Discrete Isothermic nets III

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Today Cube theorem via conceptual framework....

Discrete gauge theory

1. **Bundles** A (discrete) vector bundle of rank r over $\Omega \subseteq \mathbb{Z}^N$ is a set V & a surjection $\pi: V \to \Omega$ s.t.

$\forall i \in \Omega \quad V_i := \pi^{-1}\{i\}$ is an r-dim'l vector space (the fibre at i)

[Idea: have a vector space attached to each point & in discrete world that is all (no issues with local triviality etc.)]

Examples
- same vector space over each point (a trivial bundle) e.g.

$$\underline{\mathbb{R}}^{n+1,1} := \mathbb{R}^{n+1,1} \times \Omega \qquad \pi = \pi_2$$
$$\underline{\mathbb{R}}^{n+1,1}_i = \mathbb{R}^{n+1,1} \times \{i\} \cong \mathbb{R}^{n+1,1}$$

} fibres also have inner product of signature $n+1,1$

- $f: \Omega \to S^n = P(\mathcal{L}) \subseteq P(\mathbb{R}^{n+1,1})$ can be described equivalently as a discrete line subbundle of the trivial $\underline{\mathbb{R}}^{n+1,1}$ bundle:

$$f \leq \underline{\mathbb{R}}^{n+1,1} \quad \text{via} \quad f_i = f(i) \leq \mathbb{R}^{n+1,1}$$

We tacitly identify these two viewpoints (same not^n f).

- Any construction on vector spaces applies to vector bundles e.g. given line subbundle $f \leq \underline{\mathbb{R}}^{n+1,1}$ have $f^\perp/f$ a bundle of Euclidean vector spaces (signature $n,0$)

Defns. A **section** of a vector bundle V is $\sigma: \Omega \to V$ s.t. $\pi \circ \sigma = \mathrm{id}_\Omega$
i.e. $\sigma(i) \in V_i \quad \forall i \in \Omega$. $[V = \bigsqcup_i V_i]$

- For vector bundles V, W over Ω, a map $\varphi: V \to W$ is a **morphism** if

$$\pi_W \circ \varphi = \pi_V \qquad (\text{i.e. } \varphi(V_i) \leq W_i)$$
$$\varphi|_{V_i}: V_i \to W_i \quad \text{is linear.}$$

Also φ is an isomorphism if each $\varphi|_{V_i}$ is a linear isom..

2. **Connections** (modelling parallel transport for smooth connections)

A (discrete) **connection** on V is a map

$$\Gamma: \{\text{oriented edges}\} \to \bigsqcup_{i,j} \mathrm{Hom}(V_i, V_j)$$

s.t. $\Gamma_{ji} : V_i \to V_j$ is a linear isom. with

$\Gamma_{ij} = \Gamma_{ji}^{-1}$ (here ji is edge $\underset{(i,j)}{i \to j}$)

Examples 1. Trivial bundles have a trivial connection Γ^{triv}

e.g. $\Gamma^{triv}_{ji} : \mathbb{R}^{n+1,1} \times \{i\} \to \mathbb{R}^{n+1,1} \times \{j\}$ on $\underline{\mathbb{R}}^{n+1,1}$.

$(v,i) \mapsto (v,j)$

2. $f : \Omega \to P(\mathcal{L})$ with $f(i) \neq f(j)$ for any edge (i,j).

$\Gamma_{ji} : (f^\perp/f)_i \cong (f_i \oplus f_j)^\perp \cong (f^\perp/f)_j$ canonically,

via the projections $(f_i \oplus f_j)^\perp \to f_i^\perp/f_i$, $\to f_j^\perp/f_j$) so have a connection on $f^\perp/f$.

3. For $\alpha \in \mathbb{R}^\times$ or $\alpha : \{edges\} \to \mathbb{R}^\times$ have

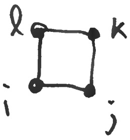
$\Gamma^{\alpha}_{ji} := \Gamma^{f(j)}_{f(i)}(\alpha_{ij}) : \underline{\mathbb{R}}^{n+1,1}_i \to \underline{\mathbb{R}}^{n+1,1}_j$.

NB. Each of these examples is a metric connection i.e. Γ_{ji}'s are isometries.

