MA40254: DIFFERENTIAL & GEOMETRIC ANALYSIS

Contents

Motivation: the problem with grad, curl and div

Gradient. Let $U \subseteq \mathbb{R}^3$ be open and $f\colon U \to \mathbb{R}$ differentiable. Then the partial derivatives of f define a *vector field*

$$\operatorname{grad} f\colon U \to \mathbb{R}^3; \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \\ \partial f/\partial x_3 \end{pmatrix}$$

i.e., $(\operatorname{grad} f)(x)$ is a vector at each $x \in U$.

If $\gamma\colon \mathbb{R} \to U;\ t \mapsto \gamma(t)$ is a curve with $\gamma(0) = x$, we can ask if

$$\frac{d\gamma}{dt}(0) = (\operatorname{grad} f)(x)\ ?$$

Now suppose we change to spherical polar coordinates by the map

$$\varphi\colon (0,\infty) \times (0,\pi) \times (-\pi,\pi) \to \mathbb{R}^3; \quad \begin{pmatrix} r \\ \theta \\ \psi \end{pmatrix} \mapsto \begin{pmatrix} r\sin\theta\,\cos\psi \\ r\sin\theta\,\sin\psi \\ r\cos\theta \end{pmatrix}$$

If $U \subseteq \operatorname{im}\varphi$ then x, f and γ are represented in spherical polar coordinates by $\tilde{x} = \varphi^{-1}(x)$, $\tilde{f} = f \circ \varphi\colon \varphi^{-1}(U) \to \mathbb{R}$ and $\tilde{\gamma} = \varphi^{-1} \circ \gamma\colon \mathbb{R} \to \varphi^{-1}(U)$.

Problem. To have

$$\frac{d\gamma}{dt}(0) = (\operatorname{grad} f)(x) \ \Leftrightarrow\ \frac{d\tilde{\gamma}}{dt}(0) = (\operatorname{grad} \tilde{f})(\tilde{x})$$

we cannot define

$$\operatorname{grad} \tilde{f} = \begin{pmatrix} \partial \tilde{f}/\partial r \\ \partial \tilde{f}/\partial \theta \\ \partial \tilde{f}/\partial \psi \end{pmatrix}$$

but instead must set

$$\operatorname{grad} \tilde{f} = \begin{pmatrix} \partial \tilde{f}/\partial r \\ \frac{1}{r^2}\partial \tilde{f}/\partial \theta \\ \frac{1}{r^2(\sin\theta)^2}\partial \tilde{f}/\partial \psi \end{pmatrix}.$$

To understand this problem, recall that $\operatorname{grad} f(x)$ is related to the (Fréchet) derivative Df_x of f at x by

$$Df_x(v) = v \cdot (\operatorname{grad} f)(x)$$

where $v \in \mathbb{R}^3$ and $\cdot$ denotes the Euclidean scalar/inner/dot product. Recall $Df_x\colon \mathbb{R}^3 \to \mathbb{R}$ is the best linear approximation to f near x, hence $Df_x \in \mathbb{R}^{3*} = \mathcal{L}(\mathbb{R}^3, \mathbb{R})$, the *dual space* of linear forms $\mathbb{R}^3 \to \mathbb{R}$.

Root of problem. $\operatorname{grad} f$ depends on the inner product, so only transforms nicely for diffeomorphisms which preserve the inner product, and φ above does not!

Solution. Work instead with

$$Df = df\colon U \to \mathbb{R}^{3*}; \quad x \mapsto Df_x\ !$$

Divergence and curl. If $v\colon U \to \mathbb{R}^3$ is a vector field, we may define

$$\operatorname{div} v = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \quad \text{and} \quad \operatorname{curl} v = \begin{pmatrix} \partial v_2/\partial x_3 - \partial v_3/\partial x_2 \\ \partial v_3/\partial x_1 - \partial v_1/\partial x_3 \\ \partial v_1/\partial x_2 - \partial v_2/\partial x_1 \end{pmatrix}$$

but the transformation rules into spherical polar coordinates are even more horrible.[1]

We can resolve this problem by introducing *differential forms*: functions α on U with values in $\mathbb{R} = \mathrm{Alt}^0(\mathbb{R}^3)$, $\mathbb{R}^{3*} = \mathrm{Alt}^1(\mathbb{R}^3)$, $\mathrm{Alt}^2(\mathbb{R}^3)$ and $\mathrm{Alt}^3(\mathbb{R}^3)$, where $\mathrm{Alt}^k(\mathbb{R}^3)$ denotes the vector space of alternating k-multilinear forms on $\mathbb{R}^3$. Then we replace grad, curl and div by the exterior derivative d between functions with values in these spaces. This more sophisticated algebra simplifies the transformation law to

$$\mathrm{d}\widetilde{\alpha} = \widetilde{\mathrm{d}\alpha}.$$

(Also $\mathrm{d}^2 := \mathrm{d} \circ \mathrm{d} = 0$ captures in a memorable way the rules relating grad, curl and div, and there is an obvious generalisation from $\mathbb{R}^3$ to $\mathbb{R}^n$.)

