THE CR GEOMETRY OF WEIGHTED EXTREMAL KÄHLER AND SASAKI METRICS

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Abstract. We establish an equivalence between conformally Einstein–Maxwell Kähler 4-manifolds recently studied in [5, 10, 35, 44, 48, 49, 50] and extremal Kähler 4-manifolds in the sense of Calabi [20] with nowhere vanishing scalar curvature. The corresponding pairs of Kähler metrics arise as transversal Kähler structures of Sasaki metrics compatible with the same CR structure and having commuting Sasaki–Reeb vector fields. This correspondence extends to higher dimensions using the notion of a weighted extremal Kähler metric [7, 11, 45, 46, 47], illuminating and uniting several explicit constructions in Kähler and Sasaki geometry. It also leads to new existence and non-existence results for extremal Sasaki metrics, suggesting a link between the notions of relative weighted K-stability of a polarized variety introduced in [11, 47], and relative K-stability of the Kähler cone corresponding to a Sasaki polarization, studied in [18, 26].

Introduction

The famous Calabi problem [20], which seeks the existence of canonical Kähler metrics, is a central and very active topic of current research in Kähler geometry. As a candidate for a canonical metric on a complex manifold (M, J), Calabi proposed a notion of *extremal Kähler metric* g, meaning that its scalar curvature $Scal(g)$ is a *Killing potential*, i.e., the vector field $J\, grad_g\, Scal(g)$ is a Killing vector field for g. Examples include constant scalar curvature (CSC) Kähler metrics on (M, J), and hence also Kähler–Einstein metrics.

More recently, in real dimension 4, another natural generalization of CSC Kähler metrics has been studied [5, 48, 49, 50]: Kähler metrics g admitting a positive Killing potential f for which the scalar curvature of the conformal metric $\tilde{g} = (1/f^2)g$ is a constant c, i.e.,

$$Scal(\tilde{g}) = f^2 Scal(g) - 6f\Delta_g f - 12|df|_g^2 = c, \tag{1}$$

where Δ_g is the riemannian laplacian and $|\cdot|_g$ the norm defined by g. The metric $\tilde{g}$ satisfies a riemannian analogue of the Einstein–Maxwell equations with cosmological constant in generally relativity [59, 29], and thus we say g is a *conformally Einstein–Maxwell Kähler metric*. Many explicit examples of such metrics have been exhibited [5, 6, 10, 35, 44, 49, 50], and they have a striking resemblance to similar explicit examples of extremal Kähler metrics, see e.g. [10, Prop. 3]. Elucidating the connection suggested by these examples was the main motivation for this article, and our main result implies in particular an equivalence between the classes of conformally Einstein–Maxwell Kähler 4-manifolds and extremal Kähler 4-manifolds of nowhere zero scalar curvature. Our approach was suggested in part by [6, App. C], which implies that both kinds of metric can arise as quotients of a common strictly pseudo-convex CR 5-manifold $(N, \mathcal{D}, J)$ of Sasaki type. It also generalizes to complex manifolds (M, J) of real dimension $2m$, using the notion of a *weighted extremal metric* [7, 10, 45], as we now explain.

Let (g, ω) be a Kähler metric on (M, J), f a function on M, and $\nu \in \mathbb{R}$ a real number (which we call the *weight*). Then the (f, ν) *scalar curvature* of g is defined to be

$$Scal_{f,\nu}(g) := f^2 Scal(g) - 2(\nu - 1)f\Delta_g f - \nu(\nu - 1)|df|_g^2, \tag{2}$$

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Definition 1. Let (g,ω) be a Kähler metric on (M,J) let f be a Killing potential for g and let $\nu\in\mathbb{R}$. We say that g is *(f,ν)-extremal* if its (f,ν) scalar curvature, $Scal_{f,\nu}(g)$, given by (2), is also a Killing potential.

When M is compact and $f>0$ on M, the (f,ν) scalar curvature (2) is the momentum map associated to a formal GIT problem on the space $\mathcal{K}_\omega(M)^{\mathbb{T}}$ of $\mathbb{T}$-invariant ω-compatible Kähler metrics for any torus $\mathbb{T}$ in the group $Ham(M,\omega)$ of hamiltonian symplectomorphisms of (M,ω) which contains the flow of K [7, 10, 45]. This is similar to the framework found by Donaldson [33] and Fujiki [34] for Calabi extremal Kähler metrics. Indeed, the latter can be recovered from the weighted generalization by setting $f\equiv 1$.

For $m=2$ and $\nu=4$, (2) reduces to (1), so that Definition 1 includes the conformally Einstein–Maxwell Kähler metrics already discussed. This case was extended to the weight $\nu=2m$ (for any m) in [10], where it was noted that (2) then computes the scalar curvature of the hermitian metric $(1/f^2)g$. Thus examples of $(f,2m)$-extremal metrics include the conformally Einstein Kähler metrics studied in [31, 32]. The weight most relevant here is instead $\nu=m+2$, which first appeared in [7], where it was discovered that certain quotients, of an m-fold product $\mathbb{S}^3\times\cdots\times\mathbb{S}^3$ of CR 3-spheres by an m-torus, are $(f,m+2)$-extremal for a suitable f. However, an intrinsic geometric interpretation of $(f,m+2)$-extremality with $m>2$ has so far been lacking. Our main result rectifies this by providing an interpretation in CR geometry, whose basic notions we now recall (see also Section 1).

Let $(N,\mathcal{D})$ be a contact $(2m+1)$-manifold and denote by $L_{\mathcal{D}}\colon \mathcal{D}\times\mathcal{D}\to TN/\mathcal{D}$ the Levi form of $\mathcal{D}$, defined, via local sections $X,Y\in C^\infty_N(\mathcal{D})$, by the tensorial expression $L_{\mathcal{D}}(X,Y)=-\eta_{\mathcal{D}}([X,Y])$, where $\eta_{\mathcal{D}}\colon TN\to TN/\mathcal{D}$ is the quotient map. A *contact vector field* is a vector field X such that $\mathcal{L}_X(C^\infty_N(\mathcal{D}))\subseteq C^\infty_N(\mathcal{D})$. We make fundamental use of the following basic fact in the theory of contact manifolds (see e.g. [13]).

Lemma 1. *The map $X\mapsto\eta_{\mathcal{D}}(X)$ from contact vector fields to sections of $TN/\mathcal{D}$ is a linear isomorphism, whose inverse $\xi\mapsto X_\xi$ is a first order linear differential operator.*

There is thus a *contact Lie algebra* $\mathfrak{con}(N,\mathcal{D})$ of sections ξ of $TN/\mathcal{D}$ under the *Jacobi bracket*

$$[\xi,\chi]:=\eta_{\mathcal{D}}([X_\xi,X_\chi])=\mathcal{L}_{X_\xi}\chi=-\mathcal{L}_{X_\chi}\xi.$$

Now suppose $J\in End(\mathcal{D})$ is a CR structure on $(N,\mathcal{D})$; then we obtain a second order linear differential operator $\xi\mapsto\mathcal{L}_{X_\xi}J$ on $\mathfrak{con}(N,\mathcal{D})$. Its kernel

$$\mathfrak{cr}(N,\mathcal{D},J):=\{\xi\in\mathfrak{con}(N,\mathcal{D}):\mathcal{L}_{X_\xi}J=0\}$$

is a Lie subalgebra of $\mathfrak{con}(N,\mathcal{D})$, whose elements ξ correspond to CR vector fields X_ξ on N. If moreover $(\mathcal{D},J)$ is strictly pseudo-convex then $TN/\mathcal{D}$ has an orientation whose positive sections χ are those for which $\chi^{-1}L_{\mathcal{D}}(\cdot,J\cdot)$ is positive definite. Note that $\chi^{-1}L_{\mathcal{D}}=d\eta_\chi|_{\mathcal{D}}$ where $\eta_\chi:=\chi^{-1}\eta_{\mathcal{D}}$ is the contact form defined by χ. We let $\mathfrak{con}_+(N,\mathcal{D})\subseteq\mathfrak{con}(N,\mathcal{D})$ be the open cone of positive sections χ of $TN/\mathcal{D}$. We then have the following fundamental definitions (see e.g. [14]).

Definition 2. Let $(N,\mathcal{D},J)$ be a strictly pseudo-convex CR manifold. Then the *Sasaki cone* of $(N,\mathcal{D},J)$ is $\mathfrak{cr}_+(N,\mathcal{D},J):=\mathfrak{cr}(N,\mathcal{D},J)\cap\mathfrak{con}_+(N,\mathcal{D})$. If $\mathfrak{cr}_+(N,\mathcal{D},J)$ is nonempty then $(N,\mathcal{D},J)$ is said to be of *Sasaki type*, an element $\chi\in\mathfrak{cr}_+(N,\mathcal{D},J)$ is called a *Sasaki structure* on $(N,\mathcal{D},J)$, with *Sasaki–Reeb vector field* X_χ, and $(N,\mathcal{D},J,\chi)$ is called a *Sasaki manifold*. We say χ is *quasi-regular* if the flow of X_χ generates an $\mathbb{S}^1$ action on N, and moreover *regular* if this action is free.

The following well-known construction provides a standard way (see e.g. [13]) to extend geometric notions on Kähler manifolds to Sasaki manifolds.

Example 1. Let (M, J, g, ω) be a Kähler manifold such that $[\omega/2\pi]$ is an integral de Rham class. Then there is a principal $\mathbb{S}^1$-bundle $\pi\colon N \to M$ with a connection 1-form η satisfying $d\eta = \pi^*\omega$. Thus $(N, \mathcal{D}, J, \chi)$ is a Sasaki manifold, where $\mathcal{D} = \ker\eta \leqslant TN$, J is the pullback of the complex structure on TM to $\mathcal{D} \cong \pi^*TM$ and χ is the image in $TN/\mathcal{D}$ of the generator X_χ of the $\mathbb{S}^1$ action (with $\eta(X_\chi) = 1$, so $\eta = \eta_\chi$).

Conversely, if $\chi \in \mathfrak{cr}_+(N, \mathcal{D}, J)$ is (quasi-)regular, then N is a principal $\mathbb{S}^1$-bundle (or orbibundle) $\pi\colon N \to M$ over a Kähler manifold (or orbifold) M. Irrespective of regularity, this correspondence between Kähler geometry and Sasaki geometry holds locally: any point of a Sasaki manifold $(N, \mathcal{D}, J, \chi)$ has a neighbourhood in which the leaf space M of the flow of X_χ is a manifold and has a Kähler structure (g, J, ω) induced, using the identification $\mathcal{D} \cong \pi^*TM$, by the *transversal Kähler structure* (g_χ, J, ω_χ) on $\mathcal{D}$, where $\omega_\chi := d\eta_\chi|_{\mathcal{D}}$ and $g_\chi := \omega_\chi(\cdot, J\cdot)$. Indeed g_χ, J, and ω_χ are all X_χ-invariant, so they all descend to M, and we refer to (M, g, J, ω) as a *Sasaki–Reeb quotient* of $(N, \mathcal{D}, J, \chi)$.

For $\chi \in \mathfrak{cr}(N, \mathcal{D}, J)$, we set

$$\mathfrak{con}^\chi := \{\xi \in \mathfrak{con}(N, \mathcal{D}) \mid [\chi, \xi] = 0\},$$
$$\mathfrak{cr}^\chi := \mathfrak{con}^\chi \cap \mathfrak{cr}(N, \mathcal{D}, J) \qquad \text{and} \qquad \mathfrak{cr}^\chi_+ := \mathfrak{con}^\chi \cap \mathfrak{cr}_+(N, \mathcal{D}, J).$$

If in addition $\chi \in \mathfrak{cr}_+(N, \mathcal{D}, J)$, then

$$C^\infty_N(\mathbb{R})^\chi := \{f \in C^\infty_N(\mathbb{R}) : df(X_\chi) = 0\}$$

is a Lie algebra under the *transversal Poisson bracket* $\{f_1, f_2\} := -\omega_\chi^{-1}(df_1|_{\mathcal{D}}, df_2|_{\mathcal{D}})$, and we have the following elementary but central lemma.

Lemma 2. *The map $f \mapsto \xi = f\chi$ is a Lie algebra isomorphism from $C^\infty_N(\mathbb{R})^\chi$ to $\mathfrak{con}^\chi$, and $\xi \in \mathfrak{cr}^\chi$ if and only if f is a transversal Killing potential for (g_χ, ω_χ), i.e., $-\omega_\chi^{-1}(df)$ is a transversal Killing vector field for g_χ.*

Thus we obtain elements of $\mathfrak{cr}^\chi$ as pullbacks of Killing potentials from (local) Sasaki–Reeb quotients of N by χ. The Levi-Civita connection on Sasaki–Reeb quotients pulls back to a connection ∇^χ on $\mathcal{D}$ preserving (g_χ, J, ω_χ), which turns out to be (see e.g. [27, §4]) the so-called *Tanaka–Webster connection* [63] of $(\mathcal{D}, J, \chi)$. Thus the scalar curvature of Sasaki–Reeb quotients pulls back to the *Tanaka–Webster scalar curvature* $Scal(g_\chi)$ of ∇^χ, and hence $(N, \mathcal{D}, J, \chi)$ is CSC, i.e., $Scal(g_\chi)$ is constant, if and only if its Sasaki–Reeb quotients are. We may define (weighted) extremal Sasaki structures similarly.

Definition 3. Let $(N, \mathcal{D}, J, \chi)$ be a Sasaki manifold and $\xi = f\chi \in \mathfrak{cr}^\chi$. The (ξ, ν) *scalar curvature $Scal_{\xi,\nu}(g_\chi)$ of χ* is the function induced on N by the (f, ν) scalar curvature (2) on Sasaki–Reeb quotients. We say that χ is (ξ, ν)*-extremal* if $Scal_{\xi,\nu}(g_\chi)\chi \in \mathfrak{cr}(N, \mathcal{D}, J)$. For constant f, this reduces to extremality of χ in the sense of [14].

Example 2. Lemma 2 shows that any quasi-regular Sasaki manifold over an (f, ν)-extremal orbifold (M, J, g, ω) is (ξ, ν)-extremal, with $\xi = f\chi$, cf. Example 1 in the regular case.

For (f, ν)-extremal metrics, we have noted that the weight $\nu = 2m$ has a special interpretation in conformal geometry. The next lemma provides an analogous interpretation in CR geometry mentioned above of the weight $\nu = m + 2$ for (ξ, ν)-extremal metrics.

Lemma 3. *For any $\xi \in \mathfrak{cr}(N, \mathcal{D}, J)$, $\sigma(\xi) := Scal_{\xi,m+2}(g_\chi)\,\chi \in \mathfrak{con}(N, \mathcal{D})$ is independent of $\chi \in \mathfrak{cr}^\xi_+$. Hence $\xi \mapsto \sigma(\xi)$ is a second order quadratic differential operator, with $\sigma(\xi) = Scal(g_\xi)\,\xi$ in the case that $\xi \in \mathfrak{cr}_+(N, \mathcal{D}, J)$.*

We now emphasise a key feature of Sasaki geometry, which was used in [25, 37, 52, 57] to construct CSC Sasaki manifolds from Kähler manifolds which are not necessarily CSC (see also [15, 35]). Namely, $\mathfrak{cr}_+(N, \mathcal{D}, J)$ is open in $\mathfrak{cr}(N, \mathcal{D}, J)$, so if $(N, \mathcal{D}, J)$ is of Sasaki type, and $\dim \mathfrak{cr}(N, \mathcal{D}, J) \geqslant 2$, we obtain a family of Sasaki structures χ on N, inducing transversal Kähler structures on $(\mathcal{D}, J)$. With this in mind, the following is our main result, which is an immediate consequence of Lemma 3, but has many ramifications.

Theorem 1. *Let $(N, \mathcal{D}, J)$ be a CR $(2m+1)$-manifold with Sasaki cone $\mathfrak{cr}_+(N, \mathcal{D}, J)$. Then, for any $\chi, \xi \in \mathfrak{cr}_+(N, \mathcal{D}, J)$ with $[\chi, \xi] = 0$, $(N, \mathcal{D}, J, \chi)$ is $(\xi, m+2)$-extremal if and only if $(N, \mathcal{D}, J, \xi)$ is extremal.*

Together with Example 2, this result shows that the constructions of $(f, m+2)$-extremal Kähler metrics available in [5, 49, 50, 35, 44, 45] yield many new extremal Sasaki metrics.

We further observe that since $\mathcal{L}_{X_\xi}(Scal(g_\xi)) = 0$, $[\xi, \sigma(\xi)] = 0$. Hence the equivalent conclusions of Theorem 1 correspond to the occurrence of $\sigma(\xi)$ in $\mathfrak{cr}^\xi$; we then have $\pm\sigma(\xi) \in \mathfrak{cr}^\xi_+$ if and only if the scalar curvature $Scal(g_\xi)$ is everywhere positive or everywhere negative, in which case we can take $\chi = \pm\sigma(\xi)$ in Theorem 1 to obtain $Scal_{\xi,m+2}(g_\chi) = \pm 1$.

Corollary 1. *Let $(N, \mathcal{D}, J, \xi)$ be an extremal Sasaki $(2m+1)$-manifold. Then the $\mathfrak{cr}^\chi_+$-family of $(\xi, m+2)$-extremal Sasaki structures on $(N, \mathcal{D}, J)$ contains a Sasaki structure $\chi := \pm\sigma(\xi)$ of constant nonzero $(\xi, m+2)$ scalar curvature if and only if the extremal Sasaki structure ξ has nowhere zero scalar curvature. Thus there is an equivalence between Sasaki manifolds $(N, \mathcal{D}, J, \chi)$ of constant nonzero $(\xi, m+2)$ scalar curvature and extremal Sasaki manifolds $(N, \mathcal{D}, J, \xi)$ with nowhere zero scalar curvature.*

The proofs of Lemmas 1–3 are straightforward rephrasings of standard results in CR geometry, but for the convenience of the reader, we indicate their proofs in Section 1. In Section 2, we define, on a compact contact manifold of Sasaki type, a formal GIT setting for the search for (ξ, ν)-extremal Sasaki structures, extending the picture in [40] and providing a conceptual explanation for the key Lemma 3 above. Then in the rest of the paper we return to Kähler geometry and applications of Theorem 1, which gives a way of relating different Kähler geometries locally or (under suitable rationality conditions) globally. We formalize this as follows.

