Smooth integrable geometry IV

1. Twistor theory & GL_2 geometry

Kodaira theory (for rational curves) Let Z be a complex mfd containing a rational curve $X \cong \mathbb{P}(H) \cong \mathbb{P}^1$ $(H \cong \mathbb{C}^2)$ with normal bundle

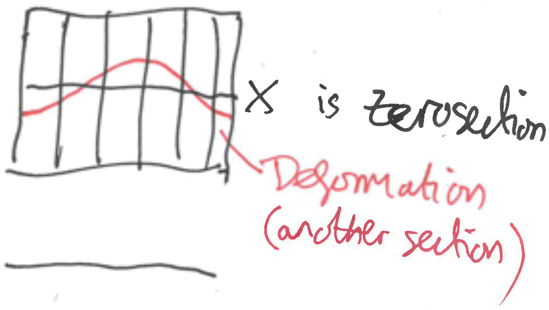
$$\mathcal{N} = \bigoplus_{k \in \mathbb{N}} \mathcal{O}(k) \otimes E_k \qquad (\text{NB } k \geq 0)$$

Then $H^1(\mathcal{N}) = 0$ so moduli space of deformations of X is (locally, near X) a complex mfd M with

$$T_X M \cong H^0(\mathcal{N}) \cong \bigoplus_{k \in \mathbb{N}} H^0(\mathcal{O}(k)) \otimes E_k$$
$$\cong \bigoplus_{k \in \mathbb{N}} S^k H^* \otimes E_k$$

Example $Z = \bigoplus_{k \in \mathbb{N}} \mathcal{O}(k) \otimes E_k$

$$\downarrow$$
$$\mathbb{P}(H)$$



The space of sections is the vector space $\bigoplus_{k \in \mathbb{N}} S^k H^* \otimes E_k$, so Kodaira moduli space M is (locally) this.

Remark To avoid confusion, think of M as (an open subset of) an affine space with $\bigoplus_{k \in \mathbb{N}} S^k H^* \otimes E_k$ as vector space of translations/constant vector fields.

G & GL_2 - geometry

Let $\rho: G \to GL(W)$ be a (faithful) repn of a complex Lie gp G on a complex n-dim'l space W.

Q What is a manifold M modelled on W?

Example (Flat model) M (an open subset of) a complex affine space with translation group W, so $TM \cong M \times W$.

More generally ~~ϕ~~ M is loc. diffeomorphic to W
$\Rightarrow$ M has principal $GL(W)$-bundle $GL(M)$ with fibre

$$GL(M)_x = \{\psi : W \to T_x M \text{ invertible linear}\}$$

'frames / framings of $T_x M$ by W'.

$$g \in GL(W),\ \psi \in GL(M)_x \mapsto \psi g := \psi \circ g : W \to T_x M.$$
$g : W \to W$ $\qquad$ $\psi : W \to T_x M$

[so far no diff. geom, just diff top.]

Defn For $G_0 \le GL(W)$, a G_0-structure on M is a principal G_0-subbundle $P(M) \subseteq GL(M)$.
('special frames' related by G_0 e.g. $G_0 = O(W)$ & orthonormal frames)

Standard (but wrong!) answer to Q M has a G_0-structure with $G_0 = \rho(G) \le GL(W)$
($G_0 \cong G$ if ρ faithful)

Better choices for G_0: groups 'preserving' G action
i.e. $G_0 = \begin{cases} C_\rho(G) & \text{centralizer of } \rho(G) \text{ in } GL(W) \\ \text{or } N_\rho(G) & \text{normalizer} \text{ ——— '' ———} \end{cases}$

Defn A $\rho(G)$-geometry on M is an $N_\rho(G)$-structure
A hyper-$\rho(G)$-geometry ——— '' ——— $C_\rho(G)$-structure.

Obs $N_\rho(G)$ acts on $\rho(G)$ by automorphisms

$$h \cdot \rho(g) = h \rho(g) h^{-1} \in \rho(G).$$

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Consequence If $P(M) \subseteq GL(M)$ is a $\rho(G)$-geometry then

$$G_M := P(M) \times_{N_\rho(G)} \rho(G) \subseteq GL(TM)$$

is a bundle of groups with fibre $(G_M)_x \leq GL(T_x M)$.
If hyper-$\rho(G)$ then $G_M \cong M \times G$, so have G-action on each $T_x M$.

Case at hand: $G = GL(H) \cong GL_2$, $W = \bigoplus_{k \in \mathbb{N}} S^k H^* \otimes E_k$
'GL_2-geometry'

2. Twistor theory of GL_2-geometries

Focus on example of $W = H^* \otimes E$, $\dim H = 2$, $\dim E = m$.
So $\rho : GL(H) \to GL(H^* \otimes E)$ has

$$N_\rho(G) = GL(H^*).GL(E) \quad \& \quad C_\rho(G) = GL(E).$$

- $\rho(GL_2)$-geometry on M: $TM \cong \mathcal{H}^* \otimes \mathcal{E}$, $\mathcal{H}$ rank 2, $\mathcal{E}$ rank m
- hyper-$\rho(GL_2)$-geometry if also $\mathcal{H} \cong M \times H$ ($\rightsquigarrow GL(H)$ action)

Origin of terminology For $m = 2\ell$ this is a complexification of
- quaternionic geometry $(GL(\mathbb{H}).GL(\mathbb{H}^\ell)$-structures$)$
- & hypercomplex geometry $(GL(\mathbb{H}^\ell)$-structures$)$

Q When is such M a moduli space of rational curves?
- Need to (re)construct Z from M.
- Expected normal bundle is $\mathcal{O}(1) \otimes E$.

