally umbilic and channel submanifolds, and we discuss the curvature spheres which can be used to describe them. In §§13.2–13.3 we study the interaction of submanifolds with the symmetry breaking induced by a constant vector or $(k + 1)$-plane (cf. §§7.2–7.3) and find homological characterizations of minimal submanifolds in a spaceform and submanifolds splitting across a product decomposition. Then, in §13.4, we give a fast analysis of orthogonal systems and Dupin's Theorem.

We next turn to Willmore and constrained Willmore surfaces. On any compact surface Σ in S^n, the square norm (with respect to the conformal metric and the metric on the weightless normal bundle), $|\mathrm{II}^0|^2$, of the tracefree second fundamental form is a section of L^{-2} (called the Willmore integrand), and so may be invariantly integrated. The integral $\int_\Sigma |\mathrm{II}^0|^2$ is called the *Willmore functional* $\mathcal{W}$.

A compact Willmore surface is a critical point for $\mathcal{W}$ on the space of all immersions of a fixed compact surface Σ, while a compact constrained Willmore surface is a critical point restricted to immersions inducing a fixed conformal structure on Σ. It is usual to extend the definition of Willmore and constrained Willmore surfaces to arbitrary surfaces by requiring that the Euler–Lagrange equations hold, with a Lagrange multiplier (which is a holomorphic quadratic differential) in the constrained case. We take this as our starting point in §14.1, where we derive the classical equation in codimension one, and obtain a spectral deformation in arbitrary codimension. Then, in §14.2, we derive the (constrained) Willmore equation from the functional, using the relation with the harmonic map equation for the central sphere congruence. Although this is well-known, our machinery does not get in the way the key idea, and reveals the homological nature of this theory.

We next turn to isothermic surfaces, which were of great classical interest partly because they are the only surfaces which admit a deformation which does not alter the induced conformal metric, normal connection and tracefree second fundamental form. This is of particular interest in our theory, because in this situation the induced Möbius structure $\mathcal{M}^\Sigma$ is the key invariant. We give a manifestly conformally-invariant definition in §15.1, then describe the deformations and associated family of flat connections. In §15.2 we give examples: products of curves in spaceforms, CMC and generalized H-surfaces, and quadrics (previously considered as 'mysterious' examples). We end with an intrinsic equation for isothermic surfaces in codimension one §15.3.

In any type of submanifold geometry, it is natural to ask when the induced intrinsic geometry is flat. In conformal submanifold geometry this has an unambiguous meaning for $m \geqslant 3$: the induced conformal metric should be (conformally) flat. For $m = 1$, flatness is automatic, so it remains to consider the case $m = 2$. Using only conformal structures, flatness would also be automatic, but the theory of Möbius structures provides an obvious nontrivial condition: flatness of the induced conformal Möbius structure or equivalently of the normal Cartan connection (on the central sphere congruence). However, a more general condition turns out to be more natural: that there is *some* enveloped sphere congruence V inducing a flat conformal Möbius structure on Σ. Imposing also flatness of the weightless normal bundle (which is automatic in codimension one), we thus develop, in section 16, a new unified theory of "Möbius-flat submanifolds". In dimension $m \geqslant 3$ these are the conformally-flat submanifolds, while in dimension $m = 2$ and codimension one, they turn out to be the classical Guichard surfaces [23, 49]. Using this theory we show that channel submanifolds and constant Gaussian curvature surfaces are Möbius-flat, as are certain extrinsic products. We rederive results of Cartan and Hertrich-Jeromin[36, 54] on conformally-flat hypersurfaces and Guichard nets [50], and end this part by placing Dupin cyclides in this context.

13. Sphere congruences and symmetry breaking

13.1. Shape operators, curvature spheres, and channel submanifolds.

Definition 13.1. A *curvature sphere* is an enveloped m-sphere congruence V such that II^V is degenerate (as an $N\Sigma$-valued bilinear form on $T\Sigma$). Let T_V denote the subbundle of

$T\Sigma$ on which II^V degenerates: we assume for simplicity that its rank, called the *multiplicity* of V, is constant on Σ.

Curvature spheres are a convenient way of describing the eigenspaces and eigenvalues of the shape operator. Indeed if V is any enveloped sphere congruence and E is a simultaneous eigenspace for the normal components of the shape operator $\mathbb{S}^V$ such that $\mathbb{S}^V(U)$ has eigenvalue $\nu(U)$ for all normal vectors U, then $V+\nu$ is a curvature sphere with $T_{V+\nu}=E$.

The existence of simultaneous eigenspaces is facilitated by the following well-known fact.

Proposition 13.2. *The normal components of $\mathbb{S}^V$ commute with each other if and only if Σ has flat (weightless) normal bundle.*

Proof. By the Ricci equation (9.3a) and the symmetry of the shape operator, we have

$$\begin{aligned}\langle R^{\nabla}_{X,Y}U_1,U_2\rangle = \langle(\mathrm{II}^V_X\mathbb{S}^V_Y-\mathrm{II}^V_Y\mathbb{S}^V_X)U_1,U_2\rangle &= \langle \mathbb{S}^V_Y U_1,\mathbb{S}^V_X U_2\rangle - \langle \mathbb{S}^V_X U_1,\mathbb{S}^V_Y U_2\rangle\\ &= \langle \mathbb{S}^V_{\mathbb{S}^V_X U_2}U_1,Y\rangle - \langle \mathbb{S}^V_{\mathbb{S}^V_X U_1}U_2,Y\rangle.\end{aligned}$$

The result follows. □

Remark 13.3. Since Q^V (viewed as an Λ^2-valued endomorphism) is a linear combination of id with normal components of $\mathbb{S}^0$ and $\mathrm{II}^0\circ\mathbb{S}^0$, we have that $Q^V\wedge\mathbb{S}^0=0$ if the normal bundle is flat.

We deduce from the proposition that if Σ has flat normal bundle, then the normal components of $\mathbb{S}^V$ are simultaneously diagonalizable, since they are symmetric. Thus if $V_1,\dots V_\ell$ are the curvature spheres, then $T\Sigma$ is the orthogonal direct sum of the T_{V_i}. The subbundles T_{V_i} are the (simultaneous) eigenspaces of the shape operator, and the conormal eigenvalues ν_i give the curvature spheres via $V_i=V+\nu_i$.

Proposition 13.4. *If a curvature sphere V has multiplicity greater than 1 then T_V is an integrable distribution and V is constant (i.e., a parallel subbundle of $\Sigma\times\mathbb{R}^{n+1,1}$) along the leaves of the corresponding foliation.*

Proof. The Codazzi equation for V implies that

$$\begin{aligned}0 &= (\nabla^D\mathrm{II}^V)_{X,Y}-(\nabla^D\mathrm{II}^V)_{Y,X}+A^{D,V}_X Y - A^{D,V}_Y X\\ &= \mathrm{II}^V_{[X,Y]}+A^{D,V}_X Y-A^{D,V}_Y X\end{aligned}\tag{13.1}$$

for X,Y in T_V. Thus $\mathrm{II}^V_{[X,Y]}$ is in T_V, but it is also in $T_V^\perp$ by the symmetry of II^V, and therefore $\mathrm{II}^V_{[X,Y]}=0$ and $[X,Y]$ is in T_V. Now contracting (13.1) with Y, we get $A^{D,V}_X\langle Y,Y\rangle - A^{D,V}_Y\langle X,Y\rangle=0$, and taking $\langle X,Y\rangle=0$, $\langle Y,Y\rangle\neq 0$, we deduce that $A^{D,V}_X=0$ for all X in T_V. Hence $\mathcal{N}^V_X=0$ for all X in T_V, *i.e.*, V is parallel in the T_V directions. □

The first conclusion of this proposition is automatic for curvature spheres of multiplicity one, but the second is not.

Definitions 13.5. Let Σ be a submanifold of S^n of dimension $m\geqslant 2$ with flat normal bundle. Then Σ is said to be:

(i) a *Dupin submanifold* iff its curvature spheres are all constant along their foliations;

(ii) a *totally umbilic submanifold* iff $\mathrm{II}^0=0$, *i.e.*, it has only one curvature sphere;

(iii) a *channel submanifold* iff it is the envelope of a 1-parameter family of m-spheres, *i.e.*, it admits an enveloped sphere congruence V whose derivative $\mathcal{N}^V\colon T\Sigma\to\mathfrak{m}_V$ has rank one (so that V is, in particular, a curvature sphere of multiplicity $m-1$).

The following well-known observations are immediate consequences of Proposition 13.4.

Corollary 13.6. *A totally umbilic m-submanifold ($m \geqslant 2$) of S^n is an open subset of its curvature sphere (hence is a Dupin submanifold).*

Corollary 13.7. *Suppose Σ has flat normal bundle and exactly two curvature spheres. Then Σ is either a surface, a channel submanifold, or a Dupin submanifold.*

13.2. Constant vectors and tangent congruences in spaceform geometries. In §11.3 we showed how to break the conformal invariance of our theory by introducing a compatible riemannian metric on M. When M is an open subset of $S^n = P(\mathcal{L})$, such a metric is given by a section of the positive light-cone $\mathcal{L}^+ \subset \Lambda$ over M. As we have discussed in §7.2, a particularly important class of such sections are the conic sections $\{\sigma \in \mathcal{L}^+ : (v_\infty, \sigma) = -1\}$ associated to nonzero vectors $v_\infty \in \mathbb{R}^{n+1,1}$. We have seen that such a section induces a constant curvature metric g on M, with Weyl structure $\hat{\Lambda}_g = \langle v_\infty + \frac{1}{2}(v_\infty, v_\infty)\sigma \rangle$ in $M \times \mathbb{R}^{n+1,1}$ (where $(v_\infty, \sigma) = -1$).

Along a submanifold Σ of M, $\hat{\Lambda}_g$ defines an ambient Weyl structure, inducing the sphere congruence $V_g = \Lambda^{(1)} \oplus \hat{\Lambda}_g = \Lambda^{(1)} \oplus \langle v_\infty \rangle$. We refer to a sphere congruence containing a constant vector v_∞ as the *tangent congruence* to Σ in the spaceform given by v_∞. When v_∞ is null and the geometry on $M \subseteq S^n \smallsetminus \langle v_\infty \rangle$ is euclidean, the tangent spheres pass through the point at infinity, and so they stereoproject to tangent planes in the usual sense.

Proposition 13.8. *For a sphere congruence V and Weyl structure $\hat{\Lambda} \subset V$ (with associated Weyl derivative D), the following are equivalent*:

(i) *$V = V_g$ and $\hat{\Lambda} = \hat{\Lambda}_g$ for a conic section of $\mathcal{L}^+$*;

(ii) *V is a tangent congruence and is enveloped by $\hat{\Lambda}$ (i.e., $\mathcal{N}|_{\hat{\Lambda}} = 0$)*;

(iii) *$r^{D,V} = \frac{1}{n} s\mathsf{c}$ for a section s of $\Lambda^2 \cong Hom(\hat{\Lambda}, \Lambda)$, and $A^{D,V} = 0$.*

Proof. By (11.9), $r^{D,V_g} = (r^g)^\top = \frac{1}{n} s^g \mathsf{c}$ and $A^{D,V_g} = (r^g)^\perp = 0$. Thus (i)$\Leftrightarrow$(ii) and these imply (iii). For the converse implication, if $r^{D,V} = \frac{1}{n} s\mathsf{c}$ and $A^{D,V} = 0$, the Gauß equation implies that s is D-parallel. Let $\hat{\sigma}$ be a D-parallel section of $\hat{\Lambda}$ and $s(\hat{\sigma})$ the induced section of Λ; then $v_\infty = \hat{\sigma} + \frac{1}{n} s(\hat{\sigma})$ is a constant vector. $\square$

If v_∞ is null, $\hat{\Lambda}_g$ is constant, so the submanifold it defines is a point, but otherwise, it is immersed. The Gauß–Codazzi–Ricci equations in this case are equivalent to the riemannian Gauß–Codazzi–Ricci equations in the corresponding spaceform geometry and the Bonnet theorem reduces to the riemannian one.

We now consider the relation between v_∞ and V_Σ. Since $V_g = \exp(H^g) V_\Sigma$ contains v_∞, it follows that V_Σ component of v_∞ is $H^g \ell$, where ℓ is dual to σ. In particular

$$\Sigma \text{ is a minimal submanifold in the spaceform iff } v_\infty \in V_\Sigma \text{ iff } V_\Sigma = V_g,$$

i.e., the tangent congruence and the central sphere congruence agree. We remark that in general the homology class of v_∞ is the pair $(\ell, H^g \ell)$ in $L \oplus L\, N^*\Sigma$, and so the minimal submanifolds are precisely those for which the homology class of v_∞ is tangential.

Finally, we ask when the Weyl structure of g provides a second envelope of the central sphere congruence, *i.e.*, $A^D = 0$ for the induced Weyl derivative D: equation (11.13) here reads $A^D = -\nabla^D H^g$, *i.e.*, the submanifold has parallel mean curvature in the given spaceform geometry.