Definition 4. Let (M, g, J, ω) be a Kähler manifold and f a positive Killing potential. We say that $(\tilde{M}, \tilde{g}, \tilde{J}, \tilde{\omega})$ is a *CR twist* of M by f, or an *f-twist* for short, if it is a Sasaki–Reeb quotient by the Sasaki structure $f\chi$ on the Sasaki manifold $(N, \mathcal{D}, J, \chi)$ corresponding (over any open subset where $[\omega/2\pi]$ is integral) to M via Example 1.

A CR f-twist can be seen as a special case of the twist construction of Swann [60] (see also [43, 58]) which has been used to study different geometric structures. In these terms, Theorem 1 shows that any extremal Kähler metric can (locally) be obtained from a $(f, m+2)$-extremal Kähler metric, via a CR f-twist, while Corollary 1 establishes an equivalence between Kähler metrics of constant $(f, m+2)$ scalar curvature and extremal Kähler metrics of nonvanishing scalar curvature.

In real dimension $2m = 4$, the latter reduces to the equivalence between conformally Einstein–Maxwell Kähler metrics (g, J, ω, f) and extremal metrics $(\tilde{g}, \tilde{J}, \tilde{\omega})$ of nonvanishing scalar curvature alluded to above: the extremal Kähler 4-manifold is obtained as the Sasaki–Reeb quotient with respect to an extremal Sasaki structure ξ of $(N, \mathcal{D}, J)$, whereas (g, J, ω, f) is obtained as the quotient with respect to the Sasaki structure defined by the scalar curvature of ξ. Thus our correspondence gives a conceptual explanation and generalization of [10, Prop. 3], and can be used to obtain new examples of extremal Sasaki and Kähler metrics from the known conformally Einstein–Maxwell Kähler ones.

More generally, as any CR twist of an extremal Kähler metric is $(f, m+2)$-extremal for some f, one of the main theses of this paper is that one can reduce the search for extremal Kähler metrics to the search of $(f, m+2)$-extremal Kähler metrics on simpler Kähler manifolds. We explore this idea in the remainder of the paper.

As a warm-up, in Section 3, we consider the simplest examples: the Bochner-flat Sasaki–Reeb quotients of CR spheres [19, 63] and products. Then, in Section 4, we turn our attention to toric geometry. While toric Kähler and Sasaki geometries have been well-studied, to apply our theory, we develop a CR-invariant viewpoint, building on [51, 52, 54, 56]. In the toric case, CR-invariance corresponds to projective invariance on the image of the momentum map, and as an interesting side benefit, we give a manifestly projectively invariant treatment of the Legendre transform, by relating it to a particular case of a Bernstein–Gelfand–Gelfand resolution [12, 53], as constructed in [21, 22]. Returning the main line of the paper, we then obtain explicit descriptions of the CR f-twists of toric manifolds and of toric bundles given by the generalized Calabi ansatz, showing that the latter are CR f-twists of a product metric. In Section 5, we recast, in terms of the general correspondence herein, some of the explicit families of $(f, m+2)$-extremal Kähler metrics, including those obtained by the regular ambitoric ansatz in [5] and by an ansatz in [7]. This leads both to a higher dimensional extension of the regular ambitoric ansatz [5] and to a complete classification of the $(f, m+2)$-extremal Kähler metrics obtained by this ansatz. In the final Section 6, we turn to global considerations. We define the Calabi problem for (ξ, ν)-extremal Sasaki metrics, which naturally generalizes the existence problem of extremal Sasaki metrics in a given Sasaki polarization [14], recently studied in many places [16, 17, 18, 26, 52, 56, 57, 62]. We end by illustrating how the (non)existence of (f, ν)-extremal Kähler metrics in a given integral Kähler class of a geometrically ruled complex surface (as studied in [11, 44, 47]) leads both to existence and non-existence results for extremal Sasaki metrics compatible with (possibly irregular) Sasaki–Reeb vector fields on the corresponding contact manifolds. In particular, we establish the following Yau–Tian–Donaldson type correspondence.

Theorem 2. *Let $(M, J) = P(\mathcal{O} \oplus \mathcal{L}) \to B$ be a compact ruled complex surface over a Riemann surface B, L a polarization of (M, J), and $\omega \in 2\pi c_1(L)$ an $\mathbb{S}^1$-invariant Kähler metric with respect to the circle action by scalar multiplication in $\mathcal{O}$. Let $(N, \mathcal{D}, J, \chi)$ be the regular Sasaki manifold over (M, J, ω) given by Example 1, $\xi \in \mathfrak{cr}_+(N, \mathcal{D}, J)$ the lift of the generator of the $\mathbb{S}^1$-action on M and $\hat{Z}_\xi$ the induced holomorphic vector field on L. Then N admits a X_χ-invariant, $\mathcal{D}$-compatible CR structure which is extremal Sasaki with respect to ξ if and only if $(M, L, \hat{Z}_\xi)$ is analytically relatively $(\hat{Z}_\xi, 4)$ K-stable with respect to admissible test-configurations in the sense of* [11].

This suggests a link between the weighted K-stability of [11, 47] for a smooth polarized variety, and K-stability of the Kähler cone of a Sasaki polarization, studied in [18, 26].

1. Proofs of Lemmas 1–3

Let $(N, \mathcal{D})$ be a contact $(2m+1)$-manifold, i.e., $\mathcal{D} \leqslant TN$ is a rank $2n$ distribution on N, with quotient map $\eta_\mathcal{D} \colon TN \to TN/\mathcal{D}$, whose Levi form $L_\mathcal{D}(X, Y) = -\eta_\mathcal{D}(X, Y)$ $(X, Y \in C^\infty_N(\mathcal{D}))$ is nondegenerate at each point of N. A *CR structure* on $(N, \mathcal{D})$ is a complex structure J on $\mathcal{D}$ such that the subbundle $\mathcal{D}^{(1,0)}$ of $(1,0)$-vectors in $\mathcal{D} \otimes \mathbb{C}$ is closed under Lie bracket. This implies in particular that J is an *almost CR structure*, i.e., a complex structure on $\mathcal{D}$ such that $L_\mathcal{D}$ has type $(1,1)$ with respect to J. We say that $(\mathcal{D}, J)$ is *strictly pseudo-convex* if $L_\mathcal{D}$ is definite (with respect to J) at each point of N, i.e., $L_\mathcal{D}(\cdot, J\cdot)$ is a definite bundle metric on $\mathcal{D}$. It then follows that $(N, \mathcal{D})$ is *co-oriented*, i.e., $TN/\mathcal{D}$ is an oriented real line bundle.

Proof of Lemma 1. We show that any section ξ of $TN/\mathcal{D}$ has a unique lift to a contact vector field X_ξ (with $\eta_\mathcal{D}(X_\xi) = \xi$). On the open subset where ξ is nonzero, the contact condition implies that X_ξ is the Reeb vector field of the contact form $\eta_\xi := \xi^{-1}\eta_\mathcal{D}$, which is characterized by $\eta_\xi(X_\xi) = 1$ and $d\eta_\xi(X_\xi, \cdot) = 0$. Now suppose $\xi = f\chi$ with χ nonvanishing and f a smooth function. Then $d\eta_\chi = df \wedge \eta_\xi + d\eta_\xi$ and the characterization of Reeb vector fields gives

$$X_\xi = fX_\chi - (d\eta_\chi|_{\mathcal{D}})^{-1}(df|_{\mathcal{D}}) \tag{3}$$

where f is nonzero, but this formula extends X_ξ smoothly over the zeroset of f. This also shows $\xi \mapsto X_\xi$ is a first order differential operator. Since a contact vector field in $\mathcal{D}$ is necessarily zero by the nondegeneracy of the Levi form, the lift is unique. □

Proof of Lemma 2. Clearly $\mathcal{L}_{X_\chi}(f\chi) = df(X_\chi)\chi$, and if $f, h \in C^\infty_M(\mathbb{R})^\chi$ then (3) implies

$$[f\chi, h\chi] = dh(X_{f\chi}) = -(\omega_\chi|_{\mathcal{D}})^{-1}(df|_{\mathcal{D}}, dh|_{\mathcal{D}})$$

so the first part is immediate. We may thus suppose that f is the pullback of a smooth function, also denoted f, on a Sasaki–Reeb quotient (M, g, J, ω), with symplectic gradient K. Then the same formula (3) shows that X_ξ is a lift of K to N. Furthermore, since X_ξ is contact and X_χ-invariant, $\mathcal{L}_{X_\xi}J$ is horizontal and X_χ-invariant, hence vanishes iff its pushforward to M vanishes, which holds iff $\mathcal{L}_K J = 0$ on M, i.e., f is a Killing potential. □

Proof of Lemma 3. We let $\xi = f\chi$ with $\chi \in \mathfrak{cr}^\xi_+$ and expand the definition of the $(\xi, m+2)$ scalar curvature of g_χ on N to obtain

$$Scal_{\xi,m+2}(g_\chi)\,\chi = \big(f^2 Scal(g_\chi) - 2(m+1)f\Delta_{g_\chi}f - (m+1)(m+2)\big|df|_{\mathcal{D}}\big|^2_{g_\chi}\big)\,\chi \tag{4}$$

$$= \Big(f Scal(g_\chi) - 2(m+1)\Delta_{g_\chi}f - \frac{(m+1)(m+2)}{f}\big|df|_{\mathcal{D}}\big|^2_{g_\chi}\Big)\,\xi. \tag{5}$$

where (5) holds on the open subset U where ξ is nonzero. Now, as noted already, $Scal(g_\xi)$ and $Scal(g_\chi)$ are the Tanaka–Webster scalar curvatures of the Tanaka–Webster connections induced by ξ and χ [27, §4], and a straightforward but tedious computation of the change of the Tanaka–Webster scalar curvature under a change of connection, which can be found e.g. in [42, (2.9)], shows that on U, (5) computes $Scal(g_\xi)\xi$, independently of f. However, on any open subset where $\xi = 0$, $f = 0$ and hence the right hand side of (4) is zero. Thus $Scal_{\xi,m+2}(g_\chi)\,\chi$ is independent of χ on a dense open subset, hence everywhere. The equality (4) now shows that this is a second order quadratic differential operator in ξ. □

2. Formal GIT picture for weighted extremal Sasaki metrics

Let $(N, \mathcal{D})$ be a compact co-oriented contact $(2m+1)$-manifold (or orbifold), and fix a torus $\mathbb{T}$ in its group $Con(N, \mathcal{D})$ of contact transformations. As explained in the introduction, we tacitly identify the Lie algebra of $Con(N, \mathcal{D})$ with the space $\mathfrak{con}(N, \mathcal{D})$ of smooth sections of $TN/\mathcal{D}$ and denote by $\mathfrak{con}_+(N, \mathcal{D})$ the open cone of positive sections in of $TN/\mathcal{D}$ with respect to its orientation. Let $Con(N, \mathcal{D})^{\mathbb{T}}$ denote the group of $\mathbb{T}$-equivariant contact transformations, with Lie algebra identified with the space $\mathfrak{con}(N, \mathcal{D})^{\mathbb{T}}$ of $\mathbb{T}$-invariant sections of $TN/\mathcal{D}$. Thus, the Lie algebra $\mathfrak{t}$ of $\mathbb{T}$ as a linear subspace of $\mathfrak{con}(N, \mathcal{D})^{\mathbb{T}}$.

Now observe that $vol_{\mathcal{D}} := \eta_{\mathcal{D}} \wedge L_{\mathcal{D}}^{\wedge m}$ is a well-defined section of $\wedge^{2m+1}T^*N \otimes (TN/\mathcal{D})^{m+1}$: indeed for any nonvanishing section χ of $TN/\mathcal{D}$, $\chi^{-m-1} vol_{\mathcal{D}} = \eta_\chi \wedge d\eta_\chi^{\wedge m}$. Fix two such sections $\chi, \xi \in \mathfrak{t} \cap \mathfrak{con}_+(N, \mathcal{D})$ and $\nu \in \mathbb{R}$. Then $\mathfrak{con}(N, \mathcal{D})^{\mathbb{T}}$ has a bi-invariant inner product

$$\langle \xi_1, \xi_2 \rangle_{\xi,\chi,\nu} := \int_N (\xi_1/\chi)(\xi_2/\chi)(\xi/\chi)^{-\nu-1}\,\eta_\chi \wedge d\eta_\chi^{\wedge m} = \int_N \xi_1\,\xi_2\,\xi^{-\nu-1}\chi^{\nu-2-m}\,vol_{\mathcal{D}}.$$

Let $\mathcal{AC}_+(N, \mathcal{D})^{\mathbb{T}}$ be the space of $\mathbb{T}$-invariant almost CR structures on $(N, \mathcal{D})$, such that $L_{\mathcal{D}}$ is of type $(1,1)$ and positive definite with respect to J and the given orientation on $TN/\mathcal{D}$. We denote by $\mathcal{C}_+(N, \mathcal{D})^{\mathbb{T}} \subseteq \mathcal{AC}_+(N, \mathcal{D})^{\mathbb{T}}$ the subset of $\mathbb{T}$-invariant compatible CR structures on $(N, \mathcal{D})$. Notice that $Con(N, \mathcal{D})^{\mathbb{T}}$ acts naturally on $\mathcal{AC}_+(N, \mathcal{D})^{\mathbb{T}}$ (preserving

$\mathcal{C}_+(N,\mathcal{D})^{\mathbb{T}}$) and the tangent space of $\mathcal{AC}_+(N,\mathcal{D})^{\mathbb{T}}$ at J is identified with the Fréchet space of smooth sections $\dot{J}$ of $End(\mathcal{D})$ satisfying

$$\dot{J}J + J\dot{J} = 0, \qquad L_{\mathcal{D}}(\dot{J}\cdot,\cdot) + L_{\mathcal{D}}(\cdot,\dot{J}\cdot) = 0,$$

so $\mathcal{AC}_+(N,\mathcal{D})^{\mathbb{T}}$ has a formal Fréchet Kähler structure $(\mathbf{J},\boldsymbol{\Omega}^{\xi,\chi,\nu})$ defined by $\mathbf{J}_J(\dot{J}) := J\dot{J}$ and

$$\boldsymbol{\Omega}^{\xi,\chi,\nu}_J(\dot{J}_1,\dot{J}_2) := \frac{1}{2}\int_N \mathrm{tr}\big(J\dot{J}_1\dot{J}_2\big)\,(\xi/\chi)^{-\nu+1}\,\eta_\chi\wedge d\eta_\chi^{\wedge m} = \frac{1}{2}\int_N \mathrm{tr}\big(J\dot{J}_1\dot{J}_2\big)\,\xi^{-\nu+1}\chi^{\nu-2-m}\,vol_{\mathcal{D}}.$$

To see this, we can take $\chi \in \mathfrak{t}\cap\mathfrak{con}_+(N,\mathcal{D})$ to be quasi-regular with a global quotient (M,ω). Then our set-up reduces to the formal GIT picture for (f,ν)-extremal ω-compatible, $\mathbb{T}/\mathbb{S}^1_\chi$-invariant almost-Kähler metrics on the symplectic orbifold (M,ω), discussed in [7, 10, 45]. The momentum map for the action of $Ham(M,\omega)^{\mathbb{T}}$ at a compatible Kähler structure is identified with the (f,ν) scalar curvature, showing that for a CR structure $J\in\mathcal{C}_+(N,\mathcal{D})^{\mathbb{T}}$, the corresponding momentum map for the action of $Con(N,\mathcal{D})^{\mathbb{T}}$ on $\mathcal{AC}_+(N,\mathcal{D})^{\mathbb{T}}$ is $Scal(g_\xi)\xi$ (where we multiply by ξ to obtain an element of $\mathfrak{con}(N,\mathcal{D})^{\mathbb{T}}$).

We now notice that for $\nu = m+2$, the bi-invariant inner product on $\mathfrak{con}(N,\mathcal{D})^{\mathbb{T}}$ and the formal Kähler structure on $\mathcal{AC}_+(N,\mathcal{D})^{\mathbb{T}}$ are independent of χ. In this case, our setting reduces to the formal GIT picture for extremal Sasaki metrics on (N,η_ξ) discussed in [40], where the momentum map for the action of $Con(N,\mathcal{D})^{\mathbb{T}}$ is the Tanaka–Webster scalar curvature $Scal(g_\xi)\xi$ (the multiplication by ξ is implicit in [40] through the identification of the Lie algebra with smooth functions).

Hence this provides another explanation as to why the weight $m+2$ is special and the transversal $(\xi,m+2)$ scalar curvature $Scal_{\xi,m+2}(g_\chi)\chi$ of g_χ is independent of χ and equal to the Tanaka–Webster scalar curvature of g_ξ, viewed as an element of $\mathfrak{con}(N,\mathcal{D})^{\mathbb{T}}$.

3. Basic Examples

3.1. **Bochner-flat $(f,m+2)$-extremal metrics.** Let us now consider the *standard CR sphere* $\mathbb{S}^{2m+1}\subseteq\mathbb{C}^{m+1}$, $m\geqslant 2$, with $\mathcal{D} = T\mathbb{S}^{2m+1}\cap J(T\mathbb{S}^{2m+1})$, J induced by the standard complex structure on $\mathbb{C}^{m+1}$, and $\mathfrak{cr}(\mathbb{S}^{2m+1},\mathcal{D},J)\cong\mathfrak{su}(1,m+1)$. By a result of Webster [63], for any $\chi\in\mathfrak{cr}_+(\mathbb{S}^{2m+1},\mathcal{D},J)$, the transversal Kähler structure (g_χ,J,ω_χ) is Bochner-flat, and thus extremal (see [19]), and any Bochner-flat Kähler manifold (M,g,J,ω) is (locally) obtained as a χ-reduction of $(\mathbb{S}^{2m+1},\mathcal{D},J)$ for some such χ. It then follows from Theorem 1 that for any $\xi\in\mathfrak{cr}^\chi_+$, $(\mathcal{D},J,\chi)$ is a $(\xi,m+2)$-extremal, and hence by Lemma 2 (see Example 2), we have the following observation.

Proposition 1. *Let (M,g,J,ω) be a Bochner-flat Kähler $2m$-manifold and $f>0$ a Killing potential. Then (g,ω) is $(f,m+2)$-extremal.*

To obtain global examples, we let $\chi_{\mathrm{w}}\in\mathfrak{cr}_+(\mathbb{S}^{2m+1},\mathcal{D},J)$ correspond to the weighted Hopf fibration $\mathbb{S}^{2m+1}\to\mathbb{C}P^m_{\mathrm{w}}$, realizing $(\mathbb{S}^{2m+1},\mathcal{D},J,\chi_{\mathrm{w}})$ as a quasi-regular Sasaki manifold over the Bochner-flat weighted projective space $(\mathbb{C}P^m_{\mathrm{w}},J,g,\omega)$ (see [19, 28]). Thus we have the following higher dimensional extension of [10, Prop. 5].

