

# CONTINUED FRACTIONS AND EINSTEIN MANIFOLDS OF INFINITE TOPOLOGICAL TYPE

DAVID M. J. CALDERBANK AND MICHAEL A. SINGER

ABSTRACT. We present a construction of complete self-dual Einstein metrics of negative scalar curvature on an uncountable family of manifolds of infinite topological type.

## 1. INTRODUCTION

1.1. **Summary.** Let  $\alpha$  be an irrational number,  $0 < \alpha < 1$ , and consider the *modified continued fraction expansion* of  $\alpha$ ,

$$\alpha = \frac{1}{e_1 - \frac{1}{e_2 - \frac{1}{e_3 - \dots}}}, \quad (1.1)$$

with  $e_j \geq 2$  for all  $j$ . We associate to  $\alpha$  a noncompact 4-manifold  $M_\alpha$ , which is connected and simply connected, but has  $b_2(M_\alpha) = \infty$ ,  $H_2(M_\alpha)$  being generated by an infinite sequence of embedded 2-spheres  $S_j$ , with

$$S_j \cdot S_{j+1} = -1, \quad S_j \cdot S_j = e_j, \quad S_j \cdot S_k = 0 \text{ for } |j - k| > 1. \quad (1.2)$$

Then, subject to the uniform bound  $3 \leq e_j \leq N$ , we construct a complete self-dual Einstein metric  $g_\alpha$ , of negative scalar curvature, on  $M_\alpha$ . The key point about the construction is that  $M_\alpha$  and  $g_\alpha$  are toric: there is a smooth action of  $T^2$  on  $M_\alpha$  which preserves  $g_\alpha$ .

The work in this paper complements that in our previous papers [3] and [4]. Indeed, in [3], we associated to each *rational* number  $\alpha$ ,  $0 < \alpha < 1$ , a noncompact 4-manifold  $M_\alpha$  and, subject to the condition  $e_j \geq 3$ , we constructed a complete SDE metric  $g_\alpha$  on  $M_\alpha$ , where, as above,  $M_\alpha$  is connected and simply connected, but now  $H_2(M_\alpha)$  is generated by a *finite* sequence of spheres  $S_j$  which satisfy (1.2).

We note a parallel in hyperkähler geometry: the  $A_n$ -gravitational instantons constructed by the Gibbons–Hawking Ansatz are analogous to the case that  $\alpha$  is rational, while the hyperkähler manifolds of infinite topological type in [1] correspond to  $\alpha$  being irrational. But there are very important differences: in the hyperkähler case, the self-intersections  $e_j$  all have to be 2, which is complementary to the condition  $e_j \geq 3$  that we impose here. On the other hand, despite the uniform bound  $e_j \leq N$ , we obtain uncountably many non-diffeomorphic SDE manifolds in this way—for this, it is enough to allow the  $e_j$  to take only the values 3 and 4, for example.

The plan of this paper is as follows. In §2 we gather some elementary facts about the continued fraction expansion (1.1). The only result that may be new here is Theorem 2.9; we are grateful to Chris Smyth for assisting us with its proof. In §3, we give the construction of  $M_\alpha$ . This is a straightforward extension of the work in [3], which, as we have indicated, corresponds to the case that  $\alpha$  is rational. (That work, in turn, rests on the combinatorial description of toric 4-manifolds due to Orlik and Raymond [9].) In §4 we write down a SDE metric  $g_\alpha$  on  $M_\alpha$ . This is defined initially on a dense open subset  $U \subset M_\alpha$  (the set on

which the  $T^2$ -action is free) but at the end of the section we show that  $g_\alpha$  extends smoothly to the whole manifold. The argument is given in detail partly to make this paper more self-contained, and also to clarify one point omitted from [3]<sup>1</sup>. In §5 we give some technical estimates on the functions which enter the definition of  $g_\alpha$ . These are needed for the smooth extension of  $g_\alpha$  from  $U$  to  $M_\alpha$ , and also pave the way for the proof that  $g_\alpha$  is complete in §6.

For the reader familiar with [3], we make some remarks about the extension of the construction of  $g_\alpha$  from finite to infinite continued fractions. From the point of view of [3, §4], one would like to allow  $k$  to go to  $\infty$  in

$$\sqrt{\rho}F(\rho, \eta) = \sum_{j=0}^{k+1} w_j \sqrt{\rho^2 + (\eta - y_j)^2},$$

where

$$w_j = m_{j+1} - m_j, \quad y_j = \frac{n_{j+1} - n_j}{m_{j+1} - m_j}$$

and the  $n_j/m_j$  are the continued fraction approximants to  $\alpha$ . But the sequence  $w_j$  increases rapidly with  $j$ , while the  $y_j$  also converge to  $\alpha$ , so the status of such a limit is unclear. However, we noted in [3, §5] that the above sum represents an eigenfunction of the hyperbolic laplacian with boundary data equal to  $\eta \mapsto m_j\eta - n_j$  for  $y_j \leq \eta \leq y_{j-1}$  and this makes perfectly good sense also for infinite continued fractions. This observation is really the key to the construction in this paper.

**1.2. Notation.** Denote by  $\mathcal{H}^2$  the hyperbolic plane, by  $\overline{\mathcal{H}^2}$  its conformal compactification. We shall always identify  $\mathcal{H}^2$  with the upper half-plane  $\{(x, y) \in \mathbb{R}^2 : y > 0\}$ ; then the hyperbolic metric is given by  $(dx^2 + dy^2)/y^2$  and  $\overline{\mathcal{H}^2} = \{(x, y) : y \geq 0\} \cup \{\infty\}$ . Note that half-space coordinates near  $\infty$  can be defined by

$$(\tilde{x}, \tilde{y}) = \left( -\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right).$$

Further, denote by  $T^2$  the standard 2-torus, identified with  $\mathbb{R}^2/(2\pi\mathbb{Z})^2$ . We shall write  $z = (z_1, z_2)$  for standard linear coordinates on  $\mathbb{R}^2$ . The circle-subgroup generated by  $m\partial_{z_1} + n\partial_{z_2}$  (for  $m, n \in \mathbb{Z}$ ) will be denoted by  $S_{(m,n)}^1$ . We shall denote by  $\varepsilon$  the standard skew form

$$\varepsilon(z', z'') = \det(z', z'') = z'_1 z''_2 - z'_2 z''_1. \quad (1.3)$$

**1.3. Acknowledgement.** We thank Chris Smyth for useful conversations on continued fractions and for the proof of Theorem 2.9. We also thank Jim Wright for useful conversations.

## 2. CONTINUED FRACTIONS AND TORIC 4-MANIFOLDS

**2.1. Continued fractions.** Let us begin with our irrational number  $\alpha$  and its continued fraction expansion (1.1). Set

$$(m_0, n_0) = (0, -1), \quad (m_1, n_1) = (1, 0), \quad (2.1)$$

and

$$(m_j, n_j) \text{ coprime with } \frac{n_j}{m_j} = \frac{1}{e_1 - \frac{1}{e_2 - \cdots - \frac{1}{e_{j-1}}}}. \quad (2.2)$$

---

<sup>1</sup>For readers of that paper, we did not show in [3, §5] that  $\lim_{\rho \rightarrow 0} \rho^{-1} |\det \Phi(\rho, \eta)| > 0$ , and this is needed for the smooth extension of the metric.

The most important properties of this sequence of pairs  $(m_j, n_j)$  are summarized as follows.