13.3. Symmetry breaking for submanifolds. We study again the metrics considered in §7.3, associated to a $(k+1)$-plane W in $\mathbb{R}^{n+1,1}$, with orthogonal $(n-k+1)$-plane $W^\perp$, where $0 \leqslant k \leqslant n$, which identify an open subset of S^n with a product of spaceforms of dimension k and $n-k$. For an m-dimensional submanifold $\Lambda \to \Sigma$ of this open subset, let

$\hat{\varLambda}$ be the ambient Weyl structure along Σ with $\varLambda \oplus \hat{\varLambda} = (\Sigma \times W \oplus \varLambda) \cap (\Sigma \times W^\perp \oplus \varLambda)$, and V the corresponding enveloped sphere congruence.

It is natural to ask when Σ is a local product of immersions of a p-manifold and an $(m-p)$-manifold into the spaceforms associated to W and $W^\perp$ (where $0 \leqslant p \leqslant k$, $0 \leqslant m-p \leqslant n-k$). It is clear that this implies that the lines $\wedge^{k+1} W$ and $\wedge^{n-k+1} W^\perp$ lie in $\wedge^{p+1} V \otimes \wedge^{k-p} V^\perp$ and $\wedge^{m-p+1} V \otimes \wedge^{n-k-(m-p)} V^\perp$ respectively. Hence a unit vector ω in $\wedge^{k+1} W$ may be written $\omega = \theta \wedge v \wedge \chi$ with $\theta \in \wedge^p \varLambda^{(1)}$, $v \in V$ and $\chi \in \wedge^{k-p} V^\perp$ (all decomposable). Since we require that $\varLambda \not\subseteq W \cup W^\perp$, we must have $\theta \notin \varLambda \wedge \wedge^{p-1} \varLambda^{(1)}$ and $v \notin \varLambda^\perp$. On the other hand, $\varLambda \oplus \hat{\varLambda}$ has nontrivial intersection with W, so without loss, we can take $v \in \varLambda \oplus \hat{\varLambda}$ and $\theta \in \wedge^p U$, where $U := (\varLambda \oplus \hat{\varLambda})^\perp \cap V$

Similarly $\wedge^{n-k+1} W^\perp$ contains a unit vector of the form $\omega^\perp = \theta^\perp \wedge v^\perp \wedge \chi^\perp$ with $\theta^\perp \in \wedge^{m-p} U$, $v^\perp \in \varLambda \oplus \hat{\varLambda}$ and $\chi^\perp \in \wedge^{n-k-(m-p)} V^\perp$ (all decomposable).

Observe that the central sphere congruence may be written $V_\Sigma = \varLambda \oplus U \oplus \hat{\varLambda}_\Sigma$, where $\hat{\varLambda}_\Sigma = \exp(-H^V)\hat{\varLambda}$ is the null complement to $\varLambda$ in $U^\perp \cap V_\Sigma$. It follows that for nontrivial splittings $(0 < p < m)$, the conditions on $W, W^\perp$ that we have obtained have a homological consequence: $[\omega]_{S^n}|_\Sigma = [\omega]_\Sigma = \theta \otimes (v \textit{ mod } \varLambda^\perp) \otimes \chi$ and is a decomposable section of

$$\begin{aligned} H_0(T^*\Sigma, \wedge^{p+1} V_\Sigma) \otimes \wedge^{k-p} V_\Sigma^\perp &= \wedge^p (T\Sigma\, \varLambda)\, L \otimes \wedge^{k-p} V_\Sigma^\perp \\ &\subset \wedge^k (TS^n\, \varLambda)\, L|_\Sigma = H_0(T^*S^n, S^n \times \wedge^{k+1} \mathbb{R}^{n+1,1})|_\Sigma, \end{aligned}$$

and similarly for $[\omega^\perp]_{S^n}|_\Sigma$. This turns out to be a characterization.

Theorem 13.9. *Let $W \subset \mathbb{R}^{n+1,1}$ be a $(k+1)$-dimensional subspace with orthogonal space $W^\perp$. Then an immersed submanifold of $S^n \smallsetminus (P(W) \cup P(W^\perp))$ splits locally across the induced product structure as a product of a p-submanifold and an $(m-p)$-submanifold $(0 < p < m)$ if and only if the homology class $[\omega]_{S^n}$ of some (hence any) nonzero $\omega \in \wedge^{k+1} W$ is a section of $\wedge^p (T\Sigma\, \varLambda)\, L \otimes \wedge^{k-p} V_\Sigma^\perp$ along Σ.*

Proof. It remains to show that the homological condition implies the splitting. For this recall from §7.3 that $[\omega]_{S^n}$ and $[\omega^\perp]_{S^n}$ are sections of the top exterior powers of the two distributions tangent to the spaceform factors (up to a line bundle), so the homological condition means that one of these distributions meets $T\Sigma \oplus N\Sigma$ (along Σ) in the sum of a tangential p-plane and a normal $(k-p)$-plane. The other distribution, being orthogonal, then meets $T\Sigma \oplus N\Sigma$ in a tangential $(m-p)$-plane and a normal $(n-k-(m-p))$-plane. Now the tangential planes are integrable, and this locally splits Σ. □

In the case $k = 1$, the above theorem has an intuitively clear meaning.

Corollary 13.10. *The vector field K on S^n associated to a 2-dimensional subspace W of $\mathbb{R}^{n+1,1}$ is tangent to Σ if and only if Σ is an open subset of a revolute, a cylinder or a cone, over a submanifold of the same codimension in $\mathcal{H}^{n-1}$, $\mathbb{R}^{n-1}$ or $\mathcal{S}^{n-1}$ respectively. In this case the homology class of ω_∞ along Σ is the corresponding tangent vector field.*

The properties of these submanifolds can easily be read off from the data on Σ by imposing the condition that ω, here equal to $\theta \wedge v$, with $\theta \in U$ and $v \in \varLambda \oplus \hat{\varLambda}$, is parallel. We obtain immediately that $\mathcal{N}^V \theta = 0 = \mathcal{N}^V v$, so that V is a curvature sphere, constant along the curvature lines in the direction of $K = \theta \otimes (v \textit{ mod } \varLambda^\perp)$, $\hat{\varLambda}$ is another envelope, and $\mathcal{N}^V$ (hence II^V) preserves the decomposition $T\Sigma = \langle K \rangle \oplus K^\perp \cap T\Sigma$ induced by the product metric on $\mathcal{S}^1 \times \mathcal{H}^{n-1}$, $\mathbb{R} \times \mathbb{R}^{n-1}$ or $\mathcal{H}^1 \times \mathcal{S}^{n-1}$.

Example 13.11. Recall that a (Dupin) cyclide in S^3 is an orbit of a two dimensional abelian subgroup of the Möbius group $M\ddot{o}b(3)$. Dupin's classification of the cyclides can be read off easily from the above. In the span of a two dimensional abelian subalgebra of $\mathfrak{so}(4,1)$

we can always find a basis of decomposable elements (just consider the Jordan normal form). These decomposables define two orthogonal 2-planes in $\mathbb{R}^{4,1}$, reducing the geometry to $\mathcal{H}^2 \times \mathcal{S}^1$, $\mathbb{R}^2 \times \mathbb{R}$, or $\mathcal{S}^2 \times \mathcal{H}^1$, and the cyclide as a surface of revolution, a cylinder, or a cone, according to whether the 2-plane is spacelike, degenerate, or timelike. At most one of the planes can be degenerate or timelike. Therefore a cyclide, if not totally umbilic, is a channel surface in two ways, and is either a circular torus of revolution, a cylinder of revolution, or a cone of revolution.

If the splitting is trivial (*i.e.*, $p = 0$ or $p = m$), then $[v_\infty]$ is no longer a pure tangential class in general. Indeed, as we have seen in the case $k = 0$ (or $k = n$) of the spaceform geometries of the previous paragraph (§13.2), $[v_\infty]$ is tangent to Σ if and only if the immersion (into the spaceform factor) is minimal.

13.4. Orthogonal systems and Dupin's Theorem. An extreme form of symmetry breaking is given by n-tuply orthogonal systems. Suppose that we have a local diffeomorphism $\Lambda \subset \Sigma \times \mathbb{R}^{n+1,1}$ of a product of 1-manifolds $\Sigma = \prod_{i=1}^n \Sigma_i$ with $S^n = P(\mathcal{L})$, such that the factors are orthogonal for the induced conformal metric. Let $U_i = d_{X_i}(\Lambda)$, where X_i is a nonvanishing tangent vector field to Σ_i pulled back to the product. Thus the U_i are mutually orthogonal rank 2 bundles with sum $\Lambda^\perp$ and intersection Λ. For each i we have a one parameter family of hypersurfaces Σ_{x_i} in S^n by fixing the ith coordinate $x_i \in \Sigma_i$ as a parameter. Dupin's Theorem is the following one.

Theorem 13.12. *For $i \neq j$, the families Σ_{x_i} and Σ_{x_j} meet each other in curvature lines.*

Proof. It suffices to show that X_j is tangent to a curvature line on Σ_{x_i}, *i.e.*, $\mathrm{II}^{0(i)}_{X_j} X_k = 0$ for $k \notin \{i, j\}$. This is a multiple of $(d_{X_j} d_{X_k} \sigma_k, u_i)$ where u_i is a unit section of U_i and σ_k is a section of Λ pulled back from Σ_k. Now we simply observe that $d_{X_j} d_{X_k} \sigma_k = d_{X_k} d_{X_j} \sigma_k$ since $[X_j, X_k] = 0$, which is a section of U_k, since $d_{X_j}\sigma_k$ is a section of Λ (for $j \neq k$). As U_k is orthogonal to U_i (for $i \neq k$), we are done. □

The proofs in the literature take a euclidean point of view of this result. Ours is quite different, and thus avoids computing the Levi-Civita connection of the induced metric.

14. Constrained Willmore surfaces

14.1. Definition and spectral deformation.

Definition 14.1. An immersion Λ of a surface Σ in S^n is said to be *constrained Willmore* if there is a holomorphic quadratic differential $q \in C^\infty(\Sigma, S^2 T_0^*\Sigma)$ (*i.e.*, $dq = 0$) such that $(\mathcal{H}^\nabla)^*(\mathrm{II}^0) = \langle q, \mathrm{II}^0 \rangle$. We refer to such a holomorphic quadratic differential q as a *Lagrange multiplier* for the constrained Willmore immersion. If $(\mathcal{H}^\nabla)^*(\mathrm{II}^0) = 0$ the surface is said to be *Willmore* or *conformally minimal.*

Let g be an ambient metric of constant curvature, and let D be the induced exact Weyl connection on Σ. Then, by (11.16) and (11.12)–(11.13), we have

$$(\mathcal{H}^\nabla)^*(\mathrm{II}^0) = -\,div^{\nabla,D} A^D + \langle r_0^{D,\Sigma}, \mathrm{II}^0 \rangle = \Delta^{\nabla,D} H^g + \langle H^g(\mathrm{II}^0), \mathrm{II}^0 \rangle$$

so that the constrained Willmore equation in a spaceform may be written

$$\Delta^{\nabla,D} H^g + \langle H^g(\mathrm{II}^0), \mathrm{II}^0 \rangle = \langle q, \mathrm{II}^0 \rangle.$$

In codimension one, $\langle H^g(\mathrm{II}^0), \mathrm{II}^0 \rangle = |\mathrm{II}^0|^2 H^g = 2(|H^g|^2 - K^g) H^g$, where K^g is the gaussian curvature $det\,\mathrm{II}^g$. This yields the usual form of the Willmore equation when $q = 0$ and g is flat [81].

An immediate consequence of this formulation is the well known fact that surfaces with parallel mean curvature in a spaceform are constrained Willmore (including CMC surfaces in codimension one), while minimal surfaces are Willmore. For this, we take $q = H^g(\mathrm{II}^0)$ and observe that q is holomorphic by virtue of the Codazzi equation (11.21c), equation (11.13), and the fact that $\nabla^D H^g = 0$.

The following proposition provides a more general class.

