Topics in Integrable Geometry 1

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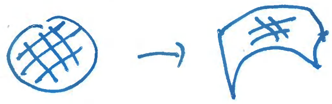
Discrete integrable geometry by examples: I Isothermic surfaces

Plan Differential geom: use calculus to study e.g. $f: \Sigma \to \mathbb{R}^3$

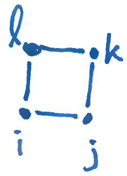


Model for this is discrete geometry:

use $f: \mathbb{Z}^2 \to \mathbb{R}^3$ instead (or, later, $f: \mathbb{Z}^N \to S^n$)

Here $\mathbb{Z}^2$ is not just a bunch of points (not enough structure):

We also have
- edges (i,j)
- elementary quadrilaterals (faces) (note labelling convention)



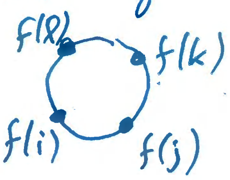
In $\mathbb{Z}^3$ we also have 3-cells (cubes)

Also have many generalizations (e.g. buildings)

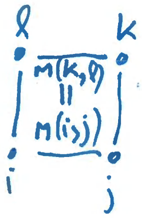
Unmotivated definition [∃ motivation: this models classical geometric notion of isothermic surface studied by Darboux, Bianchi, Christoffel, ...]

$f: \mathbb{Z}^2 \to \mathbb{R}^3$ is called an isothermic net if

- elementary quadrilaterals are concircular (Recall 3 points determine a circle, so this constrains the 4th point)



- $\exists\, m: \{\text{edges}\} \to \mathbb{R}^\times$ equal on opposite edges (a 'factorizing function') $m(i,l) = m(j,k)$ &



- The cross ratio of concircular points satisfies

$$\operatorname{cr}(f(l), f(j);\ f(i), f(k)) = \frac{m(i,l)}{m(i,j)} \quad \left(= \frac{m(j,k)}{m(l,k)}\right)$$

diagonally opposite pairs

To explain this definition, the first order of business is...

What is the cross ratio of 4 points on a circle?

Short answer (summary):

1/ A circle S^1 is a conic in P^2

2/ Any conic is a projective line $S^1 \cong \mathbb{P}^1$

3/ 4 points on $\mathbb{P}^1$ have a cross ratio

Details (instructive)

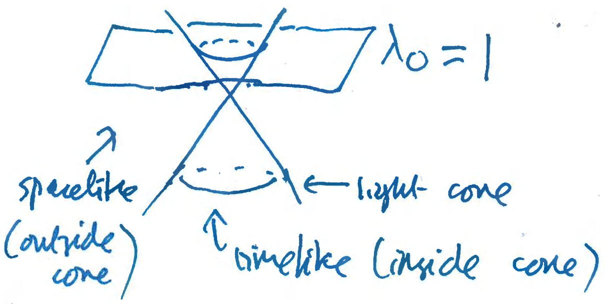
1/ $S^1 \subseteq \mathbb{R}^2$ is $\{(x_1, x_2) \mid x_1^2 + x_2^2 = 1\}$

View $\mathbb{R}^2 \subseteq \mathbb{P}^2 = \mathbb{P}(\mathbb{R}^3) = \dfrac{\mathbb{R}^3 \setminus \{0\}}{\mathbb{R}^*} = \{[\lambda_0, \lambda_1, \lambda_2] : \lambda_i \in \mathbb{R} \text{ not all } 0\}$

via $x_i = \lambda_i / \lambda_0$ & then

$$S^1 = \{[\lambda_0, \lambda_1, \lambda_2] \mid \underbrace{\lambda_1^2 + \lambda_2^2 - \lambda_0^2}_{} = 0\}$$

a quadratic form on $\mathbb{R}^3$ with signature $(2,1)$



The quadratic form vanishes on a cone, called the light cone (or null cone)

Otherwise said: $\mathbb{R}^3 = \mathbb{R}^{2,1}$ (a 3d real vector space with a symmetric bilinear form $(\,,\,)$ of signature $(2,1)$ – an 'inner product')

$\mathcal{L} \subset \mathbb{R}^{2,1}$

$= \{v \in \mathbb{R}^{2,1} \mid (v,v) = 0\}.$

2/ Veronese map – a good model of $\mathbb{R}^{2,1}$ ('spinors'):

- look at $\mathbb{R}^2$ & fix a nonzero $\omega \in \Lambda^2 \mathbb{R}^{2*}$ (a skew bilinear form) $\omega(x,y) = -\omega(y,x)$, and consider $S^2\mathbb{R}^2$ (3-dimensional) with the symmetric bilinear form

$$(\alpha\beta, \gamma\delta) = \omega(\alpha,\gamma)\omega(\beta,\delta) - \omega(\alpha,\delta)\omega(\beta,\gamma) \text{ for } \alpha,\beta,\gamma,\delta \in \mathbb{R}^2$$

[Exercise: this is well-defined & symmetric in $\alpha\beta \leftrightarrow \gamma\delta$];

Observe $(\alpha^2, \alpha^2) = 0$ & $(\alpha^2, \beta^2) = -2\omega(\alpha,\beta)^2$ (more generally)

Indeed $v \in \mathcal{L} \iff v = \pm\alpha^2$ & we have a map $\alpha \mapsto \alpha^2 \in \mathcal{L} \subseteq S^2\mathbb{R}^2$ (with $\alpha \in \mathbb{R}^2$) which induces

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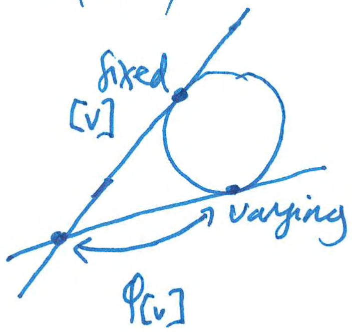
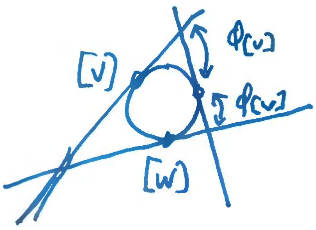
a bijection $P(\mathbb{R}^2) \to P(\mathcal{L})$.