Corollary 2. *The Bochner-flat metric on $\mathbb{C}P^m_{\mathrm{w}}$ is $(f,m+2)$-extremal for any positive Killing potential f.*

3.2. **Flat (f,ν)-extremal metrics.** The conclusion of Proposition 1 can be strengthened for flat Kähler metrics.

Proposition 2. *Let (V,g_V,ω_V) be a flat Kähler manifold and $f>0$ a Killing potential on V. Then, for any scalar-flat Kähler manifold (B,g_B,ω_B) the Kähler product (M,g,ω) of (V,g_V,ω_V) and (B,g_B,ω_B) is (f,ν)-extremal for any ν.*

Proof. As B is scalar-flat, (2) implies that the (f,ν) scalar curvature of $V \times B$ equals the (f,ν) scalar curvature of M. Thus, we need to establish the claim on $M := V$. As g is a flat metric, (2) reduces to

$$-2(\nu-1)f\Delta_g f - \nu(\nu-1)|df|_g^2 \tag{6}$$

so it suffices to show that each of the two terms in (6) is a Killing potential for g. For the first term, using that g is Ricci-flat and f is a Killing potential, the Bochner identity shows that $\Delta_g f$ is a constant, and thus $f\Delta_g f$ is a Killing potential of g. For the second term, using that g is Bochner-flat and Proposition 1, it follows that (6) with $\nu = m+2$ gives rise to a Killing potential, and hence $|df|_g^2$ is a Killing potential for g. □

3.3. $(f, m+2)$**-extremal products.** As noted in the proof of Proposition 2, the Kähler product of a scalar-flat, Kähler $2(m-\ell)$-manifold (B, g_B, ω_B) with a (f,ν)-extremal Kähler 2ℓ-manifold (V, g_V, ω_V) gives rise to a (f,ν)-extremal $2m$-manifold (M, g, ω). In [11, 45], for any given ν, large families of (f,ν)-extremal Hodge (i.e., compact, integral) Kähler manifolds (V, g_V, ω_V) of dimension 2ℓ are constructed. Taking such a (V, g_V, ω_V, f) with $\nu = m+2$ $(m \geqslant \ell)$ and considering the Kähler product of (V, g_V, ω_V) with a scalar-flat Hodge Kähler $2(m-\ell)$-manifold (B, g_B, ω_B), we obtain a compact $(f, m+2)$-extremal Kähler $2m$-manifold (M, g, ω, J), which gives rise to a compact extremal Sasaki $(2m+1)$-manifold $(N, \mathcal{D}, J, \xi)$ via Example 1. Notice that the extremal Sasaki manifold thus obtained is not in general quasi-regular, but when it is (which places a rationality condition on the positive Killing potential f of g_V), the resulting extremal Kähler orbifold is not in general a product, even though (M, g, ω) is. We detail and generalize this observation below in the setting of toric bundles.

4. Toric geometry and toric bundles

4.1. **Toric contact manifolds.** Applications of Theorem 1 depend in particular on the existence of independent commuting elements $\xi, \chi \in \mathfrak{cr}(N, \mathcal{D}, J) \leqslant \mathfrak{con}(N, \mathcal{D})$. The maximal dimension of an abelian subalgebra of $\mathfrak{con}(N, \mathcal{D})$ is $m+1$ (assuming N is connected of dimension $2m+1$). Let us therefore consider the case that we have such an $(m+1)$-dimensional abelian subalgebra $\mathfrak{h} \hookrightarrow \mathfrak{con}(N, \mathcal{D}); a \mapsto \xi_a$, in which case $(N, \mathcal{D}, \mathfrak{h})$ is said to be *toric*. We usually assume that the corresponding contact vector fields generate an effective contact action of a real $(m+1)$-torus $\mathbb{T}^{m+1}$, whose Lie algebra is thus canonically isomorphic to $\mathfrak{h}$. Hence we have an integral lattice $\varLambda \subseteq \mathfrak{h}$ with $\mathbb{T}^{m+1} \cong \mathfrak{h}/2\pi\varLambda$, and, on the dense open set N° where the $\mathbb{T}^{m+1}$-action is free, angle coordinates $t\colon N^\circ \to \mathfrak{h}/2\pi\varLambda$.

We also assume that the tautological bundle homomorphism

$$\begin{aligned} N \times \mathfrak{h} &\to TN/\mathcal{D} \\ (p,a) &\mapsto \xi_a(p) \end{aligned}$$

is surjective (as it is in the Sasaki case), so that its transpose $(TN/\mathcal{D})^* \to N \times \mathfrak{h}^*$ is injective. We thus obtain a *momentum map* $\bar\mu\colon N \to P(\mathfrak{h}^*)$, where $\bar\mu(p)$ is the image of $(TN/\mathcal{D})^*_p$ in $\mathfrak{h}^*$, for any $p \in N$. Hence $\bar\mu^*\mathcal{O}_{\mathfrak{h}^*}(-1) \cong (TN/\mathcal{D})^*$, where $\mathcal{O}_{\mathfrak{h}^*}(-1)$ is the tautological line bundle over the projective space $P(\mathfrak{h}^*)$, with fibre $\mathcal{O}_{\mathfrak{h}^*}(-1)_\tau = \tau \leqslant \mathfrak{h}^*$.

Remark 1. Alternatively observe (cf. [54]) that the annihilator $\mathcal{D}^0 \leqslant T^*N$ of $\mathcal{D}$ inherits from T^*N a closed 2-form which is nondegenerate on the complement of the zero section, and any contact vector field on N lifts to a hamiltonian vector field on $\mathcal{D}^0$. The restriction to $\mathfrak{h}$ of the momentum map of this action is $\tilde\mu\colon \mathcal{D}^0 \to \mathfrak{h}^*$ with $\langle\tilde\mu(\alpha), a\rangle = \alpha(\xi_a)$ for $\alpha \in \mathcal{D}^0$ and $a \in \mathfrak{h}$, where angle brackets denote contraction of $\mathfrak{h}^*$ with $\mathfrak{h}$. Using the natural duality $\mathcal{D}^0 \cong (TN/\mathcal{D})^*$, $(p, \tilde\mu(\alpha))$ is the element of $\bar\mu^*\mathcal{O}_{\mathfrak{h}^*}(-1) \leqslant N \times \mathfrak{h}^*$ corresponding to $\alpha \in \mathcal{D}^0_p$.

Herein, we generally work instead with the *momentum section* $\hat{\mu}\colon N \to \mathfrak{h}^* \otimes (TN/\mathcal{D})$ defined by $\langle a, \hat{\mu}(p)\rangle = \xi_a(p) \in (TN/\mathcal{D})_p$ for $a \in \mathfrak{h}$ and $p \in N$. If $z\colon P(\mathfrak{h}^*) \to \mathfrak{h}^* \otimes \mathcal{O}_{\mathfrak{h}^*}(1)$ denotes the tautological section, with $\mathcal{O}_{\mathfrak{h}^*}(1) := \mathcal{O}_{\mathfrak{h}^*}(-1)^*$ and $\langle a, z(\tau)\rangle = \langle a, \cdot\rangle|_\tau$ for $a \in \mathfrak{h}$ and $\tau \in P(\mathfrak{h}^*)$, then under the isomorphism $\bar{\mu}^*\mathcal{O}_{\mathfrak{h}^*}(1) \cong TN/\mathcal{D}$, $\hat{\mu} = \bar{\mu}^* z$.

Any nonzero $\varepsilon \in \mathfrak{h}$ defines an affine chart

$$\mathcal{A} := \{p \in \mathfrak{h}^* | \langle \varepsilon, p\rangle = 1\} \hookrightarrow P(\mathfrak{h}^*)$$

and if $U \subseteq P(\mathfrak{h}^*)$ is an open subset of the image of this chart, then $\langle \varepsilon, z\rangle$ restricts to a trivialization of $\mathcal{O}_{\mathfrak{h}^*}(1)|_U$. In this trivialization, z is the affine lift of U to $\mathcal{A} \subseteq \mathfrak{h}^*$, i.e., we have $z\colon U \to \mathfrak{h}^*$ with $\langle \varepsilon, z\rangle = 1$. Hence $\langle \varepsilon, dz\rangle = 0$, i.e.,

$$dz\colon TU \to U \times \varepsilon^0, \qquad \text{where} \qquad \varepsilon^0 = \{p \in \mathfrak{h}^* | \langle \varepsilon, p\rangle = 0\},$$

is the trivialization of TU in this affine chart. Thus $\mathfrak{h}$ is naturally identified with the space $\mathit{Aff}(U)$ of affine functions on U: $a \in \mathfrak{h}$ defines the affine function $\langle a, z\rangle$ on U, with ε corresponding to the constant function 1. If $\mathfrak{t} = \mathfrak{h}/\mathit{span}(\varepsilon)$, there is a short exact sequence

$$0 \to \mathbb{R} \xrightarrow{\varepsilon} \mathfrak{h} \xrightarrow{\delta} \mathfrak{t} \to 0, \tag{7}$$

where the duality $\mathfrak{t}^* \cong \varepsilon^0$ identifies $\mathfrak{t}$ with $T^*_p U$ for any $p \in U$, while the quotient map δ sends an affine function to its linear part (the constant value of its derivative).

Now $\bar{\mu}\colon N \to P(\mathfrak{h}^*)$ takes values in the affine chart defined by ε iff $\chi := \xi_\varepsilon \in \mathfrak{con}(N, \mathcal{D})$ is a nonvanishing section of $TN/\mathcal{D}$. In this trivialization the momentum section becomes a function $\hat{\mu}\colon N \to \mathfrak{h}^*$ with $\langle \varepsilon, \hat{\mu}\rangle = 1$. The transversal Kähler form $d\eta_\chi|_{\mathcal{D}}$ descends to a symplectic form ω on any quotient of M of N by χ. If χ is quasi-regular, we may take (M, ω) to be the global quotient, which is then a toric symplectic $2m$-orbifold under the hamiltonian action of the real m-torus $\mathbb{T}^m = \mathbb{T}^{m+1}/\mathbb{S}^1_\chi$ where $\mathbb{S}^1_\chi$ is the circle action generated by X_χ.

Concretely, we can choose a basis $e_0, \ldots e_m$ for $\mathfrak{h}$ such that $\varepsilon = e_0$, introduce coordinates $z_j = \langle e_j, z\rangle$ on U with $z_0 = 1$, and write $\langle a, z\rangle = a_0 + a_1 z_1 + \cdots a_n z_n$. We then have coordinates $\hat{\mu} = (\hat{\mu}_0, \hat{\mu}_1, \ldots \hat{\mu}_m)$ and $t = (t_0, t_1, \ldots t_m)$ on N° with $\hat{\mu}_0 = 1$ and

$$\eta_\chi = \langle \hat{\mu}, dt\rangle = dt_0 + \sum_{j=1}^m \hat{\mu}_j dt_j. \tag{8}$$

Hence $d\eta_\chi = \langle d\hat{\mu} \wedge dt\rangle$, and since $d\hat{\mu}$ takes values in $\mathfrak{t}^*$, $\langle d\hat{\mu} \wedge dt\rangle$ depends only on $\delta(dt)$, which descends to a $\mathfrak{t}$-valued 1-form on M. If we denote by μ the map $M \to \mathcal{A}$ induced by the momentum section on N in the affine chart $\mathcal{A}$, we may therefore write $\omega = \langle d\mu \wedge dt\rangle$ on the image M° of N°. The isomorphism $f \mapsto f\chi$ of Lemma 2 identifies $\mathfrak{h}$ with an abelian Lie subalgebra of $C^\infty_M(\mathbb{R})$ under the Poisson bracket induced by ω. Evidently, $\chi = \xi_\varepsilon$ corresponds to $f \equiv 1$ on M. More generally, for any $a \in \mathfrak{h}$ the function $f_a = f_a(\mu) \in C^\infty_M(\mathbb{R})$ corresponding to $\xi_a \in \mathfrak{con}(N, \mathcal{D})$ satisfies $f_a(\mu(p)) = \langle a, \mu(p)\rangle$ for all $p \in M$, i.e., $f_a(\mu) = \langle a, \mu\rangle$ is the pullback by μ of the affine function $f_a(z) = \langle a, z\rangle$ on $\mathcal{A}$. Pulling back by μ, we may thus reinterpret (7) on M: in particular, we may view δ as the restriction to $\mathfrak{h} \hookrightarrow C^\infty_M(\mathbb{R})$ of the symplectic gradient, and $\mathfrak{t}$ as its image in $\mathit{Ham}(M, \omega)$.

4.2. **Toric CR manifolds and their Sasaki–Reeb quotients.** Thus far we have only considered the toric contact geometry of N and induced toric symplectic geometry on M. According to [39], on the dense open subset M°, any toric almost Kähler structure may be written in momentum–angle coordinates (μ, t) as:

$$\begin{aligned} g &= \langle d\mu, G(\mu), d\mu\rangle + \langle dt, H(\mu), dt\rangle, \qquad & J dt &= -\langle G(\mu), d\mu\rangle, \\ \omega &= \langle d\mu \wedge dt\rangle, & J d\mu &= \langle H(\mu), dt\rangle, \end{aligned} \tag{9}$$

where H is a smooth positive definite $S^2\mathfrak{t}^*$-valued function on the momentum image $\Delta^\circ := \mu(M^\circ)$ and $G = H^{-1}$ is its pointwise inverse, a smooth $S^2\mathfrak{t}$-valued function. This local expression makes sense in the affine setting, where Δ° lies in an affine space $\mathcal{A} \subseteq \mathfrak{h}^*$, modelled on $\mathfrak{t}^*$: G is a metric on $T\Delta^\circ \cong \Delta^\circ \times \mathfrak{t}^*$, and H the inverse metric on $T^*\Delta^\circ$.

A metric of the form (9) is Kähler, i.e., J is integrable, if and only if $\langle dG \wedge dz\rangle = 0$, which in affine coordinates $z = (1, z_1, \dots z_m)$ on $\mathcal{A}$ reads

$$\partial G_{ij}/\partial z_k = \partial G_{ik}/\partial z_j$$

for all $i, j, k \in \{1, \dots m\}$. Since G is symmetric, this is the integrability condition to write $G = \mathit{Hess}(u)$ for a smooth strictly convex function u defined on the momentum image Δ°, which is called a *symplectic potential*. When M is a compact manifold (or orbifold), Delzant theory [30, 55] implies that Δ° is the interior of a (rational) Delzant polytope $\Delta \subseteq \mathfrak{t}^*$, and u satisfies the Abreu boundary conditions [1] on $\partial\Delta$.

The theory of symplectic potentials in toric Kähler geometry thus relies upon a locally exact complex of linear differential operators

$$C^\infty_{\mathcal{A}}(\mathbb{R}) \xrightarrow{\mathit{Hess}} C^\infty_{\mathcal{A}}(S^2\mathfrak{t}) \xrightarrow{\mathcal{D}} C^\infty_{\mathcal{A}}(\wedge^2\mathfrak{t} \odot \mathfrak{t}) \tag{10}$$

where $\mathcal{D}(G) = \langle dG \wedge dz\rangle$ and $\wedge^2\mathfrak{t} \odot \mathfrak{t}$ denotes the alternating-free tensors in $\wedge^2\mathfrak{t} \otimes \mathfrak{t}$ (the kernel of the projection, alternation, to $\wedge^3\mathfrak{t}$). This complex is invariant under affine transformations by construction, but can actually be made projectively invariant. To do this, observe that the kernel of the hessian consists of affine functions, which on a domain $U \subseteq P(\mathfrak{h}^*)$ in projective space are not naturally ordinary functions, but sections of $\mathcal{O}_{\mathfrak{h}^*}(1)$, as we discussed above. Also the cotangent space is naturally $T^*U \cong \mathcal{O}_{\mathfrak{h}^*}(-1)^0 \otimes \mathcal{O}_{\mathfrak{h}^*}(-1) \leqslant \mathfrak{h} \otimes \mathcal{O}_{\mathfrak{h}^*}(-1)$. With these modifications, we obtain a locally exact complex of projectively invariant linear differential operators, beginning

$$\begin{aligned}0 \to \mathfrak{h} \xrightarrow{\langle z,\cdot\rangle} C^\infty_U(\mathcal{O}_{\mathfrak{h}^*}(1)) \xrightarrow{\mathit{Hess}} C^\infty_U(S^2T^*U \otimes \mathcal{O}_{\mathfrak{h}^*}(1)) \xrightarrow{\mathcal{D}}\\ C^\infty_U(\wedge^2T^*U \odot T^*U \otimes \mathcal{O}_{\mathfrak{h}^*}(1)) \to \cdots,\end{aligned} \tag{11}$$

and which reduces to (10) in any affine chart.

The complex (11) is a simple example of a Bernstein–Gelfand–Gelfand resolution [12, 53]. Without wishing to dwell on the general machinery, we observe that the construction of this resolution in [21, 22] gives a manifestly invariant construction of the projective hessian. The main idea is to relate (11) to the $\mathfrak{h}$-valued de Rham complex

$$0 \to \mathfrak{h} \to C^\infty_U(\mathfrak{h}) \xrightarrow{d} C^\infty_U(T^*U \otimes \mathfrak{h}) \to \cdots$$

using the following construction.

Lemma 4. *For any $u \in C^\infty_U(\mathcal{O}_{\mathfrak{h}^*}(1))$ there is a unique $\mathcal{L}(u) \in C^\infty_U(\mathfrak{h})$ with $\langle z, \mathcal{L}(u)\rangle = u$ (i.e., $\mathcal{L}(u)$ is a lift of u) and $\langle z, d\mathcal{L}(u)\rangle = 0$. Furthermore, in any local affine chart with $z_0 = 1$ and coordinates $z_1, \dots z_m$, $\mathcal{L}(u) = (u_0, u_1, \dots u_m)$ where*

$$-u_0 = z_1\frac{\partial u}{\partial z_1} + \cdots + z_m\frac{\partial u}{\partial z_m} - u$$

is the Legendre transform of u, and $u_j = \partial u/\partial z_j$ for $j \in \{1, \dots m\}$.

