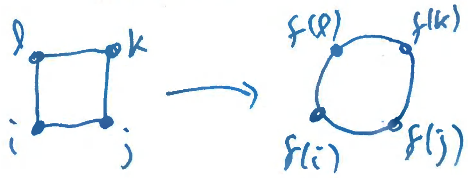
Discrete Integrable Geometry via Examples: Isothermic Nets II - Fran Burstall

Recall: $f: \mathbb{Z}^2 \to \mathbb{R}^3$ is isothermic if

- each elementary quadrilateral



is concircular

- with cross ratio $cr(f(l), f(j); f(i), f(k)) = m_{il}/m_{ij}$ for some $m: \{\text{edges}\} \to \mathbb{R}^*$ equal on opposite edges.

NB $s \longmapsto cr(p,q;r,s)$ is an affine coordinate sending $S^1 \xrightarrow{\cong} \mathbb{R} \cup \{\infty\}$ ~~to~~ $p \mapsto 0$, $q \mapsto \infty$, $r \mapsto 1$.

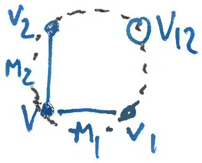
1. Existence Prescribe Cauchy data on a staircase in $\mathbb{Z}^2$ i.e.

$m' \neq m$

- put m's on edges (which prescribes m on all rows & columns) with no vertical m & horizontal m equal.
- put distinct v's $\in \mathbb{R}^3$ on vertices.

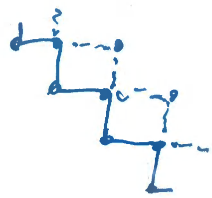
Claim: this determines a unique isothermic f

Key step:



- $\exists!$ circle through v, v_1, v_2
- then $\exists!$ v_{12} on this circle with $cr(v_1, v_2; v, v_{12}) = m_1/m_2$.

~~S~~. Thus we have a unique extension of f to a shifted staircase & we can iterate so long as the vertices remain distinct.



This gives local existence (& can arrange global existence for suitably independent initial v's)

2. Symmetry Q. For which $\phi: \mathbb{R}^3 \to \mathbb{R}^3$ do we have "f isothermic $\Rightarrow \phi \circ f$ isothermic".

Key: φ preserves circles (& then it will turn out that it preserves the cross ratio of 4 pts on a circle)

<u>A</u>: require φ <u>conformal</u> (angle-preserving)

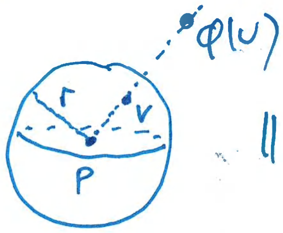
e.g.
- translations $x \mapsto x + a$ $\quad a \in \mathbb{R}^3$
- rotations / reflections $x \mapsto Ax$ $\quad A \in O(3)$
- dilations $x \mapsto \lambda x$ $\quad \lambda \in \mathbb{R}^\times$

$\Big\}$ familiar

& ... inversions in spheres:

$\varphi : \mathbb{R}^3 \setminus \{p\} \to \mathbb{R}^3 \setminus \{p\}$

$v \mapsto \varphi(v)$



where $\|v - p\| \, \|\varphi(v) - p\| = r^2$

<u>Obs</u> – this preserves the sphere of radius r centred at p

– as v approaches p, $\varphi(v)$ becomes very far away

Hence: introduce an extra point ∞ with $\varphi(p) = \infty$ & $\varphi(\infty) = p$.

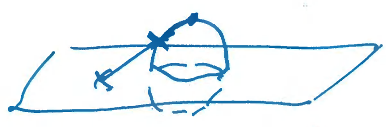
Why are these important?

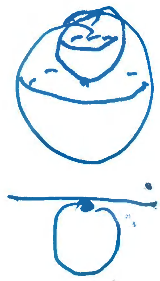
<u>Thm</u> (Liouville) Any conformal transformation of $\mathbb{R}^3 \cup \{\infty\}$ is a finite composite of inversions in spheres

[Not true in $\mathbb{R}^2$ – by complex analysis $\exists$ many conf. transformations]

[Aside: we get dilations by composing an inversion in a sphere by another with the same centre, & any Euclidean motion is a finite composite of reflections, which we can obtain by conjugating an inversion by another with centre on the sphere of the first]

This suggests we work with $S^3 = \mathbb{R}^3 \cup \{\infty\}$ via stereoprojection:





& $\varphi : \{\text{lines \& circles}\} \to \{\text{lines \& circles}\}$

(lines are circles through infinity)

[Remark: centres of spheres are not preserved by conf. transformations]

<u>Darboux model for S^n</u> (Generalizing S^1)

$\mathbb{R}^{n+1,1}$ is an $(n+2)$-dim'l real vector space with a signature $(\underset{+ve}{n+1}, \underset{-ve}{1})$ inner product $(\cdot, \cdot)$

We have the light cone $\mathbb{R}^{n+1,1} \supseteq \mathcal{L} = \{v \in \mathbb{R}^{n+1,1} \setminus \{0\} \mid (v,v) = 0\}$

$P(\mathcal{L}) \subseteq P(\mathbb{R}^{n+1,1})$ is an n-dim'l manifold (a quadric) diffeomorphic to S^n

- recover $\mathbb{R}^n$ by stereoprojection: fix $v_0, v_\infty \in \mathcal{L}$ linearly independent, so $(v_0, v_\infty) \neq 0$ & we may assume $(v_0, v_\infty) = -1$.

