1. (a) (i) Give at least one reason in each case why *normal subgroups* and *simple groups* are particularly significant in group theory.

(ii) Give an example of a nontrivial group G and G-set X for which the action of G on X is faithful but not free.

[7]

(b) Suppose that G is a group with a transitive action on a set X. Given $x \in X$ and $H \leq G$, define $\theta\colon H \to X$ by $\theta(h) = h \cdot x$. Prove that θ is surjective if and only if any $g \in G$ may be written $g = ha$ with $h \in H$ and $a \in \mathrm{Stab}_G(x)$. [7]

(c) Suppose that G has order $2m$, with m odd and $m > 1$. Prove that G cannot be simple, explaining where each hypothesis on the order of G is used in your proof. [6]

2. (a) (i) Give an example of a finite abelian group G which is a direct sum of nontrivial subgroups H_1, H_2, H_3.

(ii) Prove that a direct sum of finite cyclic groups with coprime orders is cyclic.

[6]

(b) (i) Show that $\mathbb{Z}_{30}$ is a direct sum of cyclic subgroups of order 5 and 6 and explain why this does not contradict the existence part of the Fundamental Theorem of Finite Abelian Groups.

(ii) How many abelian groups of order $p_1^5 p_2^5 \cdots p_r^5$ are there up to isomorphism, where $r \in \mathbb{Z}^+$ and $p_1, p_2, \ldots p_r$ are distinct prime numbers? You should justify your answer.

[7]

(c) Suppose that G is a finite p-group (with p prime) such that $G/Z(G)$ is cyclic and G has exactly one subgroup of order p. Prove that G is cyclic, explaining where each hypothesis on the group G is used in your proof. [7]

3. (a) (i) Give an example of a group G with a subnormal series containing two proper nontrivial subgroups which is not solvable.

(ii) Which of the permutations $(1\,2)(3\,4)$, $(1\,2)$, $(1\,4)(2\,3)$ and $(1\,2\,3)$ are conjugate in S_4? You should justify your answers.

(iii) Prove that S_4 is a solvable group.

[10]

(d) Let $V = \mathbb{F}_3^2$ be the standard 2-dimensional vector space over the field $\mathbb{F}_3 = \{0, 1, -1\}$, let $G = \mathrm{GL}(V)$, and let $X = V \setminus \{0\}$. Define an action of G on the set $P_2(X)$ of two element subsets of X with two orbits X_1 and X_2, where $|X_1| = 4$ and $|X_2| = 24$ (you should justify your definition). Hence or otherwise, prove that the quotient group $G/\{id_V, -id_V\}$ is isomorphic to S_4. [10]

[You may use results from linear algebra without proof.]

4. (a) (i) Does there exist a group G with exactly 3 Sylow 2-subgroups? Does there exist a group G with exactly 2 Sylow 3-subgroups? (You should justify your answers.)

(ii) If G is a finite simple group with a subgroup H of index $n > 1$, then the order of G divides not just $n!$ but also $n!/2$. With reference to the proof of this result, explain how the simplicity of G is used to obtain the factor 2.

[7]

(b) Let G be a finite group, let $P, H \leq G$ with P a finite p-group, and let

$$Y = \{xH \in X : P \leq {}^{x}H\}.$$

Prove that $|Y| \equiv [G : H] \bmod p$, and explain how this result can be used to prove that any two Sylow p-subgroups of G are conjugate. [7]

(c) Let G be a group of order 380. Prove that G cannot be simple. [6]