[Aside: a connection on a vector bundle induces one on an isomorphic bundle]

Defn For V with connection Γ, a section σ of V is parallel wrt Γ if $\Gamma_{ji}\sigma(i) = \sigma(j)$ $\forall$ edges (i,j)

3. Curvature If σ parallel for Γ then for any elem quad



have $\underbrace{\Gamma_{il}\Gamma_{lk}\Gamma_{kj}\Gamma_{ji}}_{R_{lkji} \in Aut(V_i)} \sigma(i) = \sigma(i)$

i.e. if $\sigma(i) \neq 0$ it is an evec of eval 1 for R_{lkji}.

Idea: if V, Γ has lots of parallel sections then...

Defn Γ flat if $R_{lkji} = 1$ on all elem. quads.

(Call R the curvature of Γ).

Exercise $f : \Omega \to P(\mathcal{L})$ with connection (2) Γ on $f^\perp/f$ is flat iff f is concircular on each elem. quad.

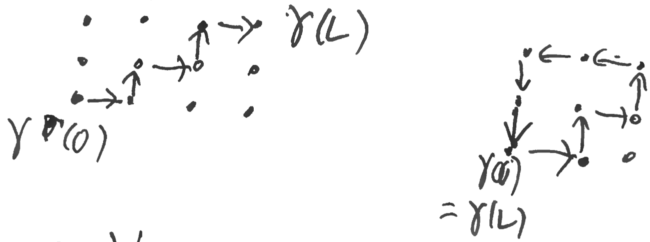
Thm Given $V \to \mathbb{Z}^N$ with connection Γ then $\exists$ parallel section through any $v \in V$ iff Γ is flat.

Proof ($\Rightarrow$) Already seen v is evec of R with eval 1.

($\Leftarrow$) Idea: show parallel transport is path-independent.

A path is $\gamma : \{0, \dots L\} \subseteq \mathbb{Z} \longrightarrow \mathbb{Z}^N$ preserving incidence i.e. $\gamma(j), \gamma(j+1)$ adjacent.

It's a loop if $\gamma(0) = \gamma(L)$.



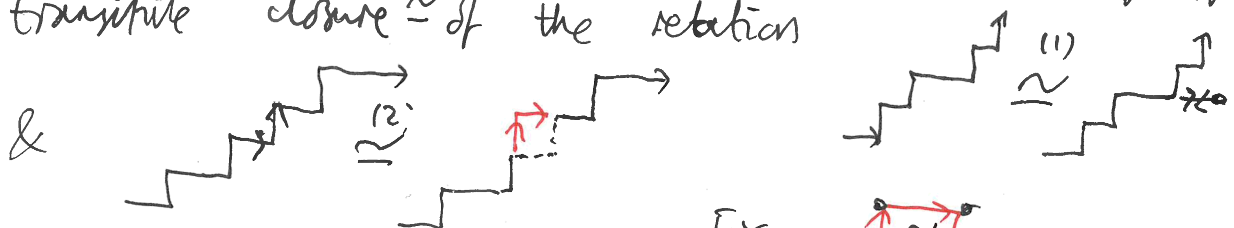
Parallel transport along γ $\quad P_\gamma : V_{\gamma(0)} \to V_{\gamma(L)}$ defined by

$$P_\gamma = \prod_{j=1}^{\ell} \Gamma_{\gamma(j)\gamma(j-1)}$$

so if σ parallel, $P_\gamma \sigma(\gamma(0)) = \sigma(\gamma(L))$.

[Aside: holonomy theorem for smooth connections — for a flat connection, parallel transport along homotopic paths coincide.]

$\gamma, \hat\gamma$ with same endpoints are homotopic if $(\gamma, \hat\gamma)$ is in transitive closure $\simeq$ of the relation



Ex

Lemma Γ flat $\Rightarrow$ $P_\gamma = P_{\hat\gamma}$ whenever $\gamma \simeq \hat\gamma$.

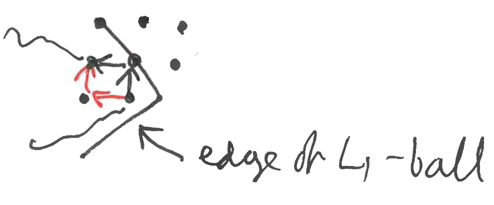
[(1) is ensured by $\Gamma_{ij} = \Gamma_{ji}^{-1}$ while (2) is ensured by flatness of Γ]

Prop $\gamma, \hat\gamma$ in $\mathbb{Z}^N$ with same endpoints are equivalent $\gamma \simeq \hat\gamma$.