Proof. In local affine coordinates with $z_0 = 1$, $\mathcal{L}(u) = (u_0, u_1, \dots u_m)$ is a lift of u iff $u = u_0 + z_1u_1 + \cdots + z_mu_m$, and we require in addition $0 = \langle z, d\mathcal{L}(u)\rangle = du_0 + z_1du_1 + \cdots + z_mdu_m$. Thus $du = u_1dz_1 + \cdots + u_mdz_m$, forcing $u_j = \partial u/\partial z_j$, which in turn determines u_0. □

This observation has some interesting consequences.

- $\mathcal{L}\colon C^\infty_U(\mathcal{O}_{\mathfrak{h}^*}(1)) \to C^\infty_U(\mathfrak{h})$ is a first order projectively invariant linear differential operator. Furthermore $\mathcal{L}(u)$ is constant if and only if u is an affine section of $\mathcal{O}_{\mathfrak{h}^*}(1)$.

- This differential lift $\mathcal{L}(u)$ of u is a "universal Legendre transform" in the sense that for any $\varepsilon \in \mathfrak{h}$ and $p \in \mathfrak{h}^*$ with $\langle \varepsilon, p\rangle = 1$, $-\langle \mathcal{L}(u), p\rangle$ is the Legendre transformation of u in the affine chart defined by ε with basepoint p (it thus depends only on p, not ε). Furthermore, the projection of $\mathcal{L}(u)$ onto $\mathfrak{t} = \mathfrak{h}/\,\mathit{span}(\varepsilon)$ gives the conjugate coordinates (the components of du in this affine chart, which depend only on ε, not p).
- Any $u \in C^\infty_U(\mathcal{O}_{\mathfrak{h}^*}(1))$ defines a congruence of affine hyperplanes $\{y \in \mathfrak{h} : \langle z(p), y\rangle = u(p)\}$ in $\mathfrak{h}$ parametrized by $p \in U$. The lift $\mathcal{L}(u)$ is the envelope of this hyperplane congruence (a classical view on the Legendre transformation): $\langle z, \mathcal{L}(u)\rangle = u$ and $\langle z, d\mathcal{L}(u)\rangle = 0$.

To complete the construction of the projectively invariant hessian, it remains to observe that since $\langle z, d\mathcal{L}(u)\rangle = 0$, $d\mathcal{L}(u)$ is a section of $T^*U \otimes \mathcal{O}_{\mathfrak{h}^*}(-1)^0$; hence $\langle dz \otimes d\mathcal{L}(u)\rangle$ is a section of $S^2T^*U \otimes \mathcal{O}_{\mathfrak{h}^*}(1)$, since dz is well-defined modulo $\mathcal{O}_{\mathfrak{h}^*}(-1)$ and we have $0 = d\langle z, d\mathcal{L}(u)\rangle = \langle dz \wedge d\mathcal{L}(u)\rangle$. In affine coordinates, $d\mathcal{L}(u) = (du_0, d(\partial u/\partial z_1), \ldots d(\partial u/\partial z_m))$ and projecting away from $\varepsilon = (1, 0, \ldots 0)$ gives the usual hessian of u.

To apply this to a toric contact manifold $(N, \mathcal{D}, \mathfrak{h})$ with $\bar\mu(N^\circ) = \Delta^\circ$, it is convenient to view the projectively invariant hessian of $u \in C^\infty_{\Delta^\circ}(\mathcal{O}_{\mathfrak{h}^*}(1))$ as the section $G = \mathit{Hess}(u)$ of $S^2\mathcal{O}_{\mathfrak{h}^*}(-1)^0 \otimes \mathcal{O}_{\mathfrak{h}^*}(-1) \leqslant S^2\mathfrak{h} \otimes \mathcal{O}_{\mathfrak{h}^*}(-1)$ with $\langle G(z), dz\rangle = d\mathcal{L}(u)$. Then we may define a CR structure J on N° by $Jdt|_{\mathcal{D}} = -\bar\mu^* d\mathcal{L}(u)|_{\mathcal{D}} = -\langle G(\hat\mu), d\hat\mu\rangle|_{\mathcal{D}}$. For any $\chi \in \mathfrak{cr}(N, \mathcal{D}, J)$, this reduces to the toric Kähler structure defined by u on local Sasaki–Reeb quotients.

Example 3. If $m = 1$ and $u = u(z_1)$ is a symplectic potential in the affine chart $z = (1, z_1)$ then the differential lift of u to $\mathfrak{h}$ is $\mathcal{L}(u) = (u(z_1) - z_1u'(z_1), u'(z_1))$ with $\langle (1, z_1), \mathcal{L}(u)\rangle = u(z_1)$ and $d\mathcal{L}(u) = u''(z_1)(-z_1, 1)\, dz_1$. Thus $Jdt_0|_{\mathcal{D}} = u''(\mu_1)\mu_1\, d\mu_1|_{\mathcal{D}}$ and $Jdt_1|_{\mathcal{D}} = -u''(\mu_1)\, d\mu_1|_{\mathcal{D}}$, in accordance with $(dt_0 + \mu_1 dt_1)|_{\mathcal{D}} = \eta_\chi|_{\mathcal{D}} = 0$.

Remark 2. A key feature of our approach is that we avoid considering compatible complex structures on the symplectic cone in $\mathcal{D}^0$ over N: such structures induce not only a CR structure J on N, but also a preferred Sasaki structure χ, a choice we wish to decouple. However, it is straightforward to compare our approach with works such as [2, 52, 57] which use the symplectic cone. First, sections of $\mathcal{O}_{\mathfrak{h}^*}(1)$ over $U \subseteq P(\mathfrak{h}^*)$ correspond bijectively to homogeneous functions of degree 1 on the inverse image of U in $\mathfrak{h}^* \setminus \{0\}$ which contains the momentum image $\tilde U$ of the symplectic cone. Thus a symplectic potential in our sense induces an ordinary function u on $\tilde U$, homogeneous of degree 1. However, the hessian of any such function is degenerate in radial directions, so does not define a metric on the symplectic cone. To get around this, we exploit the fact that symplectic potentials are not well-defined: for any $a \in \mathfrak{h}$, we can add the linear form $\langle a, z\rangle|_{\tilde U}$ to u without changing its hessian. Hence symplectic potentials are really elements of the quotient of $C^\infty_{\tilde U}(\mathbb{R})$ by $\mathfrak{h}$.

To be concrete, if $f_1, \ldots f_k$ are linear forms on $\tilde U$ corresponding to $a_{(1)}, \ldots a_{(k)} \in \mathfrak{h}$, then $u = \sum_{j=1}^k f_j \log|f_j|$ is a function on $\tilde U$ with $u(\lambda p) = \lambda u(p) + \log|\lambda| \sum_{j=1}^k f_j$. Hence it is homogeneous of degree 1 modulo $\mathfrak{h}$, but only strictly homogeneous of degree 1 if $\sum_{j=1}^k a_{(j)} = 0$. Its hessian is $\tilde G = \sum_{j=1}^k a_{(j)}^2/f_j$ with $\langle z, \tilde G\rangle = \sum_{j=1}^k a_{(j)}$, which is constant.

We can interpret this in our formalism by modifying the differential lift: for $a \in \mathfrak{h}$ and $u \in C^\infty_{\Delta^\circ}(\mathcal{O}_{\mathfrak{h}^*}(1))$, we define $\mathcal{L}_a(u)$ by $\langle z, \mathcal{L}_a(u)\rangle = u$ and $\langle z, d\mathcal{L}_a(u)\rangle = 2a$. Assuming $\mathit{Hess}(u)$ is nondegenerate and $\langle a, z\rangle$ is nonvanishing, $d\mathcal{L}_a(u)$ is nondegenerate, and defines the metric on the symplectic cone corresponding to the CR structure defined by u and the Sasaki structure ξ_a.

4.3. **The CR twist of a toric manifold.** Suppose (M, g, J, ω) is given by (9) and $(N, \mathcal{D}, J, \chi)$ is a (local) Sasaki $(2m+1)$-manifold over M corresponding to an extension (7) of $\mathfrak{t}$ by $\mathbb{R}$. We can suppose we are in the affine picture with $\langle \varepsilon, \mu\rangle = 1$ on M, where $\chi = \xi_\varepsilon$, and introduce coordinates $z_j = \langle e_j, z\rangle$ with $e_0 = \varepsilon$ so that the induced contact form on N is given by (8), where $\hat\mu$ is the pullback of μ to N.

For any $a \in \mathfrak{h}$, the affine function $f_a(z) = \sum_{j=0}^m a_j z_j$ is positive on the momentum image of M if and only if $f_a(\mu)$ is a positive Killing potential on (M, g, J, ω) if and only if $\xi_a = f_a(\hat{\mu})\chi$ is a Sasaki structure on $(N, \mathcal{D}, J)$. A CR f_a-twist of (M, g, J, ω) is then the induced toric Kähler metric on any Sasaki–Reeb quotient of $(N, \mathcal{D}, J)$ by ξ_a, which in turn has the form (9) for suitable coordinates $\tilde{\mu}, \tilde{t}$ and a symplectic potential $\tilde{u}(\tilde{\mu})$ with $\tilde{G} = \mathit{Hess}(\tilde{u})$. The new affine chart $\tilde{\mathcal{A}}$ on $P(\mathfrak{h}^*)$ has tautological affine coordinate $\tilde{z}$ with $1 = \langle a, \tilde{z}\rangle = f_a(z)\langle \varepsilon, \tilde{z}\rangle$ and hence $\tilde{z} = z/f_a(z)$.

To obtain explicit momentum-angle coordinates on $\tilde{M}$, we need to extend $\tilde{e}_0 := a$ to a basis $\tilde{e}_0, \tilde{e}_1, \dots \tilde{e}_m$ of $\mathfrak{h}$. One approach (see e.g. [52]) is to assume $a_0 \neq 0$ (which we can arrange by a translation of z) so that we may take $\tilde{e}_j = e_j$ for $j \in \{1, \dots m\}$. Since u transforms as a section of $\mathcal{O}_{\mathfrak{h}^*}(1)$, we have the following result.

Lemma 5. *Any CR f_a-twist $(\tilde{g}, \tilde{\omega})$ of the toric Kähler metric* (9) *with respect to the positive affine function $f_a(z) = a_0 + a_1 z_1 + \cdots + a_m z_m$ with $a_0 \neq 0$ is a toric Kähler metric of the form* (9) *on $\tilde{\Delta}^\circ \times \mathbb{R}^m$ with respect to momentum-angle coordinates $\tilde{\mu}, \tilde{t}$ and symplectic potential $\tilde{u}$ given by $\langle \tilde{\mu}, a\rangle = 1$,*

$$\tilde{\mu}_j = \frac{\mu_j}{f_a(z)}, \quad \tilde{t}_j := t_j - \frac{a_j}{a_0} t_0, \quad j \in \{1, \dots m\} \quad \text{and} \quad \tilde{u}(\tilde{z}) = \frac{u(z)}{f_a(z)}. \tag{12}$$

As shown in [51, 52], when M is compact, the rescaling $z \mapsto \tilde{z}$ sends the polytope Δ in $\mathcal{A}$ to a polytope $\tilde{\Delta} \subseteq \tilde{\mathcal{A}}$, and we have a compact CR f-twist $(\tilde{M}, \tilde{g}, \tilde{\omega})$ provided this polytope is rational.

Example 4. We illustrate the CR twist in the simple case of a toric Riemann surface metric

$$g_V = \frac{d\mu_1^2}{A(\mu_1)} + A(\mu_1)\, dt_1^2, \qquad \omega_V = d\mu_1 \wedge dt_1, \tag{13}$$

with *profile function* A. This is a Sasaki–Reeb quotient of a Sasaki 3-manifold $(N, \mathcal{D}, J, \chi)$ with

$$\eta_\chi = dt_0 + \mu_1 dt_1, \qquad J dt|_{\mathcal{D}} = (J dt_0|_{\mathcal{D}}, J dt_1|_{\mathcal{D}}) = -\frac{(-\mu_1, 1)}{A(\mu_1)}\, d\mu_1|_{\mathcal{D}},$$

using the affine chart $z = (1, z_1)$ as in Example 3. Thus $A(z_1) = 1/u''(z_1)$ for a symplectic potential $u = u(z_1)$, and there are straightforward integral formulae

$$\mathcal{L}(u)(z_1) = \int^{z_1} \frac{(-x, 1)}{A(x)} dx, \qquad u(z_1) = \langle \mathcal{L}(u), (1, z_1)\rangle = \int^{z_1} \frac{z_1 - x}{A(x)} dx,$$

for u and its differential lift (i.e., projective Legendre transformation) $\mathcal{L}(u)$. Any CR f_a-twist, with $f_a(z) = a_0 + a_1 z_1$, is then a Sasaki–Reeb quotient of N by the Sasaki structure $\xi = f_a(\mu)\chi$ with contact form $\eta_\xi = \eta_\chi / f_a(\mu)$. If $a \neq 0$, then as in Lemma 5 we may set $t_0 = a_0 \tilde{t}_0$ and $t_1 = \tilde{t}_1 + a_1 \tilde{t}_0$, so that $\eta_\xi = d\tilde{t}_0 + \tilde{\mu}_1 d\tilde{t}_1$, with $\tilde{\mu}_1 = \mu_1/(a_0 + a_1\mu_1)$, has Sasaki–Reeb field $X_\xi = \partial/\partial \tilde{t}_0$. We then compute

$$J d\tilde{t}|_{\mathcal{D}} = (J d\tilde{t}_0|_{\mathcal{D}}, J d\tilde{t}_1|_{\mathcal{D}}) = -\frac{(-\tilde{\mu}_1, 1)}{\tilde{A}(\tilde{\mu}_1)}\, d\tilde{\mu}_1|_{\mathcal{D}}$$

with
$$\tilde{z}_1 = \frac{z_1}{a_0 + a_1 z_1} \quad \text{and} \quad \tilde{A}(\tilde{z}_1) = \frac{a_0^2\, A(z_1)}{(a_0 + a_1 z_1)^3}.$$

However, other choices can be convenient: for example if the momentum image $\mu(V)$ is $[-1, 1]$ and $f_a \neq 0$ on $[-1, 1]$ (i.e., $|a_0| > |a_1|$) then we can instead preserve $[-1, 1]$ with

$$\tilde{z}_1 = \frac{a_0 z_1 + a_1}{a_0 + a_1 z_1} \quad \text{and} \quad \tilde{A}(\tilde{z}_1) = \frac{(a_0^2 - a_1^2)^2}{(a_0 + a_1 z_1)^3}\, A(z_1).$$

4.4. **The generalized Calabi ansatz.** We now discuss CR twists of Kähler metrics on certain toric fibre bundles $\pi\colon M \to B$ over a Kähler base manifold (B, g_B, ω_B). We follow the approach in [9], to which we refer the reader for further details, recalling here only the special case in which we are interested.

Let $(V, g_V, \omega_V, \mathbb{T}^\ell)$ be a toric Kähler 2ℓ-manifold and let $\pi\colon P \to B$ be a principal $\mathbb{T}^\ell$-bundle over a Kähler $2d$-manifold (B, g_B, ω_B), equipped with a connection 1-form $\theta \in \Omega^1(P, \mathfrak{t})$ ($\mathfrak{t}$ being the Lie algebra of $\mathbb{T}^\ell$) such that

$$d\theta = \zeta \otimes \omega_B, \quad \zeta \in \mathfrak{t}. \tag{14}$$

As before, we assume $\mathfrak{t} = \mathfrak{h}/\mathit{span}(\varepsilon)$, where $\mathfrak{h}$ may be identified with the space of affine functions on the image $\Delta \subseteq \mathfrak{h}^*$ of the momentum map of V. Using (7), we choose $a \in \mathfrak{h}$ such that $\delta a = \zeta$ and the affine function $f_a(z) := \langle a, z\rangle$ is positive on Δ.

Given these data, we can construct a Kähler $2m$-manifold (M, g, ω), with $m = \ell + d$, $M = P \times_{\mathbb{T}^\ell} V \xrightarrow{\pi} B$, and

$$\begin{aligned} g &= f_a(\mu)\pi^* g_B + \langle d\mu, G(\mu), d\mu\rangle + \langle \theta, H(\mu), \theta\rangle,\\ \omega &= f_a(\mu)\pi^*\omega_B + \langle d\mu \wedge \theta\rangle, \qquad d\theta = \delta a \otimes \omega_B, \end{aligned} \tag{15}$$

where G and H are determined by the toric Kähler metric on V in momentum–angle coordinates (9). We refer to (15) as *the generalized Calabi ansatz*, with data (V, g_V, ω_V), (B, g_B, ω_B) and $a \in \mathfrak{h}$. For any $b \in \mathfrak{h}$, the affine function $f_b(z) = \langle b, z\rangle$ pulls back to Killing potentials for both (g_V, ω_V) and (g, ω), and their CR f_b-twists are related as follows.

Proposition 3. *Let (M, g, ω) be given by the generalized Calabi ansatz for data (V, g_V, ω_V), (B, g_B, ω_B) and $a \in \mathfrak{h}$. Then for $b \in \mathfrak{h}$ with $f_b > 0$ on M, the generalized Calabi ansatz, with data a CR f_b-twist (V_b, g_b, ω_b) of V, (B, g_B, ω_B) and $a \in \mathfrak{h}$, is a CR f_b-twist of M. In particular, the Kähler product of (B, g_B, ω_B) and (V_a, g_a, ω_a) is a CR f_a-twist of M.*

Proof. It is enough to prove the result, for arbitrary V and B, in the case $a = \varepsilon$, with (M, g, ω) being the Kähler product of (B, g_B, ω_B) and (V, g_V, ω_V). Indeed, we may then recover the Kähler metric (15), associated to a given (B, g_B, ω_B), (V, g_V, ω_V) and a, as a CR twist of the Kähler product with (B, g_B, ω_B) of a CR f_a-twist (V_a, g_a, ω_a) of (V, g_V, ω_V), by taking $b = a$.

The Sasaki structure $(N, \mathcal{D}, J, \chi = \xi_\varepsilon)$ associated to the Kähler product of B and V is (locally) defined by the contact form

$$\eta_\chi = \langle \hat\mu, dt + \theta_B \otimes \varepsilon\rangle = \sum_{k=0}^{m} \hat\mu_j dt_j + \theta_B$$

where θ_B is a (local) 1-form on B with $d\theta_B = \omega_B$, and the second expression uses a basis of $\mathfrak{h}$ (for which we may assume $e_0 = \varepsilon$ so that $\hat\mu_0 \equiv 1$ and $X_\chi = \partial/\partial t_0$). The CR structure is determined from $d\eta_\chi = \pi^*(\omega_B + \omega_V)$ and $g_\chi = \pi^*(g_B + g_V)|_{\mathcal{D}}$.

