Bryant - Salamon Construction of Complete Spin(7) - metrics

Aim: To construct a complete Riemannian metric on an 8-manifold with holonomy equal to Spin(7).

Bryant - Salamon did this in 1989

1 Holonomy

Let M be a manifold, $E \to M$ a vector bundle and ∇ a connection on E. Let $\gamma : [0,1] \to M$ be a smooth curve in M. The pull-back $\gamma^*(E)$ of E to $[0,1]$ is a vector bundle over $[0,1]$ with fibre $E_{\gamma(t)}$ over $t \in [0,1]$.

Let s be a smooth section of $\gamma^*(E)$. The connection ∇ pulls back under γ to give a connection on $\gamma^*(E)$. Say s is parallel if $\nabla_{\dot\gamma(t)} s(t) = 0 \quad \forall t \in [0,1]$.

Write $\gamma(0) = x$, $\gamma(1) = y$. Then for each $e \in E_x$, $\exists!$ smooth section $s : [0,1] \to \gamma^*(E)$ satisfying

$$\begin{cases} \nabla_{\dot\gamma(t)} s(t) = 0 & t \in [0,1] \\ s(0) = e \end{cases}$$

(Existence and Uniqueness of solutions to ODEs)

Define $P_\gamma(e) = s(1)$. Then $P_\gamma : E_x \to E_y$ is the parallel transport map.

Define the holonomy group of ∇ at x to be

$$Hol_x(\nabla) = \{P_\gamma : \gamma \text{ is a loop based at } x\} \subset GL(E_x)$$

It is easy to show that this is a group with

$$P_\gamma \circ P_{\tilde\gamma} = P_{\gamma\tilde\gamma} \qquad P_\gamma^{-1} = P_{\gamma^{-1}}$$

Then $Hol_y(\nabla) = P_\gamma Hol_x(\nabla) P_\gamma^{-1}$ if γ is a curve with $\gamma(0) = x, \gamma(1) = y$

So Holonomy groups are independent of the basepoint up to conjugation.

So we can define the Holonomy group $Hol(\nabla)$.

② The group Spin(7)

Here are two equivalent definitions of Spin(7):

(i) Spin(7) is the double cover of SO(7) such that

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}(7) \longrightarrow SO(7) \longrightarrow 1$$

is a short exact sequence of Lie groups. It is connected, simply-connected and so coincides with the universal cover of SO(7). It has dimension 21.

(ii) Let $V = \mathbb{H} \oplus \mathbb{H}$ with basis given by $e_0, \dots, e_3$ the standard basis in the first summand, and $e_4, \dots, e_7$ in the second. Give V the orientation and inner product which make this an oriented orthonormal basis. Let $\alpha, \beta : V \to \mathbb{H}$ be the first, second projections respectively. Then

$$\alpha = \alpha^0 + i\alpha^1 + j\alpha^2 + k\alpha^3 \qquad \beta = \beta^0 + i\beta^1 + j\beta^2 + k\beta^3$$

where $\{\alpha^0, \dots, \alpha^3, \beta^0, \dots, \beta^3\}$ is a dual basis to $\{e_0, \dots, e_7\}$

~~Now let $\mathbb{Z}_2 = \{\pm(1,1,1)\} \subseteq S^3 \times S^3 \times S^3$ and define~~

$$H = (S^3 \times S^3 \times S^3)/\mathbb{Z}_2$$

~~where we consider $S^3 = \{x \in \mathbb{H} \mid |x|^2 = 1\}$ (unit quaternions).~~

~~H acts ~~effectively~~ faithfully on V via~~

$$[p_1, p_2, q] \cdot (x, y) = (p_1 x \bar{q}, p_2 x \bar{q})$$

Set $A = \frac{1}{2} \bar{\alpha} \wedge \alpha$, $B = \frac{1}{2} \bar{\beta} \wedge \beta$, then

$$\begin{aligned} A &= iA^1 + jA^2 + kA^3 \\ &= i(\alpha^0 \wedge \alpha^1 - \alpha^2 \wedge \alpha^3) + j(\alpha^0 \wedge \alpha^2 - \alpha^3 \wedge \alpha^1) \\ &\quad + k(\alpha^0 \wedge \alpha^3 - \alpha^1 \wedge \alpha^2) \end{aligned}$$

and similarly for $B = iB^1 + jB^2 + kB^3$

Note that $\{A^1, A^2, A^3\}$ is a basis for the anti-self-dual 2-forms on the first $\mathbb{H}$-summand, and $\{B^1, B^2, B^3\}$ is the same for the second.

