AMBITORIC GEOMETRY I: EINSTEIN METRICS AND EXTREMAL AMBIKÄHLER STRUCTURES

VESTISLAV APOSTOLOV, DAVID M. J. CALDERBANK, AND PAUL GAUDUCHON

Abstract. We present a local classification of conformally equivalent but oppositely oriented 4-dimensional Kähler metrics which are toric with respect to a common 2-torus action. In the generic case, these "ambitoric" structures have an intriguing local geometry depending on a quadratic polynomial q and arbitrary functions A and B of one variable.

We use this description to classify Einstein 4-metrics which are hermitian with respect to both orientations, as well a class of solutions to the Einstein–Maxwell equations including riemannian analogues of the Plebański–Demiański metrics. Our classification can be viewed as a riemannian analogue of a result in relativity due to R. Debever, N. Kamran, and R. McLenaghan, and is a natural extension of the classification of selfdual Einstein hermitian 4-manifolds, obtained independently by R. Bryant and the first and third authors.

These Einstein metrics are precisely the ambitoric structures with vanishing Bach tensor, and thus have the property that the associated toric Kähler metrics are extremal (in the sense of E. Calabi). Our main results also classify the latter, providing new examples of explicit extremal Kähler metrics. For both the Einstein–Maxwell and the extremal ambitoric structures, A and B are quartic polynomials, but with different conditions on the coefficients. In the sequel to this paper we consider global examples, and use them to resolve the existence problem for extremal Kähler metrics on toric 4-orbifolds with $b_2 = 2$.

Introduction

Riemannian geometry in dimension four is remarkably rich, both intrinsically, and through its interactions with general relativity and complex surface geometry. In relativity, analytic continuations of families of lorentzian metrics and/or their parameters yield riemannian ones [8, 37], while concepts and techniques in one area have analogues in the other. In complex geometry, E. Calabi's extremal Kähler metrics [12] have become a focus of attention as they provide canonical riemannian metrics on polarized complex manifolds, generalizing constant Gauss curvature metrics on complex curves. The first nontrivial examples are on complex surfaces.

This paper concerns a notion related both to relativity and complex surface geometry. An *ambikähler structure* on a real 4-manifold (or orbifold) M consists of a pair of Kähler metrics (g_+, J_+, ω_+) and (g_-, J_-, ω_-) such that

- g_+ and g_- induce the same conformal structure (i.e., $g_- = f^2 g_+$ for a positive function f on M);
- J_+ and J_- have opposite orientations (equivalently the volume elements $\frac{1}{2}\omega_+ \wedge \omega_+$ and $\frac{1}{2}\omega_- \wedge \omega_-$ on M have opposite signs).

A product of two Riemann surfaces is ambikähler. To obtain more interesting examples, we suppose that both Kähler metrics are toric, with common torus action, which we call "ambitoric". More precisely, we suppose that

- there is a 2-dimensional subspace $\mathfrak{t}$ of vector fields on M, linearly independent on a dense open set, whose elements are hamiltonian and Poisson-commuting Killing vector fields with respect to both (g_+, ω_+) and (g_-, ω_-).[1]

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[1]If ω is a symplectic form, hamiltonian vector fields $K_1 = \mathrm{grad}_\omega f_1$ and $K_2 = \mathrm{grad}_\omega f_2$ *Poisson-commute* iff the Poisson bracket $\{f_1, f_2\}$ with respect to ω is zero. This holds iff $\omega(K_1, K_2) = 0$.

The theory of hamiltonian 2-forms in four dimensions [4] implies that any orthotoric Kähler metric and certain Kähler metrics of Calabi type are ambitoric. Such metrics provide interesting examples of extremal Kähler surfaces [12, 43, 16, 25, 26, 4, 6, 34]. Here we give a local classification of ambitoric structures in general, and an explicit description of the extremal Kähler metrics thus unifying and generalizing these works.

Our examples include riemannian analogues of Plebański–Demiański metrics [38]; the latter are Einstein–Maxwell spacetimes of Petrov type D, which have been extensively studied [23], and classified by R. Debever, N. Kamran and R. G. McLenaghan [18]. In riemannian geometry, the type D condition means that both half-Weyl tensors $W^\pm$ are degenerate, i.e., at any point of M at least two of the three eigenvalues of $W^\pm$ coincide (where W^+ and W^- are viewed as symmetric tracefree operators acting on the three-dimensional spaces of selfdual and antiselfdual 2-forms respectively). Einstein metrics g with degenerate half-Weyl tensors have been classified when $W^+ = 0$ or $W^- = 0$ [3]—otherwise, the riemannian Goldberg–Sachs theorem [40, 10, 35, 2] and the work of A. Derdziński [19] imply that g is ambikähler, with compatible Kähler metrics $g_\pm = |W^\pm|_g^{2/3} g$; conversely $g = s_\pm^{-2} g_\pm$, where $s_\pm$ are the scalar curvatures of $g_\pm$. From the $J_\pm$-invariance of the Ricci tensor of g, it follows that $\mathrm{grad}_{\omega_\pm} s_\pm$ are commuting Killing vector fields for $g_\pm$, which means that $g_\pm$ are both extremal Kähler metrics. A little more work yields the following result.

Theorem 1. *Let (M, g) be an oriented Einstein 4-manifold with degenerate half-Weyl tensors $W^\pm$. Then g admits compatible ambitoric extremal metrics $(g_\pm, J_\pm, \omega_\pm, \mathfrak{t})$ near any point in a dense open subset of M. Conversely, an ambikähler structure is conformally Einstein on a dense open subset if and only if its Bach tensor vanishes.*

This suggests classifying such Einstein metrics within the broader context of *extremal ambikähler metrics* or, equivalently, ambikähler metrics for which the Bach tensor is *diagonal*, i.e., both J_+ and J_- invariant. We also discuss riemannian metrics of "Plebański–Demiański type", for which the tracefree Ricci tensor satisfies $ric_0^g(X, Y) = c\, g(\omega_+(X), \omega_-(Y))$ for some constant c. In particular ric^g is diagonal. These two curvature generalizations also give rise to ambitoric structures.

Theorem 2. *An ambikähler structure $(g_\pm, J_\pm, \omega_\pm)$, not locally a Kähler product, nor of Calabi type, nor conformal to a $\pm$-selfdual Ricci-flat metric, is locally*:

- *ambitoric if and only if there is a compatible metric g with ric^g diagonal*; *further, g has Plebański–Demiański type if and only if it has constant scalar curvature*;
- *extremal and ambitoric if and only if the Bach tensor of c is diagonal.*

Thus motivated, we study ambitoric structures in general and show that in a neighbourhood of any point, they are either of Calabi type (hence classified by well-known results), or "regular". Our explicit local classification in the regular case (Theorem 3) relies on subtle underlying geometry which we attempt to elucidate, although some features remain mysterious. For practical purposes, however, the classification reduces curvature conditions (PDEs) on ambitoric structures to systems of functional ODEs. We explore this in greater detail in section 5, where we compute the Ricci forms and scalar curvatures for an arbitrary regular ambitoric pair (g_+, g_-) of Kähler metrics. This leads to an explicit classification of the extremal and conformally Einstein examples (Theorem 4). We also identify the metrics of Plebański–Demiański type among ambitoric structures (Theorem 5)—their relation to Killing tensors is discussed in Appendix B. We summarize the main results from Theorems 3–5 loosely as follows.

Main Theorem. *Let $(g_\pm, J_\pm, \omega_\pm, \mathfrak{t})$ be a regular ambitoric structure. Then*:

- *there is a quadratic polynomial q and functions A and B of one variable such that the ambitoric structure is given by* (19)–(21) (*and these are regular ambitoric*);

- (g_+, J_+) *is an extremal Kähler metric* $\Leftrightarrow$ (g_-, J_-) *is an extremal Kähler metric* $\Leftrightarrow$ A *and* B *are quartic polynomials constrained by three specific linear conditions*;
- $g_\pm$ *are conformally Einstein* (*i.e., Bach-flat*) *if and only if they are extremal, with an additional quadratic relation on the coefficients of* A *and* B;
- $g_\pm$ *are conformal to a constant scalar curvature metric of Plebański–Demiański type if and only if* A *and* B *are quartic polynomials constrained by three specific linear conditions* (*different, in general, from the extremality conditions*).

Corollary 1. *Let* (M, g) *be an Einstein* 4*-manifold for which the half-Weyl tensors* W^+ *and* W^- *are everywhere degenerate. Then on a dense open subset of* M*, the metric* g *is locally homothetic to one of the following*:

- *a real space form*;
- *a product of two Riemann surfaces with equal constant Gauss curvatures*;
- *an Einstein metric of the form* $s_+^{-2} g_+$*, where* g_+ *is a Bach-flat Kähler metric with nonvanishing scalar curvature* s_+*, described in Proposition* 10 *or Theorem* 4.

In the second part of this work we shall obtain global consequences of these local classification results. In particular, we shall resolve the existence problem for extremal Kähler metrics on toric 4-orbifolds with $b_2 = 2$.

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1. Conformal hermitian geometry

1.1. **Conformal hermitian structures.** Let M be a 4-dimensional manifold. A *hermitian metric* on M is defined by a pair (g, J) consisting of a riemannian metric $g \in C^\infty(M, S^2T^*M)$ and an integrable almost complex structure $J \in C^\infty(M, \mathrm{End}(TM))$, which are *compatible* in the sense that $g(J\cdot, J\cdot) = g(\cdot, \cdot)$.

The *fundamental* 2*-form* or *Kähler form* $\omega^g \in \Omega^2(M)$ of (g, J) is defined by $\omega^g(\cdot, \cdot) := g(J\cdot, \cdot)$; it is a J-invariant 2-form of square-norm 2. The volume form $v_g = \frac{1}{2}\omega^g \wedge \omega^g$ induces an orientation on M (the complex orientation of J) for which ω^g is a section of the bundle $\wedge^+ M$ of *selfdual* 2-forms; the bundle $\wedge^- M$ of *antiselfdual* 2-forms is then identified with the bundle of J-invariant 2-forms orthogonal to ω^g.

For any metric $\tilde{g} = f^{-2} g$ conformal to g (where f is a positive function on M), the pair $(\tilde{g}, J)$ is also hermitian. The *Lee form* $\theta^g \in \Omega^1(M)$ of (g, J) is defined by

$$d\omega^g = -2\theta^g \wedge \omega^g,$$

or equivalently $\theta^g = -\frac{1}{2} J\delta^g \omega^g$, where δ^g is the co-differential with respect to the Levi-Civita connection D^g of g. Since J is integrable, $d\omega^g$ measures the deviation of (g, J) from being a *Kähler structure* (for which J and ω^g are parallel with respect to D^g). Thus a hermitian metric g is Kähler iff $\theta^g = 0$. Indeed

$$D^g_X \omega^g = J\theta^g \wedge X^\flat + \theta^g \wedge JX^\flat, \tag{1}$$

where $X^\flat := g(X, \cdot)$ denotes the 1-form dual to the vector field X (see e.g., [2]).

If $\tilde{g} = f^{-2}g$, the corresponding Lee forms are linked by $\theta^{\tilde{g}} = \theta^g + d\log f$; it follows that there is a Kähler metric conformal to g iff θ^g is exact; locally, this is true iff $d\theta^g = 0$, and g is then uniquely determined up to homothety.

Remark 1. A conformally invariant (and well known) interpretation of the Lee form may be obtained from the observation that a conformal class of riemannian metrics determines and is determined by an oriented line subbundle of S^2T^*M whose positive sections are the riemannian metrics in the conformal class. Writing this line subbundle as $\mathit{\Lambda}^2 := \mathit{\Lambda} \otimes \mathit{\Lambda}$ (with $\mathit{\Lambda}$ also oriented), it is thus equivalently a bundle metric c on $\mathit{\Lambda} \otimes TM$ and the volume form of this bundle metric identifies $\mathit{\Lambda}^4$ with $\wedge^4 T^*M$. A metric in the conformal class may be written $g = \ell^{-2}c$ for a positive section ℓ of the line bundle $L = \mathit{\Lambda}^*$; such an ℓ is called a *length scale*.

Any connection on TM induces a connection on $L = (\wedge^4 TM)^{1/4}$; for example, the Levi-Civita connection D^g of $g = \ell^{-2}c$ induces the unique connection (also denoted D^g) on L with $D^g\ell = 0$. A connection D on TM is said to be *conformal* if $Dc = 0$. It is well known (see e.g. [14]) that taking the induced connection on L is an affine bijection from the affine space of torsion-free conformal connections on TM (the *Weyl connections*) to the affine space of connections on L (modelled on $\Omega^1(M)$).

If J is hermitian with respect to c, the connection $D^g + \theta^g$ on L is independent of the choice of metric $g = \ell^{-2}c$ in the conformal class. Equation (1) then has the interpretation that D^J is the unique torsion-free conformal connection with $D^J J = 0$, while $d\theta^g$ is the curvature of the corresponding connection on L.

In view of this remark, we will find it more natural in this paper to view a hermitian structure as a pair (c, J) where c is a *conformal metric* as above, and J is a complex structure which is orthogonal with respect to c (i.e., $c(J\cdot, J\cdot) = c(\cdot,\cdot)$). We refer to (M, c, J) as a hermitian complex surface. A *compatible* hermitian metric is then given by a metric $g = \ell^{-2}c$ in the corresponding conformal class.

1.2. **Conformal curvature in hermitian geometry.** If (M, c) is an oriented conformal 4-manifold, then the curvature of c, measured by the *Weyl tensor* $W \in \Omega^2(M, \mathrm{End}(TM))$, decomposes into a sum of *half-Weyl tensors* $W = W^+ + W^-$ called the *selfdual* and *antiselfdual* Weyl tensors, which have the property that for $g = \ell^{-2}c$, $(V \wedge W, X \wedge Y) \mapsto g(W^{\pm}_{V,W}X, Y)$ is a section $W^{\pm}_g$ of $S^2_0(\wedge^{\pm}M) \subset \wedge^2T^*M \otimes \wedge^2T^*M$, where S^2_0 denotes the symmetric tracefree square. A half-Weyl tensor $W^{\pm}$ is said to be *degenerate* iff $W^{\pm}_g$ is a pointwise multiple of $(\omega^{\pm} \otimes \omega^{\pm})_0$ for a section $\omega^{\pm}$ of $\wedge^{\pm}M$, where $(\cdot)_0$ denotes the tracefree part—equivalently, the corresponding endomorphism of $\wedge^{\pm}M$ has degenerate spectrum.

If (M, c, J) is hermitian, with the complex orientation, then (with respect to any compatible metric $g = \ell^{-2}c$) the selfdual Weyl tensor has the form

$$W^+_g = \tfrac{1}{8}\kappa^g\,(\omega^g \otimes \omega^g)_0 + J(d\theta^g)_+ \odot \omega^g,$$

for a function κ^g, where $J(d\theta^g)_+(X, Y) = (d\theta^g)_+(JX, Y)$, $(d\theta^g)_+$ denotes the selfdual part, and $\odot$ denotes symmetric product.

Proposition 1. *If (c, J) admits a compatible Kähler metric, or more generally* [2] *a compatible metric $g = \ell^{-2}c$ with J-invariant Ricci tensor ric^g, then W^+ is degenerate.*

This is a riemannian analogue of the Goldberg–Sachs theorem in relativity [21, 41]. For Einstein metrics, more information is available [40, 35, 19, 2, 32].

Proposition 2. *For an oriented conformal 4-manifold (M, c) with a compatible Einstein metric $g = \ell^{-2}c$, the following three conditions are equivalent*:

- *the half-Weyl tensor W^+ of c is degenerate*;

- *every point of* (M,c) *has a neighbourhood with a hermitian complex structure* J;
- *every point of* (M,c) *has a neighbourhood on which either* W^+ *is identically zero or there is a complex structure* J *for which* $\hat g = |W^+|_g^{2/3} g$ *is a Kähler metric.*

Proof. The equivalence of the first two conditions is the riemannian Goldberg–Sachs theorem [40, 10, 35, 2]. Derdziński [19] shows: if a half-Weyl tensor $W^\pm$ is degenerate, then on each connected component of M it either vanishes identically or has no zero (hence has two distinct eigenvalues, one simple and one of multiplicity two); in the latter case $|W^+|_g^{2/3} g$ is a Kähler metric. If W^+ is identically zero on an open set U, there exist hermitian complex structures on a neighbourhood of any point in U. □

1.3. **The Bach tensor.** The *Bach tensor* B of a 4-dimensional conformal metric is a co-closed tracefree section B of $L^{-2}\otimes S^2T^*M$ which is the gradient of the L_2-norm $\int_M |W|_c^2$ of the Weyl tensor under compactly supported variations of the conformal metric c. For any compatible riemannian metric $g=\ell^{-2}c$, $B^g=\ell^2 B$ is a symmetric bilinear form on TM defined by the well-known expressions [9, 4]

$$B^g = \delta^g\delta^g W + \tfrac12 W *_g ric_0^g = 2\delta^g\delta^g W^\pm + W^\pm *_g ric_0^g, \tag{2}$$

where we use the action of Weyl tensors W (or $W^\pm$) on symmetric bilinear forms b given by $(W *_g b)(X,Y) = \sum_{i=1}^4 b(W_{X,e_i}Y, e_i)$ where $\{e_i\}$ is a g-orthonormal frame. Here $ric_0^g = ric^g - \frac14 s_g\, g$ is the tracefree part of the Ricci tensor; the trace part does not contribute. It immediately follows from (2) that if W^+ or W^- is identically zero then c is *Bach-flat* (i.e., B is identically zero).