Now let $f_b(z) = \sum_{k=0}^m b_k z_k$ be a positive affine function defining new affine coordinates $\tilde z_j = z_j/f_b(z)$ on $P(\mathfrak{h}^*)$. The symplectic form $\tilde\omega$ on any Sasaki–Reeb quotient $\tilde M$ of N by ξ_b pulls back to $d(\eta_\chi/f_b(\hat\mu)) = d\langle\hat\mu, dt + \theta_B \otimes \varepsilon\rangle$ and hence is given by

$$\tilde\omega = \langle\tilde\mu, d\theta\rangle + \langle d\tilde\mu \wedge \theta\rangle = \langle\tilde\mu, \varepsilon\rangle\omega_B + \langle d\tilde\mu \wedge \theta\rangle,$$

where $\tilde\mu$ is the pullback to $\tilde M$ of $\tilde z$ on $P(\mathfrak{h}^*)$ and $\theta \in \Omega^1(\tilde N, \tilde{\mathfrak{t}})$, with $\tilde{\mathfrak{t}} = \mathfrak{h}/\mathit{span}(b)$, pulls back to $dt + \theta_B \otimes \varepsilon$ mod b on N. The complex structure on B is unaffected by the CR twist, while consideration of the action of the CR structure on $d\tilde\mu|_{\mathcal{D}}$ allows us to identify the toric fibres of $\tilde M$ over B with the CR f_b-twist of V, as in Lemma 5. As in that Lemma, we can make the momentum–angle coordinates more explicit in a basis of $\mathfrak{h}$ with $e_0 = \varepsilon$ and $b_0 \neq 0$. In any case, the result now follows. $\square$

Corollary 3. *If (B, g_B, ω_B) is an extremal Kähler manifold and (V, g_V, ω_V) is an $(f_a, \ell+2)$-extremal Kähler manifold then (g, ω) given by* (15) *is $(f_a, m+2)$-extremal. In particular, a $\mathbb{C}P^\ell$-bundle $(M, J) = P(L_0 \oplus \cdots \oplus L_\ell) \to B$ over an extremal Hodge Kähler manifold (B, g_B, ω_B) has a natural 1-parameter family of $(f_a, m+2)$-extremal Kähler metrics.*

Proof. The CR f_a-twist of the Kähler metric (15) defined by Proposition 3 is the Kähler product of (B, g_B, ω_B) with the CR f_a-twist (V_a, g_a, ω_a) of (V, g_V, ω_V). As (V, g_V, ω_V) is $(\ell+2, f_a)$-extremal, (V_a, g_a, ω_a) is extremal by Theorem 1. It follows that the CR f_a-twist of (15) is extremal, so by Theorem 1 again, we conclude that (15) is $(f_a, m+2)$-extremal.

In the special case $M = P(L_0 \oplus \cdots \oplus L_\ell) \to B$, we can apply the above construction with $(V, g_V, \omega_V) = (\mathbb{C}P^\ell, g_{FS}, \omega_{FS})$, where (g_{FS}, ω_{FS}) is a Fubini–Study metric on $\mathbb{C}P^\ell$. By Corollary 2, g_{FS} is $(f_a, \ell+2)$-extremal, so the claim follows. $\square$

4.5. **The Calabi ansatz.** We now specialize to the case that (V, g_V, ω_V) in the generalized Calabi ansatz is a toric 2-manifold or orbifold (13); this is the original Calabi ansatz when $V = \mathbb{C}P^1$, and (15) reduces to

$$g = (a_0 + a_1\mu_1)g_B + \frac{d\mu_1^2}{A(\mu_1)} + A(\mu_1)\theta^2, \qquad \omega = (a_0 + a_1\mu_1)\omega_B + d\mu_1 \wedge \theta, \tag{16}$$

where $d\theta = a_1\omega_B$. By Proposition 3, (M, g, ω) has a CR f_a-twist $(\tilde{M}, \tilde{g}, \tilde{\omega})$ given by the the Kähler product of (B, g_B, ω_B) and $(\tilde{V}, g_{\tilde{V}}, \omega_{\tilde{V}})$, where the latter is a CR f_a-twist of (V, g_V, ω_V) as in Example 4. Furthermore, by Theorem 1, for any $b \in \mathfrak{h}$, g is $(f_b, m+2)$-extremal if and only if $\tilde{g}$ is $(\tilde{f}_b, m+2)$-extremal, where $f_b(\mu_1)$ and $\tilde{f}_b(\tilde{\mu}_1)$ are the Killing potentials induced by b on M and $\tilde{M}$ respectively.

If a and b are linearly independent then $\tilde{f}_b$ is nonconstant and, up to homothety, we may assume that $\tilde{f}_b(\tilde{z}_1) = \tilde{z}_1 + \tilde{b}_0$. Then $\tilde{g}$ is $(\tilde{f}_b, m+2)$-extremal iff g_B has constant scalar curvature s_B and $g_{\tilde{V}}$ has profile function $\tilde{A}$ with

$$\tilde{A}(\tilde{z}_1 - \tilde{b}_0) = p_0\tilde{z}_1^{m+2} + p_1\tilde{z}_1^{m+1} - s_B\tilde{z}_1^m + p_3\tilde{z}_1 + p_4. \tag{17}$$

If (B, g_B, ω_B) is a CSC Hodge Kähler manifold (where we may assume without loss that $[\omega_B/2\pi]$ is primitive) then this picture globalizes in a couple of ways as follows.

First, we may start from a weighted projective line $\tilde{V} = \mathbb{C}P^1_{\mathrm{w}}$, where $\mathrm{w} = (w_-, w_+)$ is a pair of positive integers. We equip $\mathbb{C}P^1_{\mathrm{w}}$ with the toric symplectic structure ω_{w} induced by the quasi-regular Sasaki structure $(\mathbb{S}^3, \mathcal{D}, J, \chi_{\mathrm{w}})$ on the 3-sphere $\mathbb{S}^3 \subseteq \mathbb{C}^2$. By (7), the rational Delzant polytope [30, 55] of $(\mathbb{C}P^1_{\mathrm{w}}, \omega_{\mathrm{w}})$ in the affine chart defined by χ_{w} is given by $\{(z_0, z_1) : z_i \geqslant 0,\ w_-z_0 + w_+z_1 = 1\}$, but we use instead the parametrization $z_0 = (1+\tilde{z}_1)/(2w_+)$, $z_1 = (1-\tilde{z}_1)/(2w_-)$ to realize this rational Delzant polytope as the interval $[-1, 1]$ with inward normals $1/(2w_+)$ and $-1/(2w_-)$. As explained in [13], for any positive integer k, the product Kähler orbifold $(B, g_B, \omega_B) \times (\mathbb{C}P^1_{\mathrm{w}}, kg_{\mathrm{w}}, k\omega_{\mathrm{w}})$ gives rise to a compact quasi-regular extremal Sasaki orbifold $(N_{\mathrm{w},k}, \mathcal{D}, J, \chi_{\mathrm{w},k})$, which is the Sasaki join of the regular Sasaki manifold $(N_B, \mathcal{D}_B, J_B, \chi_B)$ associated to (B, g_B, ω_B) and $(\mathbb{S}^3, \mathcal{D}, J, \frac{1}{k}\chi_{\mathrm{w}})$. There are well-understood conditions in terms of the integers (w_+, w_-, k) ensuring that $N_{\mathrm{w},k}$ is a smooth manifold, see [13]. Now any $\tilde{A}$ given by (17) with $|\tilde{b}_0| > 1$, which satisfies the well-known positivity and boundary conditions

$$\tilde{A}(\tilde{z}_1) > 0 \quad \text{on} \quad (-1, 1), \quad \tilde{A}(\pm 1) = 0 \quad \text{and} \quad \tilde{A}'(\pm 1) = \mp 4w_\mp/k, \tag{18}$$

gives rise to a toric, $k\omega_{\mathrm{w}}$-compatible Kähler metric $\tilde{g}_{\mathrm{w}}$ on $\mathbb{C}P^1_{\mathrm{w}}$, such that the product metric $g_B + \tilde{g}_{\mathrm{w}}$ is $(\tilde{f}_b, m+2)$-extremal. We thus get a new Sasaki structure $(N_{\mathrm{w},k}, \mathcal{D}, J_{\mathrm{w}}, \xi_b)$ which is extremal by Theorem 1. Note that ξ_b is not quasi-regular if $\tilde{b}_0$ is irrational.

For a fixed a_0, the endpoint conditions (18) determine the unknown coefficients p_0, p_1, p_3 and p_4 of a polynomial $\tilde{A}$ satisfying (17), and it remains to examine the positivity condition for $\tilde{A}$. This is therefore an effective tool for generating compact examples of extremal Sasaki metrics, providing an explanatory framework for the constructions in [16, 17].

Secondly, we may begin instead with $M = P(\mathcal{O} \oplus \mathcal{L})$ where $\mathcal{L}$ is a holomorphic line bundle over B such that $c_1(\mathcal{L}) = \ell[\omega_B/2\pi]$ for $\ell \in \mathbb{Z}^+$ (and $\mathcal{O}$ denotes the trivial line bundle). Then (16) defines a Kähler metric on M such that the $\mathbb{S}^1$-action induced by scalar multiplication in $\mathcal{O}$ is isometric and hamiltonian with momentum map μ_1 and momentum image $\mu_1(M) = [-1, 1] \subset \mathbb{R}$ if and only if $A(z_1)$ is a smooth function on $[-1, 1]$ satisfying the boundary conditions

$$A(\pm 1) = 0, \quad A'(\pm 1) = \mp 2, \tag{19}$$

and the positivity condition

$$A(z_1) > 0 \quad \text{on} \quad (-1, 1), \tag{20}$$

and $a_0 + a_1 z_1$ is positive on $[-1, 1]$ with $|a_1| = \ell$ (and we may assume $a_1 = \ell$ by replacing z_1 with $-z_1$ if necessary). Here θ is the connection form associated to a principal $\mathbb{S}^1$-connection on the unit circle bundle in $M \to B$ and

$$[\omega/2\pi] = c_1(\mathcal{O}_{\mathcal{O}\oplus\mathcal{L}}(2)) + (a_0 + a_1)c_1(\pi^*\mathcal{L}). \tag{21}$$

For any positive integers k, n such that $n/k > \ell$, $L_{k,n} := \mathcal{O}_{\mathcal{O}\oplus\mathcal{L}}(k) \otimes \pi^*\mathcal{L}^{n/\ell}$ is a polarization on M, $c_1(L_{k,n})$ being homothetic to a Kähler class of the form (21) with $a_0 = (2n/k) - \ell$ and $a_1 = \ell$. We thus let $(N_{k,n}, \mathcal{D}, J, \chi)$ be the smooth Sasaki manifold corresponding to the Kähler manifold $(M, \frac{k}{2}g, \frac{k}{2}\omega)$ via Example 1, where (g, ω) is given by (16) (with $a_0 = (2n/k) - \ell$ and $a_1 = \ell$). Up to a covering, $(N_{k,n}, \mathcal{D}, \chi)$ is determined by the ratio n/k, so we assume henceforth that k and n are coprime positive integers. In [17, (37)], the contact manifold $(N_{k,n}, \mathcal{D})$ is identified with the Sasaki join $(N_{\mathrm{w},k}, \mathcal{D})$ constructed over $B \times \mathbb{C}P^1_{\mathrm{w}}$ above, with weights $w_+ = n, w_- = n - k\ell$. The theory of CR twists further identifies the CR structure J on $(N_{k,n}, \mathcal{D})$ induced by (16) with the CR structure J_{w} on $(N_{\mathrm{w},k}, \mathcal{D})$ induced by

$$\tilde{g} = g_B + \frac{d\tilde{\mu}_1^2}{\tilde{A}(\tilde{\mu}_1)} + \tilde{A}(\tilde{\mu}_1)dt^2, \qquad \tilde{\omega} = \omega_B + d\tilde{\mu}_1 \wedge dt,$$

where $\quad \tilde{A}(\tilde{z}_1) = \dfrac{(a_0^2 - a_1^2)^2 A(z_1)}{(a_0 + a_1 z_1)^3}, \quad z_1 = \dfrac{a_0\tilde{z}_1 - a_1}{a_0 - a_1\tilde{z}_1}, \quad a_0 = \dfrac{2n}{k} - \ell \quad$ and $\quad a_1 = \ell.$

5. Separable toric geometries

5.1. **Regular ambitoric structures.** In [5, 6], the following 4-dimensional geometric structure was studied.

Definition 5. An *ambikähler structure* on a real 4-manifold or orbifold M consists of a pair of Kähler metrics (g_-, J_-, ω_-) and (g_+, J_+, ω_+) such that

- g_- and g_+ are conformally equivalent;
- J_- and J_+ have opposite orientations.

The structure is said to be *ambitoric* if in addition there is a 2-dimensional subspace $\mathfrak{t}$ of vector fields on M, linearly independent on a dense open set, whose elements are hamiltonian and Poisson-commuting Killing vector fields with respect to both (g_-, ω_-) and (g_+, ω_+)—i.e., both Kähler structures are locally toric.

It was shown in [5] that any ambitoric structure is locally either a product, of Calabi type, or a *regular* ambitoric structure given by the following ansatz. Let $q(x) = q_0 + 2q_1x + q_2x^2$ be a quadratic polynomial and let M be a 4-manifold or orbifold with real-valued functions $(x_1, x_2, \tau_0, \tau_1, \tau_2)$ such that $x_1 > x_2$, $2q_1\tau_1 = q_0\tau_2 + q_2\tau_0$, and their exterior derivatives span each cotangent space. Let $\mathfrak{t}$ be the 2-dimensional space of vector fields K on M

with $dx_1(K) = 0 = dx_2(K)$ and $d\tau_j(K)$ constant, and let $A(x)$ and $B(x)$ be positive functions on open neighbourhoods of the images of x_1 and x_2 in $\mathbb{R}$, on whose product $f_q(x_1, x_2) := q_0 + q_1(x_1 + x_2) + q_2 x_1 x_2$ is positive. Then M is ambitoric with

$$\begin{aligned} g_\pm &= \Big(\frac{x_1 - x_2}{f_q(x_1,x_2)}\Big)^{\pm 1}\Big(\frac{dx_1^2}{A(x_1)} + \frac{dx_2^2}{B(x_2)} + A(x_1)\alpha_1^2 + B(x_2)\alpha_2^2\Big),\\ \omega_\pm &= \Big(\frac{x_1 - x_2}{f_q(x_1,x_2)}\Big)^{\pm 1}(dx_1 \wedge \alpha_1 \pm dx_2 \wedge \alpha_2), \qquad \begin{aligned} J_\pm dx_1 &= A(x_1)\alpha_1,\\ J_\pm dx_2 &= \pm B(x_2)\alpha_2,\end{aligned}\\ \alpha_1 &= \frac{d\tau_0 + 2x_2 d\tau_1 + x_2^2 d\tau_2}{(x_1 - x_2) f_q(x_1,x_2)}, \qquad \alpha_2 = \frac{d\tau_0 + 2x_1 d\tau_1 + x_1^2 d\tau_2}{(x_1 - x_2) f_q(x_1,x_2)} \end{aligned} \tag{22}$$

There is a gauge freedom to make a simultaneous projective transformation of the coordinates x_1, x_2, with q transforming as a quadratic polynomial, and A, B as quartics [5]. If q has repeated roots, we may use this freedom to set $q = 1$, and then g_+ is a 2-dimensional *orthotoric metric*, as studied in [3, 4]. We then refer to g_- as a *negative orthotoric metric*.

Ambitoric structures are examples of *separable toric geometries*, i.e., they admit *separable coordinates* $x_1, \dots x_m$ in which the metric is determined by m functions of 1 variable (and some explicit data, such as q here). We now explore CR twists for some separable toric geometries. While we could simply apply the general approach given in Lemma 5, this is not expedient for a couple of reasons. On a practical level, we would need to compute: the transformation from separable coordinates to momenta, the symplectic potential, its CR twist in terms of the new momenta, and finally the transformation from these momenta back to separable coordinates. This is rather involved, and unnecessarily so, because whereas a CR twist involves a change of momentum coordinates due to the change of affine chart, the separable coordinates remain fixed. We illustrate this first in the simplest separable situation: Kähler products of toric Riemann surfaces.

5.2. **The CR twisted toric product ansatz.** A Kähler product of toric Riemann surfaces has a Kähler metric of the form:

$$g = \sum_{i=1}^m \Big(\frac{{dx_i}^2}{A_i(x_i)} + A_i(x_i){dt_i}^2\Big), \quad \omega = \sum_{i=1}^m dx_i \wedge dt_i, \quad J dx_i = A_i(x_i) dt_i, \tag{23}$$

where $A_1, \dots A_m$ are arbitrary functions of 1 variable. In this case the separable coordinates and momenta coincide: the toric Killing potentials have the form

$$f_b(x_1, \dots x_m) = b_0 + b_1 x_1 + \dots + b_m x_m.$$

It is straightforward to compute the CR structure associated to (g, ω, J) as in Example 1. Denoting by t_i, x_i also their pullbacks to N, we have $\mathcal{D} = \ker \eta$ and $J \colon \mathcal{D}^* \to \mathcal{D}^*$ given by

$$\eta = dt_0 + \sum_{i=1}^m x_i dt_i, \qquad J\big(dx_i|_{\mathcal{D}}\big) = A_i(x_i) dt_i|_{\mathcal{D}}.$$

We now lift $f_b(x_1, \dots x_m)$ to a new Sasaki structure ξ_b on N and compute the new Sasaki–Reeb quotient. The new contact form is $\eta_b := \eta_{\xi_b} = \eta/f_b$, with

$$d\eta_b = \sum_{i=1}^m dx_i \wedge \frac{\partial \eta_b}{\partial x_i} = \frac{1}{f_b(x_1, \dots x_m)} \sum_{i=1}^m dx_i \wedge (dt_i - b_i \eta_b).$$

Since $J\big(dx_i|_{\mathcal{D}}\big) = A_i(x_i)(dt_i - b_i\eta_b)|_{\mathcal{D}}$, the Sasaki–Reeb quotient is given by the following toric ansatz, originally proposed in [7], which we refer to here as a *twisted toric product*:

$$\begin{aligned} g_b &= \frac{1}{f_b(x_1,\dots x_m)}\sum_{i=1}^m\Big(\frac{dx_i{}^2}{A_i(x_i)} + A_i(x_i)\alpha_i{}^2\Big), \qquad \omega_b = \frac{1}{f_b(x_1,\dots x_m)}\sum_{i=1}^m dx_i\wedge\alpha_i, \\ J_b dx_i &= \alpha_i, \qquad d\alpha_i = -b_i\omega, \qquad f_b(x_1,\dots x_m) = b_0 + b_1x_1 + \cdots + b_mx_m. \end{aligned} \tag{24}$$

For $b_0 \neq 0$, we may obtain more explicit angle coordinates by setting $\tau_i = t_i - b_it_0/b_0$ and $\alpha_i = d\tau_i - (b_i/f_b)\sum_{j=1}^m x_jd\tau_j$. We may take as momenta

$$\mu_0 = \frac{1}{f_b(x_1,\dots x_m)}, \qquad \mu_j = \frac{x_j}{f_b(x_1,\dots x_m)}, \qquad j\in\{1,\dots m\}.$$

Hence the momentum coordinates and separable coordinates no longer agree. The original product metric (23) is a CR twist of (24) by $\mu_0 = f_b^{-1} = 1/f_b$. It was shown in [7] that when $m = 2$, this construction unifies the ambitoric product, Calabi and negative orthotoric ansatz of [5] in a single family.