**2.2. Lemma.** For each  $j \geq 1$ ,

$$(m_{j+1}, n_{j+1}) = e_j(m_j, n_j) - (m_{j-1}, n_{j-1}), \quad (2.3)$$

and for  $j \geq 0$ ,

$$m_j n_{j+1} - m_{j+1} n_j = 1. \quad (2.4)$$

*Proof.* If

$$M_j = \begin{pmatrix} 0 & 1 \\ -1 & e_j \end{pmatrix} \quad (2.5)$$

then one has, for each  $j$ :

$$\begin{pmatrix} n_{j-1} \\ m_{j-1} \end{pmatrix} = M_1 M_2 \cdots M_{j-1} \begin{pmatrix} -1 \\ 0 \end{pmatrix}; \quad (2.6)$$

$$\begin{pmatrix} n_j \\ m_j \end{pmatrix} = M_1 M_2 \cdots M_{j-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \quad (2.7)$$

$$\begin{pmatrix} n_{j+1} \\ m_{j+1} \end{pmatrix} = M_1 M_2 \cdots M_{j-1} \begin{pmatrix} 1 \\ e_j \end{pmatrix}. \quad (2.8)$$

These are all essentially equivalent to each other and are easily proved by induction. Combining these, we have

$$\begin{pmatrix} n_{j+1} & n_j \\ m_{j+1} & m_j \end{pmatrix} = M_1 M_2 \cdots M_{j-1} \begin{pmatrix} 1 & 0 \\ e_j & 1 \end{pmatrix}, \quad \begin{pmatrix} n_{j+1} & n_{j-1} \\ m_{j+1} & m_{j-1} \end{pmatrix} = M_1 M_2 \cdots M_{j-1} \begin{pmatrix} 1 & -1 \\ e_j & 0 \end{pmatrix}, \quad (2.9)$$

and, taking determinants, we obtain (2.4) and the identity

$$m_{j-1} n_{j+1} - m_{j+1} n_{j-1} = e_j. \quad (2.10)$$

From (2.4) (with  $j$  replaced by  $j - 1$ ) it follows that  $(m_{j-1}, n_{j-1})$  and  $(m_j, n_j)$  form a  $\mathbb{Z}$ -basis of  $\mathbb{Z} \oplus \mathbb{Z}$ , so that  $(m_{j+1}, n_{j+1})$  is an integer linear combination of these vectors. The coefficients are determined as in (2.3) by using the identities (2.4) and (2.10).  $\square$

**2.3. Definition.** Set

$$a_j = \frac{n_{j+1} - n_j}{m_{j+1} - m_j}, \quad b_j = \frac{1}{m_{j+1} - m_j}. \quad (2.11)$$

These are well-defined by the following lemma.

**2.4. Lemma.** (i) For all  $j \geq 1$ ,  $m_{j+1} > m_j$ ,  $n_{j+1} > n_j$ , and  $(n_j/m_j)$  is strictly increasing with limit  $\alpha$ . Hence the sequences  $(a_j)$  and  $(b_j)$  are positive. Furthermore,  $\lim_{j \rightarrow \infty} a_j = \alpha$ .

(ii) Suppose  $e_j \geq 3$  for all  $j$ . Then  $m_{j+1} > \varphi^2 m_j$  and  $n_{j+1} > \varphi^2 n_j$ , where  $\varphi = (1 + \sqrt{5})/2 > 1$  is the golden ratio. Furthermore,  $(a_j)$  and  $(b_j)$  are strictly decreasing, and  $\lim_{j \rightarrow \infty} b_j = 0$ .

*Proof.* (i) To prove that the sequences  $(m_j)$  and  $(n_j)$  are strictly increasing, use (2.3) and induction on  $j$ . In particular  $m_j > 0$  for  $j > 0$ . The monotonicity of the sequence  $n_j/m_j$  now follows from the identity

$$\frac{n_{j+1}}{m_{j+1}} - \frac{n_j}{m_j} = \frac{1}{m_j m_{j+1}}, \quad (2.12)$$

where we have used (2.4). The fact that  $n_j/m_j \rightarrow \alpha$  as  $j \rightarrow \infty$  is standard. Now note that

$$a_j - \frac{n_j}{m_j} = \frac{1}{m_j(m_{j+1} - m_j)} > 0 \quad (2.13)$$

so that  $\lim_{j \rightarrow \infty} a_j = \lim_{j \rightarrow \infty} n_j/m_j = \alpha$  as required.

(ii) We again use (2.3) and induction on  $j$ : e.g., if  $m_j > \varphi^2 m_{j-1}$ , then

$$m_{j+1} = e_j m_j - m_{j-1} > (3 - \varphi^{-2}) m_j = \varphi^2 m_j,$$

since  $\varphi^2 = (3 + \sqrt{5})/2$ . Now from (2.3) and the nonnegativity of  $(m_j)$ ,

$$m_{j+1} - m_j = (e_j - 1)m_j - m_{j-1} \geq (e_j - 1)(m_j - m_{j-1}).$$

Since  $e_j - 1 > 1$ , the statements about  $(b_j)$  are immediate. On the other hand,

$$a_{j-1} - a_j = \frac{e_j - 2}{(m_{j+1} - m_j)(m_j - m_{j-1})} > 0 \quad (2.14)$$

which gives the monotonicity of the  $a_j$ .  $\square$

**2.5. Assumption.** From now on, suppose that for all  $j$ ,  $3 \leq e_j \leq N$  for some  $N > 0$ .

**2.6. Definition.** The *envelope*  $\eta_\alpha$  of  $\alpha$  is defined as

$$\eta_\alpha(x) = \begin{cases} m_j x - n_j & \text{for } x \in [a_j, a_{j-1}], \\ 0 & \text{for } x \leq \alpha. \end{cases} \quad (2.15)$$

(Here we set  $a_{-1} = \infty$ , so that  $\eta_\alpha(x) = 1$  for  $x \geq a_0 = 1$ .)

**2.7. Proposition.**  $\eta_\alpha$  is continuous, and for  $x > \alpha$  it is strictly increasing (hence positive), concave (i.e.,  $\eta_\alpha'' \leq 0$  in the sense of distributions) and is the linear interpolant of the points  $(a_j, b_j)$  (so that  $\eta_\alpha(a_j) = b_j$  for all  $j$ ).

*Proof.* Trivial, given the previous results.  $\square$

**2.8. Example.** Suppose  $e_j = 3$  for all  $j$ . Then

$$m_j = n_{j+1} = \frac{\varphi^{2j} - \varphi^{-2j}}{\sqrt{5}}, \quad a_j = \frac{\varphi^{2j-1} + \varphi^{-2j+1}}{\varphi^{2j+1} + \varphi^{-2j-1}}, \quad b_j = \frac{\sqrt{5}}{\varphi^{2j+1} + \varphi^{-2j-1}},$$

where  $\varphi = (1 + \sqrt{5})/2$  as above. Then  $\alpha = \varphi^{-2} = 0.381966\dots$  and

$$\frac{a_j - \varphi^{-2}}{b_j^2} = \frac{1}{\sqrt{5}} \left( \frac{\varphi^{4j+2} + 2 + \varphi^{-4j-2}}{\varphi^{4j+2} + 1} \right).$$

In particular, by Proposition 2.7,

$$\eta_\alpha(x) \lesssim \sqrt{\sqrt{5}(x - \varphi^{-2})} \quad (2.16)$$

for  $x - \varphi^{-2}$  small. The two functions in (2.16) are shown in Figure 1.

