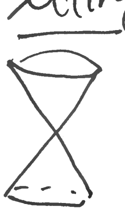
Smooth integrable geometry via examples I: isothermic surfaces

1. Motivation: limits of discrete isothermic nets

Setting $V = \mathbb{R}^{n+1,1}$ $(n+2)$-dim'l with $(\cdot,\cdot)$ of signature $(n+1,1)$.



$\mathcal{L} = \{ v \in V \mid (v,v) = 0 \}$ light cone

$P(\mathcal{L}) \subseteq P(V)$ projective light cone / conformal sphere

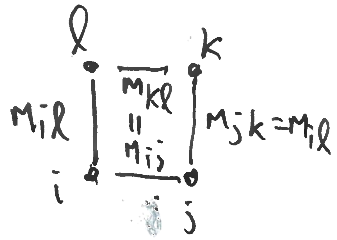
$O(V) = O(n+1,1)$ acts on $P(\mathcal{L}) \cong S^n$ by conf. transformations

Key example $v_0, v_\infty \in \mathcal{L}$, $\alpha \in \mathbb{R}^\times$, $v_0 \neq v_\infty$

$$\Gamma_{\langle v_0 \rangle}^{\langle v_\infty \rangle}(\alpha) = \begin{cases} \alpha & \text{on } \langle v_\infty \rangle \\ 1 & \text{on } v_0^\perp \cap v_\infty^\perp \\ 1/\alpha & \text{on } \langle v_0 \rangle \end{cases}$$

$i \mapsto f_i \leq V$

Defn/Thm $f : \mathbb{Z}^2 \to P(\mathcal{L})$ is isothermic with factorizing function $m : \{\text{edges}\} \to \mathbb{R}^\times$ iff for all $t \in \mathbb{R}$, Γ^t with



$$\Gamma^t_{ji} = \Gamma_{f_i}^{f_j}\left(1 - \frac{t}{m_{ij}}\right)$$

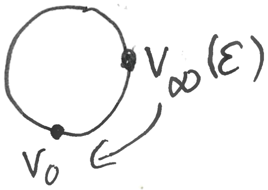
is a flat discrete connection on $\mathbb{Z}^2 \times V$.

i.e. $\Gamma_{f_j}^{f_k}\left(1 - \frac{t}{m_{il}}\right) \circ \Gamma_{f_i}^{f_j}\left(1 - \frac{t}{m_{ij}}\right) = \Gamma_{f_l}^{f_k}\left(1 - \frac{t}{m_{ij}}\right) \circ \Gamma_{f_i}^{f_l}\left(1 - \frac{t}{m_{il}}\right)$

on every elementary quadrilateral.

Idea Let lattice spacing $\to 0$. Then $\mathbb{Z}^2$ should 'converge' to a smooth surface Σ & Γ^t should 'converge' to a family of smooth flat connections on $V \times \Sigma \to \Sigma$

Completely unclear how to make this precise but we can see what the limit looks like by letting $f_j^\varepsilon, f_l^\varepsilon \to f_i$ as $\varepsilon \to 0$ and differentiating at $\varepsilon = 0$!



Let $v_\infty(\varepsilon)$ be a curve in $\mathcal{L}$ with

$$v_\infty(0) = v_0 \quad \& \quad v_\infty'(0) := \frac{d}{d\varepsilon} v_\infty(\varepsilon)\Big|_{\varepsilon=0} = w \in V.$$

Concretely can take $v_\infty(\varepsilon) = v_0 + \varepsilon w + \frac{1}{2}\varepsilon^2 (w,w) v_\infty$

where $v_\infty \in \mathcal{L}$ with $(v_0, v_\infty) = -1$.

$(v_\infty(\varepsilon), v_\infty(\varepsilon)) = 0 \ \forall \varepsilon \Rightarrow \quad 0 = \frac{d}{d\varepsilon}(v_\infty(\varepsilon), v_\infty(\varepsilon))\Big|_{\varepsilon=0} = 2(v_0, w)$

i.e. $w \in v_0^\perp$.

Now consider $\Gamma_{\langle v_0\rangle}^{\langle v_\infty(\varepsilon)\rangle}\left(1 - \frac{t}{M}\right)$ as $\varepsilon \to 0$

Problem: Not defined at $\varepsilon = 0$ as $\langle v_\infty(0)\rangle = \langle v_0\rangle$.

Soln Let $t \to 0$ quadratically in ε, say $t = \lambda \varepsilon^2$.

Claim (Exercise!) If $\Gamma(\varepsilon) = \Gamma_{\langle v_0\rangle}^{\langle v_\infty(\varepsilon)\rangle}\left(1 - \frac{\lambda\varepsilon^2}{M}\right)$ then

(1) $\forall v \in V \quad \Gamma(\varepsilon).v \to v$ as $\varepsilon \to 0$

(2) $\Gamma'(0).v = \frac{d}{d\varepsilon}\Gamma(\varepsilon).v\Big|_{\varepsilon=0} = \frac{2\lambda}{M(w,w)}\Big[\underbrace{(v_0,v)w - (w,v)v_0}_{\text{call this } (v_0\wedge w)(v)}\Big]$.