The conformal invariance of B implies that $B^{f^{-2}g} = f^2 B^g$, while the second Bianchi identity implies $\delta^g W = -\frac12 d^{D^g}(ric^g - \frac16 s_g\, g)$ (as T^*M-valued 2-forms). Thus c is also Bach-flat if it has a compatible Einstein metric.

If J is a complex structure compatible with the chosen orientation and $\hat g$ is Kähler with respect to J, then $W^+ = \frac18 s_{\hat g}(\omega^{\hat g}\otimes\omega^{\hat g})_0$, and the Bach tensor is easily computed by using (2): if $B^{\hat g,+}$ and $B^{\hat g,-}$ denote the J-invariant and J-anti-invariant parts of $B^{\hat g}$, respectively, then (see [19])

$$B^{\hat g,+} = \tfrac16(2D^+ds_{\hat g} + ric^{\hat g}\, s_{\hat g})_0, \qquad B^{\hat g,-} = -\tfrac16 D^- ds_{\hat g},$$

where, for any real function f, D^+df, resp. D^-df, denotes the J-invariant part, resp. the J-anti-invariant part, of the Hessian $D^{\hat g}df$ of f with respect to $\hat g$, and b_0 denotes the tracefree part of a bilinear form b. Hence the following hold [19, 32, 2].

Proposition 3. *Let* $(\hat g, J)$ *be Kähler and let* $g = s_{\hat g}^{-2}\,\hat g$ *(defined wherever* $s_{\hat g}$ *is nonzero). Then:*

- $(\hat g, J)$ *is* extremal *(i.e.,* $J\operatorname{grad}_{\hat g} s_{\hat g}$ *is a Killing vector field) iff* $B^{\hat g}$ *is* J*-invariant;*
- $\delta^g W^+ = 0$ *wherever* g *is defined, and hence* $B^g = W^+ *_g ric_0^g$*, i.e.,* $B^{\hat g} = \frac12 ric_0^g\, s_{\hat g}$;
- g *is an Einstein metric, wherever it is defined, iff* $B^{\hat g}$ *is identically zero there.*

Thus away from zeros of $s_{\hat g}$, $\hat g$ is extremal iff ric^g is J-invariant; this generalizes.

Proposition 4. *Let* $(\hat g, J)$ *be Kähler and suppose* $g=\varphi^{-2}\hat g$ *has* J*-invariant Ricci tensor. Then* $J\operatorname{grad}_{\hat g}\varphi$ *is a Killing vector field with respect to both* g *and* $\hat g$*.*

This follows by computing that $ric_0^g = ric_0^{\hat g} + 2\varphi^{-1}(D^{\hat g}d\varphi)_0$.

1.4. **The Einstein–Maxwell condition.** Let ω_+ and ω_- be closed (hence harmonic) selfdual and antiselfdual 2-forms (respectively) on an oriented riemannian 4-manifold (M,g). Then the Einstein–Maxwell condition in general relativity has a riemannian analogue in which the traceless Ricci tensor ric_0^g satisfies

$$ric_0^g(X,Y) = c\, g(\omega_+(X), \omega_-(Y)) \tag{3}$$

for constant c [33]. If $c = 0$, g is Einstein, while in general, the right hand side is divergence-free, and so (3) implies $\delta^g ric_0^g = 0$, or equivalently, by the contracted Bianchi identity, g is a CSC metric. A converse is available when g is conformal to a Kähler metric $(\hat{g}, J)$ with Kähler form $\omega^{\hat{g}} = \omega_+$ (cf. [33] for the case $g = \hat{g}$).

Proposition 5. *Let (M, g, J) be a hermitian 4-manifold with g conformal to a Kähler metric $(\hat{g}, \omega^{\hat{g}})$. Then g satisfies the Einstein–Maxwell equation (3), for some $\omega_\pm$ with $d\omega_- = 0$ and $\omega_+ = \omega^{\hat{g}}$, iff g is a CSC metric with J-invariant Ricci tensor.*

Proof. Clearly, (3) implies that ric^g is J-invariant. Writing $\hat{g} = f^{-2}g$, (3) with $\omega_+ = \omega^{\hat{g}}$, is then equivalent to $\omega_+(f^4 \operatorname{Ric}_0^g(\cdot), \cdot)$ being a constant multiple of ω_-, where $ric_0^g(X, Y) = g(\operatorname{Ric}_0^g(X), Y)$. Thus we require that $\omega^{\hat{g}}(f^4 \operatorname{Ric}_0^g(\cdot), \cdot)$ is closed, or equivalently co-closed. However, the conformal invariance of the divergence on symmetric traceless tensors of weight -4 implies that $\delta^{\hat{g}}(f^4 \operatorname{Ric}_0^g) = f^6\, \delta^g \operatorname{Ric}_0^g$. Hence (since $\omega^{\hat{g}}$ is $D^{\hat{g}}$-parallel) (3) holds iff ric_0^g is J-invariant and divergence-free. □

2. Ambikähler 4-manifolds and Einstein metrics

2.1. Ambihermitian and ambikähler structures.

Definition 1. Let M be a 4-manifold. An *ambihermitian structure* is a triple (c, J_+, J_-) consisting of a conformal metric c and two c-orthogonal complex structures $J_\pm$ such that J_+ and J_- induce opposite orientations on M.[2]

A compatible metric $g = \ell^{-2}c$ is called an *ambihermitian metric* on (M, J_+, J_-) and we denote by $\omega_\pm^g$ (resp. $\theta_\pm^g$) the fundamental 2-forms (resp. the Lee forms) of the hermitian metrics $(g, J_\pm)$. A symmetric tensor $S \in C^\infty(M, S^2T^*M)$ is *diagonal* if it is both J_+ and J_- invariant.

The following elementary and well-known observation will be used throughout.

Lemma 1. *Let M be a 4-manifold endowed with a pair (J_+, J_-) of almost complex structures inducing different orientations on M. Then M admits a conformal metric c for which both J_+ and J_- are orthogonal iff J_+ and J_- commute. In this case, the tangent bundle TM splits as a c-orthogonal direct sum*

$$TM = T_+M \oplus T_-M$$

of $J_\pm$-invariant rank 2 subbundles $T_\pm M$ defined as the ± 1-eigenbundles of $-J_+J_-$. (Thus a tangent vector X belongs to $T_\pm M$ iff $J_+X = \pm J_-X$.)

It follows that an ambihermitian metric g is equivalently given by a pair of commuting complex structures on M and hermitian metrics on each of the complex line subbundles T_+M and T_-M. Also any diagonal symmetric tensor S may be written $S(X, Y) = f\, g(X, Y) + h\, g(J_+J_-X, Y)$ for functions f, h.

Definition 2. An ambihermitian conformal 4-manifold (M, c, J_+, J_-) is called *ambikähler* if it admits ambihermitian metrics g_+ and g_- such that (g_+, J_+) and (g_-, J_-) are Kähler metrics.

With slight abuse of notation, we denote henceforth by ω_+ and ω_- the corresponding (symplectic) Kähler forms, thus omitting the upper indices indicating the corresponding Kähler metrics g_+ and g_-. Similarly we set $v_\pm = \frac{1}{2}\omega_\pm \wedge \omega_\pm$.

[2] The prefix *ambi-* means "on both sides", often left and right: ambihermitian structures have complex structures of either handedness (orientation); they should be contrasted (and not confused) with *bihermitian structures* where $J_\pm$ induce the *same* orientation on M.

2.2. **Type D Einstein metrics and Bach-flat ambikähler structures.** Proposition 1 shows that ambikähler structures have degenerate half-Weyl tensors. A converse is available for 4-dimensional Einstein metrics with degenerate half-Weyl tensors $W^{\pm}$ (riemannian analogues of Petrov type D vacuum spacetimes).

If $W^{\pm}$ both vanish, then g has constant curvature, i.e., is locally isometric to $S^4, \mathbb{R}^4$ or H^4, hence locally ambikähler. If instead g is half conformally-flat but not conformally-flat, we can assume (reversing orientation if necessary) that $W^- = 0$, $W^+ \neq 0$. Then, W^+ is degenerate iff g is an selfdual Einstein hermitian metric (see [3] for a classification). In either case, the underlying conformal structure of the Einstein metric is ambikähler with respect to some hermitian structures $J_{\pm}$ (see also the proof of Theorem 1 below). In the case that W^+ and W^- are both nonvanishing and degenerate, we may apply Proposition 2 to obtain a canonically defined ambikähler structure. The following proposition summarizes the situation.

Proposition 6. *For an oriented conformal* 4*-manifold* (M, c) *with a compatible Einstein metric* $g = \ell^{-2}c$*, the following three conditions are equivalent*:

- *both half-Weyl tensors* W^+ *and* W^- *are degenerate*;
- *about each point of* M *there exists a pair of complex structures* J_+ *and* J_- *such that* (c, J_+, J_-) *is ambihermitian*;
- *about each point* M *there exists a pair of complex structures* J_+ *and* J_- *such that* (c, J_+, J_-) *is ambikähler.*

If M *is simply connected and* $W^{\pm}$ *are both nonzero, then the compatible ambikähler structure* (J_+, J_-) *is unique* (*up to signs of* $J_{\pm}$) *and globally defined.*

We now characterize Einstein metrics among ambikähler structures.

Proposition 7. *Let* (M, c, J_+, J_-) *be a connected ambikähler* 4*-manifold. Then* c *is Bach-flat iff there is a compatible Einstein metric* $g = \ell^{-2}c$ *defined on a dense open subset of* M*.*

Proof. If c is Bach-flat then by Proposition 3 both of the Kähler metrics (g_+, J_+) and (g_-, J_-) are extremal, so their scalar curvatures, s_+ and s_- have holomorphic gradients. By the unique continuation principle, each of $s_{\pm}$ is either nonvanishing on an open dense subset of M or is identically zero. Hence if neither of the conformal Einstein metrics $s_+^{-2}g_+$ and $s_-^{-2}g_-$ are defined on a dense open subset of M, $s_{\pm}$ are both identically zero, which implies $W^{\pm} = 0$; then c is a flat conformal structure and there are compatible Einstein metrics on any simply connected open subset of M.

Conversely if there is compatible Einstein metric on a dense open subset, then, as already noted, B vanishes identically there, hence everywhere by continuity. □

The following lemma provides a practical way to apply this characterization.

Lemma 2. *Let* (M, c, J_+, J_-) *be a connected ambikähler conformal* 4*-manifold which is not conformally-flat and for which the corresponding Kähler metrics* g_+, g_- *are extremal, but not homothetic. Then* c *is Bach-flat iff the scalar curvatures* $s_{\pm}$ *of* $g_{\pm}$ *are related by*

$$C_+ s_- = C_- \Big(\frac{-v_-}{v_+}\Big)^{1/4} s_+, \tag{4}$$

where $C_{\pm}$ *are constants not both zero and* $v_{\pm}$ *are the volume forms of* $(g_{\pm}, J_{\pm})$.

Proof. If s_+ or s_- is identically zero, (M, c) is half-conformally-flat and (with C_+ or C_- zero) the result is trivial. Otherwise, if c is Bach-flat, $s_+^{-2}g_+$ and $s_-^{-2}g_-$ are Einstein metrics defined on open sets with dense intersection, so they must be homothetic, since (M,c) is not conformally-flat. Thus

$$s_+^{-2} g_+ = C\, s_-^{-2} g_-, \tag{5}$$

for a positive real number C, and (4) holds with $(C_-/C_+)^2 = C$.

Conversely, with $s_\pm \neq 0$, (4) implies (5), and we may choose $g_\pm$ so that $g := s_+^{-2} g_+ = s_-^{-2} g_-$ (i.e., $C = 1$). By Proposition 3, $\delta^g W^+ = 0 = \delta^g W^-$ and

$$B^g = W^\pm *_g ric_0^g. \tag{6}$$

Moreover, since (g_+, J_+) and (g_-, J_-) are both extremal by assumption, ric_0^g is diagonal, hence a pointwise multiple κ of $(J_+J_-)_g := g(J_+J_-\cdot,\cdot)$. Relation (6) can then be rewritten as

$$B^g = \kappa\, W^\pm *_g (J_+J_-)_g = \kappa\, W^\pm *_{g_\pm} (J_+J_-)_{g_\pm} = \tfrac{1}{6}\kappa\, s_\pm (J_+J_-)_{g_\pm} = \tfrac{1}{6}\kappa\, s_\pm^3 (J_+J_-)_g.$$

We deduce that $\kappa\, s_+^3 = \kappa\, s_-^3$. Since g_+ and g_- are not homothetic, s_+ and s_- are not identical; since they have holomorphic gradients, they are then not equal on a dense open set. Thus $\kappa = 0$, g is Einstein and $B^g = 0$. □

3. Ambitoric geometry

Ambitoric geometry concerns ambikähler structures for which both Kähler metrics are toric with respect to a common T^2-action; the pointwise geometry is the following.

Definition 3. An ambikähler 4-manifold (M, c, J_+, J_-) is said to be *ambitoric* iff it is equipped with a 2-dimensional family $\mathfrak{t}$ of vector fields which are linearly independent on a dense open set, and are Poisson-commuting hamiltonian Killing vector fields with respect to both Kähler structures $(g_\pm, J_\pm, \omega_\pm)$.

Hamiltonian vector fields $K = \mathrm{grad}_\omega f$ and $\tilde K = \mathrm{grad}_\omega \tilde f$ Poisson commute (i.e., $\{f, \tilde f\} = 0$) iff they are *isotropic* in the sense that $\omega(K, \tilde K) = 0$; it then follows that K and $\tilde K$ commute (i.e., $[K, \tilde K] = 0$). Thus $\mathfrak{t}$ is an abelian Lie algebra under Lie bracket of vector fields.

We further motivate the definition by examples in the following subsections.

3.1. Orthotoric Kähler surfaces are ambitoric.

Definition 4. [4] A Kähler surface (M, g, J) is *orthotoric* if it admits two independent hamiltonian Killing vector fields, K_1 and K_2, with Poisson-commuting momenta $x+y$ and xy, respectively, where x and y are smooth functions with dx and dy orthogonal.

The following result is an immediate corollary to [4, Props. 8 & 9].

Proposition 8. *Any orthotoric Kähler surface* (M, g_+, J_+, K_1, K_2) *admits a canonical opposite hermitian structure* J_- *(up to sign) with respect to which* M *is ambitoric with* $\mathfrak{t} = <\{K_1, K_2\}>$.

3.2. Ambitoric Kähler surfaces of Calabi type.

Definition 5. [4] A Kähler surface (M, g_+, J_+) is said to be of *Calabi type* if it admits a nonvanishing hamiltonian Killing vector field K such that the negative almost-hermitian pair (g_+, J_-)—with J_- equal to J_+ on the distribution spanned by K and J_+K, but $-J_+$ on the orthogonal distribution—is conformally Kähler.

Thus, any Kähler surface of Calabi type is canonically ambikähler. An explicit formula for Kähler metrics of Calabi type, using the LeBrun normal form [31] for a

Kähler metric with a hamiltonian Killing vector field, is obtained in [4, Prop. 13]: (g_+, J_+, ω_+) is given locally by

$$
\begin{aligned}
g_+ &= (az - b)g_\Sigma + w(z)dz^2 + w(z)^{-1}(dt + \alpha)^2, \\
\omega_+ &= (az - b)\omega_\Sigma + dz \wedge (dt + \alpha), \quad d\alpha = a\omega_\Sigma,
\end{aligned}
\tag{7}
$$

where z is the momentum of the Killing vector field, t is a function on M with $dt(K) = 1$, $w(z)$ is function of one variable, g_Σ is a metric on a 2-manifold Σ with area form ω_Σ, α is a 1-form on Σ and a, b are constant.

The second conformal Kähler structure is then given by

$$
\begin{aligned}
g_- &= (az - b)^{-2}g_+, \\
\omega_- &= (az - b)^{-1}\omega_\Sigma - (az - b)^{-2}dz \wedge (dt + \alpha).
\end{aligned}
$$

Note that the $\big(\Sigma, (az - b)\omega_\Sigma, (az - b)g_\Sigma\big)$ is identified with the *Kähler quotient* of (M, g_+, ω_+) at the value z of the momentum. We conclude as follows.

Proposition 9. *An ambikähler structure of Calabi type is ambitoric—with respect to Killing vector fields K_1, K_2 with $K \in <\{K_1, K_2\}>$—iff $(\Sigma, g_\Sigma, \omega_\Sigma)$ admits a hamiltonian Killing vector field.*

We shall refer to ambitoric 4-manifolds arising locally from Proposition 9 as *ambitoric Kähler surfaces of Calabi type.* A more precise description is as follows.