Idea

$$\begin{tikzcd} & F \arrow[dl, "\pi"'] \arrow[dr, "\tau"] & \\ M & & Z \end{tikzcd}$$

$\hat{\Pi} \subseteq TF$ tangent to fibres of τ,
$\Pi = d\pi(\hat{\Pi}) \leq \pi^* TM$, so need to build F, Π & $\hat{\Pi}$.

Key observation From $TM \cong \mathcal{H}^* \otimes \mathcal{E}$ we already get

$$F = P(\mathcal{H}) \qquad \& \qquad \Pi \le \pi^* TM \quad \text{with}$$
$$\pi \downarrow \mathbb{CP}^1$$
$$M$$

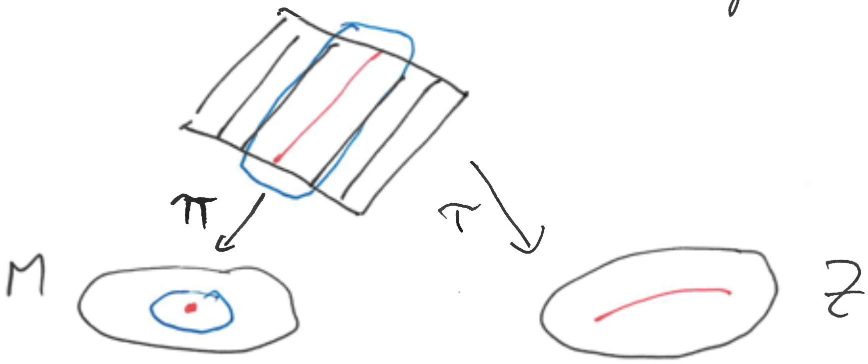
$$\Pi_{(x,\ell)} = \ell^0 \otimes \mathcal{E}_x \le \mathcal{H}_x^* \otimes \mathcal{E}_x = (\pi^* TM)_{(x,\ell)}$$

(here $x \in M$, $\ell \in P(\mathcal{H}_x)$ i.e. $\ell \le \mathcal{H}_x$ 1-diml)

So $\hat{\Pi}$ is an integrable lift of Π to TF.

Defn A Lax-integrable GL_2-geometry is a GL_2-geometry equipped with a lift $\hat{\Pi}$ of Π to TF which is Frobenius integrable. Its twistor space "is" the local leaf space of induced foliation.

Picture



[In general only have Z's in nbhd of point in M i.e. nbhds of twistor lines - but this is enough for Kodaira theory]

Fibres of τ map to rational curves X in Z & we can compute normal bundle

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & V_{(x,\ell)}F \arrow[r, "\cong"] \arrow[d] & T_{\tau(x,\ell)}X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \hat{\Pi}_{(x,\ell)} \arrow[r] \arrow[d, "(\text{lift})"', "\cong"] & T_{(x,\ell)}F \arrow[r] \arrow[d] & T_{\tau(x,\ell)}Z \arrow[r] \arrow[d, "\leftarrow \text{defn of normal bundle}"] & 0 \\
0 \arrow[r] & \Pi_{(x,\ell)} \arrow[r] \arrow[d] & T_xM \arrow[r, "\uparrow \text{compute this}"'] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

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Key to computation of normal bundle: 'Euler sequence'

$\ell \in P(\mathcal{H})_x$ i.e. $\ell \leq \mathcal{H}_x$ $\rightsquigarrow$ annihilator $\ell^\circ \leq \mathcal{H}_x^*$

& $0 \to \ell^\circ \to \mathcal{H}_x^* \to \ell^* \to 0$

[i.e. $0 \to \mathcal{O}(-1) \otimes \Lambda^2 \mathcal{H}_x^* \to \mathcal{H}_x^* \to \mathcal{O}(1) \to 0$ on $P(\mathcal{H}_x)$]

so
$$0 \to \underbrace{\ell^\circ \otimes \mathcal{E}_x}_{= T\Pi_{(x,\ell)}} \to \underbrace{\mathcal{H}_x^* \otimes \mathcal{E}_x}_{= T_x M} \to \ell^* \otimes \mathcal{E}_x \to 0$$

$\therefore$ Normal bundle is $\mathcal{O}(1) \otimes \mathcal{E}_x$, as expected.

Remarks
- In practice construct integrable lift $\tilde{\Pi}$ by introducing a connection on $\mathcal{H}$.
- For $m \geq 3$ integrability condition is existence of connections $\nabla^{\mathcal{E}}$ & $\nabla^{\mathcal{H}}$ s.t. $\nabla^{\mathcal{H}} \otimes \nabla^{\mathcal{E}}$ is torsion free on $TM = \mathcal{H}^* \otimes \mathcal{E}$.
- For $m=2$ $TM = \mathcal{H}^* \otimes \mathcal{E}$ is a (complexified) conformal structure & integrability condition is (anti) selfduality of Weyl tensor $*W = \pm W$.