Observe $\Gamma(\varepsilon) \in O(V)$ i.e. $(\Gamma(\varepsilon).v_1, \Gamma(\varepsilon).v_2) = (v_1, v_2)$

$\therefore 0 = \frac{d}{d\varepsilon}(\Gamma(\varepsilon)v_1, \Gamma(\varepsilon)v_2)\Big|_{\varepsilon=0} = (\Gamma'(0)v_1, v_2) + (v_1, \Gamma'(0)v_2)$.

Thus $\Gamma'(0) = \frac{2\lambda}{M(w,w)} v_0 \wedge w$ belongs to the Lie algebra

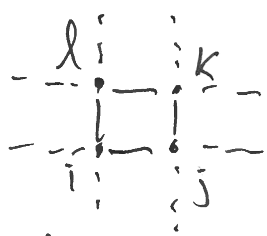
$$\mathfrak{o}(V) = \{A \in \mathfrak{gl}(V) \mid (Av_1, v_2) + (v_1, Av_2) = 0\}$$

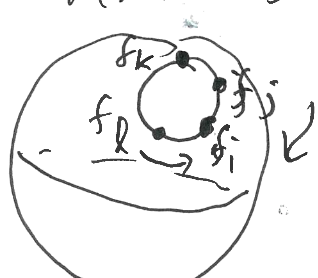
of $O(V)$ i.e. $\mathfrak{o}(V) = T_{id_V} O(V)$.

In addition $\Gamma'(0) \in \langle v_0\rangle \wedge \langle v_0\rangle^\perp$ meaning that

$\Gamma'(0)v_0 = 0$ & $\Gamma'(0)(\langle v_0\rangle^\perp) \subseteq \langle v_0\rangle$.

Return now to $f : \mathbb{Z}^2 \longrightarrow P(L) \cong S^n$





& $\Gamma_{f_i}^{f_j}\left(1 - \frac{t}{M_{ij}}\right)$, $\Gamma_{f_i}^{f_l}\left(1 - \frac{t}{M_{il}}\right)$. Let $v_0 \in f := f_i$

& $f_j \to f$, $f_l \to f$

to get elements $\frac{2\lambda}{M_1(w_1,w_1)} v_0 \wedge w_1 =: \lambda\eta_1$, $\lambda\eta_2 = \frac{2\lambda}{M_2(w_2,w_2)} v_0 \wedge v_2$

Smooth int. geom I – isothermic surfaces (ctd)

Idea $\eta_1, \eta_2 \in f \wedge f^\perp \leq \underline{o}(V)$ are data needed to define components of an $\underline{o}(V)$-valued 1-form

$$\eta = \eta_1 dx_1 + \eta_2 dx_2 \in \Omega^1_\Sigma(\underline{o}(V))$$

at each point of a surface Σ with coordinates x_1, x_2.

Hence.... Motivated but unexplained defn

$$\begin{aligned} f : \Sigma &\to P(\mathcal{L}) \\ x &\mapsto f_x \leq V \end{aligned}$$

is isothermic with 1-form $\eta \in \Omega^1_\Sigma(\underline{o}(V))$, if

(1) η takes values in $f \wedge f^\perp$ i.e. $\forall X \in T_x\Sigma$, $\eta(X) \in f_x \wedge f_x^\perp \leq \underline{o}(V)$.

(2) For all $\lambda \in \mathbb{R}$, $\nabla^\lambda := d + \lambda\eta$ is a flat connection on $V \times \Sigma$

2. Smooth gauge theory

Idea: a vector bundle over Σ is a vector space W_x depending smoothly on $x \in \Sigma$. Formalize by requiring

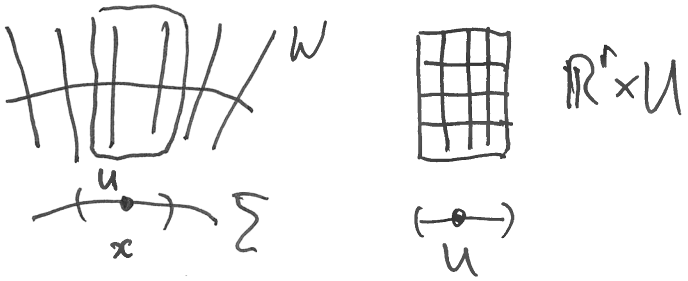
$$W := \bigsqcup_{x \in \Sigma} W_x = \{(x, w) \mid x \in \Sigma, w \in W_x\}$$

is a manifold....