Definition 6. An ambitoric 4-manifold (M, c, J_+, J_-) is said to be of Calabi type if the corresponding 2-dimensional family of vector fields contains one, say K, with respect to which the Kähler metric (g_+, J_+) (equivalently, (g_-, J_-)) is of Calabi type on the dense open set where K is nonvanishing; without loss, we can then assume that $J_+ = J_-$ on $<\{K, J_+K\}>$.

Note that this definition includes the case of a local Kähler product of two Riemann surfaces each admitting a nontrivial Killing vector field (when we have $a = 0$ in (7)). In the non-product case we can assume without loss $a = 1, b = 0$; hence

$$
\begin{aligned}
g_+ &= zg_\Sigma + \frac{z}{V(z)}dz^2 + \frac{V(z)}{z}(dt + \alpha)^2, \\
\omega_+ &= z\omega_\Sigma + dz \wedge (dt + \alpha), \qquad d\alpha = \omega_\Sigma,
\end{aligned}
\tag{8}
$$

while the other Kähler metric $(g_- = z^{-2}g_+, J_-)$ is also of Calabi type with respect to $K = \partial/\partial t$, with momentum $\bar{z} = z^{-1}$ and $\bar{V}(\bar{z}) = \bar{z}^4 V(1/\bar{z}) = V(z)/z^4$.

The form (8) of a non-product Kähler metric of Calabi type is well adapted to curvature computations. For this paper, we need the following local result.

Proposition 10. *Let (M, g_+, J_+) be a non-product Kähler surface of Calabi type with respect to K. Denote by J_- the corresponding negative hermitian structure and by g_- the conformal Kähler metric with respect to J_-.*

- *(g_+, J_+) is extremal iff (g_-, J_-) is extremal and this happens precisely when (Σ, g_Σ) in (8) is of constant Gauss curvature k and $V(z) = a_0z^4 + a_1z^3 + kz^2 + a_3z + a_4$. In particular, (c, J_+, J_-) is locally ambitoric.*
- *The conformal structure is Bach-flat iff, in addition, $4a_0a_4 - a_1a_3 = 0$.*
- *(g_+, J_+) is CSC iff it is extremal with $a_0 = 0$, and Kähler–Einstein iff also $a_3 = 0$.*

Proof. The result is well-known under the extra assumption that the scalar curvature s_+ of the extremal Kähler metric g_+ is a Killing potential for a multiple of K (see e.g., [4, Prop. 14]). However, one can show [7, Prop. 5] that the later assumption is, in fact, necessary for g_+ to be extremal. □

3.3. **Ambikähler metrics with diagonal Ricci tensor.** If an ambihermitian metric (g, J_+, J_-) has diagonal Ricci tensor ric^g, then by Proposition 1, $W^\pm$ are degenerate, and hence the Lee forms $\theta^g_\pm$ of $J_\pm$ have the property that $d\theta^g_+$ is antiselfdual, while $d\theta^g_-$ is selfdual. Let us suppose that $d\theta^g_\pm = 0$, so that (g, J_+, J_-) is locally ambikähler. (We have seen that this holds if g is Einstein, but it is also automatic if M is compact, or if $\theta^g_+ + \theta^g_-$ is closed.)

On an open set where the Kähler metrics $g_\pm = \varphi_\pm^2 g$—with Kähler forms $\omega_\pm = g_\pm(J_\pm\cdot,\cdot)$—are defined, Proposition 4 implies that $\varphi_\pm$ are Killing potentials with respect to $(g_\pm, J_\pm)$ respectively. The corresponding hamiltonian Killing vector fields $Z_\pm = \mathrm{grad}_{\omega_\pm} \varphi_\pm$ are also Killing vector fields of g, since they preserve $\varphi_\pm$ respectively. Hence they also preserve ric^g, W^+ and W^-. We shall further suppose that Z_+ preserves J_-, which is automatic unless g is selfdual Einstein, and that Z_- preserves J_+, which is similarly automatic unless g is antiselfdual Einstein.

Proposition 11. *Let $(g_\pm, J_\pm, \omega_\pm)$ be ambikähler, and suppose $g = \varphi_\pm^{-2} g_\pm$ is a compatible metric with diagonal Ricci tensor such that $Z_\pm = \mathrm{grad}_{\omega_\pm} \varphi_\pm$ preserve both J_+ and J_-. Then precisely one of the following cases occurs:*

(i) *Z_+ and Z_- are both identically zero and then (M, c, J_+, J_-) is a locally a Kähler product of Riemann surfaces;*

(ii) *$Z_+ \otimes Z_-$ is identically zero, but Z_+ and Z_- are not both identically zero, and then (M, c, J_+, J_-) is either orthotoric or of Calabi type;*

(iii) *$Z_+ \wedge Z_-$ is identically zero, but $Z_+ \otimes Z_-$ is not, and then (M, c, J_+, J_-) is either ambitoric or of Calabi type;*

(iv) *$Z_+ \wedge Z_-$ is not identically zero, and then (M, c, J_+, J_-) is ambitoric.*

In particular (M, c, J_+, J_-) is either a local product, of Calabi type, or ambitoric.

Proof. We first note that Z_+ and Z_- preserve both Lee forms $\theta^g_\pm = \varphi_\pm^{-1} d\varphi_\pm$, and hence $\theta^g_\pm(Z_\mp)Z_\pm + [Z_\mp, Z_\pm] = 0$, with $\theta^g_\pm(Z_\mp) = c_\pm$ constant. Hence $c_+Z_+ + c_-Z_- = 0$, so $[Z_+, Z_-] = 0$ and $c_\pm Z_\pm = 0$, which forces $c_\pm = 0$ (since $Z_\pm = 0$ implies $\theta^g_\pm = 0$). We now have $d\varphi_\pm(Z_\mp) = 0$, so $\omega_\pm(Z_+, Z_-) = 0$.

By connectedness and unique continuation for holomorphic vector fields, conditions (i)–(iv) are mutually exclusive and the open condition in each case holds on a dense open set. Case (i) is trivial: here $g = g_+ = g_-$ is Kähler and J_+J_- is a D^g-parallel product structure.

In case (ii) either Z_+ or Z_- is zero on each component of the dense open set where they are not both zero. Suppose, without loss that $Z_+ = 0$ so that $g = g_+$ and $Z_- = J_- \mathrm{grad}_{g_-} \varphi_- = J_- \mathrm{grad}_g \lambda$ with $\lambda = -1/\varphi_-$. However, since Z_- also preserves ω_+, $J_+J_- d\lambda$ is closed, hence locally equal to $\frac{1}{2} d\sigma$ for a smooth function σ. According to [4, Remark 2], the 2-form $\varphi := \frac{3}{2}\sigma\omega_+ + \lambda^3\omega_-$ is hamiltonian with respect to the Kähler metric (g_+, J_+); by [4, Theorems 1 & 3], this means that $g = g_+$ is either orthotoric (on a dense open subset of M), or is of Calabi type.

In case (iii) Z_+ and Z_- are linearly dependent, but are both nonvanishing on a dense open set. Hence, we may assume, up to rescaling on each component of this dense open set, that $Z := Z_+ = Z_-$. This is equivalent to

$$J_+\Big(\frac{d\varphi_+}{\varphi_+^2}\Big) = J_-\Big(\frac{d\varphi_-}{\varphi_-^2}\Big), \tag{9}$$

and hence also

$$2J_\pm d\Big(\frac{1}{\varphi_+\varphi_-}\Big) = J_\mp d\Big(\frac{1}{\varphi_+^2} + \frac{1}{\varphi_-^2}\Big).$$

Since $h\,g$, with $h = 1/\varphi_+\varphi_-$, is the barycentre of g_+ and g_-, it follows (cf. [29] and Appendix B.2) that the symmetric tensor $g(S\cdot,\cdot)$, where $S = f\,Id + hJ_+J_-$ and $2f = 1/\varphi_+^2 + 1/\varphi_-^2$, is a Killing tensor with respect to g. Clearly $\mathcal{L}_Z S = 0$, and it follows from (9) that $D^g Z^\flat$ is both J_+ and J_- invariant. Thus $X \mapsto D^g_X Z$ commutes with S and $D^g_Z S = 0$. Straightforward computations now show that SZ is a Killing field with respect to g, and hamiltonian with respect to $\omega_\pm$.

Moreover, Z and SZ commute and span a isotropic subspace with respect to $\omega_\pm$, so define an ambitoric structure on the open set where they are linearly independent. Clearly Z and SZ are linearly dependent only where J_+J_-Z is proportional to Z, in which case $g_\pm$ is of Calabi type.

Case (iv) follows by definition. □

Proof of Theorem 2. For the first part, if $g = \varphi_\pm^{-2} g_\pm$ has diagonal Ricci tensor, Proposition 11 implies the existence of an ambitoric structure once we show that $\mathcal{L}_{Z_+} J_- = 0 = \mathcal{L}_{Z_-} J_+$ where $Z_\pm = \mathrm{grad}_{\omega_\pm} \varphi_\pm = -J_\pm \,\mathrm{grad}_g \varphi_\pm^{-1}$ are the corresponding Killing vector fields of g. As already observed, this is automatic unless g is Einstein and (anti)selfdual. By assumption and without loss of generality, we may suppose g is a selfdual Einstein metric with nonzero scalar curvature s_g which is not antiselfdual. As W^+ does not vanish identically, it determines J_+ up to sign, and so $\mathcal{L}_{Z_-} J_+ = 0$. Since $Z_+ = -J_+ \,\mathrm{grad}_g |W^+|_g^{-1/3}$ it follows that $[Z_-, Z_+] = 0$. In order to show $\mathcal{L}_{Z_+} J_- = 0$, we recall that negative Kähler metrics g_- in the conformal class are in a bijection with antiselfdual twistor 2-forms ψ (see [39] and Appendix B), the latter being defined by the property that there is a 1-form α such that $D^g_X \psi = (\alpha \wedge X^\flat)^-$ for any vector field X, where $(\cdot)^-$ denotes the antiselfdual part. Specifically, in our case, $\psi = \varphi_-^{-1}\omega_-$ and $\alpha = 2Z_-^\flat$. Since $\mathcal{L}_{Z_+} Z_-^\flat = 0$, $\mathcal{L}_{Z_+}\psi$ is a parallel antiselfdual 2-form. As g is selfdual with nonzero scalar curvature, the Bochner formula shows there are no non-trivial parallel antiselfdual 2-forms; hence $\mathcal{L}_{Z_+}\psi = 0$ and so $\mathcal{L}_{Z_+} J_- = 0$.

In the other direction, we shall see later in Proposition 13 that any regular ambitoric structure admits compatible metrics with diagonal Ricci tensor. The characterization of the Plebański-Demiański case now follows from Proposition 5.

For the second part, Proposition 3 implies that an ambikähler structure $(g_\pm, \omega_\pm, J_\pm)$ has diagonal Bach tensor iff both Kähler metrics are extremal. The assumption on the conformal class ensures that it is not conformally flat and hence the corresponding scalar curvatures $s_\pm$ do not both vanish identically, so that, using Proposition 3 again, the metric $g = s_+^{-2} g_+$ say is well-defined with diagonal Ricci tensor on a dense open subset of M. By Proposition 11 (noting that $Z_\pm = J_\pm \,\mathrm{grad}_{g_\pm} s_\pm$ are well-defined on M) we conclude that $(g_\pm, \omega_\pm, J_\pm)$ is ambitoric. □

3.4. **Ambihermitian Einstein 4-manifolds are locally ambitoric.** Proposition 6 implies that any Einstein metric with degenerate half Weyl tensors—in particular, any ambihermitian Einstein metric—is ambikähler and Bach-flat. Conversely, Bach-flat ambikähler metrics $(g_\pm, J_\pm)$ are conformal to an Einstein metric g on a dense open set by Proposition 7

In the generic case that $W^\pm$ are both nonzero, the ambikähler metrics conformal to g are $g_\pm = |W^\pm|_g^{2/3} g$, and the Einstein metric is recovered up to homothety as $g = s_\pm^{-2} g_\pm$, where $s_\pm$ is the scalar curvature of $g_\pm$. We have already noted that the vector fields $Z_\pm := J_\pm \,\mathrm{grad}_{g_\pm} s_\pm$ are Killing with respect to $g_\pm$ (respectively) and hence also g. More is true.

Proposition 12. *Let (M, c, J_+, J_-) be a Bach-flat ambikähler manifold such that the Kähler metrics $g_\pm$ have nonvanishing scalar curvatures $s_\pm$. Then the vector fields*

$Z_\pm = J_\pm \operatorname{grad}_{g_\pm} s_\pm$ *are each Killing with respect to both* g_+ *and* g_-*, holomorphic with respect to both* J_+ *and* J_-*, and isotropic with respect to both* ω_+ *and* ω_- *(i.e.,* $\omega_\pm(Z_+, Z_-) = 0$*); in particular* Z_+ *and* Z_- *commute.*

Furthermore (M, c, J_+, J_-) *is ambitoric in a neighbourhood of any point in a dense open subset, and on a neighbourhood of any point where* Z_+ *and* Z_- *are linearly independent, we may take* $\mathfrak{t} = <\{Z_+, Z_-\}>$*.*

Proof. Z_+ and Z_- are conformal vector fields, so they preserve $W^\pm$ and its unique simple eigenspaces. One readily concludes [2, 19] that the Lie derivatives of g_+, g_-, J_+, J_- (and hence also ω_+ and ω_-) all vanish. Consequently, $\mathcal{L}_{Z_+} s_- = 0 = \mathcal{L}_{Z_-} s_+$— or equivalently $\omega_\pm(Z_+, Z_-) = 0$. This proves the first part.

Since we are now in the situation of Proposition 11, it remains to show that (M, c, J_+, J_-) is locally ambitoric even in cases where Proposition 11 only asserts that the structure has Calabi type. In case (i) this is easy: $g = g_+ = g_-$ is Kähler–Einstein with D^g-parallel product structure, so is the local product of two Riemann surfaces with constant Gauss curvatures.

In case (ii) $g = g_+$ is Kähler–Einstein, Proposition 10 implies that the quotient Riemann surface (Σ, g_Σ) has constant Gauss curvature.

In case (iii) $g_\pm$ are extremal, so we have either a local product of two extremal Riemann surfaces or, in Proposition 10, the quotient Riemann surface (Σ, g_Σ) has constant Gauss curvature; it follows that g_+ is locally ambitoric of Calabi type. □

Remark 2. The case $Z_+ = 0$ above yields the following observation of independent interest: let (M, g, J, ω) be a Kähler–Einstein 4-manifold with everywhere degenerate antiselfdual Weyl tensor W^-, and trivial first deRham cohomology group. Then (M, g, J, ω) admits a globally defined hamiltonian 2-form in the sense of [4] and, on a dense open subset M^0, the metric is one of the following: a Kähler product metric of two Riemann surfaces of equal constant Gauss curvatures, or a Kähler–Einstein metric of Calabi type, described in Proposition 10, or a Kähler–Einstein ambitoric metric of parabolic type (see section 5.4).

Proof of Theorem 1. For the first part, if W^+ and W^- identically vanish, we have a real space form and g is locally conformally-flat (and is obviously locally ambitoric).

If g is half-conformally-flat but not flat, then g admits a canonically defined hermitian structure $J = J_+$, i.e., g is an Einstein, hermitian self-dual metric (see [3] for a classification). In particular, g is an Einstein metric conformal to a self-dual (or, equivalently, Bochner-flat) Kähler metric (g_+, J_+). We learn from [11, 4] that such a Kähler metric must be either orthotoric or of Calabi type over a Riemann surface (Σ, g_Σ) of constant Gauss curvature. In both cases the metric is locally ambitoric by the examples discussed in the previous subsections.

In the generic case, the result follows from Propositions 6, 7 and 12.

The last part follows directly from Proposition 7. □

4. Local classification of ambitoric structures

To classify ambitoric structures on the dense open set where the (local) torus action is free (cf. [24] for the toric case), let (M, c, J_+, J_-) denote a connected, simply connected, ambihermitian 4-manifold and $\boldsymbol{K}: \mathfrak{t} \to C^\infty(M, TM)$ a 2-dimensional family of pointwise linearly independent vector fields. Let $\varepsilon \in \wedge^2\mathfrak{t}^*$ be a fixed area form.

4.1. **Holomorphic lagrangian torus actions.** We denote by K_λ the image of $\lambda \in \mathfrak{t}$ under $\boldsymbol{K}$, by $\mathfrak{t}_M$ the rank 2 subbundle of TM spanned by these vector fields, and by $\boldsymbol{\theta} \in \Omega^1(M, \mathfrak{t})$ the $\mathfrak{t}$-valued 1-form vanishing on $\mathfrak{t}_M^\perp \subset TM$ with $\boldsymbol{\theta}(K_\lambda) = \lambda$.