Proposition 14.2. *Let Σ be a surface in a spaceform with ambient metric g. Suppose that $H^g = H_+ + H_-$ where $H_\pm$ are complex conjugate sections of $N^*\Sigma \otimes \mathbb{C}$ such that $(\nabla^D H_-)^{0,1} = 0$ and $\langle \nabla^D H_-, H_- \rangle = 0$. Then Σ is constrained Willmore.*

Proof. Since $(\nabla^D H_-)^{0,1} = 0 = (\nabla^D H_+)^{1,0}$, we have

$$\begin{aligned}\Delta^{\nabla,D} H^g &= div^{\nabla,D}\big((\nabla^D H_+)^{1,0} + (\nabla^D H_-)^{1,0} + (\nabla^D H_+)^{0,1} + (\nabla^D H_-)^{0,1}\big)\\ &= div^{\nabla,D}\big(-(\nabla^D H_+)^{1,0} + (\nabla^D H_-)^{1,0} + (\nabla^D H_+)^{0,1} - (\nabla^D H_-)^{0,1}\big)\\ &= \boldsymbol{i}\, div^{\nabla,D} *\nabla^D (H_+ - H_-)\\ &= \boldsymbol{i}{*}R^\nabla \cdot (H_+ - H_-).\end{aligned}$$

Hence, using the Ricci equation (11.29), we have

$$\Delta^{\nabla,D} H^g + \langle H^g(\mathrm{II}^0), \mathrm{II}^0 \rangle = \langle q, \mathrm{II}^0 \rangle$$

where $q = 2H_-(\mathrm{II}_0)^{2,0} + 2H_+(\mathrm{II}_0)^{0,2}$. It remains to see that q (or equivalently $q^{2,0}$) is holomorphic. Since $(\nabla^D H_-)^{0,1} = 0$, it suffices to show that the contraction of $\overline{\partial}^\nabla(\mathrm{II}^0)^{2,0}$ with H_- is zero. But $\overline{\partial}^\nabla(\mathrm{II}^0)^{2,0} = (d^{\nabla,D}\mathrm{II}^0)^{2,1}$, which is a multiple of $(\nabla^D H^g)^{1,0} \wedge \mathsf{c}$ by (11.21c) and (11.13). Since $(\nabla^D H^g)^{1,0} = \nabla^D H_-$, the contraction with H_- vanishes by the second equation on H_-. $\square$

Corollary 14.3. *Let Σ be cooriented codimension two surface in a spaceform with ambient metric g, and suppose H^g is holomorphic with respect to the natural complex structure on $N^*\Sigma$ induced by its conformal structure and orientation, and the holomorphic structure induced by ∇^D. Then Σ is constrained Willmore.*

Proof. Let $H_+ = (H^g)^{0,1}$ and $H_- = (H^g)^{1,0}$ be the projections of H^g onto the eigenspaces in $N^*\Sigma \otimes \mathbb{C}$ of the complex structure. Since H^g is holomorphic, $(\nabla^D H_-)^{0,1} = 0$. On the other hand, these eigenspaces are null, so $0 = d\langle H_-, H_- \rangle = 2\langle \nabla^D H_-, H_- \rangle$. Hence we can apply the proposition. $\square$

In the case that g is flat, this result is due to Bohle [9].

One reason to be interested in constrained Willmore surfaces is that they admit a spectral deformation (in arbitrary codimension) and hence an integrable systems interpretation.

Theorem 14.4. *Let $\Lambda \subset \Sigma \times \mathbb{R}^{n+1,1}$ be an immersed surface in S^n with Gauß–Codazzi–Ricci data $(\mathsf{c}, \mathcal{M}^\Sigma, \nabla, \mathrm{II}^0)$, and quadratic differential $q \in C^\infty(\Sigma, S^2_0 T^*\Sigma)$. Then the following conditions are equivalent.*

(i) *Σ is constrained Willmore with Lagrange multiplier q.*

(ii) $(\mathsf{c}, \mathcal{M}^\Sigma + \frac{1}{2}(e^{2tJ} - 1)q, \nabla, e^{tJ}\mathrm{II}^0)$ *satisfies the Gauß–Codazzi–Ricci equations for some $t \notin \pi\mathbb{Z}$.*

(iii) $(\mathsf{c}, \mathcal{M}^\Sigma + \frac{1}{2}(e^{2tJ} - 1)q, \nabla, e^{tJ}\mathrm{II}^0)$ *satisfies the Gauß–Codazzi–Ricci equations and defines a constrained Willmore immersion of Σ in S^n for all $t \in \mathbb{R}$.*

Proof. The Gauß–Codazzi–Ricci equations hold for the data in (ii)–(iii) iff

$$\begin{aligned} 0 &= C^\Sigma + \tfrac{1}{2}(e^{2tJ}-1)dq + \langle \mathcal{B}^\nabla(\mathrm{I\!I}^0)\rangle \\ 0 &= (\mathcal{H}^\nabla + \tfrac{1}{2}(e^{2tJ}-1)q)^*(e^{tJ}J\mathrm{I\!I}^0) \\ &= \cos t(\mathcal{H}^\nabla)^*(J\mathrm{I\!I}^0) - \sin t(\mathcal{H}^\nabla)^*(\mathrm{I\!I}^0) + \tfrac{1}{2}\langle (e^{2tJ}-1)q, e^{tJ}J\mathrm{I\!I}^0\rangle \\ 0 &= R^\nabla - \mathrm{I\!I}^0 \wedge \mathbb{S}^0 \end{aligned}$$

(using (11.24)), and we observe that $\frac{1}{2}\langle (e^{2tJ}-1)q, e^{tJ}J\mathrm{I\!I}^0\rangle = \sin t\langle q, \mathrm{I\!I}^0\rangle$. Hence the Gauß–Codazzi–Ricci equations hold for all t iff they hold for some $t \notin \pi\mathbb{Z}$ iff $dq = 0$ and $(\mathcal{H}^\nabla)^*(\mathrm{I\!I}^0) = \langle q, \mathrm{I\!I}^0\rangle$. □

One way to realize this integrable system is to lift the spectral deformation to a family of flat connections. This yields an alternative definition of constrained Willmore surfaces.

Theorem 14.5. *Let $\Lambda \subset \Sigma \times \mathbb{R}^{n+1,1}$ and $(\mathsf{c}, \mathcal{M}^\Sigma, \nabla, \mathrm{I\!I}^0, q)$ be as in the previous Theorem. Then the following conditions are equivalent.*

(i) *Σ is constrained Willmore with Lagrange multiplier q.*

(ii) *There is a T^*M-valued 1-form $\xi \in \Omega^1(\Sigma, \mathfrak{stab}(\Lambda)^\perp) \subset \Omega^1(\Sigma, \mathfrak{so}(V_M))$ with $[\xi] = q$ such that $d^\mathfrak{h}\xi = 0$ and $d^{\mathfrak{D}^\mathfrak{h}-\xi,\nabla} {*\mathcal{N}} = 0$.*

(iii) *There is a $T^*\Sigma$-valued 1-form $\xi \in \Omega^1(\Sigma, \mathfrak{stab}(\Lambda)^\perp \cap \mathfrak{so}(V_\Sigma)) \subset \Omega^1(\Sigma, \mathfrak{so}(V_\Sigma))$ with $[\xi] = q$ such that $d + (e^{tJ}-1)\mathcal{N} + \frac{1}{2}(e^{2tJ}-1)\xi$ is flat for all $t \in \mathbb{R}$.*

Proof. The constrained Willmore equation may be written $0 = (\mathcal{H}^\nabla - q)^*\mathrm{I\!I}^0 = -(\mathcal{H}^\nabla - q)^*J^2\mathrm{I\!I}^0$. Now by Proposition 12.1, ${*\mathcal{N}}$ is the differential lift of $J\mathrm{I\!I}^0$ with respect to $\mathfrak{D}^\Sigma$, hence also with respect to $\mathfrak{D}^\Sigma - \xi$ provided ξ is a T^*M-valued 1-form. It follows that the constrained Willmore equation is equivalent to $d^{\mathfrak{D}^\mathfrak{h}-\xi,\nabla} {*\mathcal{N}} = 0$, where ξ is the differential lift of q (with respect to $\mathfrak{D}^\Sigma$), the equation $d^\mathfrak{h}\xi = 0$ then being equivalent to $dq = 0$. The rest follows by computing the curvature of the connections in (iii), and applying Proposition 12.2. □

14.2. Harmonicity, the functional and duality. Theorem 14.5, with $q = \xi = 0$, has two more-or-less immediate corollaries. It also provides a simple way to obtain the constrained Willmore equation from the functional. These results are all well known [18, 10], but they are derived with particular ease in our formalism.

Corollary 14.6. *A immersed surface in S^n is Willmore if and only if its central sphere congruence is harmonic.*

Proof. The central sphere congruence is harmonic if and only if $d^{\mathfrak{D}^\nabla} {*\mathcal{N}} = 0$. However, since Q is tracelike and $\mathrm{I\!I}^0$ is trace-free, $d^{\mathfrak{D}^\nabla} {*\mathcal{N}} = d^\mathfrak{h} {*\mathcal{N}}$. □

Corollary 14.7. *If the central sphere congruence of a Willmore surface has a second envelope (which is generic in codimension one) then this envelope is also a Willmore surface.*

Proof. Since $d^\mathfrak{h}\mathcal{N} = 0$ and $d^\mathfrak{h}{*\mathcal{N}} = 0$ we have that $r^{D,V_\Sigma} \wedge \mathbb{S}^0 = 0$ and $r^{D,V_\Sigma} \wedge J\mathbb{S}^0 = 0$. Away from umbilics (*i.e.*, on a dense open set because Σ is not totally umbilic, since it has a second envelope, and $\mathrm{I\!I}^0$ is analytic) it follows that the symmetric trace-free part of r^{D,V_Σ} is zero. Hence the second envelope has the same conformal structure and the same central sphere congruence. Since this is harmonic, the second envelope is also Willmore. □

Theorem 14.8. *A compact surface is (constrained) Willmore, as in Definition 14.1, if and only if it is a critical point of $\mathcal{W}$ for immersions of Σ (inducing the same conformal metric on Σ in the constrained case).*

Proof. Since $\mathcal{N}$ vanishes on Λ, $\mathcal{W} = \int_\Sigma |\mathcal{N}|^2 = \int_\Sigma |dS|^2$, where S is the central sphere congruence (viewed as a map from Σ into the grassmannian of $(3,1)$-planes in S^n). The first variation of this functional is $d\mathcal{W}(\dot S) = \int_\Sigma \langle d^{\mathfrak{D}^\nabla} {*}\mathcal{N}, \dot S\rangle$, where $\dot S \colon \Sigma \to \mathfrak{m}$ denotes the variation of S. Now $d^{\mathfrak{D}^\nabla}{*}\mathcal{N} = d^{\mathfrak{h}}{*}\mathcal{N}$ and, by Proposition 12.1, $\partial d^{\mathfrak{h}}{*}\mathcal{N} = 0$, so that $\mathcal{N} \in \Omega^2(\Sigma, \mathfrak{m} \cap \mathfrak{stab}(\Lambda)^\perp)$ and hence $d\mathcal{W}(\dot S)$ vanishes automatically when $\dot S$ takes values in $C^\infty(\Sigma, \mathfrak{m} \cap \mathfrak{stab}(\Lambda)) = im\,\partial$ (this is really just a special case of Poincaré duality for Lie algebra homology). Hence $d\mathcal{W}(\dot S) = \int_\Sigma \langle [d^{\mathfrak{h}}{*}\mathcal{N}], [\dot S]\rangle = \int_\Sigma \langle d_{BGG}(J\mathrm{I\!I}^0), [\dot S]\rangle$, with $[\dot S] \in C^\infty(\Sigma, N\Sigma)$ and $d_{BGG}(J\mathrm{I\!I}^0)$ is ${*}(\mathcal{H}^\nabla)^*(\mathrm{I\!I}^0)$, up to a sign.

For variations of S coming from variations of the immersion, $[\dot S]$ is essentially the normal variation vector field. For general variations, $[\dot S]$ can therefore be arbitrary, whereas for conformal variations are characterized by $\mathbb{S}^0[\dot S] \in C^\infty(\Sigma, sym_0(T\Sigma))$ being in the image of the conformal Killing operator $X \mapsto \mathcal{L}_X \mathsf{c}$. This last is adjoint to d on quadratic differentials. The result now follows. □

15. Isothermic surfaces

15.1. Definition and spectral deformation. The classical definition of an isothermic surface is a surface admitting a conformal curvature line coordinate, *i.e.*, a conformal coordinate $z = x + \boldsymbol{i}y$ with respect to which the second fundamental form is diagonal. This is a conformally-invariant condition, and there is a manifestly invariant way to state it, which we take as our fundamental definition.

Definition 15.1. An immersion Λ of a surface Σ in S^n is said to be *isothermic* if there is a nonzero quadratic differential $q \in C^\infty(\Sigma, S^2T^*_0\Sigma) \subset \Omega^1(\Sigma, T^*\Sigma)$ which is holomorphic (*i.e.*, $dq = 0$) and commutes with shape operators in the sense that $q \wedge \mathbb{S}^0 = 0$.

Remarks 15.2. Thus $q^{(2,0)} = dz^2$ for some conformal curvature line coordinates $z = x + \boldsymbol{i}y$.

This definition has the merit of using just the Gauß–Codazzi–Ricci data, which only determine Σ locally up to Möbius transformation. Hence, in this approach, one can develop a theory of isothermic surfaces modulo Möbius transformation.

Also note that because the normal components of $\mathbb{S}^0$ commute with q, they commute with each other, and so the weightless normal bundle is flat by the Ricci equation.

We now show that the new Gauß–Codazzi–Ricci data $(\mathsf{c}, \mathcal{M}^\Sigma + rq, \nabla, \mathrm{I\!I}^0)$ also define an isothermic surface for any r, with the same quadratic differential q (up to a constant multiple). The new surfaces are sometimes called the *T-transforms* of Σ: note that only the Möbius structure has changed here.