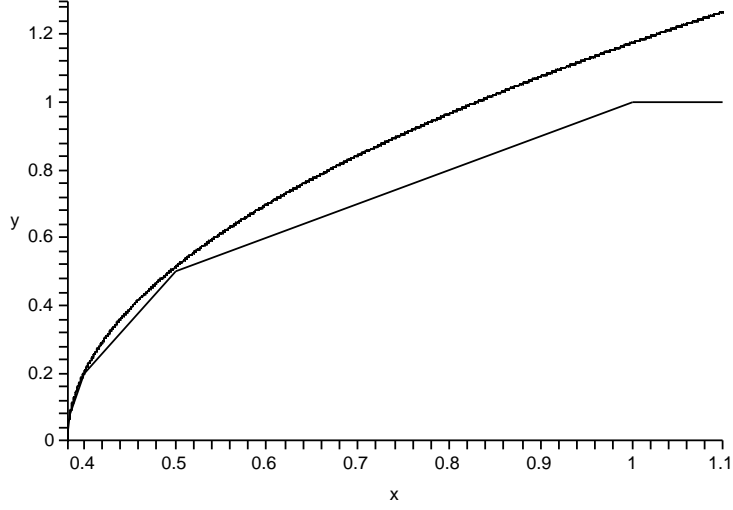
We shall now show that the behaviour of  $\eta_\alpha(x)$  is bounded by a multiple of  $\sqrt{x - \alpha}$  in general—we are indebted to Chris Smyth for the proof of this fact.

**2.9. Theorem.** *There is a constant  $\Omega > 0$  such that*

$$\eta_\alpha(x) \leq \Omega \sqrt{x - \alpha} \text{ for } x \geq \alpha. \quad (2.17)$$

*Proof.* This is a sequence of elementary deductions, based on Lemmas 2.2 and 2.4. Start by substituting  $m_{j+1} = e_j m_j - m_{j-1}$  into (2.14), to get

$$a_{j-1} - a_j = \left[ \frac{e_j - 2}{(1 - m_{j-1}/m_j)(e_j - 1 - m_{j-1}/m_j)} \right] \frac{1}{m_j^2}. \quad (2.18)$$

FIGURE 1. The envelope and the square-root function with  $\alpha = \varphi^{-2}$ 

Because  $0 < m_{j-1}/m_j < \varphi^{-2} < 1/2$  and  $e_j \geq 3$  it is elementary to bound the quantity in square brackets between  $1/2$  and  $2$ . Hence

$$\frac{1}{2m_j^2} < a_{j-1} - a_j < \frac{2}{m_j^2}. \quad (2.19)$$

Now

$$a_n - \alpha = \sum_{j=n+1}^{\infty} (a_{j-1} - a_j) \quad (2.20)$$

and since  $m_j > 2m_{j-1}$  for all  $j$ , we obtain

$$\frac{1}{2m_{n+1}^2} < a_n - \alpha < \frac{2}{m_{n+1}^2} \sum_{j=0}^{\infty} \varphi^{-4j} < \frac{3}{m_{n+1}^2}. \quad (2.21)$$

(The lower bound is the trivial one coming from the first term in the sum.)

Since  $b_{j-1} - b_j = m_j(a_{j-1} - a_j)$  we find from (2.19)

$$\frac{1}{2m_{j+1}} < b_{j-1} - b_j < \frac{2}{m_{j+1}}; \quad (2.22)$$

summing as before, we obtain

$$\frac{1}{2m_{n+1}} < b_n < \frac{2}{m_{n+1}} \sum_{j=0}^{\infty} \varphi^{-2j} < \frac{4}{m_{n+1}}. \quad (2.23)$$

Combining (2.21) and (2.23) in the obvious way we obtain

$$\frac{1}{2\sqrt{3}} \sqrt{a_n - \alpha} < b_n < 4\sqrt{2} \sqrt{a_n - \alpha}. \quad (2.24)$$

Since  $\eta_\alpha$  is piecewise linear,  $\eta_\alpha(a_n) = b_n$ , and the square-root function is concave, the bound (2.17) follows at once.  $\square$

The following technical result will be needed in §6. In order to state it, set

$$\mu(x) = m_j \text{ if } a_j < x < a_{j-1}, \quad \nu(x) = n_j \text{ if } a_j < x < a_{j-1}, \quad (2.25)$$

and

$$D(x_1, x_2) = (\mu(x_1)\nu(x_2) - \mu(x_2)\nu(x_1))(x_1 - x_2). \quad (2.26)$$

**2.10. Lemma.** *If  $3 \leq e_j \leq N$  for all  $j$ , then*

$$D(\alpha + x, \alpha + x\eta) \geq x(1 - \eta) \text{ for } x \in (0, 1 - \alpha), \eta \in (0, (4N)^{-2}). \quad (2.27)$$

*Proof.* We have

$$D(\alpha + x, \alpha + x\eta) = [\mu(\alpha + x)\nu(\alpha + x\eta) - \mu(\alpha + x\eta)\nu(\alpha + x)]x(1 - \eta); \quad (2.28)$$

since the quantity in square brackets is  $\geq 1$  if  $\alpha + x$  and  $\alpha + x\eta$  are in disjoint intervals  $[a_j, a_{j-1}]$ , it is enough to show that if  $x$  and  $\eta$  are as in (2.27) and

$$a_j < \alpha + x < a_{j-1}, \quad (2.29)$$

then  $\alpha + x\eta < a_j$ . Now if (2.29) is satisfied, then we have  $x < 4/m_j^2$  from (2.21), and so

$$x\eta < \frac{1}{4N^2m_j^2} < \frac{1}{4m_{j+1}^2} < a_j - \alpha; \quad (2.30)$$

this follows from (2.27), (2.3) (which implies  $m_{j+1} < Nm_j$ ) and (2.21). In view of the previous remarks, the proof is now complete.  $\square$

### 3. CONSTRUCTION OF $M_\alpha$

In this section, we associate to any irrational number  $\alpha$ ,  $0 < \alpha < 1$ , a toric 4-manifold  $M_\alpha$  of infinite topological type.

**3.1. Notation and set-up.** The two points  $\alpha$  and  $\infty$  decompose  $\partial\mathcal{H}^2$  as

$$\partial\mathcal{H}^2 = \partial\mathcal{H}_+^2 \cup \partial\mathcal{H}_-^2 \cup \{\alpha\} \cup \{\infty\},$$

where

$$\partial\mathcal{H}_+^2 = \{(x, 0) \in \overline{\mathcal{H}^2} : x > \alpha\}, \quad \partial\mathcal{H}_-^2 = \{(x, 0) \in \overline{\mathcal{H}^2} : x < \alpha\}.$$

Choose a smooth simple arc  $\overline{Z}$  in  $\overline{\mathcal{H}^2}$  which joins  $\alpha$  to  $\infty$  and is such that

$$\overline{Z} = Z \cup \{\alpha\} \cup \{\infty\}, \quad Z \subset \mathcal{H}^2.$$

Then  $Z$  decomposes  $\mathcal{H}^2$  as a disjoint union

$$\mathcal{H}^2 = D_+ \cup Z \cup D_-,$$

where  $D_\pm$  contains  $\partial\mathcal{H}_\pm^2$  in its closure. Finally, put

$$\overline{D}_\pm = D_\pm \cup \partial\mathcal{H}_\pm^2.$$

The reader is urged to note that  $\overline{D}_+$  does not contain  $Z$  or either of the points  $\alpha$  and  $\infty$ .