We first impose the condition that $\boldsymbol{K}$ is an infinitesimal $J_\pm$-holomorphic and $\omega_\pm$-isotropic (hence lagrangian) torus action. We temporarily omit the $\pm$ subscript, since we are studying the complex structures separately. The lagrangian condition means that $\mathfrak{t}_M$ is orthogonal and complementary to its image $J\mathfrak{t}_M$ under the complex structure J; thus $J\mathfrak{t}_M = \mathfrak{t}_M^\perp$. The remaining conditions (including the integrability of J) imply that the vector fields $\{K_\lambda : \lambda \in \mathfrak{t}\}$ and $\{JK_\lambda : \lambda \in \mathfrak{t}\}$ all commute under Lie bracket, or equivalently that the dual 1-forms $\boldsymbol{\theta}$ and $J\boldsymbol{\theta}$ are both closed. Thus we may write $\boldsymbol{\theta} = d\boldsymbol{t}$ with $dd^c\boldsymbol{t} = 0$, where $d^c\boldsymbol{t} = Jd\boldsymbol{t}$ and the "angular coordinate" $\boldsymbol{t}\colon M \to \mathfrak{t}$ is defined up to an additive constant. Conversely, if $dd^c\boldsymbol{t} = 0$ then $d\boldsymbol{t} - \sqrt{-1}d^c\boldsymbol{t}$ generates a closed differential ideal $\Omega^{(1,0)}$ for J so that J is integrable.

4.2. **Regular ambitoric structures.** We now combine this analysis for the complex structures $J_\pm$. It follows that $J_+\mathfrak{t}_M$ and $J_-\mathfrak{t}_M$ coincide and that $\mathfrak{t}_M$ is preserved by the involution $-J_+J_-$. Since the eigenbundles (pointwise eigenspaces) of $-J_+J_-$ are $J_\pm$-invariant, $\mathfrak{t}_M$ cannot be an eigenbundle and hence decomposes into $+1$ and -1 eigenbundles $\boldsymbol{\xi}_M$ and $\boldsymbol{\eta}_M$: the line bundles $\boldsymbol{\xi}_M$, $\boldsymbol{\eta}_M$, $J_+\boldsymbol{\xi}_M = J_-\boldsymbol{\xi}_M$ and $J_+\boldsymbol{\eta}_M = J_-\boldsymbol{\eta}_M$ provide an orthogonal direct sum decomposition of TM.

We denote the images of $\boldsymbol{\xi}_M$ and $\boldsymbol{\eta}_M$ under $d\boldsymbol{t}$ by $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ respectively. We thus obtain a smooth map $(\boldsymbol{\xi}, \boldsymbol{\eta})\colon M \to \mathrm{P}(\mathfrak{t}) \times \mathrm{P}(\mathfrak{t}) \setminus \Delta(\mathfrak{t})$ where $\Delta(\mathfrak{t})$ is the diagonal.

The derivatives $d\boldsymbol{\xi} \in \Omega^1(M, \boldsymbol{\xi}^*T\mathrm{P}(\mathfrak{t}))$ and $d\boldsymbol{\eta} \in \Omega^1(M, \boldsymbol{\eta}^*T\mathrm{P}(\mathfrak{t}))$ vanish on $\mathfrak{t}_M$ (since $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are $\mathfrak{t}$-invariant). In fact, more is true: they span orthogonal directions in T^*M. (Note that $\boldsymbol{\xi}^*T\mathrm{P}(\mathfrak{t}) \cong \mathrm{Hom}(\boldsymbol{\xi}, \underline{\mathfrak{t}}/\boldsymbol{\xi})$, with $\underline{\mathfrak{t}} := M \times \mathfrak{t}$, is a line bundle on M, and similarly for $\boldsymbol{\eta}^*T\mathrm{P}(\mathfrak{t})$.)

Lemma 3. *$d\boldsymbol{\xi}$ vanishes on $J_\pm\boldsymbol{\eta}_M$ and $d\boldsymbol{\eta}$ vanishes on $J_\pm\boldsymbol{\xi}_M$; hence $0 = d\boldsymbol{\xi} \wedge d\boldsymbol{\eta} \in \Omega^2(M, \boldsymbol{\xi}^*T\mathrm{P}(\mathfrak{t}) \otimes \boldsymbol{\eta}^*T\mathrm{P}(\mathfrak{t}))$ only on the subset of M where $d\boldsymbol{\xi} = 0$ or $d\boldsymbol{\eta} = 0$.*

Proof. The 1-form $(J_+ + J_-)d\boldsymbol{t}$ is closed, vanishes on $J_\pm\boldsymbol{\eta}_M$ and $\mathfrak{t}_M$, and takes values in $\boldsymbol{\xi} \subset \underline{\mathfrak{t}}$ (it is nonzero on $J_\pm\boldsymbol{\xi}_M$). Hence for any section u of $\boldsymbol{\xi} \subset M \times \mathfrak{t}$,

$$0 = d\left(\varepsilon(u, (J_+ + J_-)d\boldsymbol{t})\right) = \varepsilon(du \wedge (J_+ + J_-)d\boldsymbol{t})$$

and so $(du \mod \boldsymbol{\xi}) \wedge (J_+ + J_-)d\boldsymbol{t} = 0$. This implies that $d\boldsymbol{\xi}$ is a multiple F of $\frac{1}{2}(J_+ + J_-)d\boldsymbol{t} \in \Omega^1(M, \boldsymbol{\xi})$. Similarly $d\boldsymbol{\eta}$ is a multiple G of $\frac{1}{2}(J_+ - J_-)d\boldsymbol{t}$. $\square$

Corollary 2. *If $(M, g_\pm, J_\pm, \omega_\pm)$ is ambitoric with $(\boldsymbol{\xi}, \boldsymbol{\eta})$ as above, then there is a dense open set M^0 such that on each connected component, the ambitoric structure is either of Calabi type, or $d\boldsymbol{\xi} \wedge d\boldsymbol{\eta}$ is nonvanishing.*

Indeed, if $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are functionally dependent on an connected open set U, then one of the two is a constant $[\lambda] \in \mathrm{P}(\mathfrak{t})$ and U has Calabi type with respect to K_λ.

Definition 7. If $d\boldsymbol{\xi} \wedge d\boldsymbol{\eta}$ vanishes nowhere, we say $(M, c, J_+, J_-, \boldsymbol{K})$ is *regular.*

In the regular case $d\boldsymbol{\xi} = \frac{1}{2}F(\boldsymbol{\xi})(J_+ + J_-)d\boldsymbol{t}$ and $d\boldsymbol{\eta} = \frac{1}{2}G(\boldsymbol{\eta})(J_+ - J_-)d\boldsymbol{t}$, where F, G are local sections of $\mathcal{O}(3)$ over $\mathrm{P}(\mathfrak{t})$; more precisely $F(\boldsymbol{\xi})\colon M \to \mathrm{Hom}(\boldsymbol{\xi}, \boldsymbol{\xi}^*T\mathrm{P}(\mathfrak{t}))$ and similarly for $G(\boldsymbol{\eta})$, but $T\mathrm{P}(\mathfrak{t}) \cong \mathcal{O}(2)$ using ε. We let $\boldsymbol{\xi}^\natural$ denote the composite of $\boldsymbol{\xi}$ with the natural section of $\mathcal{O}(1) \otimes \mathfrak{t}$ over $\mathrm{P}(\mathfrak{t})$, and similarly $\boldsymbol{\eta}^\natural$. We construct from these $J_\pm$-related orthogonal 1-forms

$$\frac{d\boldsymbol{\xi}}{F(\boldsymbol{\xi})}, \qquad \frac{\varepsilon(d\boldsymbol{t}, \boldsymbol{\eta}^\natural)}{\varepsilon(\boldsymbol{\xi}^\natural, \boldsymbol{\eta}^\natural)}, \qquad \frac{d\boldsymbol{\eta}}{G(\boldsymbol{\eta})}, \qquad \frac{\varepsilon(\boldsymbol{\xi}^\natural, d\boldsymbol{t})}{\varepsilon(\boldsymbol{\xi}^\natural, \boldsymbol{\eta}^\natural)}$$

(with values in the line bundles $\boldsymbol{\xi}^*$ or $\boldsymbol{\eta}^*$) which may be used to write any $\mathfrak{t}$-invariant metric g in the conformal class as

$$\frac{d\boldsymbol{\xi}^2}{F(\boldsymbol{\xi})U(\boldsymbol{\xi}, \boldsymbol{\eta})} + \frac{d\boldsymbol{\eta}^2}{G(\boldsymbol{\eta})V(\boldsymbol{\xi}, \boldsymbol{\eta})} + \frac{F(\boldsymbol{\xi})}{U(\boldsymbol{\xi}, \boldsymbol{\eta})}\left(\frac{\varepsilon(d\boldsymbol{t}, \boldsymbol{\eta}^\natural)}{\varepsilon(\boldsymbol{\xi}^\natural, \boldsymbol{\eta}^\natural)}\right)^2 + \frac{G(\boldsymbol{\eta})}{V(\boldsymbol{\xi}, \boldsymbol{\eta})}\left(\frac{\varepsilon(\boldsymbol{\xi}^\natural, d\boldsymbol{t})}{\varepsilon(\boldsymbol{\xi}^\natural, \boldsymbol{\eta}^\natural)}\right)^2.$$

Here U and V are local sections of $\mathcal{O}(1,0)$ and $\mathcal{O}(0,1)$ over $\mathrm{P}(\mathfrak{t}) \times \mathrm{P}(\mathfrak{t}) \setminus \Delta(\mathfrak{t})$.

More concretely, in a neighbourhood of any point, a basis $\lambda = 1, 2$ for $\mathfrak{t}$ may be chosen to provide an affine chart for $\mathrm{P}(\mathfrak{t})$ so that $K_\xi := \xi K_1 - K_2$ and $K_\eta := \eta K_1 - K_2$ are sections of $\boldsymbol{\xi}_M$ and $\boldsymbol{\eta}_M$ respectively, where $\xi > \eta$ are functionally independent coordinates on M. The components of $\boldsymbol{t}\colon M \to \mathfrak{t}$ in this basis complete a coordinate system (ξ, η, t_1, t_2) with coordinate vector fields

$$\frac{\partial}{\partial \xi} = \frac{J_+ K_\xi}{F(\xi)}, \qquad \frac{\partial}{\partial \eta} = \frac{J_+ K_\eta}{G(\eta)}, \qquad \frac{\partial}{\partial t_1} = K_1, \qquad \frac{\partial}{\partial t_2} = K_2.$$

Replacing (J_+, J_-) with $(-J_+, -J_-)$ if necessary, we can assume without loss that F and G (now functions of one variable) are both positive, and thus obtain the following description of $\mathfrak{t}$-invariant ambihermitian metrics in the conformal class.

Lemma 4. *An ambihermitian metric (g, J_+, J_-) which is regular with respect to a 2-dimensional family of commuting, $J_\pm$-holomorphic lagrangian Killing vector fields is given locally by*

$$g = \frac{d\xi^2}{F(\xi)U(\xi,\eta)} + \frac{d\eta^2}{G(\eta)V(\xi,\eta)} + \frac{F(\xi)(dt_1 + \eta\, dt_2)^2}{U(\xi,\eta)\,(\xi-\eta)^2} + \frac{G(\eta)(dt_1 + \xi\, dt_2)^2}{V(\xi,\eta)\,(\xi-\eta)^2}, \tag{10}$$

$$\omega^g_\pm = \frac{d\xi \wedge (dt_1 + \eta\, dt_2)}{U(\xi,\eta)\,(\xi-\eta)} \pm \frac{d\eta \wedge (dt_1 + \xi\, dt_2)}{V(\xi,\eta)\,(\xi-\eta)}, \tag{11}$$

$$d^c_+\xi = d^c_-\xi = F(\xi)\frac{dt_1 + \eta\, dt_2}{\xi - \eta}, \quad d^c_+\eta = -d^c_-\eta = G(\eta)\frac{dt_1 + \xi\, dt_2}{\xi - \eta} \tag{12}$$

for some positive functions U and V of two variables, and some positive functions F and G of one variable. (*Here and later, $d^c_\pm h = J_\pm dh$ for any function h.*)

We now impose the condition that (c, J_+) and (c, J_-) admit $\mathfrak{t}$-invariant Kähler metrics g_+ and g_-. Let f be the conformal factor relating $g_\pm$ by $g_- = f^2 g_+$. Clearly f is $\mathfrak{t}$-invariant and so, therefore, is the metric

$$g_0 := f\, g_+ = f^{-1}\, g_-$$

which we call the *barycentric metric* of the ambitoric structure. The Lee forms, $\theta^0_\pm$, of $(g_0, J_\pm)$ are given by $\theta^0_\pm = \mp\frac{1}{2}\log f$. Conversely, suppose there is an invariant ambihermitian metric g_0 in the conformal class whose Lee forms $\theta^0_\pm$ satisfy

$$\theta^0_+ + \theta^0_- = 0 \tag{13}$$

$$d(\theta^0_+ - \theta^0_-) = 0. \tag{14}$$

Then writing locally $\theta^0_+ = -\frac{1}{2} d\log f = -\theta^0_-$ for some positive function f, the metrics $g_\pm := f^{\mp 1} g_0$ are Kähler with respect to $J_\pm$ respectively.

Thus, regular ambitoric conformal structures are defined by ambihermitian metrics g_0 given locally by Lemma 4, and whose Lee forms $\theta^0_\pm$ satisfy (13) and (14).

Lemma 5. *For an ambihermitian metric given by Lemma 4 the relation* (13) *is satisfied* (*with $g_0 = g$*) *iff $U = U(\xi)$ is independent of η and $V = V(\eta)$ is independent of ξ. In this case* (14) *is equivalent to $U(\xi)^2 = R(\xi)$ and $V(\eta)^2 = R(\eta)$, where $R(s) = r_0 s^2 + 2r_1 s + r_2$ is a polynomial of degree at most two.*

Under both conditions, the conformal factor f with $g_- = f^2 g_+$ is given—up to a constant multiple—by

$$f(\xi,\eta) = \frac{R(\xi)^{1/2} R(\eta)^{1/2} + R(\xi,\eta)}{\xi - \eta} \tag{15}$$

where $R(\xi,\eta) = r_0\xi\eta + r_1(\xi+\eta) + r_2$ is the "polarization" of R.

Proof. The Lee forms $\theta^g_\pm$ are given by $2\theta^g_\pm = u_\pm d\xi + v_\pm d\eta$, with

$$u_\pm = \frac{V_\xi}{V} \pm \frac{V}{(\xi-\eta)U}, \qquad v_\pm = \frac{U_\eta}{U} \mp \frac{U}{(\xi-\eta)V}.$$

In particular, $u_+ + u_- = 2V_\xi/V$ and $v_+ + v_- = 2U_\eta/U$. It follows that $\theta^g_+ + \theta^g_- = 0$ iff $U_\eta = 0$ and $V_\xi = 0$. This proves the first part of the lemma.

If (13) is satisfied, then

$$\theta^g_+ = \frac{1}{2}\Big(\frac{V(\eta)}{(\xi-\eta)U(\xi)}\,d\xi - \frac{U(\xi)}{(\xi-\eta)V(\eta)}\,d\eta\Big).$$

It follows that $d\theta^g_+ = 0$ iff

$$2U^2(\xi) - (\xi-\eta)(U^2)'(\xi) = 2V^2(\eta) + (\xi-\eta)(V^2)'(\eta) \tag{16}$$

where $U^2(\xi) = U(\xi)^2$ and $V^2(\eta) = V(\eta)^2$. Differentiating twice with respect to ξ, we obtain $(\xi-\eta)(U^2)'''(\xi) = 0$, and similarly $(\xi-\eta)(V^2)'''(\eta) = 0$. Thus U^2 and V^2 are both polynomials of degree at most two. We may now set $\xi = \eta$ in (16) to conclude that U^2 and V^2 coincide. Without loss of generality, we assume that U and V are both positive everywhere, so that $U(\xi) = R(\xi)^{1/2}$ and $V(\eta) = R(\eta)^{1/2}$ for a polynomial R of degree at most two. By using the identity

$$R(\xi) - R(\eta) - \tfrac{1}{2}(\xi-\eta)(R'(\xi) + R'(\eta)) \equiv 0$$

we easily check (15). □

Note that the quadratic R is, more invariantly, a homogeneous polynomial of degree 2 on $\mathfrak{t}$ (an algebraic section of $\mathcal{O}(2)$ over $\mathrm{P}(\mathfrak{t})$). However the parameterization of ambitoric structures by R and the local sections F and G of $\mathcal{O}(3)$ is not effective because of the $\mathrm{SL}(\mathfrak{t})$ symmetry and homothety freedom in the metric. Modulo this freedom, there are only three distinct cases for R: no real roots ($r_1^2 < r_0 r_2$), one real root ($r_1^2 = r_0 r_2$) and two real roots ($r_1^2 > r_0 r_2$). We shall later refer to these cases as *elliptic, parabolic* and *hyperbolic* respectively.