Theorem 15.3. *Let $\Lambda \subset \Sigma \times \mathbb{R}^{n+1,1}$ be an immersed surface in S^n with Gauß–Codazzi–Ricci data $(\mathsf{c}, \mathcal{M}^\Sigma, \nabla, \mathrm{I\!I}^0)$, and quadratic differential $q \in C^\infty(\Sigma, S^2_0T^*\Sigma)$. Then the following conditions are equivalent.*

(i) *Λ is isothermic with holomorphic commuting quadratic differential q.*

(ii) *$(\mathsf{c}, \mathcal{M}^\Sigma + rq, \nabla, \mathrm{I\!I}^0)$ satisfies the Gauß–Codazzi–Ricci equations for some $r \neq 0$.*

(iii) *$(\mathsf{c}, \mathcal{M}^\Sigma + rq, \nabla, \mathrm{I\!I}^0)$ satisfies the Gauß–Codazzi–Ricci equations and defines a isothermic immersion of Σ in S^n for all $r \in \mathbb{R}$.*

Proof. The Gauß–Codazzi–Ricci equations hold for the data in (ii)–(iii) iff

$$0 = C^\Sigma + r\,dq + \langle \mathcal{B}^\nabla(\mathrm{I\!I}^0)\rangle$$

$$0 = (\mathcal{H}^\nabla)^*(J\mathrm{I\!I}^0) + r\langle q, J\mathrm{I\!I}^0\rangle$$

$$0 = R^\nabla - \mathrm{I\!I}^0 \wedge \mathbb{S}^0$$

(using (11.24)). Clearly these equations hold for all r iff they hold for $r=0$ and some nonzero r, since then $\mathit{div}\, q = 0$ and q is orthogonal to $J\mathrm{I\!I}^0$: since $S^2_0 T^*\Sigma$ has rank 2 this last condition means that every component of $\mathrm{I\!I}^0$ commutes with q. $\square$

The T-transforms give rise an integrable systems interpretation of isothermic surfaces: the Gauß–Codazzi–Ricci data $(\mathsf{c}, \mathcal{M}^\Sigma + rq, \nabla, \mathrm{I\!I}^0)$ defines a pencil of flat connections on $V_\Sigma \oplus V_\Sigma^\perp$. It also yields an alternative definition of an isothermic surface.

Theorem 15.4. *Let $\Lambda \subset \Sigma \times \mathbb{R}^{n+1,1}$ and $(\mathsf{c}, \mathcal{M}^\Sigma, \nabla, \mathrm{I\!I}^0, q)$, be as in the previous Theorem. Then the following conditions are equivalent.*

(i) *Λ is isothermic with holomorphic commuting quadratic differential q.*

(ii) *There is a T^*M-valued 1-form $\eta \in \Omega^1(\Sigma, \mathfrak{stab}(\Lambda)^\perp) \subset \Omega^1(\Sigma, \mathfrak{so}(V_M))$ with $[\eta] = q$ such that $d\eta = 0$.*

(iii) *There is a $T^*\Sigma$-valued 1-form $\eta \in \Omega^1(\Sigma, \mathfrak{stab}(\Lambda)^\perp \cap \mathfrak{so}(V_\Sigma)) \subset \Omega^1(\Sigma, \mathfrak{so}(V_\Sigma))$ with $[\eta] = q$ such that $d + r\eta$ is flat for all $r \in \mathbb{R}$.*

Proof. Since d is flat and T^*M is abelian, $d + r\eta$ is flat if and only if $d\eta = 0$. This condition certainly implies that $\partial d^{\mathfrak{h}}\eta = 0$ so that $\eta = j^{\mathfrak{h}} q = q$ by Proposition 12.2. Now $d\eta = 0$ if and only if $d^{\mathfrak{D}^\Sigma}\eta = 0$ and $[\mathcal{N} \wedge \eta] = 0$, *i.e.*, if and only if $dq = 0$ and $q \wedge \mathbb{S}^0 = 0$. $\square$

In subsequent work, we shall focus on (ii) above as our main definition of an isothermic surface (this definition also generalizes [17]). The classical definition of isothermic surfaces is essentially a homological version of this 'integrable systems' definition. In the formulation given by (iii), the T-transform of Λ, with parameter r, is Λ itself, but viewed as a (local) immersion in the space of $(d + r\eta)$-parallel null lines in $\Sigma \times \mathbb{R}^{n+1,1}$. The corresponding map into the space of d-parallel null lines is obtained from this via a gauge transformation.

15.2. Examples and symmetry breaking. The examples of isothermic surfaces which are easy to describe fall into three classes, all of which exhibit some sort of symmetry breaking in the sense of §§13.2–13.3.

Cones, cylinders, revolutes and their generalizations. The simplest examples of isothermic surfaces in S^3 are cones over a curve in $\mathcal{S}^2$, cylinders over a curve in $\mathbb{R}^2$, and revolutes (surfaces of revolution) over a curve on $\mathcal{H}^2$. These are the 2-dimensional case of the submanifolds described in §13.3. In higher codimension, we obtain a larger class than just the cones, cylinders and revolutes in this way. We focus on the nondegenerate case (generalizing cones and revolutes), but the degenerate case is similar.

Suppose that

$$\mathbb{R}^{n+1,1} = \mathbb{R}^{k,1} \oplus \mathbb{R}^{n-k+1}$$

and so $S^n \smallsetminus S^{k-1} = \mathcal{H}^k \times \mathcal{S}^{n-k}$. We claim that any Cartesian product of curves $\gamma_1(s)$ in $\mathcal{H}^k$ and $\gamma_2(t)$ in $\mathcal{S}^{n-k}$ is isothermic. The product surface $\Sigma = \Sigma_1 \times \Sigma_2$ has a natural lift $\sigma(s,t) := \gamma_1(s) + \gamma_2(t)$ into the light-cone. Clearly s and t are curvature line coordinates, and if γ_1 and γ_2 are parameterized by arc-length, then, differentiating $\sigma(s,t)$, we see that they are also conformal.

We observe that we can take $\eta = \sigma \mathbin{\vartriangle} (d\gamma_1 - d\gamma_2)$, where $\vartriangle$ denotes the pairing $\mathbb{R}^{n+1,1} \times \mathbb{R}^{n+1,1} \to \mathfrak{so}(n+1,1)$ with $(u \mathbin{\vartriangle} w)(v) = (u,v)w - (u,w)v$. Indeed η takes values in the nilradical of Λ, so it remains to show that it is closed. Well:

$$d\eta = (d\gamma_1 + d\gamma_2) \mathbin{\overset{\vartriangle}{\wedge}} (d\gamma_1 - d\gamma_2)$$

(wedge product, using $\vartriangle$ to multiply coefficients). Now $d\gamma_i \mathbin{\overset{\vartriangle}{\wedge}} d\gamma_i = 0$ since these are wedges of (pullbacks of) 1-forms on 1-manifolds so we are left with $-d\gamma_1 \mathbin{\overset{\vartriangle}{\wedge}} d\gamma_2 + d\gamma_2 \mathbin{\overset{\vartriangle}{\wedge}} d\gamma_1$, which vanishes because the wedge product of 1-forms and $\vartriangle$ are both skew.

This class of surfaces is stable under T-transform: for this we need to find $(d + r\eta)$-stable subspaces U_1 and $U_2 = U_1^\perp$ of $\Sigma \times \mathbb{R}^{n+1,1}$ with ranks $k+1$ and $n-k+1$, such that σ induces the same splitting of $\Sigma = \Sigma_1 \times \Sigma_2$. To do that we make an Ansatz that $U_1 = im\, d\gamma_1 \oplus \langle \gamma_1 + a(r)(\gamma_1+\gamma_2)\rangle$ and hence $U_2 = im\, d\gamma_2 \oplus \langle \gamma_2 + a(r)(\gamma_1+\gamma_2)\rangle$ (with $a(r)$ a constant for fixed r). Let X_1 and X_2 be vector fields on Σ_1 and Σ_2 respectively, pulled back to the product. Then we compute

$$(d + r\eta)_{X_2}(\gamma_1 + a(r)(\gamma_1+\gamma_2)) = a(r) d_{X_2}\gamma_2 - r(\gamma_1,\gamma_1) d_{X_2}\gamma_2.$$

This lies in U_1 iff it is zero, which (assuming $(\gamma_1,\gamma_1) = -1$) gives $a(r) = -r$. For the stability of U_1, it remains to consider $v \in im\, d\gamma_1$:

$$(d + r\eta)v \mod im\, d\gamma_1 = -(dv,\gamma_1)\gamma_1 - r(d\gamma_1, v)(\gamma_1+\gamma_2) = (v, d\gamma_1)(\gamma_1 - r(\gamma_1+\gamma_2)),$$

which lies in U_1 as required. The stability of U_2 follows by orthogonality. Notice that U_1 and U_2 become degenerate and intersecting at $r = 1/2$, then change signature, so that at $r = 1$ we recover the original product in reverse order.

We remark that Tojeiro's definition [77] of isothermic submanifolds of higher dimension generalizes these examples: his submanifolds are extrinsic products of curves in spaceforms.

CMC surfaces in spaceforms. Another well-known (and classical) class of isothermic surfaces are the CMC surfaces in $\mathbb{R}^3$ (*i.e.*, surfaces of constant mean curvature H with respect to the induced metric, also called H-surfaces) or in 3-dimensional spaceforms [7, 6]. The induced metric identifies $\mathrm{I\!I}^0$ with a quadratic differential called the Hopf differential, and the Codazzi equation (in the spaceform) shows that this is holomorphic (and of course, it commutes with itself).

In higher codimension, the mean curvature is a conormal vector, and the most natural notion of 'constant mean curvature' is parallel mean curvature (using the connection on the conormal bundle induced by the spaceform metric), but this essentially forces Σ to have codimension one or two; however, as shown in [13], a more general class of surfaces, called *generalized H-surfaces*, are isothermic. We present a novel analysis of this case.

Suppose the spaceform geometry in S^n is defined by $v_\infty \in \mathbb{R}^{n+1,1}$ and let $\sigma \in C^\infty(\Sigma, \Lambda)$ be the spaceform lift of a submanifold of S^n (so $(\sigma, v_\infty) = -1$), which is the gauge induced by the spaceform metric g. Let ξ be a parallel unit section of $V_\Sigma^\perp$. Now set $\eta = \sigma \vartriangle d\xi$. This clearly takes values in the nilradical of Λ, so it remains to determine when it is closed, *i.e.*, when $d\sigma \mathbin{\curlywedge} d\xi = 0$. For this, let $V_g = \exp(H^g)V_\Sigma$ be the tangent sphere congruence in the spaceform, so that $\xi_g = \exp(H^g)\xi = \xi + \langle H^g\ell, \xi\rangle\sigma$, where ℓ is dual to σ. Thus $d\xi = d\xi^g - d(\langle H^g\ell,\xi\rangle\sigma)$ and $d\xi^g = \mathcal{N}^g\xi = -\mathbb{S}^g\xi = -\mathbb{S}^0\xi - \langle H^g\ell,\xi\rangle d\sigma$, since $A^g = 0$. Now $d\sigma \mathbin{\curlywedge} \mathbb{S}^0\xi$ is identically zero, so $d\sigma \mathbin{\curlywedge} d\xi = 0$ if and only if $d\langle H^g\ell,\xi\rangle = 0$, *i.e.*, the ξ component of the mean curvature covector H^g is constant.

These are the generalized H-surfaces. The quadratic differential q is the corresponding (ξ) component of $\mathrm{I\!I}^0$, although it is not so obvious to see directly that this commutes with the other components. Observe that by construction $\eta \cdot v_\infty + d\xi = 0$. It follows that for all $r \in \mathbb{R}$, $v_\infty + r\xi$ is parallel with respect to $d + r\eta$. This conserved quantity provides another way to characterize generalized H-surfaces, and we shall explore this further in the sequel to this paper.

Quadrics. Quadrics are natural generalizations of spheres in euclidean space. After the study of surfaces of constant nonzero gaussian curvature (K-surfaces), which are isometric to spheres or their hyperbolic analogues, surfaces isometric to quadrics were of great interest in classical differential geometry. An important classical observation is that quadrics are isothermic (see [43]), and this was generalized to conformal quadrics by Darboux [39, pp. 212–219]. This is often considered to be surprising [13, 56], since quadric surfaces live most naturally in the projective geometry of $\mathbb{R}P^3$, not the conformal geometry of S^3. Darboux's argument fits well into our formalism, so we sketch it here: in fact Darboux restricts attention to diagonal quadrics, so we complement his analysis by describing some of the geometry in the general case.

A quadric Q in S^3 is determined by a matrix $A \in sym(\mathbb{R}^{4,1})$, which is not a multiple of the identity: then $Q = \{<\sigma> \in P(\mathcal{L}) : (\sigma, \sigma) = 0 = (A\sigma, \sigma)\}$. Conversely, the quadric Q only determines A up to affine transformations $A \mapsto aA + bI$ $(a, b \in \mathbb{R},\ a \neq 0)$. The geometric idea behind Darboux's construction is to extend this affine action to a projective one: the matrices $(aA + bI)(cA + dI)^{-1}$ $(a, b, c, d \in \mathbb{R},\ ad - bc \neq 0)$ determine a one parameter family of quadrics. (In fact these quadrics are confocal, but we do not need this here.) Given A, we can parameterize the family by the matrices $A(I - uA)^{-1}$.