It was also shown in [7] that a twisted toric product metric (24) is $(f_b^{-1}, m+2)$-extremal if and only if A_j is a cubic polynomial for all $j \in \{1,\dots m\}$. We see here that this follows straightforwardly from Theorem 1, as (24) is $(f_b^{-1}, m+2)$-extremal if and only if (23) is extremal, and a toric Kähler product is extremal if and only if the factors are, meaning that each A_i is a cubic. In this case, we may also identify the CR manifold which has these metrics as its Sasaki–Reeb quotients. Indeed, straightforward computation shows that the Cartan tensor of a Sasaki 3-manifold vanishes precisely when the (transversal, i.e., Tanaka–Webster) scalar curvature is transversally holomorphic (see e.g. [41]). It then follows from [24] that the $(f_b^{-1}, m+2)$-extremal metrics given by (24) are obtained as Sasaki–Reeb quotients with respect to the CR structure of a (local) Sasaki join [13] of m copies of the standard CR structure $(\mathcal{D}_0, J_0)$ on the 3-sphere $\mathbb{S}^3 \subseteq \mathbb{C}^2$, with respect to a (local) Sasaki structure on each factor.

We next consider the extremality condition for the Kähler metrics (24), using again Theorem 1 to infer that (24) is extremal if and only if the product metric (23) is $(f_b, m+2)$-extremal. We thus have

$$\begin{aligned} &\sum_{j=1}^m\bigg(-f_b^2A_j''(x_j) + 2(m+1)f_b\frac{\partial f_b}{\partial x_j}A_j'(x_j) - (m+1)(m+2)\Big(\frac{\partial f_b}{\partial x_j}\Big)^2A_j(x_j)\bigg) \\ &\qquad = -\sum_{j=1}^m f_b^{m+3}\frac{\partial^2}{\partial x_j^2}\bigg(\frac{A_j(x_j)}{f_b^{m+1}}\bigg) = Scal_{f_b,m+2}(g) = c_0 + c_1x_1 + \cdots + c_mx_m, \end{aligned} \tag{25}$$

where $c_0, c_1, \dots c_m$ are some real constants. For $m = 1$ we get that A_1 must be a polynomial of degree $\leqslant 3$ and for $m = 2$ (24) is given by the ambitoric product, Calabi or negative orthotoric ansatz of [5], and the extremality condition (25) can be solved [5] in terms of two polynomials A_1 and A_2 of degree $\leqslant 4$. We thus assume from now on that $m \geqslant 3$.

Proposition 4. *For $m \geqslant 3$ the Kähler metric (24) is extremal if and only if it is a product of extremal Riemann surfaces, or is given by the Calabi ansatz over a product of $m-1$ CSC Riemann surfaces, or is the Kähler product of a scalar-flat product of Riemann surfaces with a product of flat Riemann surfaces, as in Proposition 2.*

Proof. Differentiating (25) $(m+1)$ times with respect to x_j yields

$$f_b^2A_j^{(m+3)}(x_j) = 0,$$

showing that each A_j must be a polynomial of degree $\leqslant m+2$. Thus, both sides of (25) are polynomials in x_i, so we may compare coefficients. Taking two derivatives in x_j gives

$$0 = f_b^m \frac{\partial^2}{\partial x_j^2}\left(\frac{A_j^{(2)}(x_j)}{f_b^{m-1}}\right) = f_b^2 A_j^{(4)}(x_j) - 2(m-1)b_j f_b A_j^{(3)}(x_j) + m(m-1)b_j^2 A_j^{(2)}(x_j).$$

If $b_i \neq 0$ for some $i \neq j$, the vanishing of the polynomial coefficients containing x_i^2 in the above relation show that A_j has degree $\leqslant 3$; if furthermore $b_j \neq 0$, then the coefficients containing x_i show that A_j has degree $\leqslant 2$. Substituting back in (25) and comparing coefficients, this yields the following three possibilities for the solutions to (25) with $m \geqslant 3$:

- $f_b(x_1, \ldots x_m) = b_0$. Then the A_j are polynomials of degree $\leqslant 3$ and the corresponding extremal metric (24) is a product of extremal Riemann surfaces;
- $f_b(x_1, \ldots x_m) = b_0 + b_j x_j$ with $b_j \neq 0$. Then (24) is given by the Calabi ansatz over the product of $(m-1)$ Riemann surfaces indexed by $i : i \neq j$. In particular, for each $i : i \neq j$, A_i is a polynomial of degree $\leqslant 2$ whereas A_j is a polynomial of degree $\leqslant m+2$, as described by (17).
- there are $0 \neq j_1 \neq j_j \neq 0$ with $b_{j_1} \neq 0 \neq b_{j_2}$. Then for each j with $b_j \neq 0$, A_j is a polynomial of degree $\leqslant 1$, and for each i with $b_i = 0$, A_i is a polynomial of degree $\leqslant 2$ with $\sum_{i:b_i=0} A_i'' = 0$. Thus, in this case, g is the product metric of a scalar-flat product of (CSC) Riemann surfaces (indexed by $\{i : b_i = 0\}$) with a product of flat Riemann surfaces (indexed by $\{j : b_j \neq 0\}$). □

5.3. **CR twists of positive regular ambitoric structures.** We return now to regular ambitoric structures (22), for which it was shown in [5] that g_+ is extremal if and only if g_- is extremal if and only if

$$A = pq + P, \qquad B = pq - P, \tag{26}$$

where p is a quadratic polynomial orthogonal to q, and P is polynomial of degree $\leqslant 4$. Note that the orthogonality condition $\langle p, q\rangle := p_0q_2 - p_1q_1 + p_2q_0 = 0$ means that the roots of p and the roots of q harmonically separate each other.

When q has distinct roots, the positive and negative structures are equivalent, cf. [6, Remark 5], while in the case of repeated roots the negative orthotoric structures are twisted toric products [7], as noted above. Hence we only need to consider the positive ambitoric metrics. The CR structure associated to (g_+, J_+, ω_+) in (22) was computed in [6, App. C], which implies that $\mathcal{D} = \ker \eta$ and $J \colon \mathcal{D}^* \to \mathcal{D}^*$ are given by

$$\eta = \frac{dt_0 + (x_1+x_2)dt_1 + x_1x_2\, dt_2}{x_1 - x_2}, \qquad \begin{aligned} J(dx_1|_{\mathcal{D}}) &= A(x_1)\frac{dt_0 + 2x_2 dt_1 + x_2^2 dt_2}{(x_1-x_2)^2}\Big|_{\mathcal{D}},\\ J(dx_2|_{\mathcal{D}}) &= B(x_2)\frac{dt_0 + 2x_1 dt_1 + x_1^2 dt_2}{(x_1-x_2)^2}\Big|_{\mathcal{D}}, \end{aligned} \tag{27}$$

independently of q, while the toric Killing potentials of (g_+, J_+, ω_+) have the form f_w/f_q, where w is quadratic polynomial, and lift to Sasaki structures $\xi_w := f_w(x_1, x_2)\chi/(x_1 - x_2)$, where $\chi \in \mathfrak{con}_+(N, \mathcal{D})$ with $\eta = \chi^{-1}\eta_{\mathcal{D}}$.

The CR structures arising from an extremal positive ambitoric metric thus have A and B of degree $\leqslant 4$, such that the roots of $A+B$ have harmonic cross-ratio (i.e., in $\{-1, 1/2, 2\}$). We may then write

$$A = p_1p_2 + P, \quad B = p_1p_2 - P, \quad \text{with} \quad \deg P \leqslant 4, \quad \deg p_j \leqslant 2, \quad \langle p_1, p_2\rangle = 0. \tag{28}$$

Here we have renamed the quadratics compared to (26) so that we are free to use q to define an arbitrary Sasaki–Reeb quotient of (27). Indeed for any quadratic q, the Sasaki structure ξ_q is $(f_{p_j}\chi, 4)$-extremal for $j \in \{1, 2\}$ by Theorem 1 and is extremal if $q = p_1$ or $q = p_2$. We obtain in particular a result of [10], as any Sasaki–Reeb quotient by ξ_q, given explicitly by (22), subject to (28), is $(f_{p_j}/f_q, 4)$-extremal for $j \in \{1,2\}$, i.e., the scalar curvature of $\tilde g_j = (f_q/f_{p_j})^2 g_+$ is a Killing potential of g_+; in fact one can compute [5, 10]

$$Scal(\tilde g_j) = -\frac{f_w}{f_q} \quad \text{with} \quad w := \{p_j, (p_j, P)^{(2)}\}, \tag{29}$$

where the Poisson bracket is given by $\{p, r\} := p' r - p r'$ and

$$(p, P)^{(2)} := p P'' - 3p' P + 6p'' P$$

is a transvectant of p and P. Special choices of q give special metrics in this family of Sasaki–Reeb quotients [5, 10].

- If $q = p_j$, then $\tilde g_j = g_+$, recovering the case that g_+ is extremal.
- If $\langle q, p_j\rangle = 0$, then g_- is also $(f_{p_j}/f_q, 4)$-extremal, and $\tilde g_j$ has diagonal Ricci tensor; if in addition $\langle (p_j, P)^{(2)}, q\rangle = 0$ then $\{p_j, (p_j, P)^{(2)}\}$ is a multiple of q; hence $\tilde g_j$ is CSC, so g_+ is conformally Einstein–Maxwell (in fact to a riemannian Plebański–Demiański metric [5, 29, 59]).
- Combining these observations, if say $q = p_1$, then $\langle q, p_2\rangle = 0$ so $g_+ = \tilde g_1$ is extremal, while $\tilde g_2$ has diagonal Ricci tensor; if in addition $\langle (p_2, P)^{(2)}, q\rangle = 0$ then $\tilde g_2$ is Einstein.
- Finally, taking $q = 1$, we obtain an orthotoric metric in the family.

Corollary 4. *Any regular positive ambitoric Kähler metric* (g_+, ω_+, J_+) *given by* (22) *can be obtained as a* f_q*-twist of an orthotoric metric.*

5.4. **The CR twisted orthotoric ansatz.** Corollary 4 immediately suggests a higher dimensional extension of the positive regular ambitoric ansatz (22). For this, we start with an orthotoric $2m$-manifold M with Kähler structure [4]:

$$\begin{gathered} g = \sum_{j=1}^m \Big(\frac{\Delta_j}{A_j(x_j)}\, dx_j^2 + \frac{A_j(x_j)}{\Delta_j} \Big(\sum_{r=1}^m \sigma_{r-1}(\hat x_j)\, dt_r \Big)^2 \Big), \\ \omega = \sum_{j=1}^m dx_j \wedge \Big(\sum_{r=1}^m \sigma_{r-1}(\hat x_j) dt_r \Big) = \sum_{r=1}^m d\mu_r \wedge dt_r, \\ J dx_j = \frac{A_j(x_j)}{\Delta_j} \sum_{r=1}^m \sigma_{r-1}(\hat x_j)\, dt_r, \qquad\qquad J dt_r = (-1)^r \sum_{j=1}^m \frac{x_j^{m-r}}{A_j(x_j)}\, dx_j, \end{gathered} \tag{30}$$

where each A_j is a smooth function of 1 variable, $\mu_r = \sigma_r(x_1, \dots x_m)$ are the momentum coordinates (σ_r being the r-th elementary symmetric function with $\sigma_0 = 1$) $\hat x_j = (x_k : k \neq j)$, and $\Delta_j = \prod_{k\neq j}(x_j - x_k)$. The separable coordinates $\boldsymbol{x} := (x_1, \dots x_m)$ are called *orthotoric* and have a natural gauge freedom under simultaneous affine changes $\tilde x_j = a x_j + b$. The toric Killing potentials in orthotoric coordinates are

$$f_q(\boldsymbol{x}) = q_0 + q_1\mu_1 + \cdots + q_m\mu_m \quad \text{with} \quad \mu_r = \sigma_r(\boldsymbol{x}), \tag{31}$$

which can be viewed as the polarized form of a degree $\leqslant m$ polynomial

$$q(x) := f_q(x, \dots x) = \sum_{j=0}^m \binom{j}{m} q_j x^j. \tag{32}$$

As usual, to write down the CR structure $(N, \mathcal{D}, J)$ over M, it is convenient to view $dt = (dt_0, dt_1, \dots dt_m)$ as a 1-form with values in the Lie algebra $\mathfrak{h} \cong \mathbb{R}^{m+1}$ of Killing potentials (with basis $\sigma_0, \sigma_1, \dots \sigma_m$). Then $\mathcal{D} = \ker \eta$ and $J \colon \mathcal{D}^* \to \mathcal{D}^*$ are given by

$$\eta = dt(\boldsymbol{x}) := \sum_{r=0}^m \mu_r dt_r, \qquad J(dx_j|_{\mathcal{D}}) = \frac{A_j(x_j)}{\Delta_j} \frac{\partial(dt(\boldsymbol{x}))}{\partial x_j}\Big|_{\mathcal{D}} \tag{33}$$

with $\mu_0 = 1$ (omitting pullbacks to N).

Proposition 5. *Let (g,ω,J) be the orthotoric Kähler metric* (30) *and let f_q be a positive function of the form* (31). *Then a CR f_q-twist of (g,ω,J) has toric Kähler metric*

$$
\begin{aligned}
g_q &= \sum_{j=1}^{m}\Big(\frac{\Delta_j}{A_j(x_j)f_q(\boldsymbol{x})}\,dx_j^2 + \frac{A_j(x_j)f_q(\boldsymbol{x})}{\Delta_j}\Big(\frac{\partial}{\partial x_j}\frac{dt(\boldsymbol{x})}{f_q(\boldsymbol{x})}\Big)^2\Big),\\
\omega_q &= \sum_{j=1}^{m} dx_j\wedge\Big(\frac{\partial}{\partial x_j}\frac{dt(\boldsymbol{x})}{f_q(\boldsymbol{x})}\Big), \qquad J_q dx_j = \frac{A_j(x_j)f_q(\boldsymbol{x})}{\Delta_j}\Big(\frac{\partial}{\partial x_j}\frac{dt(\boldsymbol{x})}{f_q(\boldsymbol{x})}\Big).
\end{aligned}
\tag{34}
$$

Proof. The CR f_q-twist is the Sasaki–Reeb quotient of (33) by the Sasaki structure $\xi_q = f_q\chi$ with contact form $\eta/f_q(\boldsymbol{x})$ (i.e., χ is the Sasaki structure with contact form $\eta=\chi^{-1}\eta_{\mathcal{D}}$), and

$$d\Big(\frac{\eta}{f_q(\boldsymbol{x})}\Big) = \sum_{j=1}^{m} dx_j\wedge\frac{\partial}{\partial x_j}\frac{dt(\boldsymbol{x})}{f_q(\boldsymbol{x})}.$$

Now we observe that

$$J(dx_j|_{\mathcal{D}}) = \frac{A_j(x_j)}{\Delta_j}\frac{\partial(dt(\boldsymbol{x}))}{\partial x_j}\Big|_{\mathcal{D}} = \frac{A_j(x_j)}{\Delta_j}f_q(\boldsymbol{x})\frac{\partial}{\partial x_j}\frac{dt(\boldsymbol{x})}{f_q(\boldsymbol{x})}\Big|_{\mathcal{D}}$$

and the 1-forms dx_j and $\frac{\partial}{\partial x_j}\frac{dt(\boldsymbol{x})}{f_q(\boldsymbol{x})}$ are basic with respect to ξ_q—in the latter case because $\big(\frac{dt(\boldsymbol{x})}{f_q(\boldsymbol{x})}\big)(X_{\xi_q})=1$. Hence the transversal Kähler structure of ξ_q is the pullback of (34). □

We now turn to the extremality condition of the Kähler metrics given by (34). By Theorem 1, such a metric is extremal iff the orthotoric metric (30) is $(f_q, m+2)$-extremal, a condition studied in [11, App. A]. Using standard formulae for the scalar curvature and laplacian of an orthotoric metric [4], this condition is

$$
\begin{aligned}
&\sum_{j=1}^{m}\Big(-f_q(\boldsymbol{x})^2\frac{A_j''(x_j)}{\Delta_j} + 2(m+1)f_q(\boldsymbol{x})\frac{\partial f_q}{\partial x_j}\frac{A_j'(x_j)}{\Delta_j} - (m+1)(m+2)\Big(\frac{\partial f_q}{\partial x_j}\Big)^2\frac{A_j(x_j)}{\Delta_j}\Big)\\
&\qquad = -\sum_{j=1}^{m}\frac{f_q(\boldsymbol{x})^{m+3}}{\Delta_j}\frac{\partial^2}{\partial x_j^2}\Big(\frac{A_j(x_j)}{f_q(\boldsymbol{x})^{m+1}}\Big) = Scal_{f_q,m+2}(g) = \sum_{k=0}^{m} c_k\mu_k.
\end{aligned}
\tag{35}
$$

for some constants $c_0,\dots c_m$. If we let $A_j(x)=P(x)$ for a j-independent polynomial P of degree $\leqslant m+2$, the metric (30) is Bochner-flat by [4], and we obtain (for any q) a solution of (35) by Proposition 1. When $q=1$, (35) describes the extremality condition for (30), which is studied in [4], where the solutions are given as $A_j(x)=P(x)+p_{j1}x+p_{j0}$ for a j-independent polynomial P of degree $\leqslant m+2$ and arbitrary real constants p_{j1},p_{j0} ($j\in\{1,\dots m\}$). Another special case is $q(x)=x^m$, i.e., $f_q=\sigma_m$, which is studied in [11, Prop. A2], where the solutions are given as $A_j(x)=P(x)+p_{j1}x^{m+1}+p_{j0}x^{m+2}$ for a j-independent polynomial P of degree $\leqslant m+2$ and arbitrary real constants p_{j1},p_{j0}. However, we notice that in this case the corresponding extremal Kähler metric g_q given by (34) is orthotoric with respect to the variables $\tilde{x}_j=1/x_j$ and functions $\tilde{A}_j(\tilde{x}_j)=\tilde{x}_j^{m+2}A_j(1/\tilde{x}_j)$, so the corresponding extremal Kähler metrics are not new. More generally, we may extend arguments from [4, Lemma 6] and [11, Prop. A2] as follows.