Remark 3. The emergence of a homogeneous polynomial of degree 2 on $\mathfrak{t}$ merits a more conceptual explanation. It also seems to be connected with a curious symmetry breaking phenomenon between ω_+ and ω_-. In (11), $\omega^g_\pm$ are interchanged on replacing V by $-V$. This is compatible with the equality $U^2 = V^2$ derived in the above lemma. However, the choice of square root of R to satisfy positivity of g breaks this symmetry.

4.3. **Local classification in adapted coordinates.** The square root in the general form of an ambitoric metric is somewhat awkward: although we are interested in real riemannian geometry, the complex analytic continuation of the metric will be branched. This suggests pulling back the metric to a branched cover and making a coordinate change to eliminate the square root. This is done by introducing rational functions ρ and σ of degree 2 such that

$$R(\sigma(z)) = \rho(z)^2. \tag{17}$$

If we then write $\xi = \sigma(x)$, $\eta = \sigma(y)$, $A(x) = F(\sigma(x))\rho(x)/\sigma'(x)^2$ and $B(y) = G(\sigma(y))\rho(y)/\sigma'(y)^2$, the barycentric metric may be rewritten as

$$\begin{aligned} g_0 &= \frac{dx^2}{A(x)} + \frac{dy^2}{B(y)} \\ &\quad + A(x)\left(\frac{\sigma'(x)(dt_1 + \sigma(y)dt_2)}{(\sigma(x)-\sigma(y))\rho(x)}\right)^2 + B(y)\left(\frac{\sigma'(y)(dt_1 + \sigma(x)dt_2)}{(\sigma(x)-\sigma(y))\rho(y)}\right)^2. \end{aligned} \tag{18}$$

There are many solutions to (17). We seek a family that covers all three cases for R and yields metrics that are amenable to computation. We do this by solving the equation geometrically. Let W be a 2-dimensional real vector space equipped (for convenience) with a symplectic form $\kappa \in \wedge^2 W^*$. Thus we have to do with the geometry of the projective line $\mathrm{P}(W)$ and the representation theory of $\mathfrak{sl}(W)$, which we summarize in Appendix A (cf. [36]). In particular, the space S^2W^* of quadratic forms p on W is a Lie algebra under Poisson bracket $\{,\}$ and has a quadratic form $p \mapsto Q(p)$ given by the discriminant of p; the latter polarizes to give an inner product $\langle p, \tilde{p}\rangle$ of signature $(2,1)$. For $u \in W$, we denote by $u^\flat \in W^*$ the linear form $v \mapsto \kappa(u,v)$.

Our construction proceeds by fixing a quadratic form $q \in S^2W^*$. The Poisson bracket $\{q,\cdot\}\colon S^2W^* \to S^2W^*$ vanishes on the span of q and its image is the 2-dimensional subspace $S^2_{0,q}W^* := q^\perp$. We thus obtain a map

$$\mathrm{ad}_q\colon S^2W^*/{<}q{>} \to S^2_{0,q}W^*.$$

We now define $\sigma_q\colon W \to S^2W^*/{<}q{>}$ via the Veronese map

$$\sigma_q(\boldsymbol{z}) = \boldsymbol{z}^\flat \otimes \boldsymbol{z}^\flat \mod q$$

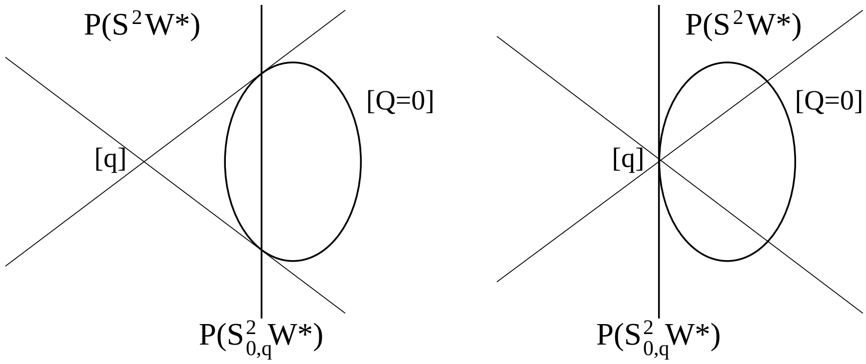
and let $R_q = \mathrm{ad}_q^* Q$. Thus $R_q(\sigma_q(\boldsymbol{z})) = Q(\{q, \boldsymbol{z}^\flat \otimes \boldsymbol{z}^\flat\}) = \langle q, \boldsymbol{z}^\flat \otimes \boldsymbol{z}^\flat\rangle^2$ (see Appendix A (33) with $p = q$ and $\tilde{p} = \boldsymbol{z}^\flat \otimes \boldsymbol{z}^\flat$, which is null) and so

$$R_q(\sigma_q(\boldsymbol{z})) = q(\boldsymbol{z})^2.$$

A geometrical solution to (17) is now given by identifying $\mathfrak{t}$ with $S^2W^*/{<}q{>}$, and R with R_q. This can have arbitrary type (elliptic, parabolic or hyperbolic): R_q is positive definite if $Q(q) < 0$, signature $(1,1)$ if $Q(q) > 0$, or semi-positive degenerate if $Q(q) = 0$. This geometrical solution represents $\boldsymbol{\xi}$ as $\sigma_q(\boldsymbol{x})$ and $\boldsymbol{\eta}$ as $\sigma_q(\boldsymbol{y})$, where

$$(\boldsymbol{x}, \boldsymbol{y})\colon M \to \mathrm{P}(W) \times \mathrm{P}(W) \setminus \Delta(W).$$

For $Q(q) \neq 0$, σ_q defines a branched double cover of $\mathrm{P}(\mathfrak{t})$ by $\mathrm{P}(W)$. For $Q(q) = 0$, the projective transformation appears to be singular for $q \in {<}\boldsymbol{z}^\flat \otimes \boldsymbol{z}^\flat{>}$, but this singularity is removable (by sending such $\boldsymbol{z}$ to ${<}\boldsymbol{z}^\flat{>} \odot W^* \mod q$) and σ_q identifies $\mathrm{P}(W)$ with $\mathrm{P}(\mathfrak{t})$ via the pencil of lines through a point on a conic. The following figure illustrates the two cases:



An area form $\varepsilon \in \wedge^2\mathfrak{t}^*$ is given by $\varepsilon(\lambda, \mu) = \langle \mathrm{ad}_q\lambda, \mu\rangle$. In particular

$$\varepsilon(\sigma_q(\boldsymbol{z}_1), \sigma_q(\boldsymbol{z}_2)) = \langle \{q, {\boldsymbol{z}_1}^\flat \otimes {\boldsymbol{z}_1}^\flat\}, {\boldsymbol{z}_2}^\flat \otimes {\boldsymbol{z}_2}^\flat\rangle = 2\kappa(\boldsymbol{z}_1, \boldsymbol{z}_2) q(\boldsymbol{z}_1, \boldsymbol{z}_2),$$

where $q(\boldsymbol{z}_1, \boldsymbol{z}_2)$ is the symmetric bilinear form obtained by polarization. It follows that the barycentric metric g_0 may be written invariantly as

$$\frac{d\boldsymbol{x}^2}{A(\boldsymbol{x})} + \frac{d\boldsymbol{y}^2}{B(\boldsymbol{y})} + A(\boldsymbol{x}) \left(\frac{\langle d\boldsymbol{\tau}, \boldsymbol{y} \otimes \boldsymbol{y}\rangle}{\kappa(\boldsymbol{x}, \boldsymbol{y}) q(\boldsymbol{x}, \boldsymbol{y})}\right)^2 + B(\boldsymbol{y}) \left(\frac{\langle d\boldsymbol{\tau}, \boldsymbol{x} \otimes \boldsymbol{x}\rangle}{\kappa(\boldsymbol{x}, \boldsymbol{y}) q(\boldsymbol{x}, \boldsymbol{y})}\right)^2,$$

where A, B are local sections of $\mathcal{O}(4)$ over $\mathrm{P}(W)$, $d\boldsymbol{\tau} = \frac{1}{2}\{q, d\boldsymbol{t}\}$, and we omit to mention use of the natural lift $(\cdot)^\natural$ to $\mathcal{O}(1) \otimes W$ over $\mathrm{P}(W)$. Note that $\langle q, d\boldsymbol{\tau}\rangle = 0$.

A more concrete expression may be obtained by introducing a symplectic basis e_1, e_2 of W (so that $\kappa(e_1, e_2) = 1$) and hence an affine coordinate z on $\mathrm{P}(W)$: see Appendix A. In particular, any quadratic form $p \in S^2W^*$ may be written

$$p(z) = p_0z^2 + 2p_1z + p_2$$

and its polarization is given by

$$p(x, y) = p_0xy + p_1(x + y) + p_2.$$

Elements of $\mathfrak{t}$ may thus represented by triples $[w] = [w_0, w_1, w_2] \in S^2W^*/{<}q{>}$, or by the corresponding elements $p = (p_0, p_1, p_2)$ of $S^2_{0,q}W^*$ where $p = \frac{1}{2}\{q, w\}$. The corresponding vector field on M will be denoted $K^{[w]}$ or $K^{(p)}$, so that $d\boldsymbol{t}(K^{[w]}) = [w]$ and $d\boldsymbol{\tau}(K^{(p)}) = p$. (The factor 1/2 in the formula $d\boldsymbol{\tau} = \frac{1}{2}\{q, d\boldsymbol{t}\}$ is a convenience.)

Theorem 3. *Let $(M, c, J_+, J_-, \mathfrak{t})$ be an ambitoric 4-manifold with barycentric metric g_0 and Kähler metrics (g_+, ω_+) and (g_-, ω_-). Then, about any point in a dense open subset of M, there is a neighbourhood in which (c, J_+, J_-) is either of Calabi type with respect to some $\lambda \in \mathfrak{t}$, or there there are $\mathfrak{t}$-invariant functions x, y, a quadratic polynomial $q(z) = q_0z^2 + 2q_1z + q_2$, and functions $A(z)$ and $B(z)$ of one variable with respect to which:*

$$\begin{aligned} g_0 &= \frac{dx^2}{A(x)} + \frac{dy^2}{B(y)} \\ &\quad + A(x)\left(\frac{y^2d\tau_0 + 2yd\tau_1 + d\tau_2}{(x-y)q(x,y)}\right)^2 + B(y)\left(\frac{x^2d\tau_0 + 2xd\tau_1 + d\tau_2}{(x-y)q(x,y)}\right)^2, \end{aligned} \tag{19}$$

$$\begin{aligned} \omega_+ &= \frac{x-y}{q(x,y)}\left(\frac{dx\wedge d^c_+x}{A(x)} + \frac{dy\wedge d^c_+y}{B(y)}\right) \\ &= \frac{dx\wedge(y^2d\tau_0 + 2yd\tau_1 + d\tau_2) + dy\wedge(x^2d\tau_0 + 2xd\tau_1 + d\tau_2)}{q(x,y)^2}, \end{aligned} \tag{20}$$

$$\begin{aligned} \omega_- &= \frac{q(x,y)}{x-y}\left(\frac{dx\wedge d^c_-x}{A(x)} + \frac{dy\wedge d^c_-y}{B(y)}\right) \\ &= \frac{dx\wedge(y^2d\tau_0 + 2yd\tau_1 + d\tau_2) - dy\wedge(x^2d\tau_0 + 2xd\tau_1 + d\tau_2)}{(x-y)^2}. \end{aligned} \tag{21}$$

where $2q_1d\tau_1 = q_0d\tau_2 + q_2d\tau_0$ and $q(x, y) = q_0xy + q_1(x + y) + q_2$.

Conversely, for any data as above, the above metric and Kähler forms do define an ambitoric Kähler structure on any simply connected open set where $\omega_\pm$ are nondegenerate and g_0 is positive definite.

Proof. The fact that regular ambitoric conformal structures have this form follows easily from Lemmas 4 and 5. One can either substitute into the invariant form of the metric, or carry out the coordinate transformation explicitly using (18). We deduce from (12) that

$$\begin{aligned} d^c_+x = d^c_-x &= \frac{A(x)}{(x-y)q(x,y)}(y^2d\tau_0 + 2yd\tau_1 + d\tau_2), \\ d^c_+y = -d^c_-y &= \frac{B(y)}{(x-y)q(x,y)}(x^2d\tau_0 + 2xd\tau_1 + d\tau_2). \end{aligned} \tag{22}$$

The computation of the conformal factor

$$f(x,y) = \frac{q(x,y)}{x-y} \tag{23}$$

with $\omega_- = f^2\omega_+$ requires more work, but it is straightforward to check that $\omega_\pm$ are closed, and hence deduce conversely that any metric of this form is ambitoric. □

Definition 8. A regular ambitoric structure (19) is said to be of *elliptic*, *parabolic*, or *hyperbolic* type if the number of distinct real roots of $q(z)$ (on $\mathrm{P}(W)$) is zero, one or two respectively.

For later use, we compute the momentum maps $\mu^\pm$ (as functions of x and y) of the (local) toric action with respect to $\omega_\pm$. Since

$$\omega_- = -d\chi, \qquad \chi = \frac{xy\,d\tau_0 + (x+y)d\tau_1 + d\tau_2}{x-y},$$

we have, for any $p \in S^2_{0,q}W^*$ (so $2p_1q_1 = p_0q_2 + p_2q_0$) and any $c \in \mathbb{R}$, a Killing potential

$$\mu^-_{p,c} = -\frac{p(x,y) + c(x-y)}{x-y} = -\frac{p_0xy + p_1(x+y) + p_2 + c(x-y)}{x-y} \tag{24}$$

for $K^{(p)}$.

For ω_+, we use the fact that $d\boldsymbol{\tau} = \frac{1}{2}\{q, d\boldsymbol{t}\}$ and compute, for any $w \in S^2W^*$,that

$$\iota_{K^{[w]}}\omega_+ = \frac{\frac{1}{2}\{q,w\}(y)\,dx + \frac{1}{2}\{q,w\}(x)\,dy}{q(x,y)},$$

and so

$$\mu^+_w = -\frac{w(x,y)}{q(x,y)} = -\frac{w_0xy + w_1(x+y) + w_2}{q_0xy + q_1(x+y) + q_2}. \tag{25}$$

is a Killing potential for $K^{[w]}$.

5. Extremal and conformally Einstein ambitoric surfaces

We now compute the Ricci forms and scalar curvatures of a regular ambitoric Kähler surface (cf. [1] for the toric case), and hence give a local classification of extremal ambitoric structures. By considering the Bach tensor, we also identify the regular ambitoric structures which are conformally Einstein.