Now observe that for fixed σ, $det(I - uA)(A(I - uA)^{-1}\sigma, \sigma) = 0$ is a quartic in u, whose leading coefficient vanishes if $(\sigma, \sigma) = 0$. The roots u_1, u_2, u_3 of this cubic determine three quadrics in the family passing through $<\sigma> \in P(\mathcal{L})$. Now observe that the conditions

$$(A(I - u_1 A)^{-1}\sigma, \sigma) = 0 = (A(I - u_2 A)^{-1}\sigma, \sigma)$$

imply that

$$\begin{aligned} 0 &= \big(\big(A(1 - u_2 A) - A(1 - u_1 A)\big)(I - u_1 A)^{-1}(I - u_2 A)^{-1}\sigma, \sigma\big) \\ &= (u_1 - u_2)(A(I - u_1 A)^{-1}\sigma, A(I - u_2 A)^{-1}\sigma) \end{aligned}$$

so that distinct quadrics in the family meet each other orthogonally at $<\sigma> \in P(\mathcal{L})$. By Dupin's Theorem, they therefore meet in curvature lines.

To see that this gives rise to conformal curvature line coordinates, we now follow Darboux and assume A is diagonalized, with distinct eigenvalues $(1/a_0, 1/a_1, 1/a_2, 1/a_3, 1/a_4)$ in coordinates $x = (x_0, x_1, x_2, x_3, x_4)$ on $\mathbb{R}^{4,1}$ such that $(x, x) = -x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2$. Thus $A(I - uA)^{-1}$ has eigenvalues $1/(a_i - u)$. Then, for x null, we can solve for x_i in terms of the parameters (u_1, u_2, u_3) of the three quadrics passing through that point to get $x_0^2 = -c(u_1 - a_0)(u_2 - a_0)(u_3 - a_0)/f'(a_0)$ and $x_i^2 = c(u_1 - a_i)(u_2 - a_i)(u_3 - a_i)/f'(a_i)$ for $i = 1, \ldots 4$, where $f(u) = (u - a_0)(u - a_1)(u - a_2)(u - a_3)(u - a_4)$ and c is an overall scale.

By differentiating these formulae for x_i, it follows that the conformal metric on S^3 has a representative metric

$$\frac{(u_2 - u_1)(u_3 - u_1)}{f(u_1)} du_1^2 + \frac{(u_3 - u_2)(u_1 - u_2)}{f(u_2)} du_2^2 + \frac{(u_1 - u_3)(u_2 - u_3)}{f(u_3)} du_3^2 \tag{15.1}$$

in these coordinates. Thus (u_1, u_2) are conformal curvature line coordinates on the quadrics with u_3 constant, which are therefore isothermic.

If A is not diagonalizable as above, the coordinates (u_1, u_2, u_3) are still well defined, and so a continuity argument gives the same representative metric (15.1) for the conformal structure on S^3, where $f(u)$ is the characteristic polynomial of A^{-1}.

This argument almost provides the conceptual explanation sought in [13]: it shows that conformal geometry equips a quadric with a natural conjugate net; then, perhaps not so surprisingly, this net is orthogonal, and gives rise to the conformal curvature line coordinates which show that the quadric is isothermic.

We end by remarking that the eigenvectors of A can be used to break the symmetry to that of a spaceform, and then the quadric is realized as a quadric in a spaceform. In particular, for matrices with a null eigenvector, we recover the quadrics in euclidean space.

15.3. When is a surface isothermic? The definition we have given of an isothermic surface has the disadvantage that it does not characterize when a surface admits such a quadratic differential q. There is also a subtle global issue here: q might exist locally (near any point, perhaps only in a dense open set), but not globally on Σ. However, q is unique up to a constant multiple away from umbilic points: it must be a pointwise multiple of any nonvanishing component $\langle \mathrm{II}^0, \xi\rangle$ of II^0 and this multiple is determined up to a constant since q is holomorphic. Hence q is globally defined on the universal cover of the open subset U where II^0 is non-vanishing (or equivalently it may be viewed as a holomorphic quadratic differential with values in a flat line bundle $\mathcal{L}$ on U).

In the case that the central sphere congruence has a second envelope, *i.e.*, there is a Weyl derivative D^0 with $0 = A^{D^0} = -\,div^{\nabla,D^0}\,\mathbb{S}^0$, an intrinsic characterization is available. Writing $\mathbb{S}^0 = S\otimes\xi$ for a weightless unit normal vector ξ we first see that such a second envelope can only exist if ξ is parallel, and then D^0 is the Weyl derivative of a second envelope iff $div^{D^0} S = 0$. If the weightless normal bundle is flat, it follows that $V \oplus <\xi>$ is constant, and so can assume without loss that we are in codimension one, with $S = \mathbb{S}^0$.

Since $A^{D+\gamma} = A^D + \gamma\circ\mathbb{S}^0$, it then follows that such a second envelope exists precisely on the set where either $\mathcal{N} = 0$ or $\mathbb{S}^0 \neq 0$. In the latter case, the second envelope is unique, and D^0 can be determined from an arbitrary Weyl structure D via $D + (\mathbb{S}^0)^{-1}\,div^D\,\mathbb{S}^0$.

Now any commuting quadratic differential may be written $q = \sigma\mathbb{S}^0$, and it is holomorphic if and only if $D^0\sigma = 0$. Such a (nonzero) σ exists if and only if D is exact. This turns out to be the classical assertion that

a surface is isothermic iff its central sphere congruence is Ribaucour.

Furthermore, this gives an equation for isothermic surfaces. Let D be the unique Weyl derivative with $D(|\mathbb{S}^0|^2) = 0$. Then D is exact and so, since $(\mathbb{S}^0)^{-1} = 2\mathbb{S}^0/|\mathbb{S}^0|^2$, $D^0 = D + 2\mathbb{S}^0\,div^D\,\mathbb{S}^0/|\mathbb{S}^0|^2$, which is locally exact if and only if $d(\mathbb{S}^0\,div^D\,\mathbb{S}^0/|\mathbb{S}^0|^2) = 0$. This may be written in a conformally invariant way as $d(*\langle\mathcal{B}^\nabla(\mathrm{II}^0)\rangle/|\mathrm{II}^0|^2) = 0$ (simply expand the definition of $\mathcal{B}$ using the Weyl derivative D).

16. Möbius-flat submanifolds

16.1. Definition and spectral deformation. Let Σ be a submanifold of S^n with flat weightless normal bundle. We wish to impose the condition that there is a enveloped sphere congruence V inducing a flat conformal Möbius structure on Σ. For $m \geqslant 3$, all enveloped sphere congruences induce the same conformal Möbius structure on Σ, since they induce the same conformal structure. On the other hand by equation (9.10), when $m = 2$, the conformal Möbius structure induced by V has $\mathcal{H}^V = \mathcal{H}^\Sigma - q$ with $q = H^V(\mathrm{II}^0)$, hence it is flat iff q is a *Cotton–York potential*, *i.e.*, $dq = C^\Sigma$. Since the weightless normal bundle is flat q *commutes* with the shape operators in the sense that $q\wedge\mathbb{S}^0 = 0$. (Conversely, if q is nonzero, the flatness of ∇ follows from this.)

This motivates the following definition.

Definition 16.1. An immersion Λ of Σ ($dim\,\Sigma \geqslant 2$) in S^n is called *Möbius-flat* if it has flat weightless normal bundle, and either $m \geqslant 3$ and Σ is conformally flat, or $m = 2$ and Σ admits a *commuting Cotton–York potential* (CCYP), *i.e.*, there exists $q \in C^\infty(\Sigma, S^0T^*\Sigma)$ with $dq = C^\Sigma$ and $q\wedge\mathbb{S}^0 = 0$. (For $m \geqslant 3$, we set $q = 0$.)

Conversely, if Σ is Möbius-flat, then for $m \geqslant 3$ any enveloped sphere congruence induces this flat conformal structure; for $m = 2$, an enveloped sphere congruence V induces the flat conformal Möbius structure $\mathcal{H}^\Sigma - q$ iff $q = H^V(\mathrm{II}^0)$. If $q = 0$, we can take V to be the central sphere congruence; otherwise, on the open set where q is nonzero, we can write $\mathrm{II}^0 = q\otimes U$ for a normal vector U and set $H^V = \langle U,\cdot\rangle/\langle U,U\rangle$. In codimension one the enveloped sphere congruence inducing $\mathcal{H}^\Sigma - q$ is clearly unique. (Note that H^V extends by zero to the zeros of q unless II^0 is also zero there.)

Möbius-flat submanifolds admit a spectral deformation and hence an integrable systems interpretation.

Theorem 16.2. *Let $\Lambda \subset \Sigma \times \mathbb{R}^{n+1,1}$ be a submanifold of S^n of dimension $m \geqslant 2$ with Gauß–Codazzi–Ricci data $(\mathsf{c}, \mathcal{M}^\Sigma, \nabla, \mathrm{I\!I}^0)$ and a quadratic differential $q \in C^\infty(\Sigma, S^2_0 T^*\Sigma)$ with $q = 0$ if $m \geqslant 3$. Then the following conditions are equivalent.*

(i) *Σ is Möbius-flat (with CCYP q if $m = 2$).*

(ii) *$(\mathsf{c}, \mathcal{M}^\Sigma + (t^2-1)q, \nabla, t\mathrm{I\!I}^0)$ satisfies the Gauß–Codazzi–Ricci equations for some $t \notin \{0, \pm 1\}$.*

(iii) *$(\mathsf{c}, \mathcal{M}^\Sigma + (t^2-1)q, \nabla, t\mathrm{I\!I}^0)$ satisfies the Gauß–Codazzi–Ricci equations and defines a Möbius-flat immersion of Σ in S^n (with CCYP $t^2 q$ if $m = 2$) for all $t \in \mathbb{R}$.*

Proof. The Gauß–Codazzi–Ricci equations hold for the data in (ii)–(iii) iff

$$\begin{aligned}
&0 = W^\Sigma - t^2[\![id \wedge Q]\!] - t^2 \mathbb{S}^0 \wedge \mathrm{I\!I}^0 && m \geqslant 4\\
&0 = C^\Sigma + (t^2-1)dq + t^2\langle \mathcal{B}^\nabla(\mathrm{I\!I}^0)\rangle && m = 2,3\\
&0 = t\, CCoda^\nabla \mathrm{I\!I}^0 && m \geqslant 3\\
&0 = t(\mathcal{H}^\nabla)^*(J\mathrm{I\!I}^0) + t(t^2-1)\langle q, J\mathrm{I\!I}^0\rangle && m = 2\\
&0 = R^\nabla - t^2 \mathrm{I\!I}^0 \wedge \mathbb{S}^0
\end{aligned}$$

(using (11.24)). These hold for all t iff they hold for $t = 1$, $W^\Sigma = 0$, $C^\Sigma = dq$, $R^\nabla = 0$ and $q \wedge \mathbb{S}^0 = 0$, and it suffices that the equations hold for $t = 1$ and some $t \notin \{0, \pm 1\}$. □

As in the isothermic case, the integrable systems viewpoint is made most transparent in terms of an associated family of flat connections. This also reveals an alternative definition which will be very useful to us in the sequel to this paper.

Theorem 16.3. *Let $\Lambda \subset \Sigma \times \mathbb{R}^{n+1,1}$ and $(\mathsf{c}, \mathcal{M}^\Sigma, \nabla, \mathrm{I\!I}^0, q)$ be as in the previous Theorem. Then the following conditions are equivalent.*

(i) *Σ is Möbius-flat (with CCYP q if $m = 2$).*

(ii) *There is a sphere congruence V enveloped by Λ and a 1-form $\chi^V \in \Omega^1(\Sigma, \mathfrak{stab}(\Lambda)^\perp)$ such that $d\chi^V = R^{\mathfrak{D}^{\nabla,V}}$.*

(iii) *For any sphere congruence V enveloped by Λ, there is $\chi^V \in \Omega^1(\Sigma, \mathfrak{stab}(\Lambda)^\perp \cap \mathfrak{so}(V))$ such that $\mathrm{d}^V_t := \mathfrak{D}^{\nabla,V} + t\mathcal{N}^V + (t^2-1)\chi^V$ is flat for all $t \in \mathbb{R}$ (or equivalently, since d_1 is flat, $\mathfrak{D}^{\nabla,V} - \chi^V$ is flat and $[\mathcal{N}^V \wedge \chi^V] = 0$).*

When $m = 2$, χ^V is a lift of $q - H^V(\mathrm{I\!I}^0)$ for a CCYP q. (Notice that the condition in (ii) *only needs to be verified for one sphere congruence V; then the—apparently stronger—condition in* (iii) *holds for all V.)*

Proof. We prove (i)⇒(iii)⇒(ii)⇒(i). Clearly (iii)⇒(ii) by taking $t = 0$ and observing that

$$d\chi^V = d^{\mathfrak{D}^{\nabla,V}}\chi^V + [\mathcal{N}^V \wedge \chi^V],$$

whereas the curvature of $\mathfrak{D}^{\nabla,V} - \chi^V$ is $R^{\mathfrak{D}^{\nabla,V}} - d^{\mathfrak{D}^{\nabla,V}}\chi^V$.

