Smooth integrable geometry by examples III

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1. Twistor correspondences for rational curves

Recall basic setting of twistor/lax approach to integrable systems.

Double fibration

$$\begin{tikzcd} & F \text{ corresp. space} \arrow[dl, "\pi"'] \arrow[dr, "\tau"] & \\ \text{space-time } M & & Z \text{ twistor space} \end{tikzcd}$$

where π has cpct fibres.

So given $x \in M$ $\quad \tau\pi^{-1}\{x\}$ cpct in Z "twistor submanifold" (twistor line)

$z \in M \rightsquigarrow \pi(\tau^{-1}\{z\})$ "α-submanifold"

Focus on case where $\pi^{-1}\{x\} \cong \mathbb{CP}^1$ so twistor lines are rational curves.

Example:

$$\begin{tikzcd} & F_{1,2}(\mathbb{C}^n) = \{(x,z) \mid z \leq x\} \arrow[dl, "\pi"'] \arrow[dr] & \\ M = G_2(\mathbb{C}^n) & & \mathbb{P}(\mathbb{C}^n) = Z \end{tikzcd}$$

$\shortparallel$

moduli space of proj. lines (deg 1 rational curves in Z)

For $U \subset M$ we have (Ward-) correspondence (Penrose-Ward transf)

between $\quad W \to \hat{U} = \tau(\pi^{-1}U)$ vector bundles, trivial on twistor lines

$$\updownarrow$$

$V \to U$ vector bundles with connection ∇, flat on α-submanifolds

with $\quad \pi^* V \cong \tau^* W$

$n=3$: $\quad G_2(\mathbb{C}^3) \cong \mathbb{P}(\mathbb{C}^{3*}) \rightsquigarrow$ twistor corresp is proj. duality.

α submanifs 1-dim $\rightsquigarrow$ flatness is vacuous.

$V \to U$ vectors w connection $\longleftrightarrow$ $W \to \hat{U}$ vb's trivial on lines.

$n=4$: $\quad Gr_2(\mathbb{C}^4)$ is Klein quadric $\rightsquigarrow$ flatness $\Leftrightarrow$ SDYM eq^{n}s.

$n \geq 4$: $\quad$ flatness sometimes called SDYM hierarchy.

Subexample $\quad U \subset Gr_2(\mathbb{C}^n)$ is affine chart

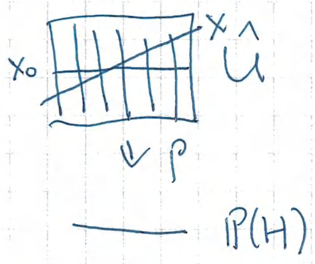
Fix $E \subseteq \mathbb{C}^n$ with $\dim E = \cancel{2}n-2$ & set

$U = \{x \in Gr_2(\mathbb{C}^n) \mid x \cap E = \{0\}\}$

$\hat{U} = \{z \in \mathbb{P}(\mathbb{C}^n) \mid z \cap E = \{0\}\} = \mathbb{P}(\mathbb{C}^n) \setminus \mathbb{P}(E)$

Let $H = \mathbb{C}^n/E$ so $\mathbb{C}^n \to H$ linear, kernel E

proj map $\hat{U} = \mathbb{P}(\mathbb{C}^n) \setminus \mathbb{P}(E) \to \mathbb{P}(H) \cong \mathbb{CP}^1$



$x \in U \iff p(x) = \mathbb{P}(H)$

& x is image of section of p

Fix $x_0 \in U$ so $\mathbb{C}^n = x_0 \oplus E$ & any other $x \in U$ is graph of linear map $H \to E$ with $x_0 \to 0$ linear map.

So U is affine space modelled on $\mathrm{Hom}(H,E) \cong E \otimes H^*$ & x_0 is a basepoint making

$$\begin{array}{c} \hat{U} \cong E \otimes \mathcal{O}(1) \\ \downarrow \\ \mathbb{P}(H) \end{array}$$

& space of sections $H^0(\mathbb{P}(H), E \otimes \mathcal{O}(1)) = E \otimes H^0(\mathbb{P}(H), \mathcal{O}(1))$

$= E \otimes H^*$ as predicted.

Otherwise said: $T_{x_0} U \cong H^0(\mathbb{P}(H), E \otimes \mathcal{O}(1))$

2. Vector bundles and sheaves on $\mathbb{CP}^1$

H 2-dim^l vector space

$\mathbb{P}(H) \cong \mathbb{CP}^1$

Have tautological line bundle $\mathcal{O}(-1) \subseteq H \times \mathbb{P}(H)$

$= \{(\ell, v) \mid v \in \ell\}$ i.e. $\mathcal{O}(-1)_\ell = \ell$

$k \in \mathbb{Z} \qquad \mathcal{O}(k) = \begin{cases} \mathcal{O}(-1)^{\otimes(-k)} & k<0 \\ \mathcal{O} & k=0 \\ (\mathcal{O}(-1)^*)^{\otimes k} & k>0 \end{cases}$ with fibre $\ell \otimes \dots \otimes \ell$, trivial, $\ell^* \otimes \dots \otimes \ell^*$