Integration. In vector calculus, integration is as important as differentiation, and there are line integrals, surface integrals and volume integrals: for example if $x\colon U \to \mathbb{R}^3$ parametrises a surface $S \subseteq \mathbb{R}^3$ (where $U \subseteq \mathbb{R}^2$), and $z\colon U \to \mathbb{R}^3$ describes a vector field along the surface, then the surface integral of z along S is defined by

$$\int_S z \cdot dS := \int_{(u,v)\in U} z(u,v) \cdot \Big(\frac{\partial x}{\partial u} \times \frac{\partial x}{\partial v}\Big) du\, dv,$$

which again involves Euclidean geometry (not just the dot product, but the cross product).

Differential forms provide coordinate invariant reformulations of these definitions. In addition, the fundamental theorem of calculus, Stokes' theorem for surfaces, and the divergence theorem for volumes are all special cases of Stokes' theorem for differential forms α on submanifolds M with boundary ∂M:

$$\int_M \mathrm{d}\alpha = \int_{\partial M} \alpha.$$

This includes in particular the Fundamental Theorem of Calculus when $M = [a,b]$ is a closed interval:

$$\int_{[a,b]} \frac{\mathrm{d}f}{\mathrm{d}x}\mathrm{d}x = \int_{\{a,b\}} f = f(b) - f(a).$$

[1] Formulae omitted to reduce risk of ongoing post-traumatic stress from MA20223!

1. Smooth functions on $\mathbb{R}^n$

1.1. Differentiation.

Definition 1.1. Let V, W be finite dimensional normed vector spaces (we will often take $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$ with the Euclidean norm). Let $U \subseteq V$ open. Then $f : U \to W$ is differentiable at $x \in U$ if there exists a linear map $Df_x : V \to W$, called the *derivative* of f at x, such that

$$f(x + v) = f(x) + Df_x(v) + g(v)\|v\|$$

where $\lim_{v\to 0} g(v) = 0$.

Remarks 1.2. It is easy to show that Df_x is unique if it is exists. Since all norms are equivalent on finite dimensional V, W, the definition is independent of the chosen norms.

Definition 1.3. If $f : U \to W$ is differentiable at every $x \in U$, then we say f is *differentiable* (on U). Then the derivative of f is the function

$$Df : U \to \mathcal{L}(V, W),\ x \mapsto Df_x,$$

where $\mathcal{L}(V, W)$ is the vector space of linear maps $V \to W$.

Remarks 1.4. Observe the distinction (conceptionally and notationally) between derivative of f at a point (the linear map $Df_x : V \to W$) and the derivative function $Df : U \to \mathcal{L}(V, W)$.

For $U \subset \mathbb{R}^n$ and $f : U \to \mathbb{R}^m, (x_1, \dots, x_n) \mapsto (y_1, \dots, y_m)$ and $x \in U$, the linear map $Df_x : \mathbb{R}^n \to \mathbb{R}^m$ is represented, with respect to the standard basis, by the matrix whose entries are the partial derivatives $\frac{\partial y_i}{\partial x_j}$.

Example 1.5. Suppose $f : U \to W$ is the restriction to $U \subset V$ of a linear map $\alpha : V \to W$. Then

$$f(x + v) = \alpha(x + v) = \alpha(x) + \alpha(v) = f(x) + \alpha(v) + 0.$$

Thus $Df_x = \alpha$ and $Df : U \to \mathcal{L}(V, W)$ is a constant function with constant value α.

Remark 1.6. Observe that $\mathcal{L}(V, W)$ is also a finite dimensional normed vector space, a convenient norm being the *operator norm*

$$\|\phi\|_{op} := \sup_v \|\phi(v)\|,$$

taking the supremum over all $v \in V$ with $\|v\|_V = 1$.

Hence we can iterate: f is twice differentiable if Df is differentiable.

Notation 1.7. For vector spaces V, W and $k \in \mathbb{N}$ let:

- $\mathcal{M}^k(V; W) = \{k\text{-linear maps } V^k \to W\}$;
- $\mathcal{M}^k(V) = \mathcal{M}^k(V; \mathbb{R})$ and $\mathcal{M}^0(V; W) = W$;
- $\mathrm{Sym}^k(V; W) \subseteq \mathcal{M}^k(V; W)$ be the subspace of fully symmetric k-linear maps.

For $\eta \in \mathcal{L}(V, \mathcal{M}^{k-1}(V; W))$ let $\eta^\vee \in \mathcal{M}^k(V; W)$ be defined by

$$\eta^\vee(v_1, \dots, v_k) = (\eta(v_1))(v_2, \dots, v_k).$$

If $f : U \to W$ is $k \geq 1$ times differentiable and $x \in U$, define (recursively) $D^k f_x = D(D^{k-1}f)^\vee_x \in \mathcal{M}^k(V; W)$, where $D^0 f = f$ (thus $D^1 f = Df$).