Suppose $d\chi^V = R^{\mathfrak{D}^{\nabla,V}}$ with χ^V as in (ii). By considering the $\mathfrak{m}$-component of this equation, we see that $[\beta \wedge (\chi^V)^\perp] = 0$, hence $\chi^V \in \mathfrak{so}(V)$ and $\mathfrak{D}^V - \chi^V$ is a flat connection on V. Since $\exp(-H_V)\cdot(\mathfrak{D}^V - \chi^V)$ is a flat (hence normal) Cartan connection on V_Σ inducing the same soldering form as $\mathfrak{D}^\Sigma$, it must equal $\mathfrak{D}^\Sigma - q$, where $q = 0$ if $m \geqslant 3$ and q is a quadratic differential for $m = 2$. Hence $\chi^V = Q^V + q$ and q is the given quadratic differential by assumption when $m = 2$.

Conversely, given (i) we can define $\chi^V = Q^V + q$ so that $\exp(H_V)\cdot(\mathfrak{D}^\Sigma - q) = \mathfrak{D}^V - \chi^V$.

To complete the proof of (ii)⇒(i)⇒(iii), we note that $Q^V \wedge \mathbb{S}^0 = Q \wedge \mathbb{S}^0 = 0$ since the weightless normal bundle is flat (the latter equality follows either from the explicit formula for Q or by computing that $\Box_{\mathfrak{h}}[Q \wedge \mathcal{N}] = 0$ so that $[Q \wedge \mathcal{N}]$ is the differential lift of its homology class, which is zero because $[Q \wedge \mathcal{N}] \in im\,\partial$). Now it is clear that $[\mathcal{N}^V \wedge \chi^V] = 0$ iff $q \wedge \mathbb{S}^0 = 0$, and $\mathfrak{D}^{\nabla,V} - \chi^V$ is flat iff the normal bundle is flat and $\mathfrak{D}^\Sigma - q$ is flat, *i.e.*, $m \geqslant 3$ and Σ is conformally-flat or $m = 2$ and $dq = C^\Sigma$. Since

$$R^{d_t^V} = (1 - t^2)\big(R^{\mathfrak{D}^{\nabla,V}} - d^{\mathfrak{D}^V}\chi^V - t[\mathcal{N}^V \wedge \chi^V]\big),$$

the equivalence of these conditions with the flatness of d_t^V is immediate. □

16.2. General theory and examples. The following observation makes precise the idea that Möbius-flat submanifolds are those with a enveloped sphere congruence inducing a flat Möbius structure, and leads to many examples of Möbius-flat submanifolds.

Theorem 16.4. *Let Σ be an m-submanifold of S^n with flat weightless normal bundle. Then Σ is Möbius-flat (with CCYP q if $m = 2$) if and only if* there is an enveloped sphere congruence V (with $q = H^V(\mathrm{II}^0)$ if $m = 2$) such that $\mathbb{S}^V \wedge \mathrm{II}^V + [\![id \wedge Q^V]\!] = 0$ (which is automatic for $m \leqslant 3$) and for some (hence any) Weyl derivative D, we have*

$$d^D Q^V + A^{D,V} \wedge \mathrm{II}^V = 0 \qquad (m = 3), \tag{16.1}$$

$$\tfrac{1}{2} DK^V \wedge \mathsf{c} = A^{D,V} \wedge \mathrm{II}^V \qquad (m = 2). \tag{16.2}$$

*For $m \geqslant 3$, it then follows that any enveloped sphere congruence satisfies these conditions. (*The 'only if' part holds only away from umbilic points when $m = 2$.)*

Proof. Let V be an enveloped sphere congruence. Then Σ is Möbius-flat if and only if it has flat weightless normal bundle, $W^\Sigma = 0$ (this equation is vacuous for $m \leqslant 3$) and $C^\Sigma = dq$ for $m \leqslant 3$. Now $W^\Sigma = \mathbb{S}^V \wedge \mathrm{II}^V + [\![id \wedge Q^V]\!]$ by the Gauß equation (11.4), so it is zero if and only if $\mathbb{S}^V \wedge \mathrm{II}^V + [\![id \wedge Q^V]\!] = 0$. When $m \leqslant 3$, the Gauß equation (11.5) instead gives $C^\Sigma = -A^{D,V} \wedge \mathrm{II}^V - d^D Q^V$ (with respect to any Weyl derivative D). For $m = 2$ we have $Q^V = -H^V(\mathrm{II}^0) - \frac{1}{2}K^V\mathsf{c}$. The result easily follows. □

Corollary 16.5. *Channel submanifolds are Möbius-flat.*

Proof. By definition, a channel submanifold admits an enveloped sphere congruence V such that $\mathcal{N}^V : T\Sigma \to Hom(V, V^\perp) \oplus Hom(V^\perp, V)$ has codimension one kernel. Hence $\mathbb{S}^V \wedge \mathrm{II}^V = 0$ (so the weightless normal bundle is flat), $\mathrm{II}^V \wedge \mathbb{S}^V = 0$ (and so $Q^V = 0$ for $m = 3$ and $K^V = 0$ for $m = 2$) and $A^{D,V} \wedge \mathrm{II}^V = 0$ for any Weyl derivative D. Hence Σ is Möbius-flat. □

The *gaussian* (*sectional*) *curvature* of a 2-plane $U \subseteq T\Sigma$ is defined to be $K_U = \langle det\,\mathrm{II}^V|_U\rangle$. It is a section of Λ^2. We say that a submanifold of S^n has *constant gaussian curvature* with respect to a compatible metric g on S^n if $K_U = K^g$ for all U and $D^g K^g = 0$.

Corollary 16.6. *A submanifold of a spaceform with flat weightless normal bundle and constant gaussian curvature is Möbius-flat and the induced metric has constant curvature.*

Proof. Take $V = V_g$, the tangent congruence, and $D = D^g$, the induced Weyl derivative, so that $A^{V_g,D^g} = 0$. For $m = 2$ we then have $K^{V_g} = K^g$, hence $D^g K^{V_g} = 0$. More generally, $\mathbb{S}^g \wedge \mathrm{II}^g = -\frac{1}{2}K^g \sum_{i,j} \varepsilon_i \wedge \varepsilon_j \otimes [\![\varepsilon_i, e_j]\!]$ with $D^g K^g = 0$ where e_i is a local frame of $T\Sigma\,\Lambda$ with dual frame ε_i. (This can be seen, for instance by viewing both sides as Λ^2-valued quadratic forms on $\bigwedge^2(T\Sigma\,\Lambda)$.) Therefore for $m \geqslant 3$, $Q^{V_g} = -\frac{1}{2}K^g\mathsf{c}$, hence $r^{D^g,\Sigma} = r^{D,V_g} - Q^{V_g} = (\frac{1}{n}s^g + \frac{1}{2}K^g)\mathsf{c}$. The result easily follows. □

Of course, it is well-known that for $m \geqslant 3$, m-manifolds of constant sectional curvature are conformally-flat, so the point of the above Corollary is to take a submanifold point of view on this result, via gaussian (sectional) curvature in a spaceform, and extend it to the case $m = 2$.

It is also well known that for $m \geqslant 3$, a product of a p-manifold and an $(m-p)$-manifold of constant sectional curvatures c_p and c_{m-p} is conformally flat provided $c_p + c_{m-p} = 0$. Note that the sectional curvature of a 1-manifold is undefined, and a product of a 1-manifold and an $(m-1)$-manifold of constant sectional curvature is always conformally flat.

Let us consider the case that this product is an extrinsic product of submanifolds of spaceforms inside S^n, as in §13.3. As usual we focus on the nondegenerate case $S^n \smallsetminus S^{k-1} = \mathcal{H}^k \times \mathcal{S}^{n-k}$, but the degenerate case is similar. We thus obtain the following submanifold analogue of the $m \geqslant 3$ result, together with an extension to the case $m = 2$.

Theorem 16.7. *Let $\Sigma_1^p \to \mathcal{H}^k$ and $\Sigma_2^{m-p} \to \mathcal{S}^{n-k}$ be immersed submanifolds of spaceforms (with $1 \leqslant p \leqslant k$, $1 \leqslant m-p \leqslant n-k$). Then the product embedding $\Sigma_1 \times \Sigma_2 \to S^n$ is Möbius-flat if and only if Σ_1 and Σ_2 have flat normal bundles and constant gaussian curvatures K_1 and K_2 (the latter condition being automatic on a 1-dimensional manifold) such that $K_1 + K_2 = 0$ if $p > 1$ and $m - p > 1$.*

Proof. Pulling back by the obvious projections we carry out all computations on $\Sigma = \Sigma_1 \times \Sigma_2$, and let σ_1, σ_2 be the induced maps $\Sigma \to \mathcal{H}^k$, $\Sigma \to \mathcal{S}^{n-k}$. Then $\Sigma \times \mathbb{R}^{n+1,1} = V \oplus V^\perp$, with $V = <\sigma_1, d\sigma_1> \oplus <\sigma_2, d\sigma_2>$, $V^\perp = V_1^\perp \oplus V_2^\perp$ (corresponding to the normal bundles of Σ_1 and Σ_2), and $\Lambda = <\sigma_1 + \sigma_2>$. $\hat{\Lambda} := <\sigma_1 - \sigma_2>$ is the Weyl structure induced by the product geometry. Clearly $\mathfrak{D}^V$, $\mathcal{N}^V$ and ∇^V split across the decomposition of V and $V^\perp$. We deduce two things: first, ∇^V is flat if and only if Σ_1 and Σ_2 have flat normal bundle; second, V is enveloped by both Λ and $\hat{\Lambda}$, so $A^{D,V} = 0$ (for the Weyl derivative D induced by $\hat{\Lambda}$) and $\mathcal{N}^V = \mathrm{II}^V - \mathbb{S}^V$, with $\mathrm{II}^V = \mathrm{II}^1 + \mathrm{II}^2$ being the (direct) sum of the second fundamental forms of the factors (and $\mathbb{S}^V$ similarly). Now

$$\mathbb{S}^V \wedge \mathrm{II}^V = \mathbb{S}^1 \wedge \mathrm{II}^1 + \mathbb{S}^2 \wedge \mathrm{II}^2$$

and the equation $\mathbb{S}^V \wedge \mathrm{II}^V + [\![id \wedge Q^V]\!]$, which is necessary for Möbius-flatness and determines Q^V for $m \geqslant 3$, implies that $m = 2$ or $Q^V = f(\mathsf{c}_1 - \mathsf{c}_2)$ for some function f: indeed we first observe that Q^V must split as $Q_1 + Q_2$, with no cross term, but then we obtain

$$[\![id \wedge Q^V]\!] = [\![id_1 \wedge Q_1]\!] + [\![id_2 \wedge Q_2]\!] + [\![id_1 \wedge Q_2]\!] + [\![id_2 \wedge Q_1]\!]$$

and the last two terms can only cancel with $Q_1 = f\mathsf{c}_1$, $Q_2 = -f\mathsf{c}_2$. For $m \geqslant 4$, f is constant, while for $m = 3$, $d^D Q^V = 0$ if and only if f is constant. By Theorem 16.4, this completes the proof for $m \geqslant 3$.

For $m = 2$, we write $\mathrm{II}^1 = e_1^2 \otimes \nu_1$ and $\mathrm{II}^2 = e_2^2 \otimes \nu_2$ with respect to a weightless orthonormal basis e_1, e_2. Then $K^V = \langle det\, \mathrm{II}^V \rangle = \langle \nu_1, \nu_2 \rangle = 0$, so our product surface is always Möbius-flat (provided the normal bundles are flat), with CCYP $q = H^V(\mathrm{II}^0) = \frac{1}{4}(e_1^2 - e_2^2) \otimes (|\nu_1|^2 - |\nu_2|^2)$. ☐

Remark 16.8. There is a fast way to see that these submanifolds are Möbius-flat, using the equation $d\chi^V = R^{\mathfrak{D}^\nabla, V}$: simply take $\chi^V = c(\sigma_1 + \sigma_2) \vartriangle (d\sigma_1 - d\sigma_2)$. When $m = 2$, $d\chi^V = 0$ (as we have seen, these product manifolds are isothermic) and $R^{\mathfrak{D}^\nabla, V} = 0$. In higher dimensions $d\chi^V = c d\sigma_1 \mathbin{\text{\textdelta}} d\sigma_1 - c d\sigma_2 \mathbin{\text{\textdelta}} d\sigma_2$, and so we take c proportional to the gaussian curvatures of the factors to get $d\chi^V = R^{\mathfrak{D}^\nabla, V}$.