5.1. **Ricci forms and scalar curvatures.** As in [12, 4], we adopt a standard method for computing the Ricci form of a Kähler metric as the curvature of the connection on the canonical bundle: the log ratio of the symplectic volume to any holomorphic volume is a Ricci potential. For regular ambitoric metrics, $d\boldsymbol{t} + \sqrt{-1}d^c_\pm\boldsymbol{t}$ is a $J_\pm$-holomorphic $\mathfrak{t}$-valued 1-form. From (22) we obtain that for any $w \in S^2W^*$,

$$\langle d^c_\pm\boldsymbol{\tau}, w\rangle = \frac{\{q,w\}(x)}{A(x)}dx \mp \frac{\{q,w\}(y)}{B(y)}dy$$

(since $\langle q, d\boldsymbol{\tau}\rangle = 0$), and deduce (using $d\boldsymbol{\tau} = \frac{1}{2}\{q, d\boldsymbol{t}\}$) that for any $p \in S^2_{0,q}W^*$,

$$\langle d^c_\pm\boldsymbol{t}, p\rangle = -\frac{p(x)}{A(x)}dx \pm \frac{p(y)}{B(y)}dy.$$

Using an arbitrary basis for $S^2_{0,q}W^*$ we find that

$$v_0 = \frac{(x-y)^2q(x,y)^2}{A(x)^2B(y)^2}dx \wedge d^c_+x \wedge dy \wedge d^c_+y.$$

can be taken as a holomorphic volume for both J_+ and J_- (up to sign). The symplectic volumes $v_\pm$ of $\omega_\pm$ are

$$v_+ = \frac{(x-y)^2}{q(x,y)^2 A(x)B(y)} dx \wedge d^c_+ x \wedge dy \wedge d^c_+ y,$$
$$v_- = \frac{q(x,y)^2}{(x-y)^2 A(x)B(y)} dx \wedge d^c_- x \wedge dy \wedge d^c_- y.$$

Hence the Ricci forms $\rho_\pm = -\frac{1}{2} dd^c_\pm \log |v_\pm / v_0|$ of $\omega_\pm$ are given by

$$\rho_+ = -\tfrac{1}{2} dd^c_+ \log \frac{A(x)B(y)}{q(x,y)^4}, \qquad \rho_- = -\tfrac{1}{2} dd^c_- \log \frac{A(x)B(y)}{(x-y)^4}.$$

The 2-forms $dd^c x$ and $dd^c y$ are obtained by differentiating the two sides of (22). After some work, we obtain

$$\begin{aligned} dd^c_\pm x &= \Big(A'(x) - \frac{q(x) - q_0\,(x-y)^2}{(x-y)q(x,y)} A(x) \Big) \frac{dx \wedge d^c_\pm x}{A(x)} \\ &\quad \pm \frac{q(y)A(x)}{(x-y)q(x,y)} \frac{dy \wedge d^c_\pm y}{B(y)}, \\ dd^c_\pm y &= \mp \frac{q(x)B(y)}{(x-y)q(x,y)} \frac{dx \wedge d^c_\pm x}{A(x)} \\ &\quad + \Big(B'(y) + \frac{q(y) - q_0\,(x-y)^2}{(x-y)q(x,y)} B(y) \Big) \frac{dy \wedge d^c_\pm y}{B(y)}. \end{aligned}$$

Hence for any $\mathfrak{t}$-invariant function $\phi = \phi(x,y)$,

$$\begin{aligned} dd^c_\pm \phi &= \phi_{xx}\, dx \wedge d^c_\pm x + \phi_{yy}\, dy \wedge d^c_\pm y + \phi_{xy}(dx \wedge d^c_\pm y + dy \wedge d^c_\pm x) \\ &\quad + \phi_x\, dd^c_\pm x + \phi_y\, dd^c_\pm y \\ &= \Big(\big(A(x)\phi_x\big)_x - \frac{q(x) - q_0\,(x-y)^2}{(x-y)q(x,y)} A(x)\phi_x \Big) \frac{dx \wedge d^c_\pm x}{A(x)} \\ &\quad \pm \frac{q(y)A(x)\phi_x}{(x-y)q(x,y)} \frac{dy \wedge d^c_\pm y}{B(y)} \mp \frac{q(x)B(y)\phi_y}{(x-y)q(x,y)} \frac{dx \wedge d^c_\pm x}{A(x)} \\ &\quad + \Big(\big(B(y)\phi_y\big)_y + \frac{q(y) - q_0\,(x-y)^2}{(x-y)q(x,y)} B(y)\phi_y \Big) \frac{dy \wedge d^c_\pm y}{B(y)} \\ &\quad + \phi_{xy}(dx \wedge d^c_\pm y + dy \wedge d^c_\pm x). \end{aligned}$$

In particular, the expression is both J_+ and J_- invariant iff $\phi_{xy} = 0$. The invariant part simplifies considerably when expressed in terms of the Kähler forms $\omega^0_\pm$ of the barycentric metric. Using the fact that $q_0 x + q_1$ and $q_0 y + q_1$ are the y and x derivatives of $q(x,y)$ respectively, we eventually obtain

$$\begin{aligned} dd^c_\pm \phi &= \frac{q(x,y)^2}{2} \bigg(\Big[\frac{A(x)\phi_x}{q(x,y)^2} \Big]_x \pm \Big[\frac{B(y)\phi_y}{q(x,y)^2} \Big]_y \bigg) \omega^0_+ \\ &\quad + \frac{(x-y)^2}{2} \bigg(\Big[\frac{A(x)\phi_x}{(x-y)^2} \Big]_x \mp \Big[\frac{B(y)\phi_y}{(x-y)^2} \Big]_y \bigg) \omega^0_- \\ &\quad + \phi_{xy}(dx \wedge d^c_\pm y + dy \wedge d^c_\pm x). \end{aligned}$$

Substituting the Ricci potentials for ϕ, we thus obtain, after a little manipulation,

$$\begin{aligned}\rho_+ = &-\frac{q(x,y)^2}{4}\left(\Big[q(x,y)^2\Big[\frac{A(x)}{q(x,y)^4}\Big]_x\Big]_x + \Big[q(x,y)^2\Big[\frac{B(y)}{q(x,y)^4}\Big]_y\Big]_y\right)\omega_+^0\\ &-\frac{(x-y)^2}{4}\left(\Big[\frac{q(x,y)^4}{(x-y)^2}\Big[\frac{A(x)}{q(x,y)^4}\Big]_x\Big]_x - \Big[\frac{q(x,y)^4}{(x-y)^2}\Big[\frac{B(y)}{q(x,y)^4}\Big]_y\Big]_y\right)\omega_-^0,\\ &+2\frac{(q_0q_2-q_1^2)(dx\wedge d_+^c y + dy\wedge d_+^c x)}{q(x,y)^2}\\ \rho_- = &-\frac{q(x,y)^2}{4}\left(\Big[\frac{(x-y)^4}{q(x,y)^2}\Big[\frac{A(x)}{(x-y)^4}\Big]_x\Big]_x - \Big[\frac{(x-y)^4}{q(x,y)^2}\Big[\frac{B(y)}{(x-y)^4}\Big]_y\Big]_y\right)\omega_+^0\\ &-\frac{(x-y)^2}{4}\left(\Big[(x-y)^2\Big[\frac{A(x)}{(x-y)^4}\Big]_x\Big]_x + \Big[(x-y)^2\Big[\frac{B(y)}{(x-y)^4}\Big]_y\Big]_y\right)\omega_-^0\\ &+2\frac{dx\wedge d_-^c y + dy\wedge d_-^c x}{(x-y)^2}.\end{aligned}$$

(In particular g_+ can only be Kähler–Einstein in the parabolic case—when q has a repeated root—while g_- is never Kähler–Einstein.) The scalar curvatures, given by $s_\pm = 2\rho_\pm \wedge \omega_\pm / v_\pm$, should be $\mathrm{SL}(W)$-invariants of A, B and q. For this we observe that for any quadratic form p with $Q(p) = 0$, and any function A of one variable,

$$p(x)^2\left(\Big[p(x)\Big[\frac{A(x)}{p(x)^2}\Big]_x\Big]_x\right) = p(x)A''(x) - 3p'(x)A'(x) + 6p''(x)A(x),$$

which is the transvectant $(p, A)^{(2)}$ when A is a quartic (or more generally, a local section of $\mathcal{O}(4)$)—see Appendix A. We apply this with $p(x) = q(x,y)^2$ and $p(x) = (x-y)^2$, and treat $B(y)$ in a similar way to obtain,

$$s_+ = -\frac{(q(x,y)^2, A(x))^{(2)} + (q(x,y)^2, B(y))^{(2)}}{(x-y)q(x,y)} \tag{26}$$

$$s_- = -\frac{((x-y)^2, A(x))^{(2)} + ((x-y)^2, B(y))^{(2)}}{(x-y)q(x,y)}, \tag{27}$$

where y is fixed when taking a transvectant with respect to x and vice versa.

5.2. **Extremality and Bach-flatness.** The Kähler metrics $g_\pm$ are extremal if their scalar curvatures $s_\pm$ are Killing potentials. Since the latter are $\mathfrak{t}$-invariant (and $\mathfrak{t}_M$ is lagrangian), this can only happen if $s_\pm$ is the momentum of some Killing vector field $K^{(p)} \in \mathfrak{t}$. The condition is straightforward to solve for g_+: equating (26) (for s_+) and (25) (for μ^+) yields

$$(q(x,y)^2, A(x))^{(2)} + (q(x,y)^2, B(y))^{(2)} = (x-y)w(x,y). \tag{28}$$

Differentiating three times with respect to x or three times with respect to y shows that A and B (respectively) are polynomials of degree at most four. We now introduce polynomials Π and P determined by $A = \Pi + P$ and $B = \Pi - P$. Since the left hand side of (28) is antisymmetric in (x, y), the symmetric part of the equation is

$$(q(x,y)^2, \Pi(x))^{(2)} + (q(x,y)^2, \Pi(y))^{(2)} = 0 \tag{29}$$

On restriction to the diagonal $(x = y)$ in this polynomial equation, we obtain

$$q^2\Pi'' - 3qq'\Pi' + 3(q')^2\Pi = 0.$$

To solve this linear ODE for Π, we set $\Pi(z) = q(z)\pi(z)$ to get $q^2(q''\pi - q'\pi' + q\pi'') = 0$, from which we deduce that π is a polynomial of degree ≤ 2 ($\pi''' = 0$) and that π is orthogonal to q. Conversely, by straightforward verification, this ensures Π solves (29).

The antisymmetric part of (28) is

$$(q(x,y)^2, P(x))^{(2)} - (q(x,y)^2, P(y))^{(2)} = (x-y)(w_0xy + w_1(x+y) + w_2).$$

The left hand side is clearly divisible by $x - y$ and since it is quadratic in both x and y, the quotient is (affine) linear in both x and y, hence the polarization of a quadratic form. To compute this quadratic form we divide the left hand side by $x - y$ and restrict to the diagonal to obtain

$$q^2P''' - 3qq'P'' + 3\big((q')^2 + qq''\big)P' - 6q'q''P = \{q, (q,P)^{(2)}\}$$

As q is nonzero, any quadratic form may be represented as $(q,P)^{(2)}$ for some quartic P, and hence any quadratic form w orthogonal to q has the form $w = \{q, (q,P)^{(2)}\}$ for some quartic P. Thus

$$s_+ = -\frac{w(x,y)}{q(x,y)},$$

where $w = \{q, (q,P)^{(2)}\}$ is orthogonal to q. Hence, except in the parabolic case (q degenerate), s_+ is constant iff it is identically zero.

Remarkably, the extremality condition for g_- coincides with that for g_+. To see this, we equate (27) (for s_-) and (24) (for μ^-) to obtain the extremality equation

$$((x-y)^2, A(x))^{(2)} + ((x-y)^2, B(y))^{(2)} = q(x,y)(p_0xy + p_1(x+y) + p_2 + c(x-y)), \tag{30}$$

which we shall again decompose into symmetric and antisymmetric parts: for this we first observe, by taking three derivatives, that A and B are polynomials of degree ≤ 4, we write $A = \Pi + P$, $B = \Pi - P$ as before.

The symmetric part, namely

$$((x-y)^2, \Pi(x))^{(2)} + ((x-y)^2, \Pi(y))^{(2)} = q(x,y)(p_0xy + p_1(x+y) + p_2),$$

immediately yields, on restricting to the diagonal ($y = x$), $\Pi(z) = q(z)\pi(z)$ with $\pi(z) = p(z)/24$. Further, since $\langle p, q\rangle = 0$, the equation is satisfied with this Ansatz.

The antisymmetric part, namely

$$((x-y)^2, P(x))^{(2)} - ((x-y)^2, P(y))^{(2)} = cq(x,y)(x-y)$$

yields $c = 0$ (divide by $x - y$ and restrict to the diagonal) and is then satisfied identically for *any* polynomial P of degree ≤ 4. Thus we again have an extremal Kähler metric with

$$s_- = -\frac{24\pi(x,y)}{x-y}.$$

Note that s_- is constant iff it is identically zero.

The Bach-flatness condition is readily found using Lemma 2: since $-v_-/v_+ = q(x,y)^4/(x-y)^4$, equation (4) holds iff $\pi(x,y)$ and $w(x,y)$ are linearly dependent.

Theorem 4. *Let $(J_+, J_-, g_+, g_-, \mathfrak{t})$ be a regular ambitoric structure as in Theorem 3. Then (g_+, J_+) is an extremal Kähler metric if and only if (g_-, J_-) is an extremal Kähler metric if and only if*

$$\begin{aligned} A(z) &= q(z)\pi(z) + P(z),\\ B(z) &= q(z)\pi(z) - P(z), \end{aligned} \tag{31}$$

where $\pi(z)$ is a polynomial of degree at most two orthogonal to $q(z)$ and $P(z)$ is polynomial of degree at most four. The conformal structure is Bach-flat if and only if the quadratic polynomials π and $\{q, (q,P)^{(2)}\}$ are linearly dependent.

5.3. **Compatible metrics with diagonal Ricci tensor.** A consequence of the explicit form (24)–(25) for the Killing potentials is that any regular ambitoric structure admits $\mathfrak{t}$-invariant compatible metrics with diagonal Ricci tensor.

Proposition 13. *Let $(g_\pm, J_\pm, \omega_\pm, \mathfrak{t})$ be a regular ambitoric structure as in Theorem 3. Then for any quadratic $p(z) = p_0z^2 + 2p_1z + p_2$ orthogonal to q,*

$$g = \frac{(x-y)^2}{p(x,y)^2} g_- = \frac{q(x,y)^2}{p(x,y)^2} g_+ = \frac{q(x,y)(x-y)}{p(x,y)^2} g_0$$

has diagonal Ricci tensor and scalar curvature

$$s^g = -\frac{(p(x,y)^2, A(x))^{(2)} + (p(x,y)^2, B(y))^{(2)}}{(x-y)q(x,y)}$$

Any $\mathfrak{t}$-invariant compatible metric with diagonal Ricci tensor arises in this way.

Proof. By Proposition 4, a compatible metric $g = \varphi_+^{-2} g_+ = \varphi_-^{-2} g_-$ has diagonal Ricci tensor iff $\varphi_\pm$ are Killing potentials with respect to $\omega_\pm$. For g to be $\mathfrak{t}$-invariant, the corresponding Killing fields must be in $\mathfrak{t}$, hence $\varphi_+ = w(x,y)/q(x,y)$ for some $w \in S^2W^*$ and $\varphi_- = p(x,y)/(x-y)+c$ for some $p \in S^2_{0,q}W^*$. The equality $\varphi_+^{-2} g_+ = \varphi_-^{-2} g_-$ is satisfied iff $w = p$ and $c = 0$. The formula for the scalar curvature is a tedious computation which we omit. □

We have seen in Theorem 2 that the riemannian analogues of Plebański–Demiański metrics are compatible CSC metrics compatible with diagonal Ricci tensor. Since the scalar curvature of g has the same form as the scalar curvature of g_+ (with q replaced by p), the calculations used for the extremality of g_+ establish the following result.

Theorem 5. *A compatible metric g with diagonal Ricci tensor is CSC if and only if*

$$\begin{aligned} A(z) &= p(z)\rho(z) + R(z), \\ B(z) &= p(z)\rho(z) - R(z), \end{aligned} \tag{32}$$

where $\rho(z)$ is a quadratic polynomial orthogonal to $p(z)$ and $R(z)$ is a quartic polynomial orthogonal to $q(z)p(z)$ (equivalently $(q,R)^{(2)}$ is orthogonal to p or, equally, $(p,R)^{(2)}$ is orthogonal to q). The metric is Einstein when $\rho(z)$ is a multiple of $q(z)$.

This is strikingly similar to, yet also different from, the extremal case. They overlap in the Einstein case, and in the parabolic case with p a multiple of q.

5.4. **Normal forms.** The projective choice of coordinate on $\mathrm{P}(W)$ can be used to set $q(z) = 1$, z or $1 + z^2$ in the parabolic, hyperbolic or elliptic cases respectively. To describe the curvature conditions in these normal forms, we write $A(z) = a_0z^4 + a_1z^3 + a_2z^2 + a_3z + a_4$ and $B(z) = b_0z^4 + b_1z^3 + b_2z^2 + b_3z + b_4$.

Parabolic type. When $q(z) = 1$, $d\tau_0 = 0$, and $S^2_{0,q}W^* = \{p(z) = 2p_1z + p_2\}$; we may represent $[w] \in S^2W^*/{<}q{>}$ by $w_0z^2 + 2w_1z$ with $\frac{1}{2}\{q, w\} = -w_0z - w_1$ and define components of $\xi \in \mathfrak{t}^*$ by $\xi(p) = 2\xi_1p_1 + \xi_2p_2$. Modulo constants, the Killing potentials for $\omega_\pm$ are spanned by

$$\mu_1^+ = x + y, \qquad \mu_2^+ = xy,$$
$$\mu_1^- = -\frac{1}{x-y}, \qquad \mu_2^- = -\frac{x+y}{2(x-y)},$$

while the barycentric metric g_0 and Kähler forms $\omega_\pm$ take the form

$$
\begin{aligned}
g_0 &= \frac{dx^2}{A(x)} + \frac{dy^2}{B(y)} + \frac{A(x)(dt_1 + y\,dt_2)^2}{(x-y)^2} + \frac{B(y)(dt_1 + x\,dt_2)^2}{(x-y)^2},\\
\omega_+ &= dx \wedge (dt_1 + y\,dt_2) + dy \wedge (dt_1 + x\,dt_2),\\
\omega_- &= \frac{dx \wedge (dt_1 + y\,dt_2)}{(x-y)^2} - \frac{dy \wedge (dt_1 + x\,dt_2)}{(x-y)^2}.
\end{aligned}
$$

The metrics $g_\pm$ are extremal iff

$$a_0 + b_0 = a_1 + b_1 = a_2 + b_2 = 0,$$

in which case

$$s_+ = -6a_1 - 12a_0\,\mu_1^+, \qquad s_- = 12(a_4 + b_4)\,\mu_1^- + 12(a_3 + b_3)\,\mu_2^-.$$

The structure is Bach-flat iff $a_1 + 4a_0 z$ and $-(a_4+b_4)+(a_3+b_3)z$ are linearly dependent, i.e.,

$$a_1(a_3 + b_3) + 4a_0(a_4 + b_4) = 0.$$

For $p(z) = z$, $g = q(x,y)^2 g_+/p(x,y)^2$ is CSC iff $a_0 + b_0 = a_2 + b_2 = a_4 + b_4 = 0$, and $a_1 = b_1$.