Definition 1.8. We say f is (*of class*) C^0 if f is continuous, and (recursively) f is (*of class*) C^k if f is differentiable and Df is C^{k-1}. We say f is *smooth* or C^∞ if f is C^k for every $k \in \mathbb{N}$.

Proposition 1.9. *f is C^1 if and only if its first order partial derivatives all exist and are continuous, and f is smooth if and only if its partial derivatives of all orders exist.*

Proposition 1.10. *If $f\colon U \to W$ is C^2 on $U \subseteq V$, then for all $x \in U$, $D^2 f_x$ is symmetric,* i.e.*, $D^2 f_x \in \mathrm{Sym}^2(V;W)$. If f is C^k then $D^k f$ takes values in $\mathrm{Sym}^k(V;W)$.*

Remark 1.11. If $U \subseteq \mathbb{R}^n$ and $f\colon U \to \mathbb{R}$ is twice differentiable, then $D^2 f_x$ is a bilinear form and the matrix H representing $D^2 f_x$ with respect to the standard basis is the Hessian, given by $H_{ij} = \dfrac{\partial^2 f}{\partial x_i \partial x_j}$. Proposition 1.10 means that $D^2 f_x$ is a symmetric bilinear form, so H is a symmetric matrix, *i.e.*, partial derivatives commute.

Proposition 1.12 (Chain rule)**.** *If $U \subseteq V$ and $\widetilde{U} \subseteq W$ are open and $f : U \to \widetilde{U}$, $g : \widetilde{U} \to X$ are differentiable at $x \in U$ and $f(x) \in \widetilde{U}$ respectively, then $g \circ f$ is differentiable at x with $D(g \circ f)_x = Dg_{f(x)} \circ Df_x$.*

Theorem 1.13 (Mean value theorem)**.** *If $f : U \to \mathbb{R}$ is differentiable and the segment $[x, y]$ is contained in U, then $\exists \xi \in [x, y]$ such that $f(y) - f(x) = Df_\xi(y - x)$.*

Corollary 1.14 (Mean value inequality)**.** *If $f : U \to W$ is differentiable and $[x, y] \subset U$ then $\exists \xi \in [x, y]$ such that $\|f(y) - f(x)\| \leq \|Df_\xi(y - x)\|$.*

Hence $\|f(y) - f(x)\| \leq \|y - x\| \sup_{\xi \in U} \|Df_\xi\|_{\mathrm{op}}$.

Recall that any normed vector space V is a metric space with $d(x, y) = \|y - x\|$. Hence any subset $S \subseteq V$ is also a metric space, whose open and closed sets are intersections with S of open and closed subsets in V (respectively).

Definition 1.15. For $S \subseteq V$, we say that $f : S \to W$ is smooth iff every $x \in S$ has an open neighbourhood $U \subseteq V$ and a smooth function $F : U \to W$ such that the restriction of F to $U \cap S$ equals f. We denote the set of such functions by $C^\infty(S, W)$.

1.2. **Inverse Function Theorem.** For $U, \widetilde{U} \subseteq \mathbb{R}^n$ open and $f : U \to \widetilde{U}$, $g : \widetilde{U} \to U$ inverses, the differentiability of one does not imply the differentiability of the other. For example, for $U = \widetilde{U} = \mathbb{R}$, $x \mapsto x^3$ is differentiable but $y \mapsto \sqrt[3]{y}$ is not (at $y = 0$).

Definition 1.16. Let $U \subseteq \mathbb{R}^n$, $\widetilde{U} \subseteq \mathbb{R}^m$ be open. Then $f\colon U \to \widetilde{U}$ is a (C^k) *diffeomorphism* if it is differentiable (C^k) and has a differentiable (C^k) inverse $g\colon \widetilde{U} \to U$.

Proposition 1.17. *Let $U \subseteq \mathbb{R}^n, \widetilde{U} \subseteq \mathbb{R}^m$ be open. If $f : U \to \widetilde{U}$ and $g : \widetilde{U} \to U$ are inverses, f is differentiable at $x \in U$, and g is differentiable at $y = f(x) \in \widetilde{U}$, then $Df_x : \mathbb{R}^n \to \mathbb{R}^m$ is an isomorphism with inverse Dg_y; in particular, $m = n$.*

Proof. Applying the chain rule to $g \circ f = \mathrm{Id}_U$ gives $D(g \circ f)_x = Dg_y \circ Df_x = D(\mathrm{Id}_U)_x = \mathrm{Id}_{\mathbb{R}^n}$, and similarly since $f \circ g = \mathrm{Id}_{\widetilde{U}}$, the chain rule gives $D(f \circ g)_y = Df_x \circ Dg_y = \mathrm{Id}_{\mathbb{R}^m}$ since $g(y) = x$. Hence Df_x is an isomorphism with inverse Dg_y by definition. □

We often make use of the following corollary to the rank-nullity theorem: if $\varphi\colon V \to W$ is a linear map between vector spaces of the same dimension and $\ker\varphi = \{0\}$, then φ is a linear isomorphism.