Define $\Phi \in \Lambda^4 V$ by

$$\Phi = -\frac{1}{6}\left((A^1)^2 + (A^2)^2 + (A^3)^2\right) + A^1B^1 + A^2B^2 + A^3B^3 - \frac{1}{6}\left((B^1)^2 + (B^2)^2 + (B^3)^2\right)$$

$$= \alpha^{0123} + (\alpha^{01} - \alpha^{23}) \wedge (\beta^{01} - \beta^{23}) + (\alpha^{02} - \alpha^{31}) \wedge (\beta^{02} - \beta^{31}) + (\alpha^{03} - \alpha^{12}) \wedge (\beta^{03} - \beta^{12}) + \beta^{0123}$$

Bryant - Salamon show that $\mathrm{Stab}(\Phi) = \mathrm{Spin}(7)$ in their paper.

③ Definite 4-forms

Let M be an 8-manifold, $x \in M$.

~~Define the set of definite 4-forms on T_xM, $\Lambda^4_+(T^*M)$ to be the $GL(T_xM)$ orbit of Φ. So $\Psi \in \Lambda^4_+(T^*M)$ if $\exists L : T^*M \to V$~~

"$GL(V)$-orbit of Φ"

Define $\Lambda^4_+(T_x^*M) = \{L^*(\Phi) \mid L : T_xM \xrightarrow{\cong} V\}$, then $\Lambda^4(T_xM)$ is diffeomorphic to $GL(V)/\mathrm{Spin}(7)$.

For any $\Psi \in \Lambda^4_+(T_xM)$, we have a uniquely determined inner product and orientation given by

$$\langle x, y\rangle_\Psi = \langle Lx, Ly\rangle \qquad *_\Psi 1 = L^*(*_\Phi 1)$$

where $L : T_xM \to V$ satisfies $L^*(\Phi) = \Psi$.

(this is well-defined since L is unique up to composition with an element of $\mathrm{Spin}(7)$).

Let $\Omega^4_+(M)$ denote the set of forms $\Psi \in \Omega^4(M)$ with the property that $\Psi_x \in \Lambda^4_+(T_xM)$ $\forall x \in M$. Then, as above, associated to such a Ψ is a canonical Riemannian metric g_Ψ and orientation $*_\Psi 1$.

__Theorem 1__ (Bryant, Fernandez [indep])

Let M be an 8-manifold, $\Psi \in \Omega^4_+(M)$. Then Ψ is parallel w.r.t. the Levi-Civita connection of g_Ψ if and only if $d\Psi = 0$.

__Theorem 2__ (Bryant)

Let M be a simply connected, connected manifold, and let g be a Riemannian metric on M whose holonomy is a subgroup of Spin(7). Then

$$\mathrm{Hol}(g) = \mathrm{Spin}(7) \iff \nexists \text{ nonzero parallel 1-forms or 2-forms on } M.$$

4 Spin bundles

Let (M^4, ds^2) be an oriented Riemannian 4-manifold. Let P denote the bundle of linear isometries $u : T_xM \to \mathbb{H}$ which preserve orientation, then P is a principal right $SO(4)$-bundle over M.

Define the canonical $\mathbb{H}$-valued 1-form ω on P by

$$\omega(v) = u(\pi_*(v)) \qquad \text{for } v \in T_uP, \ \pi : P \to M.$$

Then, if $\omega = \omega_0 + i\omega_1 + j\omega_2 + k\omega_3$, then

$$\begin{aligned}\tfrac{1}{2}(\bar{\omega} \wedge \omega) &= i(\omega_0 \wedge \omega_1 - \omega_2 \wedge \omega_3) + j(\omega_0 \wedge \omega_2 - \omega_3 \wedge \omega_1) \\ &\quad + k(\omega_0 \wedge \omega_3 - \omega_1 \wedge \omega_2) \\ &=: i\Omega_1 + j\Omega_2 + k\Omega_3.\end{aligned}$$

A spin-structure $(\tilde{P}, \tilde{\pi})$ on M is a principal bundle $\tilde{P} \to M$ with fibre Spin(4), together with a map of bundles $\tilde{\pi} : \tilde{P} \to P$ that is locally modelled on the projection $\mathrm{Spin}(4) \to SO(4)$.