Hyperbolic type. When $q(z) = 2z$, $d\tau_1 = 0$, and $S^2_{0,q}W^* = \{p(z) = p_0 z^2 + p_2\}$; we may represent $[w] \in S^2W^*/{<}q{>}$ by $w_0 z^2 + w_2$ with $\frac{1}{2}\{q, w\} = -w_0 z^2 + w_2$ and define components of $\xi \in \mathfrak{t}^*$ by $\xi(p) = \xi_1 p_2 + \xi_2 p_0$. Modulo constants, the Killing potentials for $\omega_\pm$ are spanned by

$$
\begin{aligned}
\mu_1^+ &= -\frac{1}{x+y}, & \mu_2^+ &= \frac{xy}{x+y},\\
\mu_1^- &= -\frac{1}{x-y}, & \mu_2^- &= -\frac{xy}{x-y},
\end{aligned}
$$

while the barycentric metric g_0 and Kähler forms $\omega_\pm$ then take the form

$$
\begin{aligned}
g_0 &= \frac{dx^2}{A(x)} + \frac{dy^2}{B(y)} + \frac{A(x)(dt_1 + y^2 dt_2)^2}{(x^2-y^2)^2} + \frac{B(y)(dt_1 + x^2 dt_2)^2}{(x^2-y^2)^2}\\
\omega_+ &= \frac{dx \wedge (dt_1 + y^2\,dt_2)}{(x+y)^2} + \frac{dy \wedge (dt_1 + x^2\,dt_2)}{(x+y)^2}\\
\omega_- &= \frac{dx \wedge (dt_1 + y^2\,dt_2)}{(x-y)^2} - \frac{dy \wedge (dt_1 + x^2\,dt_2)}{(x-y)^2}.
\end{aligned}
$$

The metrics $g_\pm$ are extremal iff

$$a_0 + b_0 = a_2 + b_2 = a_4 + b_4 = 0,$$

in which case

$$s_\pm = -6(a_3 \pm b_3)\,\mu_1^\pm - 6(a_1 \pm b_1)\,\mu_2^\pm.$$

The Bach-flatness condition is therefore

$$(a_3 - b_3)(a_1 + b_1) + (a_3 + b_3)(a_1 - b_1) = 0.$$

For $p(z) = 1 + \varepsilon z^2$, $g = q(x,y)^2 g_+/p(x,y)^2$ is CSC iff $a_0 + b_0 = -\varepsilon^2(a_4 + b_4)$, $a_1 + b_1 = \varepsilon(a_3 + b_3)$, $a_2 + b_2 = 0$, and $a_1 - b_1 = -\varepsilon(a_3 - b_3)$. The resulting family

$$\frac{1}{(1+\varepsilon xy)^2}\left(\frac{(x^2-y^2)dx^2}{A(x)} + \frac{(x^2-y^2)dy^2}{B(y)} + \frac{A(x)(dt_1 + y^2 dt_2)^2}{x^2-y^2} + \frac{B(y)(dt_1 + x^2 dt_2)^2}{x^2-y^2}\right)$$

of metrics, where

$$A(z) = h + \kappa + (\sigma + \delta)z + \gamma z^2 + \varepsilon(\sigma - \delta)z^3 + (\lambda - \varepsilon^2 h)z^4,$$
$$B(z) = h - \kappa + (\sigma - \delta)z - \gamma z^2 + \varepsilon(\sigma + \delta)z^3 - (\lambda + \varepsilon^2 h)z^4,$$

is an analytic continuation of the Plebański–Demiański family [38, 23].

Elliptic type. When $q(z) = 1 + z^2$, $d\tau_0 + d\tau_2 = 0$, and $S^2_{0,q}W^* = \{p(z) = p_0 z^2 + 2p_1 z + p_2 : p_2 = -p_0\}$; we may represent $[w] \in S^2W^*/<q>$ by $-w_2 z^2 + 2w_1 z + w_2$ with $\frac{1}{2}\{q, w\} = w_1 z^2 - 2w_2 z - w_1$ and define components of $\xi \in \mathfrak{t}^*$ by $\xi(p) = \xi_1 p_1 + \xi_2 p_2$. Modulo constants, the Killing potentials for $\omega_\pm$ are spanned by

$$\mu_1^+ = -\frac{1 - xy}{1 + xy}, \qquad \mu_2^+ = -\frac{x + y}{1 + xy},$$
$$\mu_1^- = -\frac{x + y}{x - y}, \qquad \mu_2^- = \frac{1 - xy}{x - y},$$

while the barycentric metric g_0 and Kähler forms $\omega_\pm$ then take the form:

$$g_0 = \frac{dx^2}{A(x)} + \frac{dy^2}{B(y)} + \frac{A(x)(dt_1 + (y^2 - 1)dt_2)^2}{(x - y)^2(1 + xy)^2} + \frac{B(y)(dt_1 + (x^2 - 1)dt_2)^2}{(x - y)^2(1 + xy)^2}$$
$$\omega_+ = \frac{dx \wedge (2y\, dt_1 + (y^2 - 1)dt_2)}{(1 + xy)^2} + \frac{dy \wedge (2x\, dt_1 + (x^2 - 1)dt_2)}{(1 + xy)^2}$$
$$\omega_- = \frac{dx \wedge (2y\, dt_1 + (y^2 - 1)dt_2)}{(x - y)^2} - \frac{dy \wedge (2x\, dt_1 + (x^2 - 1)dt_2)}{(x - y)^2}.$$

The metrics $g_\pm$ are extremal iff

$$a_2 + b_2 = 0, \qquad a_0 + b_0 + a_4 + b_4 = 0, \qquad a_1 + b_1 = a_3 + b_3,$$

in which case

$$s_+ = 6(a_3 - b_1)\mu_1^+ - 12(a_4 + b_0)\mu_2^+, \qquad s_- = 12(a_3 + b_3)\mu_1^- + 12(a_4 + b_4)\mu_2^-.$$

The Bach-flatness condition is therefore:

$$(a_3 - b_1)(a_3 + b_3) + 4(a_4 + b_4)(a_4 + b_0) = 0.$$

For $p(z) = 1 - z^2$, $g = q(x,y)^2 g_+/p(x,y)^2$ is CSC iff $a_2 + b_2 = 0$, $a_0 + b_0 = 0$, $a_4 + b_4 = 0$, and $a_1 + b_1 + a_3 + b_3 = 0$. For $p(z) = z$, we have instead $a_0 + b_0 = 0$, $a_2 + b_2 = 0$, $a_4 + b_4 = 0$ and $a_1 - b_1 + a_3 - b_3 = 0$.

Summary table. The following table summarizes the extremal metric conditions.

Condition	**Parabolic type**	**Hyperbolic type**	**Elliptic type**
$g_\pm$ extremal	$a_0 + b_0 = 0$ $a_1 + b_1 = 0$ $a_2 + b_2 = 0$	$a_0 + b_0 = 0$ $a_2 + b_2 = 0$ $a_4 + b_4 = 0$	$a_0 + b_0 + a_4 + b_4 = 0$ $a_2 + b_2 = 0$ $a_1 + b_1 = a_3 + b_3$
$g_\pm$ Bach-flat	extremal and $a_1(a_3 + b_3) =$ $-4a_0(a_4 + b_4)$	extremal and $(a_3 - b_3)(a_1 + b_1) =$ $-(a_3 + b_3)(a_1 - b_1)$	extremal and $(a_3 - b_1)(a_3 + b_3) =$ $-4(a_4 + b_4)(a_4 + b_0)$
$s_+ \equiv 0$ $(W_+ \equiv 0)$	extremal and $a_0 = 0$ $a_1 = 0$	extremal and $a_1 = b_1$ $a_3 = b_3$	extremal and $a_3 = b_1$ $a_4 = -b_0$
$s_- \equiv 0$ $(W_- \equiv 0)$	extremal and $a_3 = -b_3$ $a_4 = -b_4$	extremal and $a_1 = -b_1$ $a_3 = -b_3$	extremal and $a_3 = -b_3$ $a_4 = -b_4$

g_- is never Kähler–Einstein, and is a CSC iff $s_- \equiv 0$. The same holds for g_+ except in the parabolic case, when g_+ has constant scalar curvature iff it is extremal with $a_0 = 0$, and is Kähler–Einstein if also $a_3 + b_3 = 0$.

Appendix A. The projective line and transvectants

Let W be a 2-dimensional real vector space equipped with a symplectic form κ (a non-zero element of $\wedge^2 W^*$). This defines an isomorphism $W \to W^*$ sending $u \in W$ to the linear form $u^\flat \colon v \mapsto \kappa(u, v)$; similarly there is a Lie algebra isomorphism from $\mathfrak{sl}(W)$ (the trace-free endomorphisms of W) to S^2W^* (the quadratic forms on W, under Poisson bracket $\{,\}$) sending $a \in \mathfrak{sl}(W)$ to the quadratic form $u \mapsto \kappa(a(u), u)$.

The quadratic form $-\det$ on $\mathfrak{sl}(W)$ induces a quadratic form Q on S^2W^* proportional to the discriminant, which polarizes to give an $\mathfrak{sl}(W)$-invariant inner product $\langle p, \tilde{p}\rangle = Q(p + \tilde{p}) - Q(p) - Q(\tilde{p})$ of signature $(2, 1)$ satisfying the following identity:

$$Q(\{p, \tilde{p}\}) = \langle p, \tilde{p}\rangle^2 - 4Q(p)Q(\tilde{p}). \tag{33}$$

A more concrete expression may be obtained by introducing a symplectic basis e_1, e_2 of W (so that $\kappa(e_1, e_2) = 1$) and hence an affine coordinate z on $\mathrm{P}(W)$. A quadratic form $q \in S^2W^*$ may then be written

$$q(z) = q_0z^2 + 2q_1z + q_2$$

with polarization

$$q(x, y) = q_0xy + q_1(x + y) + q_2.$$

In these coordinates the Poisson bracket of $q(z)$ with $w(z)$ is

$$\{q, w\}(z) = q'(z)w(z) - w'(z)q(z) \qquad \text{with}$$

$$\{q, w\}_0 = 2q_0w_1 - 2q_1w_0, \quad \{q, w\}_1 = q_0w_2 - q_2w_0, \quad \{q, w\}_2 = 2q_1w_2 - 2q_2w_1,$$

and the quadratic form and inner product on S^2W^* are

$$Q(q) = q_1^2 - q_0q_2 \qquad \text{and} \qquad \langle q, p\rangle = 2q_1p_1 - (q_2p_0 + q_0p_2).$$

The elements of S^mW^* may similarly be regarded as polynomials in one variable of degree at most m. For any $n, m \in \mathbb{N}$, the tensor product $S^mW^* \otimes S^nW^*$ has the following *Clebsch-Gordan* decomposition into irreducible component:

$$S^mW^* \otimes S^nW^* = \bigoplus_{r=0}^{\min\{m,n\}} S^{m+n-2r}W^*. \tag{34}$$

For any $r = 0, \ldots, \min\{m, n\}$, the corresponding $\mathrm{SL}(W)$-equivariant map $S^mW^* \otimes S^nW^* \to S^{m+n-2r}W^*$ (well-defined up to a multiplicative constant) is called the *transvectant* of order r, and denoted $(p, q)^{(r)}$—see e.g., Olver [36]. For $m = n$, the transvectant of order r is symmetric if r is even, and skew if r is odd. When p, q are regarded as polynomials in one variable, it may be written explicitly as:

$$(p, q)^{(r)} = \sum_{j=0}^{r} (-1)^j \binom{n-j}{r-j} \binom{m-r+j}{j} p^{(j)} q^{(r-j)}, \tag{35}$$

where $p^{(j)}$ stands for the j-th derivative of p, with $p^{(0)} = p$, and similarly for $q^{(r-j)}$. In particular, $(p, q)^{(0)}$ is multiplication, and for any $p, q \in S^2W^*$, $(p, q)^{(1)}$ and $(p, q)^{(2)}$ are constant multiples of the Poisson bracket and inner product respectively.

Elements of S^mW^* (and corresponding polynomials in an affine coordinate) may be viewed as (algebraic) sections of the degree m line bundle $\mathcal{O}(m)$ over $\mathrm{P}(W)$; in particular, there is a tautological section of $\mathcal{O}(1) \otimes W$. The formula (35) for transvectants extends from algebraic sections to general smooth sections.

Appendix B. Killing tensors and ambitoric conformal metrics

The material in this appendix is related to work of W. Jelonek [27, 28, 29] and some well-known results in general relativity, see [17] and [30]. To provide a different slant, we take a conformal viewpoint (cf. [13, 15, 20, 42]) and make explicit the connection with M. Pontecorvo's description [39] of hermitian structures which are conformally Kähler. We then specialize the analysis to ambitoric structures.

B.1. **Conformal Killing objects.** Let (M, c) be a conformal manifold. Among the conformally invariant linear differential operators on M, there is a family which are overdetermined of finite type, sometimes known as twistor or Penrose operators; their kernels are variously called twistors, tractors, or other names in special cases. Among the examples where the operator is first order are the equations for twistor forms (also known as conformal Killing forms) and conformal Killing tensors, both of which include conformal vector fields as a special case. There is also a second order equation for Einstein metrics in the conformal class. Apart from the obvious presence of (conformal) Killing vector fields and Einstein metrics, conformal Killing 2-tensors and twistor 2-forms are very relevant to the present work.

Let S^k_0TM denote the bundle of symmetric $(0,k)$-tensors $\mathcal{S}_0$ which are tracefree with respect to c in the sense that $\sum_i \mathcal{S}_0(\varepsilon_i, \varepsilon_i, \cdot) = 0$ for any conformal coframe ε_i. In particular, for $k = 2$, $\mathcal{S}_0 \in S^2_0TM$ may be identified with $\sigma_0 \in L^2 \otimes \mathrm{Sym}_0(TM)$ via $\alpha \circ \sigma_0(X) = \mathcal{S}_0(\alpha, c(X, \cdot))$ for any 1-form α and vector field X. Here $\mathrm{Sym}_0(TM)$ is the bundle of tracefree endomorphisms of TM which are symmetric with respect to c; thus σ_0 satisfies $c(\sigma_0(X), Y) = c(X, \sigma_0(Y))$ and hence defines a (weighted) $(2,0)$-tensor S_0 in $L^4 \otimes S^2_0T^*M$, another isomorph of S^2_0TM (in the presence of c).

A *conformal Killing* (2-)*tensor* is a section $\mathcal{S}_0$ of S^2_0TM such that the section $\mathrm{sym}_0\, D\mathcal{S}_0$ of $L^{-2} \otimes S^3_0TM$ is identically zero, where D is any Weyl connection (such as the Levi-Civita connection of any metric in the conformal class) and sym_0 denotes orthogonal projection onto $L^{-2} \otimes S^3_0TM$ inside $T^*M \otimes S^2TM \cong L^{-2} \otimes TM \otimes S^2TM$. Equivalently $\mathrm{sym}\, D\mathcal{S}_0 = \mathrm{sym}(\chi \otimes c)$ for some vector field χ. Taking a trace, we find that $(n+2)\chi = 2\delta^D\mathcal{S}_0$, where $\delta^D\mathcal{S}_0$ denotes $\mathrm{tr}_c\, D\mathcal{S}_0$, which may be computed, using a conformal frame e_i with dual coframe ε_i, as $\sum_i D_{e_i}\mathcal{S}_0(\varepsilon_i, \cdot)$. Thus $\mathcal{S}_0$ is conformal Killing iff

$$\mathrm{sym}\, D\mathcal{S}_0 = \tfrac{2}{n+2}\, \mathrm{sym}(c \otimes \delta^D\mathcal{S}_0), \tag{36}$$

This is independent of the choice of Weyl connection D. On the open set where $\mathcal{S}_0$ is nondegenerate, there is a unique such D with $\delta^D\mathcal{S}_0 = 0$, and hence a nondegenerate $\mathcal{S}_0$ is conformal Killing iff there is a Weyl connection D with $\mathrm{sym}\, D\mathcal{S}_0 = 0$.

A *conformal Killing* 2-*form* is a section ϕ of $L^3 \otimes \wedge^2T^*M$ such that $\pi(D\phi) = 0$ (for any Weyl connection D) where π is the projection orthogonal to $L^3 \otimes \wedge^3T^*M$ and $L \otimes T^*M$ in $T^*M \otimes L^3 \otimes \wedge^2T^*M$. It is often more convenient to identify ϕ with a section Φ of $L \otimes \mathfrak{so}(TM)$ via $\phi(X, Y) = c(\Phi(X), Y)$, where $\mathfrak{so}(TM)$ denotes the bundle of skew-symmetric endomorphisms of TM with respect to c.