**3.2. Notation.** The boundary component  $\partial\mathcal{H}_+^2$  is decomposed into the intervals  $[a_j, a_{j-1}]$ . We shall refer to these as *edges* and to the  $a_j$  themselves as *corners*.

**3.3. Construction of  $M_\alpha$ .** Recall the combinatorial description of smooth toric 4-manifolds of Orlik and Raymond [9]. Let  $M$  be a compact, simply connected 4-manifold, with a smooth action of the 2-torus  $T^2$ , free on the open subset  $U \subset M$ . Let  $\pi : M \rightarrow M/T^2 = P$  be the quotient map. Then  $P$  is a topological polygon with a finite number of edges  $E_j$ , say. If  $x$  is a point in the interior of  $P$ , then  $\pi^{-1}(x) = T^2$ , while if  $x \in \text{int } E_j$ , then  $\pi^{-1}(x)$  is a circle. This circle is a special orbit of the  $T^2$ -action, with isotropy group  $S_{(m_j, n_j)}^1$ . If  $x$  is a corner  $E_j \cap E_{j+1}$  of  $P$ , then  $\pi^{-1}(x)$  is a point, fixed by the  $T^2$ -action.

More important for our purposes is the converse construction: to a polygon  $P$ , with edges  $E_j$ , labelled by coprime pairs  $(m_j, n_j)$ , one can construct a toric 4-orbifold  $M$  such that  $M/T^2$  is  $P$ , with special orbits described in the previous paragraph; for  $M$  to be a smooth manifold, we require that the pairs  $(m_j, n_j)$  and  $(m_{j+1}, n_{j+1})$  form a  $\mathbb{Z}$ -basis for  $\mathbb{Z} \oplus \mathbb{Z}$  for each  $j$ . One way to understand this construction is as follows. Starting from the closed polygon  $P$ , form the product  $P \times T^2$ . For each point  $x \in \partial P$ , the labelling gives us a subtorus  $T_x$  of  $T^2$ , which is a circle if  $x$  is in the interior of an edge and is  $T^2$  itself if  $x$  is a corner. The manifold  $M$  is formed by contracting  $T_x$  to a point. A smooth local model for these ‘contractions’ is given by the toric description of  $\mathbb{R}^4$ . Using polar coordinates, the  $T^2$ -action is

$$(z_1, z_2) \cdot (r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) = (r_1 e^{i(\theta_1 + z_1)}, r_2 e^{i(\theta_2 + z_2)}).$$

The quotient space is the closed quadrant  $Q = \{(r_1, r_2) : r_1 \geq 0, r_2 \geq 0\}$ . We have maps

$$Q \times T^2 \xrightarrow{\beta} \mathbb{R}^4 \xrightarrow{\pi_c} Q \quad (3.1)$$

where  $\pi_c$  is the quotient map  $\mathbb{R}^4 \rightarrow \mathbb{R}^4/T^2 = Q$  and

$$\beta(r_1, r_2, \theta_1, \theta_2) = (r_1 e^{i\theta_1}, r_2 e^{i\theta_2})$$

contracts  $S_{0,1}^1$  along the  $r_1$ -axis,  $S_{1,0}^1$  along the  $r_2$ -axis, and the whole  $T^2$  at the corner of  $Q$ . Returning to the construction of  $M$ , if  $x \in \partial P$  is in the interior of an edge, with isotropy group  $S_{(m,n)}^1$ , then we can change basis in the lattice so that  $(m, n)$  is mapped to  $(0, 1)$ . Then a neighbourhood  $N$  of  $x$  in  $P$  can be identified with a neighbourhood  $N'$  of  $r_1 = 1$ , say, on the  $r_1$ -axis in  $Q$ . The part of  $M$  over  $N$  can then be defined to be  $\pi_c^{-1}(N')$ . Similarly, if  $x$  is a corner of  $P$ , then we can change basis in the lattice so that the labels of the two edges through  $x$  are mapped to  $(1, 0)$  and  $(0, 1)$ . Again, we identify a small neighbourhood  $N$  of  $x$  with a neighbourhood  $N'$  of 0 in  $Q$  and construct the part of  $M$  over  $N$  as  $\pi_c^{-1}(N')$ . This concludes our outline of the reconstruction of a toric 4-manifold from a labelled polygon.

With this understood, we can describe  $M_\alpha$  using the notation introduced in §3.1. That is, we form the product  $\overline{D}_+ \times T^2$ , and contract  $S_{(m_j, n_j)}^1$  over each point of  $[a_j, a_{j-1}]$ ; the result is an open manifold that we shall call  $M_\alpha$ . Notice that a sequence of points  $p_j \in M_\alpha$  with  $\pi(p_j)$  converging to a point on  $\overline{Z}$  will have no convergent subsequence in  $M_\alpha$ . On the other hand, if  $\pi(p_j)$  converges to a point on  $\partial\mathcal{H}_+^2$ , then  $p_j$  *does* have a convergent subsequence.

**3.4. Remark.** Note that for  $j \geq 1$ ,  $S_j = \pi^{-1}[a_j, a_{j-1}]$  is a smoothly embedded 2-sphere. Suitably oriented, we have

$$S_j \cdot S_{j+1} = -1, \quad S_j \cdot S_j = e_j.$$

#### 4. A SELF-DUAL EINSTEIN METRIC ON $M_\alpha$

For background to the material presented in this section, the reader is referred to [2] and [3]. We present the essential formulae here: the proofs can be found in the cited papers.

Let  $F(x, y)$  be a solution of the equation

$$\Delta F = \frac{3}{4}F \quad (4.1)$$

where  $\Delta = y^2(\partial_x^2 + \partial_y^2)$  is the Laplacian of  $\mathcal{H}^2$ . Let

$$f(x, y) = \sqrt{y}F(x, y)$$

and introduce the auxiliary quantities

$$v_1 = (f_y, xf_y - yf_x), \quad v_2 = (f_x, xf_x + yf_y - f)$$

and

$$w = f_x^2 + f_y^2 - y^{-1}ff_y, \text{ so that } \varepsilon(v_1, v_2) = yw.$$

(Recall from (1.3) that  $\varepsilon$  stands for the standard symplectic form on  $\mathbb{R}^2$ .) Set

$$g = \frac{|w|}{f^2} \left( dx^2 + dy^2 + \frac{\varepsilon(v_1, dz)^2 + \varepsilon(v_2, dz)^2}{w^2} \right).$$

This is a self-dual Einstein metric on  $U \times T^2$ , where  $U \subset \mathcal{H}^2$  is the set on which  $w \neq 0$ ,  $f \neq 0$ . Moreover, the sign of the scalar curvature is opposite to the sign of  $w$ .

The next task is to choose the eigenfunction  $F$  so as to obtain a metric that extends smoothly over the special  $T^2$ -orbits. This is very conveniently done using the following integral formulae which show how to recover  $F$  from its (renormalized) boundary value  $u(x) = f(x, 0)$ .

**4.1. Integral formulae.** Let  $u$  be a distribution on  $\mathbb{R}$ , viewed as the finite part of  $\partial\mathcal{H}^2$ . If  $u$  does not grow too fast at  $\infty$ , we define

$$f(x, y) = \int \frac{y^2}{2((x-x_1)^2 + y^2)^{3/2}} u(x_1) dx_1, \quad (4.2)$$

which we shall also write as

$$f(x, y) = [k_y * u](x, y), \text{ where } k_y(x) = \frac{y^2}{2(x^2 + y^2)^{3/2}}. \quad (4.3)$$

Then  $F(x, y) = y^{-\frac{1}{2}}f(x, y)$  satisfies (4.1) and has boundary data given by  $u$  in the sense that

$$f(x, y) \rightarrow u(x) \text{ as } y \rightarrow 0$$

in the sense of distributions.