Definition 1.18. We say $f\colon U \to \mathbb{R}^n$ is a *local diffeomorphism* if Df_x is an isomorphism for all $x \in U$.

Thus any diffeomorphism is necessarily a local diffeomorphism. This turns out to be sufficient, at least locally.

Theorem 1.19 (Inverse function theorem)**.** *Let $U \subseteq \mathbb{R}^n$ be open, $x \in U$ and f be C^1 on U with Df_x is an isomorphism. Then x has an open neighbourhood $U' \subseteq U$ such that $\widetilde{U} := f(U') \subseteq \mathbb{R}^n$ is open and the restriction $f\colon U' \to \widetilde{U}$ is a diffeomorphism.*

We will also see later that if f is C^k then $f\colon U' \to \widetilde{U}$ is a C^k diffeomorphism.

To prove the Theorem 1.19, first note that if its conclusion holds for $\tilde{f} = L \circ f$ for any linear isomorphism L, it also holds for f. If we take $L := (Df_x)^{-1}$, then by the chain rule $D\tilde{f}_u = DL_{f(u)} \circ Df_u = L \circ Df_u$ for any $u \in U$, so in particular $D\tilde{f}_x = \mathrm{Id}_{\mathbb{R}^n}$. So without loss of generality, $Df_x = \mathrm{Id}_{\mathbb{R}^n}$.

Since f is C^1, Df is continuous, so Df_u is close to $\mathrm{Id}_{\mathbb{R}^n}$ for u close to x: concretely,

$$\exists\, r > 0 \quad \text{s.t.} \quad B_r(x) := \{u : \|u - x\| < r\} \subseteq U$$
$$\text{and} \quad \forall\, u \in B_r(x) \quad \text{we have} \quad \|\mathrm{Id}_{\mathbb{R}^n} - Df_u\|_{\mathrm{op}} < \tfrac{1}{2}.$$

The plan is now to show that $U' := B_r(x)$ satisfies the conclusions of Theorem 1.19. The key is to observe that if we define $h := \mathrm{Id}_{\mathbb{R}^n} - f$, then for $y, z \in B_r(x)$, the mean value inequality implies that

$$\|h(y) - h(z)\| \le \|y - z\| \sup_{u \in B_r(x)} \|\mathrm{Id}_{\mathbb{R}^n} - Df_u\|_{\mathrm{op}} \le \tfrac{1}{2}\|y - z\| \tag{1.1}$$

Lemma 1.20. *If $y, z \in U'$, then*

$$\|y - z\| \le 2\|f(y) - f(z)\|.$$

In particular, the restriction of f to U' is injective.

Proof. With $h(z) = z - f(z)$ as before, (1.1) implies

$$\begin{aligned}\|y - z\| = \|h(y) + f(y) - h(z) - f(z)\| &\le \|f(y) - f(z)\| + \|h(y) - h(z)\| \\ &\le \|f(y) - f(z)\| + \tfrac{1}{2}\|y - z\|\end{aligned}$$

This rearranges to give the stated inequality. □

Now recall the following theorem from Analysis 2A.

Theorem 1.21 (Contraction mapping theorem)**.** *Let $S \subseteq \mathbb{R}^n$ be closed and $H : S \to S$. If $\exists c < 1$ such that $\forall\, y, z \in S$, $\|H(y) - H(z)\| \le c\|y - z\|$* (i.e., *$H$ is a contraction*)*, then H has a unique fixed point $z \in S$.*

Now (1.1) implies h is a contraction. Furthermore, for any $w \in \mathbb{R}^n$, the same is true for h_w defined by $h_w(z) = h(z) + w = z - f(z) + w$, since $h_w(y) - h_w(z) = h(y) - h(z)$. On the other hand, z is a fixed point of h_w (*i.e.*, $h_w(z) = z$) if and only if $f(z) = w$. We use this to prove that the image of U' is open.