These also define "sheaves" by

$$\mathcal{O}(k)(U) = \{\text{holo sections of } \mathcal{O}(k)|_U\} \qquad U \subseteq \mathbb{P}(H) \text{ open.}$$

$$H^0(\mathbb{P}(H), \mathcal{O}(k)) := \mathcal{O}(k)(\mathbb{P}(H)) = \begin{cases} 0 & k<0 \\ S^k H^* & k \geq 0 \end{cases}$$

Why? A holo. section σ of $\mathcal{O}(k)$ ~~defines~~ has pullback to $\pi^{-1}(U) \subset H$ defines a homog. deg k hol f^n: given ~~sets~~ such a f^n restrict to $\ell \in U$, $\ell \subseteq \pi^{-1}U \subseteq H$ to get elt of $\ell^{*\otimes k}$

Magic of holo geometry: if $U = \mathbb{P}(H)$ then homog. deg k f^n is polynℓ — this tells us that there are no more global sections than $S^k H^*$.

Grothendieck: Let $V \to \mathbb{CP}^1$ be holo vector bundle

Then $V \cong \bigoplus_{k \in \mathbb{Z}} E_k \otimes \mathcal{O}(k)$ for some vector spaces E_k

non-canonical

Example: $0 \to \mathcal{O}(2) \to V \to \mathcal{O}(1) \to 0$

Then $V \cong \mathcal{O}(2) \oplus \mathcal{O}(1)$ splitting exist and are an affine space on $\mathrm{Hom}(\mathcal{O}(1), \mathcal{O}(2)) \cong H^0(\mathcal{O}(1)) \cong H^*$

Generalises to Harder-Narasimhan filtration for higher genus ...

Remark If we allow more general "coherent" sheaves on $\mathbb{CP}^1$ then there are more irred. objects:

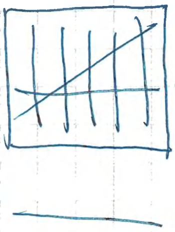
$$\mathcal{O}(k) \quad k \in \mathbb{Z}$$

"skyscraper sheaves" $\quad \mathcal{O}_p(U) = \begin{cases} \mathbb{C} & p \in U \\ 0 & \text{otherwise} \end{cases}$

eg: ~~or~~ $\mathcal{O}(k)/m_\ell^k$ $\qquad H^0(\mathcal{O}(k)/m_\ell^k) \cong J_\ell^{k-1}\mathcal{O}(k) \cong \mathbb{C}^k$

section of $\mathcal{O}(-1)$ vanishing at ℓ

3. Kodaira Thy



x_0 $E \otimes \mathcal{O}(1) = \hat{\mathcal{U}} \subset Z$

$\hat{(\mathbb{P}(H)}$

all sections of $\hat{\mathcal{U}}$

$\mathcal{U} \subseteq M$

$T_x \mathcal{U} = \mathrm{Hom}(H, E)$

$= H^0(\mathbb{P}(H), \hat{\mathcal{U}})$ ← all sections cos all are linears.

Any ~~stood~~ submanifold near a section is a section also

$\therefore$ $\mathcal{U}$ is moduli space of <u>all</u> deformations of x_0

Kodaira says this happens in great generality:

if $X \subseteq Z$ is compact cmplx submanifold
(Z not nec. cmpt)

~~what is space of~~

then the normal bdle to X in Z is $N = TZ|_X / TX$

In good cases ("$H^1(X, N) = 0$") the modul space of deformations M
(sheaf cohomology)

of X is smooth near $X \in M$ with tangent space $H^0(X, N)$

Case $X = \mathbb{P}^1$ Serre duality say $H^1(X, N) \cong H^0(X, N^* \otimes \mathcal{O}(-2))^*$
(computable.)

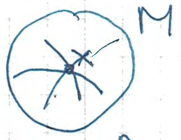
so $H^1(\mathbb{CP}^1, \mathcal{O}(k))^* = H^0(\mathbb{CP}^1, \mathcal{O}(-k-2)) \cong \begin{cases} S^{-k-2} H^* & k \leq -2 \\ 0 & \text{otherwise.} \end{cases}$

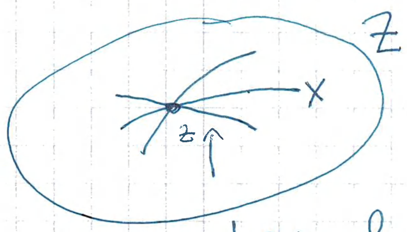
(so if $H^0(X, N) \neq 0$ $H^1(X, N) = 0$ — hooray!)

so can do $F = \{(X, z) \mid z \in X\}$

$$\begin{tikzcd} & F \arrow[dl] \arrow[dr] & \\ M & & Z \end{tikzcd}$$

Example Let Z be cmplx surface containing rational curve w/ normal bundle $\mathcal{O}(1)$. Let M be moduli of deformations: smooth near X with tangent space $T_X M = H^0(X, \mathcal{O}(1)) \cong \mathbb{C}^2$ — another surface.





gives projective structure on M

1-param. family cos sections have one zero

& conversely ...