Defn A (smooth) vector bundle of rank r over a smooth N-mfd Σ is a smooth mfd W & a smooth map $\pi : W \to \Sigma$ s.t. $\forall x \in \Sigma$

(1) $W_x := \pi^{-1}(x)$ is an r-dim'l vector space

(2) $\exists$ nbhd U of x & a diffeomorphism $\psi : \pi^{-1}(U) \to \mathbb{R}^r \times U$ s.t. $\phi|_{\pi^{-1}(x)} : \pi^{-1}(x) = \mathbb{R}^r \times \{x\}$ is linear



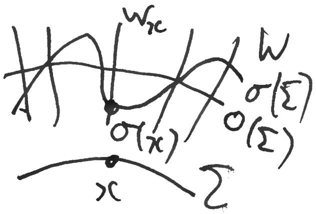
Examples • Same vector space $W_x = V$ at each $x \in \Sigma$ (trivial vector bundle): $W = V \times \Sigma$ & $\pi = \pi_2 : W \to \Sigma$.

• $f : \Sigma \to P(\mathcal{L})$ 'is' a rank 1 (line) subbundle of $V \times \Sigma$ s.t. $f_x \leq V$ is null $\forall x \in \Sigma$.

• $T\Sigma \to \Sigma$ is tangent bundle of Σ, $T_x\Sigma$ = tangent space at x.

• Any (functorial) construction on vector spaces applies 'fibrewise' to vector bundles e.g. $\mathrm{Hom}(W, W')_x = \mathrm{Hom}(W_x, W'_x)$, $\mathfrak{gl}(W) = \mathrm{Hom}(W, W)$, $(\Lambda^k W)_x = \Lambda^k(W_x)$.

Defns • A section of a vector bundle W is a smooth map $\sigma : \Sigma \to W$ s.t. $\pi \circ \sigma = \mathrm{id}_\Sigma$ i.e. $\sigma(x) \in W_x$ $\forall x \in \Sigma$.



e.g. $W = V \times \Sigma$ trivial $\Rightarrow \sigma(x) = (v(x), x)$ for a smooth fn $v : \Sigma \to V$.

• $\Omega^k_\Sigma(W) = \{ W\text{-valued } k\text{-forms on } \Sigma \}$
$= \{ \text{sections of } \mathrm{Hom}(\Lambda^k T\Sigma, W) \}$

$k = 0$ is just sections of W.

• A connection on a vector bundle W is a linear map

$$\nabla : \Omega^0_\Sigma(W) \to \Omega^1_\Sigma(W)$$
$$\sigma \mapsto_{x \mapsto} (X \mapsto \nabla_X \sigma)$$

s.t. $\forall \sigma \in \Omega^0_\Sigma(W)$, $X \in T_x\Sigma$, $x \in \Sigma$

$$\nabla_X(f\sigma) = f(x)\nabla_X\sigma + df_x(X)\,\sigma$$

• A section σ of W is parallel for ∇ if $\nabla_X \sigma = 0$ $\forall X \in T_x\Sigma$

Examples (1) Trivial bundle $\underline{V} = V \times \Sigma$ has trivial connection

$$d : \Omega^0_\Sigma(\underline{V}) \to \Omega^1_\Sigma(\underline{V})$$

$$\sigma = (v, id_\Sigma) \mapsto \big(X \mapsto d_X \sigma = (dv(X), id_\Sigma)\big)$$

$v : \Sigma \to V$

Thus σ parallel $\iff$ v constant

(2) If $\eta \in \Omega^1_\Sigma(\mathfrak{gl}(W))$ & ∇ is a connection on W then $\nabla + \eta$ is a connection on W with

$$(\nabla + \eta)_X \sigma = \nabla_X \sigma + \eta_X(\sigma)$$

(3) In case $W = V \times \Sigma$, $\mathfrak{gl}(W) = \mathfrak{gl}(V) \times \Sigma \supseteq \mathfrak{o}(V) \times \Sigma$ with $(\cdot, \cdot)$ on V as before

(1) & (2) combine: any $\eta \in \Omega^1_\Sigma(\mathfrak{o}(V))$ defines connections $\nabla^\lambda = d + \lambda\eta$ on $V \times \Sigma$ for $\lambda \in \mathbb{R}$

Flatness If σ is a ∇-parallel section of W

$$0 = \nabla_X \nabla_Y \sigma - \nabla_Y \nabla_X \sigma - \nabla_{[X,Y]}\sigma = R^\nabla_{X,Y}\sigma$$

with $R^\nabla \in \Omega^2_\Sigma(\mathfrak{gl}(W))$. Say ∇ flat if $R^\nabla = 0$.

Example $\nabla^\lambda = d + \lambda\eta$ flat if

$$0 = R^{\nabla^\lambda} = \underbrace{\cancel{R^d}}_{\substack{0 \text{ since} \\ d \text{ trivial}}} + \lambda\, d\eta + \underbrace{\tfrac{1}{2}\lambda^2 [\eta \wedge \eta]}_{\substack{0 \text{ if } \eta \text{ is valued in } f \wedge f^\perp \\ \text{since Lie algebra is abelian.}}}$$

Punchline In defn of isothermic (2) $\iff$ $d\eta = 0$.