Proposition 6. *Let $m\geqslant 3$. Then the orthotoric Kähler metric* (30) *is $(f_q, m+2)$-extremal for some positive f_q in the form* (31) *if and only if either all $A_j(x)$ are equal to a j-independent polynomial of degree $\leqslant m+2$ or else the polynomial q has a root of multiplicity m (possibly at infinity) so that, up to a simultaneous affine transformation of the x_j in* (30), *we may assume that either $q(x)=1$ or $q(x)=x^m$. Then, $A_j(x)$ are the solutions described in* [4, Prop. 17] *and* [11, Prop. A2], *respectively. In particular, each* *extremal Kähler metric of the form* (34) *is either Bochner-flat or orthotoric with respect to suitable variables.*

Proof. Multiplying (35) by $\Delta := \prod_{j<k}(x_j - x_k)$, we get the relation

$$f_q(\boldsymbol{x})^{m+3}\sum_{j=1}^{m} \pm\Delta(\hat{x}_j)\frac{\partial^2}{\partial x_j^2}\Big(\frac{A_j(x_j)}{f_q(\boldsymbol{x})^{m+1}}\Big) = \Delta\Big(\sum_{k=0}^{m} c_k\sigma_k(\boldsymbol{x})\Big),$$

where $\Delta(\hat{x}_j) = \prod_{i<k\neq j}(x_i - x_k)$ and the signs $\pm$ are left unspecified. The right hand side in the above equality is a polynomial of degree $\leqslant m$ in each variable x_j, $\Delta(\hat{x}_j)$ is a polynomial of degree $(m-2)$ in any $x_i, i \neq j$ and of degree 0 in x_j, whereas $f_q(\boldsymbol{x})$ is a polynomial of degree $\leqslant 1$ in each x_j. It follows that for $m \geqslant 2$,

$$0 = \frac{\partial^{m+1}}{\partial x_j^{m+1}}\Big(f_q(\boldsymbol{x})^{m+3}\frac{\partial^2}{\partial x_j^2}\Big(\frac{A_j(x_j)}{f_q(\boldsymbol{x})^{m+1}}\Big)\Big) = f_q(\boldsymbol{x})^2 A_j^{(m+3)}(x_j),$$

showing that each A_j must be a polynomial of degree $\leqslant m+2$.

Now let $k \neq j$ be fixed indices. Multiplying (35) by $x_j - x_k$ and letting $x_j = x = x_k$ leads to the vanishing of

$$(f_0 + xf_1)^2 P''_{jk}(x) + 2(f_0+xf_1)(f_1+xf_2)x^{m+2}\Big(\frac{P'_{jk}(x)}{x^{m+1}}\Big)' + (f_1+xf_2)^2 x^{m+3}\Big(\frac{P_{jk}(x)}{x^{m+1}}\Big)'', \tag{36}$$

where $P_{jk}(x) = A_j(x) - A_k(x)$. Here each $f_k = \sum_{r=0}^{m} q_{r+k}\hat{\sigma}_r$, with $\hat{\sigma}_r$ denoting the r-th elementary symmetric function of the variables $x_i : i \neq j, k$ (and letting $\hat{\sigma}_r = 0$ for $r > m-2$), is a polynomial of degree $\leqslant 1$ in each $x_i : i \neq j, k$. Equivalently, $f_k : k \in \{0,1,2\}$ can be viewed as affine functions in the variables $\hat{\sigma}_1, \dots \hat{\sigma}_{m-2}$ and thus (36) can be viewed as a polynomial of degree $\leqslant 2$ in $\hat{\sigma}_1, \dots \hat{\sigma}_{m-2}$. By making an simultaneous affine change of the variables x_j in (30) if necessary (which preserves the orthotoric structure of the metric, see [4]), we can assume without loss that $q_0 \neq 0$, i.e., $f_0 \neq 0$. We thus consider the following three cases.

Case 1. f_0, f_1, f_2 are linearly independent affine functions of $\hat{\sigma}_1, \dots \hat{\sigma}_{m-2}$. Then, using f_0, f_1, f_2 as independent variables, and considering the coefficients of f_2^2, $f_0 f_1$ and f_0^2 in (36) yields that $P_{jk}(x)$ must belong to the common kernel of the ODEs $P''(x) = 0$, $(P'(x)/x^{m+1})' = 0$ and $(P(x)/x^{m+1})'' = 0$. The latter is trivial, thus showing that $P_{jk}(x) \equiv 0$ in this case, i.e., $A_j(x) = P(x)$ must be a j-independent function.

Case 2. f_0, f_1, f_2 span a 2-dimensional subspace of affine functions of $\hat{\sigma}_1, \dots \hat{\sigma}_{m-2}$. In this case, (36) is a polynomial of degree 2 in two independent variables in the span of f_0, f_1, f_2, which places three relations involving $P''(x)$, $(P'(x)/x^{m+1})'$ and $(P(x)/x^{m+1})''$. Using their functional independence, we conclude again that $P''(x) = 0$, $(P'(x)/x^{m+1})' = 0$ and $(P(x)/x^{m+1})'' = 0$, i.e., $P_{jk}(x) = A_j(x) - A_k(x) = 0$ so that $A_j(x) = P(x)$ is a j-independent function.

Case 3. f_0, f_1, f_2 span a 1-dimensional subspace of affine functions in $\hat{\sigma}_1, \dots \hat{\sigma}_{m-2}$. Thus, in this case, $f_1 = \lambda_1 f_0$ and $f_2 = \lambda_2 f_0$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$. The first identity means $q_{r+1} = \lambda_1 q_r$ for $r \in \{0, \dots m-1\}$ whereas the second identity is equivalent to $q_{r+2} = \lambda_2 q_r$ for $r \in \{0, \dots m-2\}$. As we have assumed $q_0 \neq 0$, we conclude that $\lambda_2 = \lambda_1^2$ and then $q_r = \lambda_1^r q_0$, i.e., $q(x) = q_0(1+\lambda_1 x)^m$. It thus follows that either $q(x) = 1$ (i.e., $\lambda_1 = 0$) and then (35) describes the extremal Kähler condition of an orthotoric metric, which has been analysed in [4, Prop. 15]. Otherwise, by making a simultaneous affine change of x_j in (30), we can assume $q(x) = x^m$ and (35) then reduces to finding $(\sigma_m, m+2)$-extremal metrics, which has been accomplished in [11, Prop. A2].

To summarize, we have proven that one of the following holds:

- $q = 1$ and $A_j(x) = P(x) + p_{j1}x + p_{j0}$, for a j-independent polynomial P of degree $\leqslant m+2$;

- up to a simultaneous affine transformation of the x_j in (30), $q(x) = x^m$ and $A_j(x) = P(x) + p_{j1}x^{m+1} + p_{j0}x^{m+2}$ for a (j-independent) polynomial P of degree $\leqslant m+2$;
- $A_j(x) = P(x)$ are all equal to a polynomial P of degree $\leqslant m+2$.

In the third case, the orthotoric Kähler metric (30) is Bochner-flat (see e.g. [4, Prop. 17] or [19]) and any q provides a solution to (35) (see Proposition 1). By result of Webster [63], any CR q-twist of g is again a Bochner-flat Kähler metric, which completes the proof. □

Remark 3. Similar arguments yield a classification of (f_q, ν)-extremal orthotoric metrics, where $\nu \in \mathbb{R} \setminus \{1, \dots m+2\}$ and $m \geqslant 3$. Indeed, as shown in [11], in this case we have to consider the equation

$$-\sum_{j=1}^{m} \frac{f_q(\boldsymbol{x})^{\nu+1}}{\Delta_j} \frac{\partial^2}{\partial x_j^2}\left(\frac{A_j(x_j)}{f_q(\boldsymbol{x})^{\nu-1}}\right) = Scal_{f_q,\nu}(g) = \sum_{k=0}^{m} c_k \mu_k. \tag{37}$$

Multiplying by $x_j - x_k$ and letting $x_j = x = x_k$ leads again to the conclusion that one of the following three cases occurs: (1) $q = 1$ and $A_j(x) = P(x) + p_{j1}x + p_{j0}$ for a polynomial P of degree $\leqslant m$ by the classification in [4, Prop. 15], or, (2) up to a simultaneous affine change of the variables x_j in (30), $q = x^m$ and $A_j(x) = P(x) + p_{j1}x^{\nu-1} + p_{j0}x^{\nu}$ according to [11, Prop. A2], or (3) $A_j(x) = P(x)$ are j-independent. In the third case, multiplying (37) with Δ leads to the equation

$$0 = \frac{\partial^{m+1}}{\partial x_j^{m+1}}\Big(f_q(\boldsymbol{x})^{\nu+1}\frac{\partial^2}{\partial x_j^2}\Big(\frac{P(x_j)}{f_q(\boldsymbol{x})^{\nu-1}}\Big)\Big) = f_q(\boldsymbol{x})^{\nu-m}\frac{\partial^2}{\partial x_j^2}\Big(\frac{P^{(m+1)}(x_j)}{f_q(\boldsymbol{x})^{\nu-m-2}}\Big).$$

Letting $x_j = x$, $f_k = \sum_{r=0}^{m} q_r \hat{\sigma}_{r-k}$ where $\hat{\sigma}_r$ denotes the r-th elementary symmetric function of $x_i, i \neq j$ with $\hat{\sigma}_r = 0$ for $r \geqslant m$, the above conditions reduce to the vanishing of

$$(f_0+xf_1)^2 P^{(m+3)}(x) - 2(\nu-m-2)f_1(f_0+xf_1)P^{(m+2)}(x) + (\nu-m-2)(\nu-m-1)f_1^2 P^{(m+1)}(x).$$

If f_0 and f_1 are linearly independent affine functions of $\hat{\sigma}_1, \dots \hat{\sigma}_{m-1}$ (and as $\nu \neq m+1, m+2$ by assumption), this implies $P^{(m+1)}(x) = 0$, i.e., P must be a polynomial of degree $\leqslant m$ and the metric (30) is flat (see [4, Prop. 17]). These are precisely the solutions described in [11, Prop. A1]. Otherwise, either $f_0 = 0$ (i.e., $q = x^m$) or $f_1 = \lambda f_0$ (i.e., $q(x) = q_0(1+\lambda x)^m$), so we are again in a situation covered by [11, Prop. A2] and [4, Prop. 15].

6. The Calabi problem and non-existence results

6.1. **The Calabi problem for (f,ν)-extremal Kähler metrics.** Let (M, J) be a compact connected complex manifold of real dimension $2m$, and $\Omega \in H^2(M, \mathbb{R})$ a Kähler class. As observed in [10, 36, 45], many features of the theory of extremal Kähler metrics extend naturally to the (f,ν)-extremal case. In particular, one can formulate a weighted version of the Calabi problem [20] which seeks an (f,ν)-extremal Kähler metric (g, ω) with $\omega \in \Omega$. We pin down the function f indirectly by fixing first a quasi-periodic holomorphic vector field with zeroes K generating a torus $\mathbb{T} \leqslant Aut^r(M, J)$ inside the reduced group of automorphisms of (M, J) (see e.g. [38]), and secondly, a real constant $\kappa > 0$ such that for any $\mathbb{T}$-invariant Kähler metric (g, ω) with $\omega \in \Omega$, the Killing potential f of K with respect to g, normalized by $\int_M f\omega^m/m! = \kappa$, is positive on M, see [10, Lemma 1].

Problem 1. Is there a $\mathbb{T}$-invariant Kähler metric $(\tilde{g}, \tilde{\omega})$ on (M, J) with $\tilde{\omega} \in \Omega$, which is $(\tilde{f}, \nu)$-extremal where $\tilde{f}$ is the Kähler potential of K with respect to $\tilde{g}$ determined by $\int_M \tilde{f}\tilde{\omega}^m/m! = \kappa$? We refer to such metrics as $(K, \kappa, m+2)$-extremal.

Remark 4. It is easy to check that if (g, ω) and $(\tilde{g}, \tilde{\omega})$ are two $\mathbb{T}$-invariant Kähler metrics in Ω with Kähler forms $\tilde{\omega} = \omega + dd^c\varphi$ for a $\mathbb{T}$-invariant smooth function φ on M, then the corresponding κ-normalized Killing potentials $\tilde{f}$ and f of K are related by

$$\tilde{f} = f + d^c\varphi(K). \tag{38}$$

Indeed, the proof of [10, Lemma 1] shows that $\tilde{f} = f \circ \Phi$ for some diffeomorphism Φ of M. It thus follows that if f is the κ-normalized Killing potential of K with respect to g, then $\tilde{f}$ is the unique Killing potential of K with respect to $\tilde{g}$ such that $\tilde{f}(M) = f(M)$. The latter property holds for $\tilde{f}$ defined by (38), as can be seen by considering points of minima and maxima for f and $\tilde{f}$ (at which $K = J\,grad_g\, f = J\,grad_{\tilde{g}}\, \tilde{f}$ vanishes).

6.2. **The Calabi problem for (ξ,ν)-extremal Sasaki metrics.** Let N be a compact connected $(2m+1)$-manifold. Following [13, 14], one can extend the Calabi problem to an analogous problem in Sasaki geometry by fixing a nowhere zero vector field X as the candidate for the Sasaki–Reeb vector field of the extremal Sasaki structure on N, together with a complex structure J_X on the quotient bundle $\mathcal{D}_X$ of TN by the span of X. If $(\mathcal{D}, J, \chi)$ is a Sasaki structure with $X_\chi = X$ then $\mathcal{D}$ is transverse to X and so the projection onto $\mathcal{D}_X$ is a bundle isomorphism, and we can require in addition that this isomorphism intertwines J and J_X. The corresponding Sasaki structures on N are completely determined by their contact distributions, or equivalently, their contact forms η, with $\eta(X) = 1$ and $d\eta(X,\cdot) = 0$.

Definition 6. [14] The subspace $\mathcal{S}(X, J_X)$ of $\Omega^1(N)$ whose elements η are contact forms of Sasaki structures compatible with (X, J_X) is called a *Sasaki polarization* of $(N, \mathcal{D}, J, X)$. We also fix a torus $\mathbb{T}$ in the automorphism group $Aut(N, X, J_X)$ and let $\mathcal{S}(X, J_X)^{\mathbb{T}}$ denote the $\mathbb{T}$-invariant elements of $\mathcal{S}(X, J_X)$.

One can now imitate constructions in Kähler geometry by using the basic de Rham complex $\Omega^\bullet_X(N) = \{\alpha \in \Omega^\bullet(N) : \imath_X\alpha = 0 = \mathcal{L}_X\alpha\}$, with differential d_X given by restriction of d (which evidently preserves basic forms). Since $\{\alpha \in \wedge^k T^*N : \imath_X\alpha = 0\}$ is naturally isomorphic to $\wedge^k \mathcal{D}_X^*$, J_X^* is also well defined on $\Omega^\bullet_X(N)$, yielding a twisted differential d^c_X. The same holds for the subspace $\Omega^\bullet_X(N)^{\mathbb{T}}$ of $\mathbb{T}$-invariant basic forms. Following [14, Lemma 3.1] and [13, Prop. 7.5.7], $\mathcal{S}(X, J_X)^{\mathbb{T}}$ is an open subset of an affine space modelled on $C^\infty_{N,0}(\mathbb{R})^{\mathbb{T}} \times \Omega^1_{X,\mathrm{cl}}(N)^{\mathbb{T}}$, where $C^\infty_{N,0}(\mathbb{R})^{\mathbb{T}}$ denotes the quotient by constants of the space of smooth $\mathbb{T}$-invariant functions on N, and $\Omega^1_{X,\mathrm{cl}}(N)^{\mathbb{T}}$ denotes the basic $\mathbb{T}$-invariant closed 1-forms on N. Indeed, for any two elements $\eta, \tilde{\eta} \in \mathcal{S}(X, J_X)^{\mathbb{T}}$, $\tilde{\eta} - \eta$ is basic and so

$$\tilde{\eta} = \eta + d^c_X\varphi + \alpha \tag{39}$$

for a $\mathbb{T}$-invariant smooth function φ, and a basic $\mathbb{T}$-invariant closed 1-form α. It follows from (39) that the induced Kähler forms on local quotients (M, J) of N by X are linked by $\tilde{\omega} = \omega + dd^c\varphi$, i.e., belong to the same Kähler class. Moreover any K in the Lie algebra of $\mathbb{T}$ is CR for both CR structures $(\mathcal{D}, J)$ and $(\tilde{\mathcal{D}}, \tilde{J})$ induced by η and $\tilde{\eta}$, and hence induces on any such M a Killing vector field, also denoted K, for both ω and $\tilde{\omega}$, with respective Killing potentials pulling back to $f = \eta(K)$ and $\tilde{f} = \tilde{\eta}(K) = f + d^c\varphi(K) + \alpha(K)$. Notice that by the $\mathbb{T}$-invariance and closedness of α, the term $\alpha(K)$ is a constant.