Lemma 1.22. *Suppose $B_\delta(y) \subseteq U$ and for all $z \in B_\delta(y)$, $\|\mathrm{Id}_{\mathbb{R}^n} - Df_z\|_{op} < \frac{1}{2}$. Then $f(B_\delta(y))$ contains $B_{\delta/4}(f(y))$.*

Proof. Suppose $w \in B_{\delta/4}(f(y))$, and consider the restriction of h_w to $z \in \overline{B_{\delta/2}(y)}$. Since $\|w - f(y)\| < \delta/4$ and $h_w(y) = y - f(y) + w$, $\|z - y\| \leq \delta/2$ implies that

$$\|h_w(z) - y\| = \|h_w(z) - h_w(y) + w - f(y)\| \leq \|h_w(z) - h_w(y)\| + \|w - f(y)\| < \delta/2.$$

It follows that h_w maps $S := \overline{B_{\delta/2}(y)}$ to $B_{\delta/2}(y) \subseteq S$. Since $S \subseteq B_\delta(y)$, h_w is a contraction on S and so by Theorem 1.21, it has a unique fixed point $z \in S$. Since $z \in B_\delta(y)$ and $h_w(z) = z$ implies $w = f(z)$, this completes the proof. □

Now for any $v \in f(U')$ with $U' = B_r(x) \subseteq U$, take $y \in U'$ with $f(y) = v$, and $\delta > 0$ such that $B_\delta(y) \subseteq U'$. Then the above lemma applies to show $f(U')$ contains an open ball centred at $v = f(y)$. Thus $f(U')$ is open, $f\colon U' \to f(U')$ is a bijection between open sets, hence has an inverse $g\colon f(U') \to U' \subseteq \mathbb{R}^n$.

Lemma 1.23. *g is differentiable at $y = f(x)$.*

Proof. If Dg_y exists then it must be $(Df_x)^{-1} = \mathrm{Id}_{\mathbb{R}^n}$, so we want to control $g(w) - g(y) - w + y$ for w close to y. This equals $g(w) - x - f(g(w)) + f(x)$ and

$$\frac{\|g(w) - x - f(g(w)) + f(x)\|}{\|w - y\|} = \frac{\|g(w) - x\|}{\|w - y\|} \frac{\|f(g(w)) - f(x) - \mathrm{Id}_{\mathbb{R}^n}(g(w) - x)\|}{\|g(w) - x\|}.$$

However, Lemma 1.20 shows that for $w \in f(U')$, $\|g(w) - x\| \leq 2\|w - y\|$, and hence the first factor is ≤ 2. As $w \to y$, also $g(w) \to x$, so the second factor $\to 0$ because $Df_x = \mathrm{Id}$. □

We now observe that any $z \in U'$ could have been used in place of x in Lemma 1.23.

Lemma 1.24. *f is a local diffeomorphism on U'.*

Proof. For any $z \in U'$ and $v \in \mathbb{R}^m$, $\|v - Df_z(v)\| \leq \frac{1}{2}\|v\|$. In particular if $Df_z(v) = 0$, then $\|v\| = 0$, so $\ker Df_z = 0$ and Df_z is invertible by rank-nullity. □

Hence g is differentiable on $\widetilde{U} = f(U')$, which completes the proof of Theorem 1.19. In fact more is true: g is as differentiable as f.

Theorem 1.25. *Let $U \subseteq \mathbb{R}^n, \widetilde{U} \subseteq \mathbb{R}^m$ be open. If $f : U \to \widetilde{U}$ and $g : \widetilde{U} \to U$ are inverses, f is C^k, and Df_x is an isomorphism for every $x \in U$, then g is C^k.*

By Theorem 1.19, g is differentiable on $\widetilde{U}$, hence C^0. Theorem 1.25 now follows by induction on k using the following.

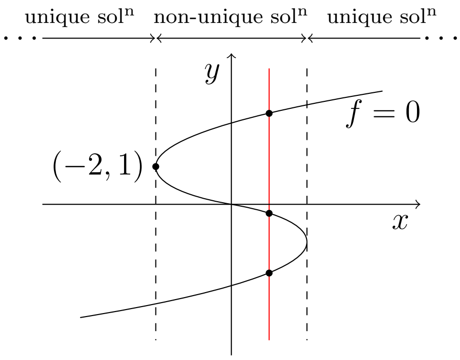
Lemma 1.26. *For $k \geq 1$, if f is C^k and g is C^{k-1}, then g is C^k.*

Proof. By the above, g is differentiable and so $Dg_w = (Df_{g(w)})^{-1} \in \mathrm{GL}_n(\mathbb{R})$ for all $w \in \widetilde{U}$ by Proposition 1.17. In other words $Dg : \widetilde{U} \to \mathrm{GL}_n(\mathbb{R}) \subset M_{n,n}(\mathbb{R})$ is a composition of

(1) $g : \widetilde{U} \to U$, which is C^{k-1} by assumption; then
(2) $Df : U \to \mathrm{GL}_n(\mathbb{R})$, which is C^{k-1} since f is C^k; and then
(3) $\mathrm{inv} : \mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})$, $A \mapsto A^{-1}$, which is C^∞. (Exercise)

By the chain rule, $Dg = \mathrm{inv} \circ Df \circ g : \widetilde{U} \to \mathrm{GL}_n(\mathbb{R})$ is C^{k-1}, so g is C^k. □

1.3. **Implicit Function Theorem.** For a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$, we can try to use the equation $f(x, y) = 0$ to "implicitly" define y as a function of x, *i.e.*, find $h : \mathbb{R} \to \mathbb{R}$ such that $f(x, h(x)) = 0$ and the level set $f^{-1}(0) = \{(x, y) \in \mathbb{R}^2 \,|\, f(x, y) = 0\}$ is precisely $\text{graph}(h) = \{(x, h(x)) \,|\, x \in \mathbb{R}\}$. The problem is that given a particular x, there could be zero or multiple solutions y to the equation $f(x, y) = 0$. For example, let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = x - y^3 + 3y$.