Define the spin bundle $S \to M$ to be $S = \tilde{P} \times_{\mathrm{Spin}(4)} \mathbb{H}$.

(Spin(4) acts on $\mathbb{H}$ on the right by multiplication with the second factor $\mathrm{Spin}(4) \cong S^3 \times S^3$).

where S^3 is unit quaternions

Now, let $a = a_0 + i a_1 + j a_2 + k a_3$ be a linear coordinate on $\mathbb{H}$, and set $r = a\bar{a} = a_0^2 + a_1^2 + a_2^2 + a_3^2$.

Since the Lie algebra of S^3 is $T_1 S^3 \cong \mathrm{Im}\,\mathbb{H}$, then the Riemannian connection on P is represented by two 1-forms ψ and ϕ with values in $\mathrm{Im}\,\mathbb{H}$

(A principal $SO(4)$-connection on P is a diff. 1-form on P with values in $\mathfrak{so}(4)$ ...)

These are uniquely determined by the equations

$$\psi + \bar{\psi} = \phi + \bar{\phi} = 0 \quad , \quad d\omega = -\psi \wedge \omega - \omega \wedge \phi$$

and we write $\psi = \psi_1 i + \psi_2 j + \psi_3 k$, $\phi = \phi_1 i + \phi_2 j + \phi_3 k$.

We are going to assume that (M, ds^2) is Einstein and self-dual. One can show that this is equivalent to the existence of a constant κ (scalar curvature) such that

$$\begin{pmatrix} d\phi_1 + 2\phi_2 \wedge \phi_3 \\ d\phi_2 + 2\phi_3 \wedge \phi_1 \\ d\phi_3 + 2\phi_1 \wedge \phi_2 \end{pmatrix} = \frac{\kappa}{2} \begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{pmatrix}$$

The horizontal forms on $\tilde{P} \times \mathbb{H}$ are spanned by the components of ω and $\alpha = da - a\phi = \alpha_0 + i\alpha_1 + j\alpha_2 + k\alpha_3$. Note the following relations

- $dr = a\bar{\alpha} + \alpha\bar{a}$
- $d\alpha = -\alpha \wedge \phi - \frac{\kappa}{4} a\bar{\omega} \wedge \omega$
- $\frac{1}{2}(\bar{\alpha} \wedge \alpha) = iA_1 + jA_2 + kA_3$ ~~[illegible]~~

Now consider the following three $\mathrm{Spin}(4)$-invariant, horizontal 4-forms:

- $\Psi_1 = -\frac{1}{6}(A_1^2 + A_2^2 + A_3^2) = \alpha_0\alpha_1\alpha_2\alpha_3$
- $\Psi_2 = A_1\Omega_1 + A_2\Omega_2 + A_3\Omega_3$
- $\Psi_3 = -\frac{1}{6}(\Omega_1^2 + \Omega_2^2 + \Omega_3^2) = \omega_0\omega_1\omega_2\omega_3$

Note the resemblence of these to the summands of $\bar{\Phi}$.

We have

- $d\Psi_1 = -\frac{k}{4} dr \wedge \Psi_2$
- $d\Psi_2 = -\frac{3k}{2} dr \wedge \Psi_3$ $\quad d\Psi_3 = 0.$

by using a number of complicated identities.

(5) ODEs determining the metric

Let $f(r), g(r)$ be two positive functions of r. Define $\Phi \in \Omega^4_+(S)$ by $\Phi = f^2\Psi_1 + fg\Psi_2 + g^2\Psi_3$ which has the associated metric

$$g_\Phi = f(\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2) + g(\omega_0^2 + \omega_1^2 + \omega_2^2 + \omega_3^2)$$

(Note we are identifying skew/symmetric forms on S with Spin(4)-invariant horizontal skew/symmetric forms on $\tilde{P} \times \mathbb{H}$).