If  $f$  is given by (4.2), there is also a formula for  $w$  in terms of the boundary value  $u$ . For this, introduce

$$(\mu(x), \nu(x)) = (u'(x), xu'(x) - u(x)),$$

so that  $(\mu, \nu)$  is the boundary value of  $v_2$  and  $\mu(x)x - \nu(x) = u(x)$ . If

$$D(x_1, x_2) = (\mu(x_1)\nu(x_2) - \mu(x_2)\nu(x_1))(x_1 - x_2), \quad (4.4)$$

then

$$w(x, y) = \frac{1}{2}y^{-2} \iint k_y(x-x_1)k_y(x-x_2)D(x_1, x_2) dx_1 dx_2. \quad (4.5)$$

This is essentially [3, (5.16)].

**4.2. Remark.** At the beginning of this section we made the assumption that  $u$  should not grow too fast at  $\infty$ . Our explicit choice of coordinates obscures the invariance of the preceding formulae. It is more natural to interpret  $u$  as a distributional section of the  $-\frac{1}{2}$ -power of the density bundle on  $\partial\mathcal{H}^2$ . Then the above integral formulae become fully invariant under  $PSL_2(\mathbb{R})$ , acting by isometries on  $\mathcal{H}^2$  and projectively on its boundary. The distributions that we shall actually use will satisfy  $u(\pm x) = \pm 1$  for all sufficiently large  $|x|$ ; it is not difficult to check that  $u(x)|dx|^{-\frac{1}{2}}$  is then smooth in a neighbourhood of  $\infty$ .



**4.3. Definition of  $g_\alpha$ .** In order to obtain a metric on  $M_\alpha$ , apply the previous formulae with  $u$  equal to the odd extension of  $\eta$  to the left of  $\alpha$ :

$$u(x) = \eta(x) - \eta(2\alpha - x).$$

It is clear from (4.2) that  $f(\alpha, y) = 0$  for all  $y \geq 0$ ; accordingly we set  $Z = \{(\alpha, y) : y > 0\}$  (cf. §3.1).

**4.4. Theorem.** *The metric  $g_\alpha$  is defined on  $D_+ \times T^2$  and extends smoothly to  $M_\alpha$ .*

*Proof.* See [3, Theorem 5.2.1]. That result applies to show that  $w > 0$  in  $\mathcal{H}^2$  and that  $f > 0$  in  $D_+$ , so that  $g_\alpha$  is defined on  $D_+ \times T^2$ . We explain why the metric extends to  $M_\alpha$  since this was not done in [3] and we did not explicitly check that  $w$  has the needed boundary behaviour. This is now fixed in Proposition 5.4 below.

For this, we need to understand the asymptotic behaviour of  $f$  and  $w$  as  $y \rightarrow 0$ . So pick  $x_0 > \alpha$ . We distinguish two cases according to whether  $x_0$  is or is not one of the  $a_j$ . The easier case corresponds to  $a_j < x_0 < a_{j-1}$  for some  $j$ . Then by Proposition 5.3, with  $m_1 = m_2 = m_j$ ,  $n_1 = n_2 = n_j$ , we have

$$f(x, y) = m_j x - n_j + y^2 f_1(x) + \dots, \quad (4.6)$$

and by part (i) of Proposition 5.4,

$$w(x, y) = w_0(x) + w_1(x)y^2 + \dots, \quad \text{where } w_0(x) > 0, \quad (4.7)$$

for  $|x - x_0|$  and  $y \geq 0$  sufficiently small. Write  $(m_j, n_j) = (m, n)$ , to simplify the notation. Then computing with these formulae,

$$\begin{aligned} v_1 &= y(2f_1, 2xf_1 - m) + O(y^3) \\ v_2 &= (m, n) + O(y^2) \\ \varepsilon(v_1, v_2) &= y(2uf_1 - (u')^2). \end{aligned}$$

Thus, for small  $y$ ,

$$g_\alpha \simeq \frac{w_0}{u^2} \left( dx^2 + dy^2 + \frac{y^2(2f_1 dz_2 + (u' - 2xf_1) dz_1)^2 + (mdz_2 - ndz_1)^2}{(2uf_1 - (u')^2)} \right)$$

In particular

$$g_\alpha(m\partial_{z_1} + n\partial_{z_2}, m\partial_{z_1} + n\partial_{z_2}) \simeq y^2.$$

If we introduce new coordinates

$$\begin{aligned} d\theta &= mdz_2 - ndz_1 \\ d\psi &= m_1 dz_2 - n_1 dz_1, \end{aligned}$$

where  $|mn_1 - m_1n| = 1$ , we have

$$g_\alpha = \frac{w_0}{u^2} (dx^2 + a(x)d\theta^2 + dy^2 + y^2 d\psi^2 + \dots), \quad (a(x) > 0)$$

which shows that  $g_\alpha$  does extend as a smooth metric to  $\pi^{-1}(U)$ , where  $U$  is a small neighbourhood of the boundary-point  $x_0$ .

To complete the proof, we must consider the behaviour of  $g_\alpha$  at one of the corners  $a_j$ . By a suitable change of variables, we may assume that  $x_0 = a_0 = 1$ , so that  $u(x) = x$  for  $1 - \delta < x \leq 1$ ,  $u(x) = 1$  for  $1 \leq x < 1 + \delta$ .

By Proposition 5.3, we have

$$f(x, y) = 1 + \frac{1}{2}(x - 1) - \frac{1}{2}((x - 1)^2 + y^2)^{1/2} + f_1(x)y^2 + \dots$$

and by part (ii) of Proposition 5.4

$$w(x, y) = \frac{1}{2}((x-1)^2 + y^2)^{-1/2} + w_1(x) + \dots \quad (4.8)$$

where the omitted terms contain only even powers of  $y$ . Using the change of variable

$$x - 1 + iy = (r_2 + ir_1)^2$$

we compute, to leading order,

$$\begin{aligned} v_1 &= -\frac{r_1 r_2}{r_1^2 + r_2^2}(1, 1) + \dots \\ v_2 &= \frac{1}{r_1^2 + r_2^2}(r_1^2, -r_2^2) + \dots \\ w &= \frac{1}{2} \frac{1}{r_1^2 + r_2^2} + \dots \\ dx^2 + dy^2 &= 4(r_1^2 + r_2^2)(dr_1^2 + dr_2^2). \end{aligned}$$

Substituting these into the formula for  $g_\alpha$  we find

$$g_\alpha = (dr_1^2 + r_1^2 dz_1^2) + (dr_2^2 + r_2^2 dz_2^2) + \dots$$

exactly as required for smooth extension to  $M_\alpha$  in a neighbourhood of the corner.  $\square$

## 5. BOUNDARY BEHAVIOUR OF $f$ AND $w$

This section is devoted to a study of the integral formulae for  $f$  and  $w$ , (4.2) and (4.5). As in the previous section, we assume that  $u$  is a distribution on  $\mathbb{R}$  such that

$$u \text{ is equal to a constant multiple of } \operatorname{sgn}(x) \text{ outside a compact set.} \quad (5.1)$$

Since

$$K_y''(x) = k_y(x), \quad K_y(x) = \frac{1}{2} \sqrt{x^2 + y^2}, \quad (5.2)$$

we can rewrite (4.2) as

$$f(x, y) = \int K_y(x - x_1) u''(x_1) dx_1, \quad (5.3)$$

the condition (5.1) being used to justify the integration by parts.