Given x_0 and y_0 such that $f(x_0, y_0) = 0$, we could try instead to define $y = h(x)$ only for x close to x_0, insisting that y is close to y_0. However, this can still fail: in the example, if we take $(x_0, y_0) = (-2, 1)$, then for $x < -2$ there is no solution for y, while for $x > -2$ the solution is not unique. The problem here is that $\frac{\partial f}{\partial y} = -3y^2 + 3 = 0$. The Implicit Function Theorem asserts (in arbitrary dimensions) that this is the only problem.

To state it, we start with a function $f\colon U \to \mathbb{R}^m$ where U is open in $\mathbb{R}^{n+m} \cong \mathbb{R}^n \times \mathbb{R}^m$, and denote by $D_1 f_z$ and $D_2 f_z$ be the restrictions of Df_z to $\mathbb{R}^n \times \{0\} \cong \mathbb{R}^n$ and $\{0\} \times \mathbb{R}^m \cong \mathbb{R}^m$ (respectively) in $\mathbb{R}^n \times \mathbb{R}^m$.

Theorem 1.27. *Let $U \subseteq \mathbb{R}^{n+m}$ be open and $f : U \to \mathbb{R}^m$ be C^k. Let $x_0 \in \mathbb{R}^n, y_0 \in \mathbb{R}^m$ and suppose that $z := (x_0, y_0) \in U$ and $f(z) = 0$. If $D_2 f_z$ is an isomorphism, then there exist open sets $U_1 \subseteq \mathbb{R}^n$, $U_2 \subseteq \mathbb{R}^m$ with $x_0 \in U_1$, $y_0 \in U_2$ and a C^k function $h : U_1 \to U_2$ such that $U_1 \times U_2 \subseteq U$ and $\{(x, y) \in U_1 \times U_2 \,|\, f(x, y) = 0\} = \{(x, h(x)) \,|\, x \in U_1\}$.*

Proof. Let $F : U \to \mathbb{R}^n \times \mathbb{R}^m$, $(x, y) \mapsto (x, f(x, y))$. Then $DF_z : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$ maps $(u, v) \mapsto (u, D_1 f_z(u) + D_2 f_z(v))$ since $Df_z(u, v) = D_1 f_z(u) + D_2 f_z(v)$. If $DF_z(u, v) = 0$ then $u = 0$ and hence $D_2 f_z(v) = 0$ (since $D_2 f_z$ is an isomorphism); hence $\ker DF_z = \{0\}$ and DF_z is an isomorphism by rank-nullity. The Inverse Function Theorem 1.19 and 1.25 now provide an C^k inverse $G : F(U') \to U'$ to F on an open neighbourhood $U' \subseteq U$ of z. Shrinking U' if necessary, we may assume, wlog, first that $U' = U_1' \times U_2$. We now set $U_1 = \{x \in U_1' : (x, 0) \in F(U')\}$, which is open (and a neighbourhood of x_0) because $F(U')$ is open in $\mathbb{R}^{n+m}$, so its intersection with $\mathbb{R}^n \times \{0\} \cong \mathbb{R}^n$ is open in $\mathbb{R}^n$.

Now by the form of F, $G(x, y) = (x, g(x, y))$ where $g(x, f(x, y)) = y$ and $f(x, g(x, y)) = y$. Hence for any $(x, y) \in U_1 \times U_2$, we may define $h\colon U_1 \to U_2$ by $h(x) := g(x, 0)$. Then $f(x, y) = 0$ implies $h(x) = g(x, 0) = g(x, f(x, y)) = y$ and conversely, $y = h(x)$ implies $f(x, y) = f(x, g(x, 0)) = 0$. □

Henceforth, we use the term "diffeomorphism" to mean smooth (C^∞) diffeomorphism, *i.e.*, a smooth map with a smooth inverse. Using Definition 1.15, we may extend this terminology to functions $f\colon S \to T$ between arbitrary subsets $S \subseteq \mathbb{R}^n$ and $T \subseteq \mathbb{R}^m$.