We now give a more explicit description of Möbius-flat submanifolds. For this, recall that the flatness of the weightless normal bundle means that the components of $\mathbb{S}^0$ commute, and so they are simultaneously diagonalizable. This means we can locally write

$$\mathbb{S}^0 = \sum_{i=1}^m \varepsilon_i \otimes e_i \otimes \nu_i$$

where ν_i are conormal vectors (whose components are the eigenvalues of the components of $\mathbb{S}^0$) with $\sum_i \nu_i = 0$, and e_i, ε_i are dual orthonormal frames of $T\Sigma\Lambda$ and $T^*\Sigma L$ (the e_i being the eigenvectors of $\mathbb{S}^0$). (Thus $\mathrm{II}^0 = \sum_{i=1}^m \varepsilon_i^2 \otimes \nu_i^\sharp$, using the identification $S^2_0 T^*\Sigma \otimes N\Sigma \cong S^2_0(T^*\Sigma \otimes L) \otimes (\Lambda^2\, N\Sigma)$, where $\nu(\nu_i^\sharp) = \langle \nu, \nu_i\rangle$.) When $m = 2$, q is also diagonalized in this frame, and we write $q = \mu^2(\varepsilon_1^2 - \varepsilon_2^2)$ with μ a section of Λ. (Up to reordering we can assume μ is real.)

We can now extend results of Cartan and Hertrich-Jeromin to arbitrary dimension and codimension for $m \geqslant 3$, and a result of [21] to a much broader context for $m = 2$.

Theorem 16.9. *Let Σ be an m-dimensional submanifold of S^n ($m \geqslant 2$) with flat normal bundle, and write $\mathbb{S}^0 = \sum_{i=1}^m \varepsilon_i \otimes e_i \otimes \nu_i$ and (for $m = 2$) $q = \mu^2(\varepsilon_1^2 - \varepsilon_2^2)$. Then Σ is Möbius-flat if and only if:*

- *$m = 2$ and the (complex) 1-forms $\alpha_1 = \sqrt{\langle \nu, \nu\rangle + \mu^2}\, \varepsilon_1$ and $\alpha_2 = \sqrt{\langle \nu, \nu\rangle - \mu^2}\, \varepsilon_2$, where $\nu = \nu_1 = -\nu_2$, are closed;*
- *$m = 3$ and the (complex) 1-forms α_i are closed, where*

$$\begin{aligned}
\alpha_1 &= \sqrt{\langle \nu_2 - \nu_1, \nu_3 - \nu_1\rangle}\, \varepsilon_1,\\
\alpha_2 &= \sqrt{\langle \nu_3 - \nu_2, \nu_1 - \nu_2\rangle}\, \varepsilon_2,\\
\alpha_3 &= \sqrt{\langle \nu_1 - \nu_3, \nu_2 - \nu_3\rangle}\, \varepsilon_3;
\end{aligned}$$

- *$m \geqslant 4$ and $\langle \nu_i - \nu_j, \nu_k - \nu_\ell\rangle = 0$ for all pairwise distinct $i, j, k, \ell \in \{1, \dots m\}$.*

Proof. For $m \geqslant 4$ Möbius-flatness is equivalent to the vanishing of the Weyl curvature W^Σ, *i.e.*, using (11.25), to the equation $\mathbb{S}^0 \wedge \mathrm{II}^0 + [\![id \wedge Q]\!] = 0$. We now compute that $\mathbb{S}^0 \wedge \mathrm{II}^0 = \sum_{i,j} \langle \nu_i, \nu_j\rangle \varepsilon_i \wedge \varepsilon_j \otimes [\![\varepsilon_i, e_j]\!]$ and also $(2 - m)[\![id \wedge Q]\!] = \sum_{i,j}\big(\langle \nu_i, \nu_i\rangle + \langle \nu_j, \nu_j\rangle - \frac{1}{m-1}\sum_\ell \langle \nu_\ell, \nu_\ell\rangle\big)\varepsilon_i \wedge \varepsilon_j \otimes [\![\varepsilon_i, e_j]\!]$, so that $W^\Sigma = 0$ is equivalent to

$$(m-1)(m-2)\langle \nu_i, \nu_j\rangle + (m-1)\langle \nu_i, \nu_i\rangle + (m-1)\langle \nu_j, \nu_j\rangle - \textstyle\sum_\ell \langle \nu_\ell, \nu_\ell\rangle = 0$$

for all $i \neq j$. (These conditions are vacuous for $m \leqslant 3$.) The sum over i ($i \neq j$) of the left hand sides is zero, so it suffices to consider the differences between pairs of equations, *i.e.*,

$$\langle \nu_j - \nu_k, (m-2)\nu_i + \nu_j + \nu_k\rangle = 0$$

for all $i \neq j, k$. Again summing the left hand sides over i (for $i \neq j, k$) gives zero, so again the equations are equivalent to their differences, which proves the theorem for $m \geqslant 4$.

For $m = 2, 3$, Möbius-flatness is instead expressed by the equation $C^\Sigma = dq$. By (11.26), the Cotton–York curvature $C^\Sigma = -\langle \mathcal{B}^\nabla(\mathrm{II}^0)\rangle$ is given by a first order quadratic differential operator applied to II^0: $\langle \mathcal{B}^\nabla(\mathrm{II}^0)\rangle = d^D Q + \langle \mathrm{II}^0, d^{\nabla,D}\mathrm{II}^0\rangle$ and Q is Cartan's tensor $\langle \mathrm{II}^0_\cdot, \mathrm{II}^0_\cdot\rangle - \frac14 |\mathrm{II}^0|^2 \mathsf{c}$, which has eigenvalues $\langle \nu_i, \nu_i\rangle - \frac14 \sum_j \langle \nu_j, \nu_j\rangle$ for $1 \leqslant i \leqslant m$. We set $\chi = Q + q$ (which is χ^{V_Σ}) and observe that, since $\sum_k \nu_k = 0$, $\chi = \sum_i \varepsilon_i^2 \otimes \mu_i$, where $\mu_i + \mu_j = -\langle \nu_i, \nu_j\rangle$ for $i \neq j$ (these being the eigenvalues of operator induced by χ on $\wedge^2 T\Sigma$).

We now compute

$$\langle d^D\chi, e_i\rangle = \mu_i d^D \varepsilon_i + D\mu_i \wedge \varepsilon_i + \textstyle\sum_j \mu_j \langle De_j, e_i\rangle \wedge \varepsilon_j. \tag{16.3}$$

On the other hand

$$\langle d^{\nabla,D}\mathrm{II}^0, \mathrm{II}^0_{e_i}\rangle = \langle \nu_i, \nu_i\rangle d^D \varepsilon_i + \tfrac12 D\langle \nu_i, \nu_i\rangle \wedge \varepsilon_i + \textstyle\sum_j \langle \nu_i, \nu_j\rangle \langle De_j, e_i\rangle \wedge \varepsilon_j \tag{16.4}$$

Now $\langle De_j, e_i\rangle = -\langle De_i, e_j\rangle$ (which is zero for $i = j$). Now Σ is Möbius-flat iff $\langle \mathcal{B}^\nabla(\mathrm{II}^0)\rangle + dq = 0$, and by adding (16.3) and (16.4), we find that this holds iff for all $i = 1, \dots m$,

$$\begin{aligned} 0 &= \langle d^D\chi, e_i\rangle + \langle d^{\nabla,D}\mathrm{II}^0, \mathrm{II}^0_{e_i}\rangle \\ &= (\mu_i + \langle\nu_i,\nu_i\rangle)d^D\varepsilon_i + D(\mu_i + \tfrac{1}{2}\langle\nu_i,\nu_i\rangle)\wedge\varepsilon_i + \textstyle\sum_j \mu_i\langle De_i, e_j\rangle\wedge\varepsilon_j \\ &= (2\mu_i + \langle\nu_i,\nu_i\rangle)d^D\varepsilon_i + \tfrac{1}{2}D(2\mu_i + \langle\nu_i,\nu_i\rangle)\wedge\varepsilon_i = \tfrac{1}{2}\lambda_i d(\lambda_i\varepsilon_i), \end{aligned}$$

where $\lambda_i^2 = 2\mu_i + \langle\nu_i,\nu_i\rangle$. When $m = 2$ and $\nu_1 = -\nu_2 = \nu$, $\lambda_1^2 = 2(\langle\nu,\nu\rangle + \mu^2)$ and $\lambda_2^2 = 2(\langle\nu,\nu\rangle - \mu^2)$. When $m = 3$, $\lambda_1^2 = \langle\nu_2 - \nu_1, \nu_3 - \nu_1\rangle$, λ_2^2 and λ_3^2 being obtained by cyclic permutations. This completes the proof. □

Remark 16.10. For $m = 2, 3$ the proof shows a little more: if we do not assume C^Σ is zero, then it is a universal nonzero constant multiple of $\sum_j d\alpha_j \otimes \alpha_j$.

We refer to the closed 1-forms α_i (determined up to signs in the cases $m = 2$ and $m = 3$) as the *conformal fundamental* 1*-forms* or the *conformal principal curvature forms*. Locally we can write $\alpha_i = dx_i$ for coordinates x_i which form a principal curvature net wherever they are functionally independent, since their differentials are orthogonal with respect to both c and II^0:

$$\mathsf{c} = \textstyle\sum_{i=1}^m \lambda_i^{-2} dx_i{}^2, \qquad \mathrm{II}^0 = \sum_{i=1}^m \lambda_i^{-2} dx_i{}^2 \otimes \nu_i.$$

When $m = 2$, $\lambda_1^2 = \langle\nu,\nu\rangle + \mu^2$ and $\lambda_2^2 = \langle\nu,\nu\rangle - \mu^2$, with $\nu = \nu_1 = -\nu_2$, whereas for $m = 3$, $\lambda_1^2 = \langle\nu_{12},\nu_{13}\rangle$, $\lambda_2^2 = \langle\nu_{23},\nu_{21}\rangle$, $\lambda_3^2 = \langle\nu_{31},\nu_{32}\rangle$, with $\nu_{ij} = \nu_i - \nu_j$.

We end this paragraph by relating the case $m = 2$ to *Guichard surfaces*. Suppose that $\langle\nu,\nu\rangle \neq \mu^2$, and write $\nu^\sharp = \lambda\xi$ for a weightless unit normal ξ. Then there are locally functions (x_1, x_2) on Σ (with x_2 real or pure imaginary) such that

$$\mathsf{c} = \frac{dx_1{}^2}{\lambda^2 + \mu^2} + \frac{dx_2{}^2}{\lambda^2 - \mu^2}, \quad \mathrm{II}^0 = \lambda S \otimes \xi, \quad q = \mu^2 S \quad \text{with} \quad S := \frac{dx_1{}^2}{\lambda^2 + \mu^2} - \frac{dx_2{}^2}{\lambda^2 - \mu^2}. \tag{16.5}$$

We now compare this (in codimension one) with the characterization by Calapso [23] of Guichard surfaces [49] in $\mathbb{R}^3$: he shows that these surfaces have induced metric $g = E_1 dx_1{}^2 + E_2 dx_2{}^2$ with $\sqrt{E_1E_2}(\kappa_1 - \kappa_2) = \sqrt{E_2 \pm E_1}$, where κ_1 and κ_2 are the principal curvatures. By allowing x_2 to be real or pure imaginary, we can reduce to the condition $\sqrt{\pm E_1E_2}(\kappa_1 - \kappa_2) = \sqrt{\pm(E_1 + E_2)}$. Now, up to reordering of (x_1, x_2), this is equivalent to its square, which yields $(\kappa_1 - \kappa_2)^2 = E_1^{-1} + E_2^{-1}$. This is exactly the condition satisfied by metrics of the form $\ell^{-2}\mathsf{c}$ in (16.5). The sign of x_2^2 distinguishes between cases known by Calapso as Guichard surfaces of the first and second kind.

16.3. Conformally-flat hypersurfaces. Conformally-flat hypersurfaces in S^{m+1} have been classified (for $m \geqslant 3$ in [36, 54, 55]. From the Theorem 16.9 we can easily reobtain this classification entirely within the realm of (Möbius) conformal geometry as follows.

For $m \geqslant 4$, the equations $(\nu_i - \nu_j)(\nu_k - \nu_\ell) = 0$ force $m - 1$ of the eigenvalues ν_i of II^0 to coincide (and conversely, these equations are then satisfied). Therefore Σ has a curvature sphere of multiplicity $m - 1 \geqslant 3$ and hence is a channel submanifold by Proposition 13.4.

For $m = 3$ there are two cases. First, if one of the conformal fundamental 1-forms vanishes, then two of the ν_i coincide and Σ is again a channel submanifold. Otherwise, two of the conformal fundamental 1-forms are nonzero and real, the third is nonzero and imaginary (so that $x = (x_1, x_2, x_3)$ are really coordinates on $\mathbb{R}^{2,1}$). Furthermore, $\sum_i \lambda_i^{-2} = (\nu_{12}\nu_{13})^{-1} + (\nu_{23}\nu_{21})^{-1} + (\nu_{31}\nu_{32})^{-1} = 0$ (multiply by $\nu_{12}\nu_{23}\nu_{31}$), so that the principal curvature net satisfies a condition studied by Guichard: the sum of the squares of its Lamé functions is zero. Conversely, if the principal curvature net is Guichard, with the ν_i given by $3\nu_1 = \lambda_1\lambda_2/\lambda_3 - \lambda_1\lambda_3/\lambda_2$ and cyclic permutations thereof, then the Guichard condition ensures that the conformal fundamental forms are the dx_i, so that Σ is conformally-flat.