We now start our study of the behaviour of (4.2) as  $y \rightarrow 0$ . For this, fix  $\delta > 0$ , and define

$$B_0 = (-\delta, \delta) \times (0, \delta), \quad B = (-\delta, \delta) \times [0, \delta]. \quad (5.4)$$

Fix also a smooth cut-off function  $\beta$ , equal to 1 for  $|x| \leq 2\delta$  and equal to 0 for all  $|x| \geq 3\delta$ .

**5.1. Lemma.** (i) *In (4.2), suppose that  $u = 0$  in  $(-2\delta, 2\delta)$ . Then  $f(x, y)$  is real-analytic for  $(x, y) \in B$  and has an expansion in this domain of the form*

$$f(x, y) = y^2 f_2(x) + y^4 f_4(x) + \dots \quad (5.5)$$

(ii) *In (4.2), suppose that  $u$  has compact support, and that  $u$  is  $C^2$  in  $(-3\delta, 3\delta)$ . Then  $f(x, y)$  is continuous for  $(x, y)$  in  $B$  and real-analytic for  $(x, y) \in B_0$ . Moreover, if  $(x, y) \in B$ ,*

$$|f(x, y) - u(x)| \leq \frac{1}{2}|y| \int |(\beta u)''(x_1)| dx_1 \text{ as } y \rightarrow 0. \quad (5.6)$$

*Proof.* For (i), note that if  $(x, y) \in B$  and  $x_1$  is in the support of  $u$ , we have  $|x - x_1| > \delta$  and so  $k_y(x - x_1)$  can be expanded as a convergent series

$$\frac{y^2}{2((x - x_1)^2 + y^2)^{3/2}} = \frac{y^2}{2|x - x_1|^3} \left( 1 - \frac{3}{2} \frac{y^2}{(x - x_1)^2} + \dots \right). \quad (5.7)$$

The result follows at once from this, using term-by-term integration.

For part (ii), note first that

$$f(x, y) = \int k_y(x - x_1)\beta(x_1)u(x_1) dx_1 + \int k_y(x - x_1)(1 - \beta(x_1))u(x_1) dx_1 = I_1 + I_2. \quad (5.8)$$

By part (i),  $I_2$  is real-analytic and  $O(y^2)$  if  $(x, y) \in B$ . On the other hand, by (5.3)

$$I_1(x, y) = \int \frac{1}{2} \sqrt{(x - x_1)^2 + y^2} (\beta u)''(x_1) dx_1. \quad (5.9)$$

Hence  $I_1$  is real-analytic in  $B_0$  and continuous in  $B$ , with

$$I_1(x, 0) = \int \frac{1}{2} |x - x_1| (\beta u)''(x_1) dx_1 = u(x). \quad (5.10)$$

The estimate (5.6) is obtained by noting

$$|I_1(x, y) - I_1(x, 0)| \leq \frac{1}{2} \int (\sqrt{(x - x_1)^2 + y^2} - |x - x_1|) |(\beta u)''(x_1)| dx_1 \quad (5.11)$$

and using the triangle inequality,

$$0 \leq \sqrt{(x - x_1)^2 + y^2} - |x - x_1| \leq y.$$

□

We turn to now to the boundary behaviour of (4.2) when  $u$  is piecewise linear near 0.

**5.2. Lemma.** *Suppose in the above that*

$$u(x) = a \operatorname{sgn}(x) + b|x| + cx \text{ for } |x| < 3\delta. \quad (5.12)$$

*Then for  $(x, y) \in B_0$ ,*

$$f(x, y) = a \frac{x}{(x^2 + y^2)^{1/2}} + b(x^2 + y^2)^{1/2} + cx + O(y^2) \quad (5.13)$$

*where  $O(y^2)$  stands for a real-analytic function of  $(x, y)$  which goes to 0 like  $y^2$  as  $y \rightarrow 0$ .*

*Proof.* With  $\beta$  as before,

$$(\beta u)''(x) = 2a\delta'(x) + 2b\delta(x) + \beta''u$$

so that, using (5.3),

$$f(x, y) = 2aK_y(x) + 2bK'_y(x) + \int K_y(x - x_1)\beta''(x_1)u(x_1) dx_1 = f_1(x, y) + f_2(x, y) + f_3(x, y).$$

Since  $\beta''(x_1)u(x_1)$  vanishes for  $|x| \leq 2\delta$ , it follows as in Lemma 5.1 that  $f_3(x, y)$  is real-analytic in  $B$ . By part (ii) of the same lemma  $f_3(x, 0) = cx$  (for we know that  $f(x, 0) = u(x)$ , at least if  $x \neq 0$ ). Since

$$f_1(x, y) = a\sqrt{x^2 + y^2}, \quad f_2(x, y) = \frac{bx}{\sqrt{x^2 + y^2}},$$

the proof is complete. □

Using these lemmas, the proof of the following result is immediate.

**5.3. Proposition.** *Suppose that*

$$u(x) = \begin{cases} m_1x - n & \text{for } 0 < x < 3\delta; \\ m_2x - n & \text{for } -3\delta < x < 0. \end{cases}$$

*Then if (5.4) is satisfied, we have*

$$f(x, y) = \frac{1}{2}(m_1 - m_2)\sqrt{x^2 + y^2} + \frac{1}{2}(m_1 + m_2)x - n + O(y^2).$$

*Proof.* Write

$$u(x) = \frac{1}{2}(m_1 - m_2)|x| + \frac{1}{2}(m_1 + m_2)x - n$$

and apply Lemma 5.2. □

We now give our result concerning the boundary behaviour of  $w(x, y)$ . Let the notation be as in §3.1.

**5.4. Proposition.** (i) *Let  $c$  be a point at which  $f(x)$  is smooth. Then  $w(x, y)$  is smooth and positive in  $B$ .*

(ii) *Let  $c$  be one of the  $a_j$ . Then in  $B_0$  we have*

$$w(x, y) = \frac{1}{\sqrt{(x-c)^2 + y^2}} + O(1),$$

where  $O(1)$  stands for a smooth positive function in  $B$ .

*Proof.* In case (i), write (4.5) in the form

$$w(x, y) = \frac{1}{4} \int \left[ k_y(x - x_1) \int ((x - x_2)^2 + y^2)^{-3/2} D(x_1, x_2) dx_2 \right] dx_1. \quad (5.14)$$

Because  $D(x_1, x_2) = 0$  if  $x_1$  and  $x_2$  are sufficiently close to  $c$ , the function

$$x \mapsto \int |x - x_2|^{-3} D(x, x_2) dx_2 \quad (5.15)$$

is smooth for  $x$  near  $c$ , and because  $k_y(x) \rightarrow \delta(x)$  as  $y \rightarrow 0$ , we obtain

$$w(x, 0) = \frac{1}{4} \int |x - x_2|^{-3} D(x, x_2) dx_2 \text{ for } x \text{ sufficiently close to } c.$$

In particular,  $w(x, 0) > 0$  by the non-negativity of  $D(x_1, x_2)$ . With a little more work it can be shown that  $w(x, y)$  is smooth in  $B$ , following the proof of Lemma 5.1.