2. Submanifolds of $\mathbb{R}^s$

2.1. Submanifolds and regular values.

Definition 2.1. $M \subseteq \mathbb{R}^s$ is an *n-dimensional submanifold* if $\forall\, p \in M$, $\exists$ an open neighbourhood $U \subseteq M$ of p and an open $U' \subseteq \mathbb{R}^n$ and a diffeomorphism $\varphi : U' \to U$.
φ is called a (*local*) *parametrisation*, while $\varphi^{-1} : U \to U'$ is called a *coordinate chart.*

Unwinding Definition 1.15, this means that there is an open neighbourhood $\widetilde{U} \subseteq \mathbb{R}^s$ of x and smooth maps $F : \widetilde{U} \to U'$ and $\varphi : U' \to U$, with $U = \widetilde{U} \cap M$, such that $F|_U = \varphi^{-1}$.

Examples 2.2. (1) Let $M \subseteq \mathbb{R}^n$ be open, then M is a submanifold (by taking $U = U' = M$ and $\varphi = \mathrm{Id}_M$).
(2) Let M be an n-dimensional vector subspace of $\mathbb{R}^s$. Take φ to be a linear isomorphism $\mathbb{R}^n \cong M$ and $F : \mathbb{R}^s \to \mathbb{R}^n$ to be any linear map such that $F|_M = \varphi^{-1}$.
(3) Let $S^2 := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. Let

$$U' := (0, \pi) \times (-\pi, \pi) \subseteq \mathbb{R}^2, \quad \widetilde{U} := \mathbb{R}^3 \backslash \{(x, y, z) \mid x \leq 0, y = 0\}$$

so $U := \widetilde{U} \cap S^2$ is an open subset of S^2. Let $\varphi : U' \to U$ be the bijection

$$(\theta, \psi) \mapsto (\sin\theta\cos\psi, \sin\theta\sin\psi, \cos\theta),$$

and $F : \widetilde{U} \to \mathbb{R}^2$ by $(x, y, z) \mapsto (\arg(z, \sqrt{x^2 + y^2}), \arg(x, y))$. Then φ and F are both smooth and $\varphi^{-1} = F|_U$. So φ^{-1} is smooth and thus $\varphi : U' \to U$ is a diffeomorphism. Although U is not the whole of S^2, there are similar parametrisations (obtained e.g., by interchanging the roles of x, y and z) which together cover the remaining points.

Finding parametrisations explicitly is usually rather tedious. Fortunately there is a more convenient general method for proving that a subset $M \subseteq \mathbb{R}^s$ is a submanifold using the implicit function theorem.

Definition 2.3. Let $P \subseteq \mathbb{R}^s$ be open and $f : P \to \mathbb{R}^m$ be differentiable. Then we call $q \in \mathbb{R}^m$ a *regular value of* f if for all p in the level set $f^{-1}(q) := \{p \in P : f(p) = q\}$, we have that $Df_p : \mathbb{R}^s \to \mathbb{R}^m$ is surjective, *i.e.*, $\mathrm{rank}(Df_p : \mathbb{R}^s \to \mathbb{R}^m) = m$.

Remark 2.4. If $q \notin \mathrm{im}(f)$, then it is (vacuously) a regular value.

Example 2.5. Let $f : \mathbb{R}^3 \to \mathbb{R}$ via $(x, y, z) \mapsto x^2 + y^2 + z^2$. Now the matrix representation of $Df_{(x,y,z)} : \mathbb{R}^3 \to \mathbb{R}$ is $[2x\ 2y\ 2z]$ which is zero only if $(x, y, z) = 0$. Therefore any $q \in \mathbb{R}$ other than $f(0) = 0$ is a regular value.

Theorem 2.6. *If $P \subseteq \mathbb{R}^{n+m}$ is open, $f : P \to \mathbb{R}^m$ is smooth and $q \in \mathbb{R}^m$ is a regular value of f, then $f^{-1}(q)$ is an n-dimensional submanifold of $\mathbb{R}^{n+m}$.*

Proof. For $p \in f^{-1}(q)$, $Df_p : \mathbb{R}^{n+m} \to \mathbb{R}^m)$ is surjective, and so its kernel has dimension n by rank-nullity. Precomposing f with an invertible linear map $\mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$, we may assume $\ker(Df_p) = \mathbb{R}^n \times \{0\} \subseteq \mathbb{R}^{n+m}$. Write $p = (x_0, y_0) \in \mathbb{R}^{n+m} \cong \mathbb{R}^n \times \mathbb{R}^m$.

Then $D_2 f_p$ is an isomorphism, so by the implicit function theorem, there are neighbourhoods U_1 of x_0 in $\mathbb{R}^n$ and U_2 of y_0 in $\mathbb{R}^m$ and a smooth function $h : U_1 \to U_2$ such that $U := (U_1 \times U_2) \cap f^{-1}(q)$ is the graph $\{(x, h(x)) \mid x \in U_1\}$ (note that U is open in $f^{-1}(q)$).