Lemma 6. *Let $(N, \mathcal{D}, J)$ be a compact CR manifold and $\chi, \xi \in \mathfrak{cr}_+(\mathcal{D}, J)$ with $[\xi, \chi] = 0$. Let $X = X_\chi$, $K = X_\xi$. Then for any $\tilde{\eta} \in \mathcal{S}(X, J_X)$ with $\mathcal{L}_K\tilde{\eta} = 0$, K is a Sasaki–Reeb vector field for the induced CR structure $(\tilde{\mathcal{D}}, \tilde{J})$.*

Proof. As K is a CR vector field by construction, we need to check that $\tilde{f} := \tilde{\eta}(K) > 0$. We let $\eta \in \mathcal{S}(X, J_X)$ be the contact form of $(\mathcal{D}, \chi)$. Since K is contact with respect to $\tilde{\mathcal{D}} := \ker\tilde{\eta}$, we have $K = \tilde{f}X - (d\tilde{\eta}|_{\tilde{\mathcal{D}}})^{-1}(d\tilde{f}|_{\tilde{\mathcal{D}}})$ and hence

$$\eta(K) = \tilde{f} - (d\tilde{\eta}|_{\tilde{\mathcal{D}}})^{-1}(d\tilde{f}|_{\tilde{\mathcal{D}}}, \eta|_{\tilde{\mathcal{D}}})$$

Evaluating this relation at a global minimum p of $\tilde{f}$ we obtain $\tilde{f}(p) = \eta(K)(p) > 0$. $\square$

We now specialize the above set-up. First, we fix a nowhere zero vector K and let $\mathbb{T}$ be the torus in $Aut(N, X, J_X)$ generated by X and K. In addition, following [37], we fix a basepoint $\eta \in \mathcal{S}(X, J_X)^{\mathbb{T}}$ with corresponding Sasaki structure $(\mathcal{D}, J, \chi)$, and restrict attention to the affine slice of $\mathcal{S}(X, J_X)^{\mathbb{T}}$ consisting of contact forms $\tilde{\eta}$ related to η by (39) with $\alpha = 0$. We may thus identify this slice with

$$\Xi(X, J_X)^{\mathbb{T}} := \{\varphi \in C^{\infty}_{N,0}(\mathbb{R})^{\mathbb{T}} \,|\, \eta_\varphi := \eta + d^c_X \varphi \text{ is a contact form}\} \tag{40}$$

We also write $(\mathcal{D}_\varphi, J_\varphi, \chi_\varphi)$ for the Sasaki structure induced by η_φ for $\varphi \in \Xi_K(X, J_X)^{\mathbb{T}}$, and let $\xi_\varphi = \eta_{\mathcal{D}_\varphi}(K)$ and $\xi = \eta_{\mathcal{D}}(K)$. In view of Lemmas 2 and 6, we now have an analogue of Problem 1 for (ξ, ν)-extremal Sasaki metrics.

Problem 2. Given a compact CR manifold $(N, \mathcal{D}, J)$ of Sasaki type and $\chi, \xi \in \mathfrak{cr}_+(N, \mathcal{D}, J)$ with $[\chi, \xi] = 0$, is there $\varphi \in \Xi(X, J_X)^{\mathbb{T}}$ such that $(\mathcal{D}_\varphi, J_\varphi, \chi_\varphi)$ is (ξ_φ, ν)-extremal?

In the case $\chi = \xi$, Problem 2 reduces to the search for extremal Sasaki metrics in a given Sasaki polarization, see [14], which has been studied in many places, see e.g. [14, 18, 26, 52, 56, 57, 62]. We notice that Problems 1 and 2 are naturally linked in the regular case, via Examples 1, 2 and Remark 4. Indeed, the parametrization (40) implies that for any $\varphi \in \Xi(X, J_X)^{\mathbb{T}}$, the Kähler potentials $\tilde{f} = \eta_\varphi(K)$ and $f = \eta(K)$ of K are linked on a Sasaki–Reeb quotient (M, J) via $\tilde{f} = f + d^c\varphi(K)$, which is consistent with (38).

Remarks 5. (i) By the equivariant Gray–Moser theorem and Lemma 6 above, for each $\varphi \in \Xi(X, J_X)^{\mathbb{T}}$ the corresponding contact form $\eta_\varphi \in \mathcal{S}(X, J_X)^{\mathbb{T}}$ is equivalent to η by an identity component $\mathbb{T}$-equivariant diffeomorphism Φ of N. Pulling back J_φ by Φ gives a CR structure $J_{\varphi,\eta}$ in the space $\mathcal{C}_+(N, \mathcal{D})^{\mathbb{T}}$ of $\mathbb{T}$-invariant $\mathcal{D}$-compatible CR structures on $(N, \mathcal{D})$ introduced in Section 2, where $\mathbb{T} \leqslant Con(N, \mathcal{D})$ is the torus generated by the CR vector fields X_χ and X_ξ. Thus, Problem 2 can be viewed as a special case of the problem of finding critical points in $\mathcal{C}_+(N, \mathcal{D})^{\mathbb{T}}$ for the square norm of the momentum map of the $Con(N, \mathcal{D})^{\mathbb{T}}$-action on $\mathcal{AC}_+(N, \mathcal{D})^{\mathbb{T}}$ defined in Section 2.

(ii) In view of Theorem 1, one may ask whether, given $(N, \mathcal{D}, J, \chi, \xi)$ as in Problem 2, there exists a $(\xi_\varphi, m + 2)$-extremal solution $\varphi \in \Xi(X, J_X)^{\mathbb{T}}$ iff there exists an extremal Sasaki structure with contact form $\eta_\varphi \in \mathcal{S}(K, J_K)^{\mathbb{T}}$. This would be useful for studying irregular extremal Sasaki structures by taking X quasi-regular and K irregular. However, it is not clear how to relate the spaces $\Xi(X, J_X)^{\mathbb{T}}$ and $\mathcal{S}(K, J_K)^{\mathbb{T}}$, since J_φ is the lift of J_K to $\mathcal{D}_\varphi$ and its projection onto $TN/\,span(X)$ does not agree with J_X in general, and so the pullback $J_{\varphi,\eta} = \Phi^* J_\varphi$ with $\Phi^*\eta_\varphi = \eta$ need not descend to a complex structure in the same Teichmüller class as J on the quotient of N by X.

6.3. **Extremal Sasaki structures from ruled complex surfaces.** We now specialize to geometrically ruled complex surfaces and the regular Sasaki manifolds they define. Let $(M, J) = \pi\colon P(\mathcal{O} \oplus \mathcal{L}) \to B$ be the underlying complex manifold of a projective $\mathbb{C}P^1$-bundle over a compact Riemann surface B, where $\mathcal{L}$ is a holomorphic line bundle over B of positive degree ℓ. Let K be the generator of the holomorphic $\mathbb{S}^1$-action on (M, J) induced by scalar multiplication in $\mathcal{O}$. We denote by (g_B, ω_B) the Kähler metric on B of constant scalar curvature $4(1 - \boldsymbol{g})$, where $\boldsymbol{g}$ denotes the genus of B. It is well-known (see e.g. [8]) that the Kähler cone of (M, J) can be parametrized up to homothety by the cohomology classes of Kähler metrics (g, ω) given by the Calabi ansatz (16) as described in Section 4.5, with $a_1 = \ell$. For convenience, in this section, we let a denote the real constant a_0/ℓ and write z for z_1 and μ for μ_1, so that $a_0 + a_1\mu_1 = \ell(\mu + a)$.

For each $b \in \mathbb{R}$ with $|b| > 1$, $f_b := \mu + b$ is a positive Killing potential for K on (M, g, ω). The existence of a $(f_b, 4)$-extremal Kähler metric of the form (16) on M (up to homothety) is studied in [11, 44, 50]. It is shown there (see e.g. [11, Thm. 1]) that such a metric must be obtained from a smooth function $A(z) = P_{a,b}(z)/(z+a)$, where $P_{a,b}$ is a polynomial of degree $\leqslant 4$ uniquely determined from (19) in terms of a, b and ℓ; thus(M, g, J, ω) is $(f_b, 4)$-extremal iff $P_{a,b}(z)$ satisfies the positivity condition (20). Conversely, we have the following result.

Proposition 7. *Let $M = P(\mathcal{O} \oplus \mathcal{L}) \to \mathbb{C}P^1$ be a ruled surface and $\Omega = \lambda[\omega]$ a Kähler class on M for $\lambda > 0$, $a > 1$. Let $|b| > 1$, $f_b = \mu + b$ and $\kappa = \frac{\lambda^3}{2}\int_M f_b\,\omega^2$. If the polynomial $P_{a,b}(z)$ is not positive on $(-1,1)$, Ω contains no $(K, \kappa, 4)$-extremal Kähler metrics.*

Proof. The proof follows from a slight modification of the arguments in [46, Cor. 1], taking into account the recent result [47, Cor. 1]. Indeed, suppose for contradiction that $P_{a,b}(p_0) = 0$ for some $p_0 \in (-1,1)$, and that $[\omega]$ admits a $(K, \kappa, 4)$-extremal Kähler metric.

Consider first the case that $P_{a,b}(z)$ is negative somewhere on $(-1,1)$. If the Kähler class $[\omega/2\pi]$ is rational (which is equivalent to $a \in \mathbb{Q}$), we derive a contradiction by [47, Cor. 1] (which implies that the relative weighted Mabuchi functional must be bounded from below) and [11, Prop. 2.7] (which concludes otherwise). If the Kähler class $[\omega/2\pi]$ is not rational, we can approximate it with rational classes $[\tilde{\omega}/2\pi]$ of the form (21) by taking rational values $\tilde{a}$ close to a, and still ensure that $P_{\tilde{a},b}(z)$ is negative somewhere on $(-1,1)$. Furthermore, by the openness of weighted extremal classes established in [45, Thm. 2], we can assume that $[\tilde{\omega}]$ admits a $(K, \tilde{\kappa}, 4)$-extremal Kähler metric with $\tilde{\kappa} = \frac{1}{2}\int_M f_b\,\tilde{\omega}^2$. We get a contradiction as before.

It thus remains to consider the case when $p_0 \in (-1,1)$ is a double root of the quadratic $P_{a,b}(z)/(1-z)^2$. In this case, we prove that there exists a sequence $\tilde{b}_i$ converging to b, such that $P_{a,\tilde{b}_i}(p_0) < 0$; by the openness result in [45, Thm. 2], we can then find $\tilde{b}$ with $P_{a,\tilde{b}}(p_0) < 0$ and such that the Kähler class $[\omega]$ admits a $(K, \tilde{\kappa}, 4)$-extremal Kähler metric with $\tilde{\kappa} = \frac{1}{2}\int_M f_{\tilde{b}}\,\omega^2$, a situation we have already ruled out.

In order to find a sequence as above, it is enough to show that $\frac{\partial P_{a,b}}{\partial b}(p_0) \neq 0$ at each double root $p_0 \in (-1,1)$. The remainder of the proof establishes this technical fact.

If $a = b$, we get the natural solution of Corollary 3, in which case $P_{a,b}(z) > 0$ on $(-1,1)$. We can thus assume that $a \neq b$, and then, adapting [44, (11) & (31)] to our notation, the polynomial $P_{a,b}$ is given by

$$\frac{2P_{a,b}(z)}{1-z^2} = 2(z+a) + (1-z^2)\frac{3c_{a,b} + a + s}{3c_{a,b}^2 - 1}, \tag{41}$$

where $s = 2(1-\boldsymbol{g})/\ell$ and $c_{a,b} = (ab-1)/(a-b)$.

Suppose that $\frac{\partial P_{a,b}}{\partial b}(p_0) = 0$ for $p_0 \neq \pm 1$. We compute that this is equivalent to

$$2c_{a,b}\,s + 3c_{a,b}^2 + 2a\,c_{a,b} + 1 = 0.$$

Since $c_{a,b} \neq 0$, we may solve for s and substitute back in (41) to obtain

$$\frac{2P_{a,b}(z)}{1-z^2} = \frac{1 + 4a\,c_{a,b} + 4c_{a,b}\,z - z^2}{2c_{a,b}}.$$

Now if $P_{a,b}$ has a double root at $p_0 \neq \pm 1$, we must have $p_0 = 2c_{a,b} = 2(ab-1)/(a-b)$, the critical point of this expression. However, $(ab-1)^2 - (a-b)^2 = (a^2-1)(b^2-1) > 0$ for $|b| > 1$ and $a > 1$, so $|p_0| > 2$ and hence $p_0 \notin (-1,1)$. □

It is shown in [11, Prop. 2.12] that for a rational Kähler class of the form (21), and for $z \in (-1,1) \cap \mathbb{Q}$, $P_{a,b}(z)$ computes a weighted notion of the relative Donaldson–Futaki invariant associated to the polarized variety $(M, L_{k,n})$, where $L_{k,n}$ is a polarization on M corresponding to $a = a_0/\ell = (2n/k\ell) - 1$ as explained in Section 4.5. This motivates the following definition.

Definition 7. [11] Let $(M, L_{k,n})$ be a polarized ruled surface as above, and $\hat{Z}_b$ the quasi-periodic (real) holomorphic vector field on $L_{k,n}$, given by the lift of K with respect to the potential f_b. We say that $(M, L_{k,n}, \hat{Z}_b)$ is *analytically relatively* $(\hat{Z}_b, 4)$ K-*stable with respect to admissible test configurations* if $P_{a,b}(z) > 0$ on $(-1, 1)$.

Thus, Proposition 7 implies that that (M, J) admits a $(K, \kappa, 4)$-extremal Kähler metric in $2\pi c_1(L_{k,n})$ iff $(M, L_{k,n}, \hat{Z}_b)$ is analytically relatively $(\hat{Z}_b, 4)$ K-stable with respect to admissible test configurations.

6.4. **Proof of Theorem 2.** Let $(N_{k,n}, \mathcal{D}, \chi)$ be a compact regular contact manifold over a ruled surface M constructed in Section 4.5, and $\xi_b \in \mathfrak{con}(N_{k,n}, \mathcal{D})$ the contact lift of the generator K of the $\mathbb{S}^1$-action on M via potential $\mu + b$. We denote by $\mathbb{T} \leqslant \mathfrak{con}(N_{k,n}, \mathcal{D})$ the torus generated by χ and ξ_b, and let $P_{a,b}$ be the polynomial corresponding to the Kähler class (21) with $a_0 = (2n/k) - \ell$ and $a_1 = \ell$. We need to show that there exists a $(\xi_b, 4)$-extremal CR structure $J \in \mathcal{C}_+(N_{k,n}, \mathcal{D})^{\mathbb{T}}$ if and only if $P_{a,b}(z) > 0$ on $(-1, 1)$.

Existence follows from Theorem 1 and the fact that when $P_{a,b}(z) > 0$ on $(-1, 1)$, $A(z) = P_{a,b}(z)/(z + a)$ in (16) defines an $(f_b, 4)$-extremal Kähler metric on M (see [11, Thm. 1]).

We now establish the non-existence claim. Suppose that $P_{a,b}(z)$ has a zero on $(-1, 1)$. Denote by J the CR-structure in $\mathcal{C}_+(N_{k,n}, \mathcal{D})^{\mathbb{T}}$ induced by a Kähler metric (g, ω) on M of the form (16) and suppose for contradiction that there exists a $\mathcal{D}$-compatible CR structure $J' \in \mathcal{C}_+(N_{k,n}, \mathcal{D})^{\mathbb{T}}$ such that $(\mathcal{D}, J', \xi_b)$ is an extremal Sasaki structure and $P_{a,b}(z)$ has a zero on $(-1, 1)$. As J' is X_{ξ_b}-invariant, on M we obtain another ω-compatible complex structure J' which is invariant under the $\mathbb{S}^1$-action generated by K. As any compact Kähler 4-manifold admitiing a holomorphic vector field with zeroes must be a rational or a ruled surface [23], (M, J') must be a ruled surface too, i.e., $(M, J') = P(\mathcal{O} \oplus \mathcal{L}') \to B'$ with B' having the same genus as B. Intersection properties of the zero set of K (which equals the zero and infinity sections in either case) yield that $\deg(\mathcal{L}) = \deg(\mathcal{L}')$. Since J and J' are compatible with the same symplectic form ω, the corresponding class $[\omega]$ is of the form (21) on either surface, with the same parameter $a = a_0/\ell > 1$; furthermore, the Killing potential of K induced by ξ_b in both cases is $f_b = \mu + b$ for the same $b > 1$. It thus follows that the respective polynomials associated to $[\omega]$ on each ruled surface (M, J) and (M, J') coincide; we denote them by $P_{a,b}$.

Now, by Theorem 1, the Kähler class $[\omega]$ on (M, J') admits a $(K, \kappa, 4)$-extremal Kähler metric which, by Proposition 7, forces $P_{a,b}(z) > 0$ on $(-1, 1)$, a contradiction. □

Remarks 6. (i) Theorem 2 yields a $(\xi_b, 4)$-extremal Sasaki metric on the Sasaki join $(N_{\mathrm{w},k}, \mathcal{D})$ over $B \times \mathbb{C}P^1_{\mathrm{w}}$ (with weights $w_+ = n, w_- = n - k\ell$, where $a = (2n/k\ell) - 1$) if $P_{a,b}(z) > 0$ on $(-1, 1)$. This always happens when B has genus 0 or 1, or when a is sufficiently large (see e.g. [11, Thm. 1]). These extremal Sasaki structures are not new (see [16, 17]) but we have seen in Section 4.5 that they can equivalently be obtained from a $(\tilde{f}_b, 4)$-extremal product Kähler metric $g_B + p g_{\mathrm{w}}$ on $B \times \mathbb{C}P^1_{\mathrm{w}}$. Taking the limit $b \to \infty$ also yields the extremal Kähler metrics on the ruled surfaces constructed in [20, 61] as a CR twist of a product metric on $B \times \mathbb{C}P^1_{\mathrm{w}}$.

(ii) To the best of our knowledge, the non-existence result obtained via Theorem 2 is new, at least for irrational values of b (in which case ξ_b generates $\mathbb{T}$, and thus is not quasi-regular). To construct specific examples, following [44], on any ruled surface M over a curve of genus $\geqslant 2$ as above, there exists an explicit $a_0(M) > 1$ such that for any $a \in (1, a_0(M)]$ the polynomial $P_{a,b_a}(z)$ has a zero on $(-1, 1)$, where $b = b_a > 1$ is the unique solution of $a = \frac{1+b^2}{2b}$ satisfying $|b| > 1$. Taking coprime positive integers k, n with $a = (2n/k\ell) - 1 \leqslant a_0(M)$, we obtain a contact manifold $(N_{k,n}, \mathcal{D})$ which admits no $(\xi_b, 4)$-extremal CR structure $J \in \mathcal{C}_+(N_{k,n}, \mathcal{D})^{\mathbb{T}}$.

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