In case (ii), we use a cut-off function  $\beta$ , equal to 1 near  $c$  and with small support, to split the integral as

$$w(x, y) = w_1(x, y) + w_2(x, y),$$

where

$$w_1(x, y) = \frac{1}{4} \int \left[ k_y(x - x_1) \int ((x - x_2)^2 + y^2)^{-3/2} (1 - \beta(x_2)) D(x_1, x_2) dx_2 \right] dx_1,$$

and

$$w_2(x, y) = \frac{1}{4} \int \left[ k_y(x - x_1) \int ((x - x_2)^2 + y^2)^{-3/2} \beta(x_2) D(x_1, x_2) dx_2 \right] dx_1.$$

Now  $w_1(x, 0)$  is smooth and positive near  $c$  by the same argument as before, and if we write

$$w_2(x, y) = w_3(x, y) + w_4(x, y)$$

where

$$w_3(x, y) = \frac{1}{4} \int \left[ k_y(x - x_1)(1 - \beta(x_1)) \int ((x - x_2)^2 + y^2)^{-3/2} \beta(x_2) D(x_1, x_2) dx_2 \right] dx_1,$$

and

$$w_4(x, y) = \frac{1}{4} \int \left[ k_y(x - x_1)\beta(x_1) \int ((x - x_2)^2 + y^2)^{-3/2} \beta(x_2) D(x_1, x_2) dx_2 \right] dx_1,$$

then  $w_3(x, 0)$  is also smooth and positive near  $c$  by symmetry. Thus it remains only to consider  $w_4(x, y)$ .

In order to simplify the notation, use a translation to set  $c = 0$  and assume that  $u$  is as in Proposition 5.3 near  $x = 0$ . Then

$$D(x_1, x_2) = -\frac{1}{2}x_2 \operatorname{sgn}(x_1) - \frac{1}{2}x_1 \operatorname{sgn}(x_2) + \frac{1}{2}|x_1| + \frac{1}{2}|x_2| \text{ for } (x_1, x_2) \in (-\delta, \delta) \times (-\delta, \delta),$$

assuming, as we may, that

$$n(m_1 - m_2) = 1.$$

Now Lemma 5.2 can be applied, giving

$$\int k_y(x - x_1)\beta(x_1)D(x_1, x_2) dx_1 = \frac{1}{2} - \frac{xx_2}{\sqrt{x^2 + y^2}} + \sqrt{x^2 + y^2} - x_1 \operatorname{sgn}(x_2) + |x_2| + O(y^2)$$

and then

$$\begin{aligned} \iint k_y(x - x_1)\beta(x_1)\beta(x_2)D(x_1, x_2) dx_1 dx_2 &= -\frac{x^2}{\sqrt{x^2 + y^2}} + \sqrt{x^2 + y^2} + O(y^2) \\ &= \frac{y^2}{\sqrt{x^2 + y^2}} + O(y^2). \end{aligned}$$

Dividing this by  $y^2$  and combining with our conclusions about the other  $w_j$  now gives the stated result.  $\square$

## 6. COMPLETENESS OF $g_\alpha$

We now have a SDE metric on  $M_\alpha$ . Our final task is to show that this metric is complete.

**6.1. Theorem.** *Let  $\Gamma : [0, 1) \rightarrow M_\alpha$  be a smooth curve with finite length,*

$$L(\gamma) = \lim_{t \rightarrow 1} d(\Gamma(0), \Gamma(t)) < \infty. \quad (6.1)$$

*Then there exists a point  $p \in M_\alpha$  such that*

$$\lim_{t \rightarrow 1} \Gamma(t) = p. \quad (6.2)$$

*Proof.* Let  $\gamma = \pi \circ \Gamma$  be the projection of the curve to  $\overline{D}_+$ . Then from the form of the metric  $g_\alpha$ , it is clear that this too has finite length with respect to the base metric

$$h = \frac{w}{f^2}(dx^2 + dy^2). \quad (6.3)$$

From the estimates proved below,  $h$  is uniformly bounded below by a multiple of the euclidean metric on  $\mathbb{R}^2$ , so that

$$\gamma(1) := \lim_{t \rightarrow 1} \gamma(t)$$

exists and lies in  $\overline{D}_+ \cup Z \cup \{\alpha\}$ . There are now three possibilities to consider. First, suppose  $\gamma(1) \in \overline{D}_+$ . Then  $\pi^{-1}(\gamma(1))$  is contained in the interior of  $M_\alpha$ , and so (6.2) must be satisfied. Next, suppose, if possible, that  $\gamma(1) \in Z$ . Since  $f$  vanishes on  $Z$ ,  $h$  has a double-pole along

$Z$ , and it follows that  $\gamma$  must have infinite length, a contradiction. The remaining possibility is that  $\gamma(1) = (\alpha, 0)$ .

From the estimates proved below, we have

$$x > \alpha, y > 0 \text{ implies that } \frac{w}{f^2} > C(x + y - \alpha)^{-2}. \quad (6.4)$$

If we set  $\xi = x - \alpha - y$ ,  $\eta = x - \alpha + y$ , then we get

$$h \geq C \frac{d\xi^2 + d\eta^2}{2\eta^2} \text{ for } |\xi| \leq \eta$$

(that is, LHS minus RHS is positive-definite). Now our curve  $\gamma$  is contained in  $\{|\xi| < \eta\}$  and  $\lim_{t \rightarrow 1} \gamma(t) = (0, 0)$ . Hence  $l(\gamma) = \infty$  and this contradiction completes the proof, modulo the estimates established in the next section.  $\square$

**6.2. Estimates.** In order to simplify the notation in this section, we shift variables so that  $\alpha$  is translated to the origin, and we aim to understand the behaviour of  $f$  near  $(0, 0)$ .

First of all, we have

**6.3. Proposition.** *Let  $u(x)$  and  $f(x, y)$  be as throughout. Then there exist  $\varepsilon > 0$  and  $C > 0$  so that*

$$f(x, y) \leq C\sqrt{x+y} \text{ if } (x, y) \in (0, \varepsilon) \times (0, \varepsilon). \quad (6.5)$$

*Proof.* The result will be established by showing that

$$f(x, \theta x) \leq C\sqrt{x} \text{ if } 0 < \theta \leq 1 \quad (6.6)$$

and

$$f(\theta_1 y, y) \leq C\sqrt{y} \text{ if } 0 < \theta_1 \leq 1. \quad (6.7)$$

Since  $u$  is odd,

$$\begin{aligned} f(x, y) &= \int_0^\infty \{k_y(x - x_1) - k_y(x + x_1)\} u(x_1) dx_1 \\ &\leq \Omega \int_0^\infty \{k_y(x - x_1) - k_y(x + x_1)\} \sqrt{x_1} dx_1 \\ &\leq \Omega \int_0^\infty k_y(x - x_1) \sqrt{x_1} dx_1, \end{aligned} \quad (6.8)$$

where we have used the upper bound of Theorem 2.9 and the positivity of  $k_y(x)$ .