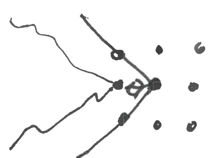
In particular any loop $\simeq$ constant loop.

Proof It suffices to consider the case of a loop γ.

Let $j \in \{0, \dots L\}$ maximize $\|\gamma(j) - \gamma(0)\|_1 = \sum_i |\gamma(j)_i - \gamma(0)_i|$ (L^1-norm). Look at path near extreme point: picture is



or



The new path is inside the L_1-ball, so new path is closer to $\gamma(0)$ near j. Now iterate $\rightsquigarrow \gamma \simeq$ const. loop at $\gamma(0)$.

Then Lemma & Prop prove the Thm: given $v \in V_{i_0}$ & $i \in \mathbb{Z}^N$ choose any path γ from i_0 to i & set $\sigma(i) := P_\gamma v$. This is well-defined by Lemma & Prop. ▨

<u>Cube Thm</u> It suffices to prove the following.

<u>Thm</u> $f : \mathbb{Z}^2 \to P(\mathfrak{h})$ is isothermic with factorizing function m (on edges) iff $\forall t \in \mathbb{R}^\times$

$$\Gamma^t_{ji} := \Gamma^{f(j)}_{f(i)}\left(1 - \frac{t}{m_{ij}}\right)$$

is a flat connection on $\underline{\mathbb{R}^{d+1,1}}$ ($\Gamma^{1-\frac{t}{m_{ij}}}_{ji}$ in not^n of Ex 3)

<u>Proof</u> Γ^t flat $\iff \underbrace{\Gamma^{f_k}_{f_j}\left(1-\frac{t}{m_{il}}\right)\Gamma^{f_j}_{f_i}\left(1-\frac{t}{m_{ij}}\right)}_{L} = \underbrace{\Gamma^{f_k}_{f_l}\left(1-\frac{t}{m_{ij}}\right)\Gamma^{f_l}_{f_i}\left(1-\frac{t}{m_{il}}\right)}_{M}$

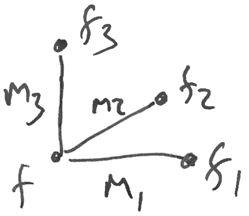
<u>Note</u> f_j is an evec of L with eval $\dfrac{1-t/m_{ij}}{1-t/m_{il}}$ & L induces identity on $f_j^\perp/f_j$ (since $(f_j \oplus f_i)^\perp \oplus f_j = f_j^\perp$) & similarly M has evec f_l & induces identity of $f_l^\perp/f_l$.

Hence $L = R \iff L f_l = f_l \iff R f_j = f_j \iff L = \Gamma^{f_j}_{f_l}\left(\dfrac{1-t/m_{ij}}{1-t/m_{il}}\right) = R$

$\iff \Gamma^{f_j}_{f_i}\left(1-\frac{t}{m_{ij}}\right) f_l = \Gamma^{f_j}_{f_k}\left(1-\frac{t}{m_{il}}\right) f_l$

These are 2 parametrizations of a circle which agree at $t = \infty$ (both are f_j) at $t=0$ (both are f_l) so they agree everywhere iff f_i, f_j, f_k, f_l are concircular & they agree a $t = m_{il}$ i.e. $\Gamma^{f_j}_{f_i}\left(1-\frac{m_{il}}{m_{ij}}\right) f_l = f_k$ which is the isothermic condition (as reformulated using Γ's in part I). ▨

Cube Thm Recall set-up:



By the theorem, completing the cube means defining

$$f_{13} = \Gamma_f^{f_1}\left(1 - \frac{m_3}{m_1}\right) f_3 \quad \text{etc.}$$

so $f_{123} = \Gamma_f^{f_{12}}(-)\,\Gamma_f^{f_1}(-)\, f_3 = \Gamma_{f_2}^{f_{12}}(-)\,\Gamma_f^{f_2}(-)\, f_3$

i.e. f_{**3} is parallel for Γ^{m_3}.

We also learn from the proof of the theorem

$$f_{123} = \Gamma_{f_1}^{f_2}\left(\frac{1 - m_3/m_1}{1 - m_3/m_2}\right) f_3$$

so the final corner of the cube is concircular with f_1, f_2, f_3 with prescribed cross-ratio.

[Thus not only faces but tetrahedra are concircular]