Alternatively



Issue: how does $\varphi_{[v]}$ depend on choice of $[v]$?



By Veronese have $v = \alpha^2$, $w = \gamma^2$ & the variable point is β^2.

Then $\varphi_{[v]}([\beta^2]) = \alpha\beta$, $\varphi_{[w]} = \beta\gamma$,

and the maps $\beta \mapsto \alpha\beta$ & $\beta \mapsto \beta\gamma$ are linear maps.

[Why: $\alpha\beta$ is orthogonal to both α^2 & β^2]

3/ Cross ratio – given $A, B, C, D \in \mathbb{P}^1 = \mathbb{P}(V)$ distinct

$\exists!$ up to common scale basis of V so that

$A = [1,0]$, $B = [0,1]$, $C = [1,1]$

i.e. $\mathbb{P}^1 \to \mathbb{R} \cup \{\infty\}$ sends $A \mapsto 0$, $B \mapsto \infty$, $C \mapsto 1$.

$[\lambda_0, \lambda_1] \mapsto \lambda_1/\lambda_0$

& if $D = [1, c]$ then $cr(A,B;C,D) := c$ is the cross ratio. This is invariant under projective transformations.

For any affine coordinate $x = \lambda_1/\lambda_0$, sending A, B, C, D to a, b, c, d, we have $cr(A,B;C,D) = \dfrac{(a-c)(b-d)}{(a-d)(b-c)}$

Alternatively given $A, B \in \mathbb{P}^1$ distinct & $c \in \mathbb{R}^+$, define

$T(c) = \begin{cases} \sqrt{c} & \text{on } A \\ 1/\sqrt{c} & \text{on } B \end{cases}$ & observe $T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}\sqrt{c}\\ 1/\sqrt{c}\end{pmatrix}$

$= \begin{pmatrix}\sqrt{c} & 0 \\ 0 & 1/\sqrt{c}\end{pmatrix}$ in $SL(V)$ so $[T]C = \begin{bmatrix}c\\1\end{bmatrix} = D$.

Now identify S^1 with $\mathbb{P}^1$ & $f(l), f(j),$ ~~[illegible]~~ $f(i), f(k)$

~~[illegible]~~ with A, B, C, D & set $cr(f(l), f(j) ; f(i), f(k))$

$= cr(A, B, C, D)$

[Aside $\dfrac{(a-c)(b-d)}{(a-d)(b-c)} = \dfrac{\omega(\tilde{A},\tilde{C})\,\omega(\tilde{B},\tilde{D})}{\omega(\tilde{A},\tilde{D})\,\omega(\hat{B},\tilde{C})}$ where $\tilde{A} = \binom{1}{a}$, $\tilde{B} = \binom{1}{b}$ etc.]

$$PSL(V) \cong SO_+(S^2V) \quad \text{with} \quad g(\alpha\beta) = g(\alpha)\,g(\beta)$$

$\quad \| \\ Aut(\mathbb{P}^1)$

$$T(c) \text{ acts as } \begin{cases} c & \text{on } A^2 \\ 1 & \text{on } A^\perp \cap B^\perp = AB \\ 1/c & \text{on } B^2 \end{cases} =: \Gamma^{A^2}_{B^2}(c).$$

Thus have $\Gamma^{A^2}_{B^2} : \mathbb{R}^\times \longrightarrow SO_+(S^2V)$, a homomorphism.

Given $f_0, f_\infty, f_1, f_t \in S^1$ distinct, we have

$$\Gamma^{f_\infty}_{f_0}(t)\, f_1 = f_t \iff cr(f_0, f_\infty ; f_1, f_t) = t.$$

Thus $t \mapsto \Gamma^{f_\infty}_{f_0}(t)\, f_1$ is a map $\mathbb{R} \cup \{\infty\} \to S^1$

with inverse $cr(f_0, f_\infty ; f_1, \cdot) : S^1 \to \mathbb{R} \cup \{\infty\}$,

the unique affine coordinate sending
$f_0 \mapsto 0$
$f_\infty \mapsto \infty$
$f_1 \mapsto 1$

Any other coordinate differs from this by a linear fractional transformation.

Note also $\Gamma^{f_\infty}_{f_0}(c) = \Gamma^{f_0}_{f_\infty}(1/c)$ so

$$cr(f_0, f_\infty ; f_1, f_t) = \frac{1}{cr(f_\infty, f_0 ; f_1, f_t)}$$

$$\Gamma^{f_\infty}_{f_0}(1-c)\, f_1 = \begin{cases} f_0 & c = 1 \\ f_\infty & c = \infty \\ f_1 & c = 0 \end{cases} \quad \text{so is} \quad \Gamma^{f_\infty}_{f_1}(c)\, f_0$$