B.2. **Conformal Killing tensors and complex structures.** In four dimensions a conformal Killing 2-form splits into selfdual and antiselfdual parts $\Phi_\pm$, which are sections of $L \otimes \mathfrak{so}_\pm(TM) \cong L^3 \otimes \wedge^2_\pm T^*M$. Following M. Pontecorvo [39], nonvanishing conformal Killing 2-forms Φ_+ and Φ_- describe oppositely oriented Kähler metrics in the conformal class, by writing $\Phi_\pm = \ell_\pm J_\pm$, where $\ell_\pm$ are sections of L and $J_\pm$ are oppositely oriented complex structures: the Kähler metrics are then $g_\pm = \ell_\pm^{-2}c$. Conversely if $(g_\pm = \ell_\pm^{-2}c, J_\pm)$ are Kähler and $D^\pm$ denote the Levi-Civita connections of $g_\pm$ then $D^\pm(\ell_\pm J_\pm) = 0$ so $\Phi_\pm = \ell_\pm J_\pm$ are conformal Killing 2-forms.

The tensor product of sections Φ_+ and Φ_- of $L\otimes\mathfrak{so}_+(TM)$ and $L\otimes\mathfrak{so}_-(TM)$ defines a section $\Phi_+\Phi_-$: as a section of $L^2\otimes\mathrm{Sym}_0(TM)$, this is simply the composite ($\Phi_+\circ\Phi_- = \Phi_-\circ\Phi_+$); as a section of $L^4\otimes S^2_0T^*M$ it satisfies $(\Phi_+\Phi_-)(X,Y) = c(\Phi_+(X),\Phi_-(Y))$.

When $\Phi_\pm = \ell_\pm J_\pm$ are nonvanishing, $\Phi_+\Phi_- = \ell_+\ell_- J_+J_-$ is a symmetric endomorphism with two rank 2 eigenspaces at each point. Conversely if σ_0 is such a symmetric endomorphism, we may write $\sigma_0 = \ell^2 J_+J_-$ for uniquely determined almost complex structures $J_\pm$ up to overall sign, and a positive section ℓ of L.

Proposition 14. *A nonvanishing section $\sigma_0 = \ell^2 J_+J_-$ of $L^2\otimes\mathrm{Sym}_0(TM)$ (as above) is associated to a conformal Killing 2-tensor $\mathcal{S}_0$ iff $J_\pm$ are integrable complex structures which are "Kähler on average" with length scale ℓ, in the sense that if $D^\pm$ denote the canonical Weyl connections of $J_\pm$, then the connection $D=\frac{1}{2}(D^+ + D^-)$ preserves the length scale ℓ (i.e., $D^+\ell + D^-\ell = 0$).*

If these equivalent conditions hold, then also $\mathrm{sym}\, D\mathcal{S}_0 = 0$.

(With respect to an arbitrary metric g in the conformal class, the "Kähler on average" condition means that the Lee forms $\theta^g_\pm$ satisfy $d(\theta^+_g + \theta^-_g) = 0$. In the case that J_+ and J_- both define conformally Kähler metrics $g_\pm$, the metric $g_0 = \ell^{-2}c$ is the barycentric metric with $g_0 = f\, g_+ = f^{-1} g_-$ for some function p.)

Proof. Let D, D^+, D^- be Weyl connections with $D = \frac{1}{2}(D^+ + D^-)$ in the affine space of Weyl connections. (Thus the induced connections on L are related by $D = D^+ + \theta = D^- - \theta$ for some 1-form θ.) Straightforward calculation shows that

$$D\sigma_0 = D(\ell^2)\otimes J_+\circ J_- + \ell^2\big(D^+J_+\circ J_- + J_+\circ D^-J_-\big) + R$$

where R is an expression (involving θ) whose symmetrization vanishes (once converted into a trilinear form using c). If $J_\pm$ are integrable and Kähler on average, then taking $D^\pm$ to be the canonical Weyl connections and ℓ the preferred length scale, $\ell^2 J_+J_-$ is thus associated to a conformal Killing tensor $\mathcal{S}_0$ with $\mathrm{sym}\, D\mathcal{S}_0 = 0$.

For the converse, it is convenient (for familiarity of computation) to work with the associated $(2,0)$-tensor S_0 with $S_0(X,Y) = \ell^2 c(J_+J_-X,Y)$. Since S_0 is nondegenerate, and associated to a conformal Killing tensor, we can let $D = D^+ = D^-$ be the unique Weyl connection with $\mathrm{sym}\, DS_0 = 0$: note that $\mathrm{sym}\colon L^4\otimes T^*M\otimes S^2T^*M \to L^4\otimes S^3T^*M$ here becomes the natural symmetrization map. Thus

$$\sum_{X,Y,Z} D_X(\ell^2)c(J_+\circ J_-Y,Z) = \sum_{X,Y,Z}\ell^2\Big(c\big((D_XJ_+)J_-Y,Z\big) + c\big(J_+(D_XJ_-)Y,Z\big)\Big),$$

where the sum is over cyclic permutations of the arguments. If X,Y,Z belong to a common eigenspace of S_0 then the right hand side is zero—this follows because, for instance, $c\big((D_XJ_\pm)J_\pm Y,Z\big)$ is skew in Y,Z whereas the cyclic sum of the two terms is totally symmetric.

It follows that $D\ell = 0$, hence the right hand side is identically zero in X,Y,Z. Additionally $c(D_XJ_\pm\cdot,\cdot)$ is $J_\pm$-anti-invariant. Thus these 2-forms vanish when their arguments have opposite types ($(1,0)$ and $(0,1)$) with respect to the corresponding complex structure. Now suppose for example that Z_1 and Z_2 have type $(1,0)$ with respect to J_+, but opposite types with respect to J_- (J_+ and J_- are simultaneously diagonalizable on $TM\otimes\mathbb{C}$). Then by substituting first $X=Y=Z_1$, $Z=Z_2$ into

$$\sum_{X,Y,Z} c\big((D_XJ_+)J_-Y,Z\big) = \sum_{X,Y,Z} c\big((D_XJ_-)Y,J_+Z\big),$$

and then $X=Y=Z_2$, $Z=Z_1$, we readily obtain

$$c\big((D_{Z_1}J_+)Z_1,Z_2\big) = 0 = c\big((D_{Z_2}J_+)Z_1,Z_2\big).$$

Thus $D_{J_+X}J_+ = J_+D_XJ_+$ for all X and J_+ is integrable. Similarly, we conclude J_- is integrable. □

Since D is the Levi-Civita connection D^g of the "barycentric" metric $g = \ell^{-2}c$, it follows that $S_0 = g(J_+J_-\cdot,\cdot)$ is a *Killing tensor* with respect to g, i.e., satisfies $\mathrm{sym}\, D^g S_0 = 0$ iff J_+ and J_- are integrable and Kähler on average, with barycentric metric g. More generally, we can use this result to characterize, for any metric g in the conformal class and any functions f, h, the case that

$$S(\cdot,\cdot) = f\, g(\cdot,\cdot) + h\, g(J_+J_-\cdot,\cdot), \tag{37}$$

is a Killing tensor with respect to g. If $\theta_\pm$ are the Lee forms of $(g, J^\pm)$, i.e., $D^\pm = D^g \pm \theta_\pm$, then we obtain the following more general corollary.

Corollary 3. *$S = f\, g + h\, g(J_+J_-\cdot,\cdot)$, with h nonvanishing, is a Killing tensor with respect to g if and only if:*

$$J_+ \textit{ and } J_- \textit{ are both integrable}; \tag{38}$$

$$\theta_+ + \theta_- = -\frac{dh}{h}; \tag{39}$$

$$J_+df = J_-\, dh. \tag{40}$$

(Obviously when h is identically zero, S is a Killing tensor iff f is constant.)

B.3. **Conformal Killing tensors and the Ricci tensor.** The tracefree part $ric_0^g = ric^g - \frac{1}{n}s_g g$ of the Ricci tensor of a compatible metric $g = \mu_g^{-2}c$ on a conformal n-manifold (M, c) defines a tracefree symmetric $(0,2)$-tensor $\mathcal{S}_0^g(\alpha,\beta) = ric_0^g(\alpha^\sharp,\beta^\sharp)$ (where for $\alpha \in T^*M$, $g(\alpha^\sharp,\cdot) = \alpha)$), where the corresponding section of $L^4 \otimes S_0^2T^*M$ is $\mu_g^4 ric_0^g$.

The differential Bianchi identity implies that $0 = \delta^g(ric^g - \frac{1}{2}s_g g) = \delta^g ric_0^g - \frac{n-2}{2n}ds_g$. Hence the following are equivalent:

- $\mathcal{S}_0^g$ is a conformal Killing tensor;
- $\mathrm{sym}\, D^g\mathcal{S}_0^g = \frac{n-2}{n(n+2)}\,\mathrm{sym}(g^{-1}\otimes ds_g)$;
- $ric^g - \frac{2}{n+2}s_g g$ is a Killing tensor with respect to g;
- $D_X^g ric^g(X,X) = \frac{2}{n+2}ds_g(X)g(X,X)$ for all vector fields X.

Riemannian manifolds (M, g) satisfying these conditions were introduced by A. Gray as $\mathcal{A}C^\perp$-manifolds [22]. Relevant for this paper is the case $n = 4$ and the assumption that ric^g has two rank 2 eigendistributions, which has been extensively studied by W. Jelonek [28, 29].

Supposing that g is not Einstein, Corollary 3 implies, as shown by Jelonek, that

$$ric^g - \tfrac{1}{3}s_g g = f\, g + h\, g(J_+J_-\cdot,\cdot)$$

is Killing with respect to g iff (38)–(40) are satisfied. Since $J_\pm$ are both integrable, Jelonek refers to such manifolds as *bihermitian Gray surfaces*. It follows from [2] that both (g, J_+) and (g, J_-) are conformally Kähler, so that in the context of the present paper, a better terminology would be *ambikähler Gray surfaces*.

However, the key feature of such metrics is that the Ricci tensor is $J_\pm$-invariant: as long as $J_\pm$ are conformally Kähler, Proposition 11 applies to show that the manifold is either ambitoric or of Calabi type; it is not necessary that the $J_\pm$-invariant Killing tensor constructed in the proof is equal to the Ricci tensor ric^g.

Jelonek focuses on the case that the ambihermitian structure has Calabi type. This is justified by the global arguments he employs. In the ambitoric case, there are strong constraints, even locally.

B.4. **Killing tensors and hamiltonian 2-forms.** The notion of hamiltonian 2-forms on a Kähler manifold (M, g, J, ω) has been introduced and extensively studied in [4, 5]. According to [5], a J-invariant 2-form ϕ is hamiltonian if it satisfies

$$D_X\phi = \frac{1}{2}\Big(d\sigma \wedge JX^\flat - Jd\sigma \wedge X^\flat\Big), \tag{41}$$

for any vector field X, where $X^\flat = g(X)$ and $\sigma = \mathrm{tr}_\omega\phi = g(\phi, \omega)$ is the trace of ϕ with respect to ω. An essentially equivalent (but not precisely the same) definition was given in the four dimensional case in [4], by requiring that a J-invariant 2-form φ is closed and its primitive part φ_0 satisfies

$$D_X\varphi_0 = -\frac{1}{2}d\sigma(X)\omega + \frac{1}{2}\Big(d\sigma \wedge JX^\flat - Jd\sigma \wedge X^\flat\Big), \tag{42}$$

for some smooth function σ. Note that, in order to be closed, φ is necessarily of the form $\frac{3}{2}\sigma\omega + \varphi_0$.

The relation between the two definitions is straightforward: $\varphi = \frac{3}{2}\sigma\omega + \varphi_0$ is closed and verifies (42) iff $\phi = \varphi_0 + \frac{1}{2}\sigma\omega$ satisfies (41).

Specializing Corollary 3 to the case when the metric g is Kähler with respect to $J = J_+$ allows us to identify J-invariant symmetric Killing tensors with hamiltonian 2-forms as follows:

Proposition 15. *Let S be a symmetric J-invariant tensor on a Kähler surface (M, g, J, ω), and $\psi(\cdot, \cdot) = S(J\cdot, \cdot)$ be the associated J-invariant 2-form. Then S is Killing iff $\phi = \psi - (\mathrm{tr}_\omega\psi)\omega$ is a hamiltonian 2-form (i.e., verifies (41)).*

Proof. As observed in [5, p. 407], ϕ satisfies (41) iff $\varphi = \phi + (\mathrm{tr}_\omega\phi)\omega$ is a closed 2-form and $\psi = \phi - (\mathrm{tr}_\omega\phi)\omega$ is the 2-form associated to a J-invariant Killing tensor (this is true in any complex dimension $m > 1$).

Noting that the 2-forms φ and ψ are related by $\varphi = \psi - \frac{2\mathrm{tr}_\omega\psi}{m-1}\omega$, we claim that in complex dimension $m = 2$, the 2-form $\varphi = \phi - 2(\mathrm{tr}_\omega\psi)\omega$ is automatically closed, provided that ψ is the 2-form associated to a J-invariant Killing tensor S. Indeed, under the Kähler assumption the conditions (38)–(39) specialize as

$$J_- \text{ is integrable,} \tag{43}$$

$$\theta_- = -\frac{dh}{h}, \tag{44}$$

It follows that $(g_- = h^{-2}\, g, J_-, \omega_- = g_-(J_-\cdot, \cdot))$ defines a Kähler metric. From (37) we have

$$\psi = f\,\omega_+ + h^3\,\omega_-, \tag{45}$$

where $\omega_+ = g(J_+\cdot, \cdot)$ denotes the Kähler form of (g, J_+). In particular, the trace of φ with respect to ω_+ is equal to $2f$ while the condition (40) and the fact that ω_- is closed imply that $\varphi = \psi - 4f\,\omega_+ = -3f\,\omega_+ + h^3\,\omega_-$ is closed too. □

B.5. **Killing tensors associated to ambitoric structures.** We have seen in the previous subsections that there is a link between Killing tensors and ambihermitian structures. We now make this link more explicit in the case of ambitoric metrics.

In the ambitoric situation, the barycentric metric g_0 (see section 4) satisfies $\theta_+^0 + \theta_-^0 = 0$. It then follows from Corollary 3 that the (tracefree) symmetric bilinear form $g_0(I\cdot, \cdot)$ (with $I = J_+ \circ J_-$) is Killing with respect to g_0. More generally, let g be any (K_1, K_2)-invariant riemannian metric in the ambitoric conformal class c, so that g can be written as $g = h\, g_0$ for some positive function $h(x, y)$, where x, y are the coordinates introduced in section 4. Then $\theta_+^g + \theta_-^g = -d\log h$. From Corollary 3 again, the symmetric bilinear form $S_0(\cdot,\cdot) = h\,g(I\cdot,\cdot)$ is conformal Killing. Moreover, by condition (40) in Proposition 3, it can be completed into a Killing symmetric bilinear form $S = f\,g + S_0$ iff the 1-form $dh \circ I$ is closed. Since $Idx = -dx$ and $Idy = dy$, $dh \circ I$ is closed iff $h_x\,dx - h_y\,dy$ is closed, iff $h_{xy} = 0$; the general solution is $h(x,y) = F(x) - G(y)$, for some functions F, G. Note that the coefficient $f(x,y)$ is determined by $df = -Idh = F'(x)dx + G'(y)dy$ (see (40)), so we can take without loss $f(x,y) = F(x) + G(y)$. Thus, S is Killing, with eigenvalues (with respect to g) equal to $2F(x)$ and $2G(y)$.

A similar argument shows that any metric of the form $g = f(z)g_0$, where g_0 is the barycentric metric of an ambikähler pair of Calabi type and z is the momentum coordinate introduced in section 3.2, admits a nontrivial symmetric Killing tensor of the form $S(\cdot,\cdot) = f(z)g(\cdot,\cdot) + f(z)g(I\cdot,\cdot)$ (and hence with eigenvalues $(2f(z),0)$).

It follows that there are infinitely many $\mathfrak{t}$-invariant metrics in a given ambitoric conformal class, which admit nontrivial symmetric Killing tensors.

There are considerably fewer such metrics with diagonal Ricci tensor. By Proposition 13 these have the form $g = h(x,y)g_0$ where $h(x,y) = (x-y)q(x,y)/p(x,y)^2$. In order for g to admit a nontrivial symmetric Killing tensor, we must have $h_{xy} = 0$. A calculation shows that this happens iff $Q(p) = 0$ (i.e., $p(z)$ has repeated roots). Since p is orthogonal to q, this can only happen if $Q(q) \geq 0$ and there are generically $(Q(q) > 0)$ just two solutions for p, which coincide if $Q(q) = 0$.

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Vestislav Apostolov, Département de Mathématiques, UQAM, C.P. 8888, Succursale Centre-ville, Montréal (Québec), H3C 3P8, Canada
E-mail address: apostolov.vestislav@uqam.ca

David M. J. Calderbank, Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK
E-mail address: D.M.J.Calderbank@bath.ac.uk

Paul Gauduchon, Centre de Mathématiques, École Polytechnique, UMR 7640 du CNRS, 91128 Palaiseau, France
E-mail address: pg@math.polytechnique.fr