Then we find

$$d\Phi = \left((fg)' - \frac{k}{4}f^2\right) dr \wedge \Psi_2 + \left((g^2)' - \frac{3k}{2}fg\right) dr \wedge \Psi_3$$

since $dr \wedge \Psi_1 = 0$.

Thus $d\Phi = 0 \iff f, g$ satisfy

$$\begin{cases} (fg)' = \frac{k}{4}f^2 \\ (g^2)' = \frac{3k}{2}fg \end{cases} \qquad (*)$$

If $k = 0$, f, g are constants. Otherwise,

$$f(r) = 4(c_0 r + c_1)^{-2/5} \qquad g(r) = 5 \times c_0^{-1}(c_0 r + c_1)^{3/5}$$

where c_0, c_1 are constants, $k/c_0 > 0$ and $c_0 r + c_1 > 0$ in the region of definition of the solution.

Remarks

(1) The system is homogeneous, so we may consider solutions of the form $(\lambda f, \lambda g)$ equivalent if λ ~~nonconstant~~ $\in \mathbb{R}^*$.

(2) Let $m_\lambda : S \to S$ be multiplication by $\lambda \in \mathbb{R}^*$, then

$$m_\lambda^* \Phi = (\lambda f(\lambda^2 r))^2 \Psi_1 + \lambda^2 f(\lambda^2 r) g(\lambda^2 r) \Psi_2 + g(\lambda^2 r) \Psi_3$$

so we may consider all solutions of the form $(\lambda^2 f(\lambda^2 r), g(\lambda^2 r))$ as equivalent.

If $\kappa = 0$, then we may take $f \equiv 1 \equiv g$ and then $S \to M$ is a trivial bundle and ~~the associated metric~~ we discard this case.

Otherwise, Bryant - Salamon give four other cases, but only one of these is complete:

Case (ii) $\kappa > 0$ $\quad f(r) = 4(1+r)^{-2/5}$ $\quad g(r) = 5\kappa(1+r)^{3/5}$

Since these satisfy $(*)$, then in this case, $d\Phi = 0$ and so by Theorem 1, $\nabla \Phi = 0$ and so the holonomy of g_Φ is contained in $\mathrm{Spin}(7)$.

Theorem 3 (Bryant - Salamon)

Let S be the spin bundle over S^4, where S^4 is given its standard metric. Let $\Phi \in \Omega^4_+(S)$ be the 4-form

$$\Phi = 16(1+r)^{-4/5} \Psi_1 + 20(1+r)^{1/5} \Psi_2 + 25(1+r)^{6/5} \Psi_3 .$$

Then $\mathrm{Hol}(g_\Phi) = \mathrm{Spin}(7)$ and g_Φ is complete.

Very Sketch proof

$$g_\Phi = 4(1+r)^{-2/5}(\alpha_0^2+\alpha_1^2+\alpha_2^2+\alpha_3^2) + 5(1+r)^{3/5}(\omega_0^2+\omega_1^2+\omega_2^2+\omega_3^2)$$

(as $\kappa = 1$ for the standard metric on S^4)

and $\displaystyle\int_0^\infty \frac{2\,d\rho}{(1+\rho^2)^{1/5}} = \infty$ so g_Φ is complete.

Theorem 1 $\Rightarrow \nabla\Phi = 0 \Rightarrow \mathrm{Hol}(g_\Phi) \subseteq \mathrm{Spin}(7)$

$\mathcal{S}$ homotopic to S^4 $\Rightarrow$ $\mathcal{S}$ simply connected.

Bryant and Salamon show that there are no nonzero g_Φ-parallel 1-forms or 2-forms. So by Theorem 2, $\mathrm{Hol}(g_\Phi) = \mathrm{Spin}(7)$. $\square$

So we have constructed a complete Spin(7)-metric!

Remark

A Theorem by Hitchin shows that S^4 with its canonical metric is (up to a constant factor) the unique simply connected, complete, self-dual, Einstein manifold which is spin and has positive scalar curvature.

Thus, the above methods cannot be used to construct other complete metrics with holonomy Spin(7).