  Set $\mathbb{R}^n = \langle\{v_0, v_\infty\}\rangle^\perp$ & $E_{v_\infty} = \{v \in \mathcal{L} \mid (v_\infty, v) = -1\}$.

  Then
  $$\begin{aligned} \mathbb{R}^n &\xrightarrow{\cong} E_{v_\infty} \xrightarrow{\cong} P(\mathcal{L}) \setminus \{\langle v_\infty \rangle\} \\ x &\longmapsto v_0 + x + \tfrac{1}{2}(x,x)\, v_\infty \\ & \qquad\qquad v \longmapsto \langle v \rangle \end{aligned}$$

- obtain circles & more generally S^k's $\subseteq S^n$ via

  $V \leq \mathbb{R}^{n+1,1}$ with $(,)|_{V \times V}$ signature $(k+1, 1)$ $(\dim V = k+2)$

  $S^k = P(\mathcal{L} \cap V) \subseteq P(\mathcal{L}) = S^n$.

  Space of S^k's in S^n is an open subset of a grassmannian.

  - S^1's $\leftrightarrow$ $(2,1)$-planes in $\mathbb{R}^{n+1,1}$.

    $\therefore$ $\exists!$ circle through any 3 distinct points of S^n (since $\exists!$ $\mathbb{R}^{2,1}$ spanned by 3 distinct null vectors).

  - S^{n-1} $\leftrightarrow$ $(n,1)$-planes in $\mathbb{R}^{n+1,1}$

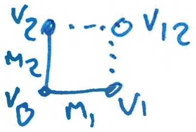
    $\therefore$ any $(n,1)$-plane V defines a reflection on $\mathbb{R}^{n+1,1}$ & this induces inversion in corresponding sphere (<u>Ex</u>: use stereoprojection to prove this)

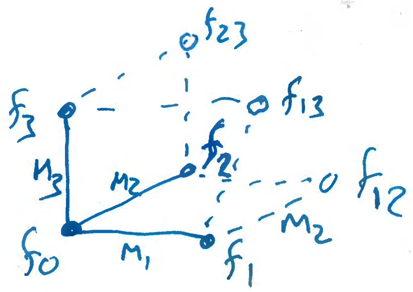
  Now reflections generate $O(n+1,1)$ (Cartan-Dieudonné) with $\pm\mathrm{id}$ acting trivially on $P(\mathcal{L})$ & have

  $$0 \to \mathbb{Z}_2 \to O(n+1,1) \to \mathrm{Conf}(S^n) \to 1.$$

Since $\mathrm{Conf}(S^n)$ acts by projective transformations it preserves cross ratios, & hence isothermic nets (for $n=3$, but can generalize).

3. Consistency (a key property, almost a definition, of integrability in discrete setting)

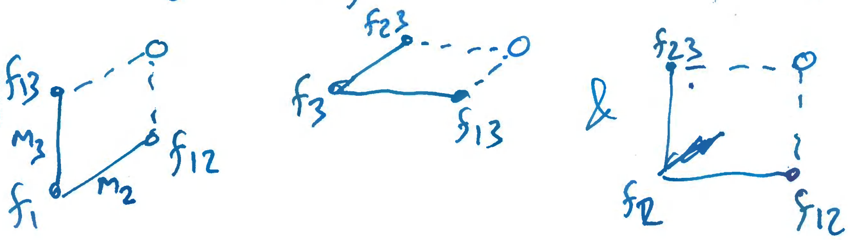
Recall 



Now add another dimension

i.e. have $f_\alpha \in S^n$ $\quad \alpha \in \{0,1,2,3\}$
& $m_i \in \mathbb{R}^*$ $\quad i \in \{1,2,3\}$ — distinct

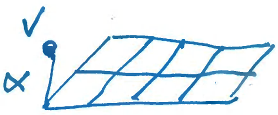
Then we can complete 3 elem. quads to get f_{ij} concircular with f_0, f_i, f_j with $cr = m_i/m_j$.

Now can complete 

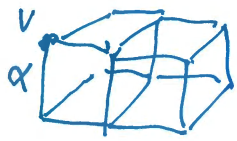
Thm $\exists!\, f_{123} \in S^n$ s.t. $f_i, f_{ij}, f_{ik}, f_{123}$ are concircular with $cr = m_j/m_k$ (for all i,j,k distinct)

(This is called 3 dim'l consistency)

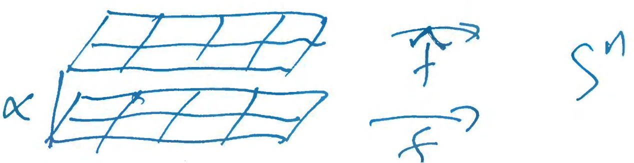
Key application Start with $f: \mathbb{Z}^2 \to S^n$ isothermic

& put $\mathbb{Z}^2 \subseteq \mathbb{Z}^3$ 

$\tilde{f}(v)$

choose $\alpha \in \mathbb{R}^*$ (different from all m's) & $\tilde{f}(v) \in S^n$. 

Consistency gives a cube & we can iterate this to get a new layer



& $\hat{f} = \tilde{f}|_{\text{new layer}}$ is also isothermic. Also have 4d, ..., nd consistency.