For $m = 2$, there are again two cases: if $\mu^2 = \langle\nu,\nu\rangle$ then $q = \nu(\mathrm{II}^0)$ and $V = V_\Sigma + \nu$ is a sphere congruence with $K^V = 0$ which induces the given flat Möbius structure, so that II^V is degenerate. It follows from Theorem 16.4 that $A^{D,V} \wedge \mathrm{II}^V = 0$ and hence Σ is a channel submanifold enveloping the sphere-curve defined by V. Otherwise, as we saw in the previous paragraph, Σ is a Guichard surface of the first or second kind (given by (16.5)) according to whether $\mu^2 > \langle\nu,\nu\rangle$ or $\mu^2 < \langle\nu,\nu\rangle$.

- When $m \geqslant 4$, a conformally-flat hypersurface in S^{m+1} is a channel submanifold, *i.e.*, $m-1$ of the eigenvalues of II^0 coincide.
- When $m = 3$, a conformally-flat hypersurface in S^4 is a either a channel submanifold or is *Guichard isothermic* in the sense that the principal curvature net is a Guichard net.
- When $m = 2$, a Möbius-flat surface in S^3 is either a channel surface or is a *Guichard surface* as described by Calapso [23].

We have a little more to say on the cases $m = 3$ and $m = 2$.

Guichard nets and the Gauß–Codazzi–Ricci equations. In [56], Hertrich-Jeromin observes that any Guichard net arises from a conformally flat hypersurface in S^4. We have already seen that Guichard nets provide solutions to the Gauß equation for such conformally-flat hypersurfaces, so it remains to verify the Codazzi equation. We then apply the conformal Bonnet theorem, thus bypassing some of the computations in [56].

Consider a Guichard net with coordinates (x_1, x_2, x_3) and Lamé functions $1/\lambda_i$, as in the previous paragraph. Define (ν_1,ν_2,ν_3) by $3\nu_1 = \lambda_1\lambda_2/\lambda_3 - \lambda_1\lambda_3/\lambda_2$ and cyclic permutations and set $\mathrm{II}^0 = \sum_i (dx_i/\lambda_i)\otimes\nu_i(dx_i/\lambda_i)$. To verify the Codazzi equation, we need to show that $d^D\mathrm{II}^0 = \alpha\wedge g$ for some 1-form α, where D is the Levi-Civita connection of the metric $g = \sum_i (dx_i/\lambda_i)^2$.

We compute D via Cartan's method using the orthonormal frame $\omega_i = dx_i/\lambda_i$ and find that $D\omega_i = \sum_j A_{ij}\otimes\omega_j$ with $A_{ij} = -(\partial_i\lambda_j)\omega_j/\lambda_j + (\partial_j\lambda_i)\omega_i/\lambda_i$ (and ∂_i is shorthand for $\partial/\partial x_i$): we just need to check that $A_{ij} = -A_{ji}$ and $d\omega_i = \sum_j A_{ij}\wedge\omega_j$. Then

$$\begin{aligned}
d^D\mathrm{II}^0 &= \sum_i D\omega_i\wedge(\nu_i\omega_i) + \omega_i\otimes d\nu_i\wedge\omega_i + \omega_i\otimes(\nu_i d\omega_i)\\
&= \sum_{i,j}\omega_j\otimes A_{ij}\wedge(\nu_i\omega_i) + \omega_i\otimes(\partial_j\nu_i)\omega_j\wedge\omega_i + \omega_i\otimes\nu_i A_{ij}\wedge\omega_j\\
&= \sum_i \omega_i\otimes\big((\partial_j\nu_i)\omega_j - (\lambda_i^{-1}\partial_j\lambda_i)(\nu_i-\nu_j)\omega_j\big)\wedge\omega_i,
\end{aligned}$$

where the last line follows by relabelling the indices in the first term and substituting for A_{ij}. We thus need the term in brackets to be independent of i (modulo ω_i). It is straightforward to verify the three conditions this entails, once one appreciates the following identity for the quantities entering into the definition of the ν's:

$$d\Big(\frac{\lambda_j\lambda_k}{\lambda_i}\Big) = -\lambda_i\Big(\frac{d\lambda_j}{\lambda_k} + \frac{d\lambda_k}{\lambda_j}\Big)$$

for i,j,k distinct. This in turn is an easy consequence of the Guichard condition $\sum_i \lambda_i^{-2} = 0$. Hence the Codazzi equation holds, and the data $([g],\mathrm{II}^0)$ define a local conformally-flat immersion into S^4.

Strictly Möbius-flat surfaces and Dupin cyclides. We turn now to Möbius-flat surfaces with CCYP $q = 0$: in this case, or more generally when q is divergence-free (*i.e.*, $q^{2,0}$ is holomorphic), the conformal Möbius structure induced by the central sphere congruence is flat, and we shall say Σ is *strictly Möbius-flat*. We obtain the Möbius structure by setting $\mu = 0$ in (16.5) (we may use this formula away from umbilic points). However, it follows from these formulae that any strictly Möbius-flat surface carries a *nonzero* holomorphic quadratic differential commuting with II^0, namely $\lambda^2 S$, and hence is also an isothermic surface. Also $g = \lambda^2 \mathsf{c}$ is flat, so these surfaces are very special. We now show that they are in fact Dupin cyclides. Although this result is presented in [21], we discuss it in more detail here to show how the properties of Dupin cyclides can be seen explicitly in this setting.

Proposition 16.11. *Away from umbilic points, a strictly Möbius-flat surface Σ in the conformal 3-sphere has conformal metric and tracefree second fundamental form given by equation (16.5) with $\mu = 0$ (for some coordinates x_1, x_2, a weightless unit normal ξ and a gauge λ), and the Möbius structure given by*

$$\mathcal{H} = sym_0(D^g)^2 + \tfrac{t}{2}(dx_1{}^2 - dx_2{}^2)$$

for a constant $t \in \mathbb{R}$, D^g being the Levi-Civita connection of the flat metric $g = \lambda^2 \mathsf{c}$. Furthermore, Σ is isothermic with holomorphic quadratic differential $q_0 = dx_1{}^2 - dx_2{}^2$, and is Willmore iff $t = 0$.

Proof. We have already obtained the formulae for c and II^0 and the conformal Möbius structure is determined by $\mathcal{H} = sym_0(D^g)^2 + \tilde{q}$ for some quadratic differential $\tilde{q}$. However, both $\mathcal{H}$ and $sym_0(D^g)^2$ are flat, so $\tilde{q}^{2,0}$ must be holomorphic, and the Codazzi equation gives $0 = (\mathcal{H}^{\nabla})^*(J\mathrm{II}^0) = \langle \tilde{q}, J\mathrm{II}^0\rangle$, so that $\tilde{q}$ commutes with II^0, and is therefore a constant multiple of $q_0 = dx_1{}^2 - dx_2{}^2$. Since q_0 is holomorphic and commutes with II^0, the surface is isothermic. It is Willmore iff $0 = (\mathcal{H}^{\nabla})^*\mathrm{II}^0 = \langle \frac{t}{2}\tilde{q}, \mathrm{II}^0\rangle$, *i.e.*, $t = 0$. $\square$

Since the vector fields $\partial/\partial x_1$ and $\partial/\partial x_2$ leave the Gauß–Codazzi–Ricci data invariant, they induce symmetries of the surface. More precisely, with respect to the decomposition $\Sigma \times \mathbb{R}^{4,1} = \Lambda \oplus U_g \oplus \hat{\Lambda}_g \oplus <\xi>$, where U_g is orthogonal to $\Lambda \oplus \hat{\Lambda}_g \oplus <\xi>$, we have

$$d = id + D^g + \nabla + \mathrm{II}^0 - \mathbb{S}^0 + \tfrac{1+t}{2}dx_1{}^2 + \tfrac{1-t}{2}dx_2{}^2, \tag{16.6}$$

(which, incidently, shows that $\hat{\Lambda}_g$ is immersed unless $|t| = 1$; furthermore, it is also strictly Möbius-flat). Here we have use the fact that the Weyl structure $\hat{\Lambda}_g$ induced by g is the second envelope of the central sphere congruence (since $A^g = -\,div^{\nabla, D^g}\,\mathrm{II}^0 = 0$). From (16.5) (with $\mu = 0$) and (16.6), we now easily check that

$$\theta_1 = \big(\partial/\partial x_1, \mathrm{II}^0_{\partial/\partial x_1}, \tfrac{1+t}{2}dx_1\big), \qquad \theta_2 = \big(\partial/\partial x_2, \mathrm{II}^0_{\partial/\partial x_2}, \tfrac{1-t}{2}dx_2\big)$$

are constant sections of $\Sigma \times \mathfrak{so}(4,1) \cong (T\Sigma \oplus N\Sigma) \oplus \mathfrak{co}(T\Sigma \oplus N\Sigma) \oplus (T^*\Sigma \oplus N^*\Sigma)$ and so are the differential lifts of $\partial/\partial x_1$ and $\partial/\partial x_2$. Thus we have two elements of $\mathfrak{so}(4,1)$ which commute and induce vector fields on S^3 (their homology classes) which are tangent to Σ. It follows that Σ is an open subset of a Dupin cyclide, cf. §13.3. Furthermore θ_1 and θ_2 are the decomposable vectors in their span: identifying $\mathfrak{so}(n+1,1)$ with $\wedge^2\mathbb{R}^{n+1,1}$, we have

$$\theta_1 = \big(-\lambda^{-1}, \xi, \tfrac{t+1}{2}\lambda\big) \wedge (0, -\lambda^{-1}dx_1, 0), \qquad \theta_2 = \quad \big(\lambda^{-1}, \xi, \tfrac{t-1}{2}\lambda\big) \wedge (0, \lambda^{-1}dx_2, 0).$$

The two planes defined by these decomposables are both spacelike for $|t| < 2$, while for $|t| = 2$ one is spacelike, the other degenerate, and for $|t| > 2$ one is spacelike, the other has signature $(1,1)$. Accordingly, Σ is (Möbius equivalent to) an open subset of a circular torus of revolution, a cylinder of revolution, or a cone of revolution.

Thus strictly Möbius-flat surfaces in S^3 form a very restricted class, in contrast to the 3-dimensional case, where there is a rich supply of conformally flat hypersurface in S^4 from cones, cylinders or revolutes over surfaces of constant gaussian curvature in $\mathcal{S}^3$, $\mathbb{R}^3$ or $\mathcal{H}^3$ respectively. This was one of our motivations for broadening the notion of Möbius-flat surface: the broader class includes arbitrary cones, cylinders, or revolutes.

Preview of Part V

The machinery of this paper reduces the theory of conformal immersions to its homological essence, and we have seen the efficiency of this in some applications. Nevertheless, it is often more expedient to work with the bundle formalism of sphere congruences, which we develop further in Part IV. One reason for this is *simplicity*: the calculus of bundles and connections is more straightforward and familiar than that of BGG operators. Another reason is that the homological approach emphasises the central sphere congruence, whereas other enveloped sphere congruences my be better adapted to the problem at hand.

In Part V of our conformal submanifold geometry project, we return to (constrained) Willmore surfaces, isothermic surfaces and Möbius-flat submanifolds to study their transformation theory. For this, sphere congruences take centre stage, with the homological viewpoint as a backdrop. In particular, we relate Ribaucour sphere congruences and Bäcklund transformations of curved flats to Darboux and Eisenhart transformations of isothermic surfaces and Möbius-flat submanifolds respectively. In the isothermic case, these links are well-known, but they are less well understood for Möbius-flat submanifolds.

Möbius-flat submanifolds arise as orthogonal submanifolds to flat spherical systems and have a rich transformation theory, generalizing the transformations of Guichard surfaces, developed by C. Guichard [49], P. Calapso [23] and L. Eisenhart [42] in dimension two and codimension one. In dimension 3 or more and codimension one, conformally flat hypersurfaces were known to be related to flat cyclic systems [55]. We provide a uniform theory in arbitrary dimension and codimension, both of transformations of Möbius-flat submanifolds, and of the relation with curved flat spherical systems, providing further justification for the definition given in this part.

We also apply the notion of polynomial conserved quantities to isothermic surfaces and Möbius-flat submanifolds. The former application has been developed extensively in [22, 67]—in particular affine conserved quantities provide a conformal approach to constant mean curvature surfaces in arbitrary spaceforms (by regarding them as isothermic surfaces). Similarly, affine conserved quantities for Möbius-flat submanifolds provide a conformal approach to constant gaussian curvature submanifolds in arbitrary spaceforms (cf. [42]).

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