To prove (6.6), introduce a natural rescaling of variables in (6.8),

$$\xi = x_1/x, \theta = y/x \quad (6.9)$$

so that (6.8) becomes

$$f(x, x\theta) \leq \Omega I = \Omega\sqrt{x} \int_0^\infty k_\theta(\xi - 1) \sqrt{\xi} d\xi. \quad (6.10)$$

Note that this yields (6.6) for each fixed positive  $\theta$ , but the uniformity as  $\theta \rightarrow 0$  needs a little further work. For this we split  $I$ , as in §5, using a bump-function  $\beta$  identically equal to 1 in a neighbourhood of  $\xi = 1$ . We have

$$I = I_1 + I_2$$

where

$$I_1 = \int_0^\infty k_\theta(\xi - 1)(1 - \beta(\xi)) \sqrt{\xi} d\xi, \quad I_2 = \int_0^\infty k_\theta(\xi - 1)\beta(\xi) \sqrt{\xi} d\xi.$$

Then since  $\lim_{\theta \rightarrow 0} k_\theta(\xi - 1) = \delta(\xi - 1)$ , we have

$$I_2 \rightarrow Cx^{1/2}\beta(1)\sqrt{1} = C\sqrt{x} \text{ as } \theta \rightarrow 0.$$

By continuity,  $I_2$  is uniformly bounded by  $C\sqrt{x}$  for all  $0 \leq \theta \leq 1$ . The estimate (6.6) now follows by noting that  $I_1 \geq 0$  and  $I_1 \rightarrow 0$  as  $\theta \rightarrow 0$ . In fact, it is easily shown that  $I_1 \leq C'\theta^2\sqrt{x}$  for  $0 \leq \theta \leq 1$ .

The complementary estimate (6.7) is somewhat simpler: return to (6.8) and make the change of variables  $x = y\theta_1$ ,  $x_1 = y\xi_1$ , so

$$f(y\theta_1, y) \leq \Omega\sqrt{y} \int_0^\infty k_1(\xi_1 - \theta_1)\sqrt{\xi_1} d\xi_1. \quad (6.11)$$

Clearly the integral is uniformly bounded for  $\theta_1 \in [0, 1]$  and this completes the proof.  $\square$

Next we need a lower bound on  $w(x, y)$ . This is given by

**6.4. Proposition.** *Suppose that in the continued fraction expansion of  $\alpha$ ,  $3 \leq e_j \leq N$ , for some  $N$ . Then there exist  $\varepsilon > 0$  and  $C > 0$  so that*

$$w(x, y) \geq C(x + y)^{-1} \text{ if } (x, y) \in (0, \varepsilon) \times (0, \varepsilon). \quad (6.12)$$

*Proof.* The argument is closely analogous to the proof of the previous proposition. In particular, (6.12) will be established by proving the separate inequalities

$$w(x, \theta x) \geq Cx^{-1} \text{ if } 0 < \theta \leq 1 \quad (6.13)$$

and

$$w(\theta_1 y, y) \geq Cy^{-1} \text{ if } 0 < \theta_1 \leq 1. \quad (6.14)$$

We use (4.5). Because  $D \geq 0$  and  $k_y(x) > 0$ , it is enough to prove the lower bound for

$$I(x, y) := \frac{1}{2}y^{-2} \int_{x_1=0}^\infty \int_{x_2=0}^{x_1} k_y(x - x_1)k_y(x - x_2)D(x_1, x_2) dx_1 dx_2 \quad (6.15)$$

where the integral has been restricted to the intersection of the positive quadrant with the region  $x_2 \leq x_1$ . In this integral, make the change of variables

$$x_1 = x\xi, x_2 = x\xi\eta, y = \theta x.$$

Then

$$\begin{aligned} I(x, \theta x) &= \frac{1}{2x^2\theta^2} \int_{\xi=0}^\infty \int_{\eta=0}^1 k_\theta(\xi - 1)k_\theta(\xi\eta - 1)D(x\xi, x\xi\eta)\xi d\xi d\eta \\ &\geq \frac{1}{2x\theta^2} \int_{\xi=0}^{a/x} \int_{\eta=0}^b k_\theta(\xi - 1)k_\theta(\xi\eta - 1)\xi^2(1 - \eta) d\xi d\eta, \end{aligned} \quad (6.16)$$

where  $a = 1 - \alpha$ ,  $b = 1/16N^2$  as in Lemma 2.10. For each  $\theta$  this gives the required  $O(1/x)$  lower bound. To see this is uniform as  $\theta \rightarrow 0$ , rewrite (6.16) as follows and take the limit:

$$\frac{1}{4x} \int_0^{a/x} k_\theta(\xi - 1) \left[ \xi^2 \int_{\eta=0}^b (1 - \eta)((1 - \xi\eta)^2 + \theta^2)^{-3/2} d\eta \right] d\xi \longrightarrow \frac{1}{4x} \int_0^b (1 - \eta)^{-2} d\eta \text{ as } \theta \rightarrow 0. \quad (6.17)$$

By continuity, the required uniform lower bound follows.

In order to obtain the other bound (6.14), return to (6.15) and make the substitutions

$$x = \theta_1 y, x_1 = y\xi, x_2 = y\xi\eta$$

to give

$$w(y\theta_1, y) \geq \frac{1}{2y} \int_{\xi=0}^{a/y} \int_0^b k_1(\xi - \theta_1)k_1(\xi\eta - \theta_1)\xi^2(1 - \eta) d\xi d\eta. \quad (6.18)$$

This gives (6.14) at once. □

**6.5. Remark.** As in [3, §5] the distribution  $u$  can be perturbed to the left of  $\alpha$  to yield an infinite-dimensional family of SDE metrics on  $M_\alpha$ . These perturbations need to preserve the monotonicity and convexity properties enjoyed by  $u$ , as well as the boundary condition  $u = -1$  for  $x \ll 0$ . Provided these perturbations are sufficiently small and supported away from  $\alpha$ , the resulting metrics will be complete; the details are left to the interested reader.

#### REFERENCES

- [1] M. T. Anderson, P. B. Kronheimer, and C. R. LeBrun *Complete Ricci-flat Kähler manifolds of infinite topological type* Comm. Math. Phys. **125** (1989) 637–642.
- [2] D. M. J. Calderbank and H. Pedersen, *Selfdual Einstein metrics with torus symmetry*, J. Diff. Geom. **60** (2002) 485–521, math.DG/0105263.
- [3] D. M. J. Calderbank and M. A. Singer, *Einstein metrics and complex singularities*, Invent. Math. **156** (2004) 405–443, math.DG/0206229.
- [4] D. M. J. Calderbank and M. A. Singer, *Toric selfdual Einstein metrics on compact orbifolds*, Duke Math. J. (to appear), math.DG/0405020.
- [5] K. Galicki, *Multi-centre metrics with negative cosmological constant*, Class. Quantum Grav. **8** (1991) 1529–1543.
- [6] G. W. Gibbons and S. W. Hawking, *Gravitational multi-instantons*, Phys. Lett. **B 78** (1978) 430–432.
- [7] C. R. Graham and J. M. Lee, *Einstein metrics with prescribed conformal infinity on the ball*, Adv. Math. **87** (1991) 186–225.
- [8] P. B. Kronheimer, *The construction of ALE spaces as hyper-Kähler quotients*, J. Diff. Geom. **29** (1989) 665–683.
- [9] P. Orlik and F. Raymond, *Actions of the torus on 4-manifolds I*, Trans. Amer. Math. Soc. **152** (1970) 531–559.

SCHOOL OF MATHEMATICS, UNIVERSITY OF EDINBURGH, KING’S BUILDINGS, MAYFIELD ROAD, EDINBURGH EH9 3JZ, SCOTLAND.

*E-mail address:* dc511@york.ac.uk

*Current address:* Department of Mathematics, University of York, Heslington, York YO10 5DD, England.

*E-mail address:* m.singer@ed.ac.uk