Smooth integrable geometry II

1. Pencils of flat connections

Recall $\quad f : \Sigma \to \mathbb{P}(\mathcal{L}) \subseteq \mathbb{P}(V) \qquad V \cong \mathbb{R}^{n+1,1}$

$\qquad f_x \leq V$

isothermic if have $\eta \in \Omega^1_\Sigma(\mathfrak{o}(V))$ s.t.

1/ $\eta_X \in f \wedge f^\perp \quad \forall X \in T\Sigma$

2/ $\forall \lambda \in \mathbb{R} \quad \nabla^\lambda = d + \lambda\eta$ flat

Focus on 2/ but

- complexify everything
- do not need to trivialise wrt ∇^0

Def^n Let $V \to \Sigma$ be holo vector bundle over cmplx manifold.
A pencil of flat connections on V is an affine linear $\nabla^\lambda = \nabla^0 + \lambda\eta$ family of flat holomorphic connections $\lambda \in \mathbb{C}$

If Σ is simply connected (or on an open such), then flat connections are trivial: V trivialised by ∇^λ-parallel sections

∴ set $W_\lambda = \{\nabla^\lambda\text{-parallel sections of } V\}$

Then $W = \coprod_{\lambda \in \mathbb{C}} W_\lambda \to \mathbb{C}$ cmplx vector bundle.

On $F := \Sigma \times \mathbb{C}$

$$\begin{tikzcd} V \arrow[dr] & & F \arrow[dl, "\pi"'] \arrow[dr, "\tau"] & & W \arrow[dl] \\ & \Sigma & & \mathbb{C} & \end{tikzcd}$$

have $\pi^* V \underset{\alpha}{\cong} \tau^* W$

$(x, \lambda, v) \mapsto \nabla^\lambda$-$\parallel$ section thru $v \in V_x$

Also $\hat{\Pi} := \ \leq TF$ is the integrable distribution tangent to fibres of τ
$\pi^* T\Sigma \cong$

∴ $\pi^* V$ flat on each leaf (fibre of τ) $\tau^{-1}\{\lambda\}$ wrt ∇_λ

2. Twistor theory & Lax distributions

General setting: double fibration or twistor correspondence

$$\begin{tikzcd} & F^{n+k} \arrow[dl, "\pi"'] \arrow[dr, "\tau"] & \\ M^n & & Z^m \end{tikzcd}$$

F^{n+k} ← correspondence space

Z^m ← "twistor space"

"space/time" → M^n

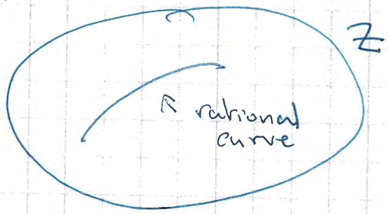
| prev. example $k = m = 1$

We keep $k=1$ mostly.

However usually require π to have cpct fibres.
Focus on $k=1$ & fibres of π are $\mathbb{CP}^1$'s.

$\forall x \in M$ $\quad \tau(\pi^{-1}(x)) \subset Z$ is required to be a rational curve (embedded $\mathbb{CP}^1$) called a twistor line.

Moving x gives family of twistor lines



Fibres of τ give foliation on F
& tangent dist $\underbrace{\hat{\Pi}}_{\ker d\tau} \leq TF$ — the Lax distribution

Have $\Pi = d\pi(\hat{\Pi}) \leq \pi^* TM$ for $TF \xrightarrow{d\pi} \pi^* TM$
need not be equality & may not have const. rank

Basic principle: If $W \to Z$ is a bundle over Z, trivial on twistor lines i.e. $W|_{\tau(\pi^{-1}\{x\})}$ trivial $\forall x \in M$

then $\pi^* W \cong \pi^* V$ on F (trivialisation unique up to constants thanks to Liouville)
for some $V \to M$ which has flat connection on $\pi(\tau^{-1}(z)) \subseteq M$ for any $z \in Z$

3. The Klein correspondence & self-dual Yang-Mills fields

Let $Z = \mathbb{CP}^3 = \mathbb{P}(\mathbb{C}^4)$ & $M = G_2(\mathbb{C}^4) \cong Q^4 \leq \mathbb{P}(\Lambda^2 \mathbb{C}^4)$ Klein quadric

So each $x \in M$ corresp to $\ell_x \subseteq \mathbb{CP}^3$

∴ $F = \{ (x,z) \in M \times Z \mid z \in \ell_x \}$ ← flag variety
↖ incidence variety.

∴

$$\begin{tikzcd} & F \arrow[dl, "\pi"'] \arrow[dr, "\tau"] & \\ M^4 & & Z^3 \end{tikzcd}$$

with $\tau(\pi^{-1}\{x\}) = \ell_x$

$\pi(\tau^{-1}(z)) = \{ x \in M \mid z \in \ell_x \}$ — α-plane in M
or α-surface (better)

Also have β-planes/β-surfaces param by $\mathbb{CP}(\mathbb{C}^{4*})$

Let $U \subseteq M$ be any open sset, $\hat{U} = \tau(\pi^{-1}(U)) \subseteq Z$

Let $W \to \hat{U}$ be a bundle trivial s.t. $W|_{\tau(\pi^{-1}(x))}$ trivial $\forall x \in U$

Then $\tau^* W = \pi^* V$ is vector bundle $V \to U$ with flat connection on each α-surface $\pi(\tau^{-1}(z))$

Now any two distinct α surfaces meet at a point & in fact $\exists!$ connection on $V \to U \subset M$ & restricting to flat connection on each α-surface. ← this is what it means to be self-dual Y-M.

[Q: is there a clean/fast proof of this?]

Example: affine chart in M^4 $= U$; $\hat{U} = \mathbb{C}^2 \otimes \mathcal{O}(1) \to \mathbb{CP}^1$ & twistor lines are sections.