Now define $\varphi : U_1 \to U$ by $x \mapsto (x, h(x))$ and $F : U_1 \times U_2 \to U_1$ by $(x, y) \mapsto x$. Then φ and F are both smooth maps, and the restriction of F to U is clearly inverse to φ. Thus φ^{-1} exists and is smooth, hence φ is a diffeomorphism. □

Example 2.7. Since $1 \in \mathbb{R}$ is a regular value of $(x, y, z) \mapsto x^2 + y^2 + z^2$, the unit sphere $S^2 \subset \mathbb{R}^3$ is two-dimensional submanifold. The local parametrisations provided by Theorem 2.6 are local graphs, such as $\varphi : U' \to U$ where

$$U' := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}, \quad U := \{(x, y, z) \in S^2 \mid z > 0\},$$
$$\text{and} \quad \varphi(x, y) := (x, y, \sqrt{1 - x^2 - y^2}).$$

2.2. **Tangent spaces and derivatives of maps between submanifolds.**

Definition 2.8. Let $M \subseteq \mathbb{R}^s$ be an n-dimensional submanifold, $p \in M$ and $v \in \mathbb{R}^s$. Then v is called a *tangent vector* to M at p if there is a smooth curve $\gamma\colon (-\varepsilon, \varepsilon) \to M$ (for $\varepsilon > 0$) such that $\gamma(0) = p$ and $\gamma'(0) := D\gamma_0(1) = v$. The set of all tangent vectors to M at p is called the *tangent space* T_pM to M at p.

Lemma 2.9. *Let $\varphi : U' \to U$ (for U' open in $\mathbb{R}^n$) be a local parametrisation of $M \subseteq \mathbb{R}^s$ with $p = \varphi(x) \in U$. Then $D\varphi_x : \mathbb{R}^n \to \mathbb{R}^s$ is an injective linear map with image T_pM. In particular T_pM is an n-dimensional vector subspace of $\mathbb{R}^s$.*

Proof. Because $\varphi : U' \to U$ is a diffeomorphism there is an open set $\widetilde{U} \subseteq \mathbb{R}^s$ that contains U and a smooth function $F : \widetilde{U} \to \mathbb{R}^n$ such that $F|_U = \varphi^{-1}$. Now $F \circ \varphi = \mathrm{Id}_{U'}$, and by the chain rule

$$D(F \circ \varphi)_x = DF_p \circ D\varphi_x = \mathrm{Id}_{\mathbb{R}^n}$$

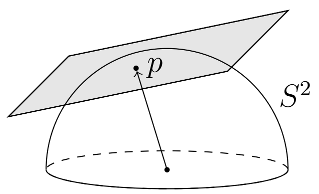
since $p = \varphi(x)$. Thus $D\varphi_x$ is injective (as it has a left-inverse).

For any $w \in \mathbb{R}^n$, let $\beta\colon (-\varepsilon, \varepsilon) \to U'$ be the curve $\beta(t) = x + tw$ and set $\gamma(t) = \varphi(\beta(t))$. Then by the chain rule $\gamma'(0) = D\varphi_x(\beta'(0)) = D\varphi_x(w)$, so $D\varphi_x(w) \in T_pM$. Conversely, if $\gamma\colon (-\varepsilon, \varepsilon) \to M$ is any smooth curve with $\gamma(0) = p$, then by decreasing ε we may assume γ has image in U, so that $\beta := F \circ \gamma$ is a smooth curve in U' with $\varphi \circ \beta = \gamma$. Hence by the chain rule $\gamma'(0) = D\varphi_x(\beta'(0))$ is in the image of $D\varphi_x$.

We conclude that $D\varphi_x\colon \mathbb{R}^n \to T_pM$ is a linear isomorphism. □

Examples 2.10.
(1) If $U \subseteq \mathbb{R}^n$ is open and $p \in U$, then parametrising by Id_U, we immediately obtain $T_pU = \mathbb{R}^n$.
(2) Let $\varphi : \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\} \to S^2$ by $(x, y) \mapsto (x, y, \sqrt{1 - x^2 - y^2})$. Now $D\varphi_{(x,y)} : \mathbb{R}^2 \to \mathbb{R}^3$ is represented by the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{-x}{\sqrt{1-x^2-y^2}} & \frac{-y}{\sqrt{1-x^2-y^2}} \end{pmatrix}$$



The columns of $D\varphi_{(x,y)}$ are linearly independent and orthogonal to $\varphi(x, y)$, so the image of $D\varphi_{(x,y)}$ is precisely $\varphi(x, y)^{\perp} \subseteq \mathbb{R}^3$. Using similar charts on other hemispheres, $T_pS^2 = p^{\perp}$ for all $p \in S^2$.

Definition 2.11. Let $M \subseteq \mathbb{R}^s$ and $N \subseteq \mathbb{R}^\ell$ be submanifolds. Let $f : M \to N$ be a smooth function, $p \in M$. The *derivative of f at p* is the map

$$Df_p : T_pM \to T_{f(p)}N$$

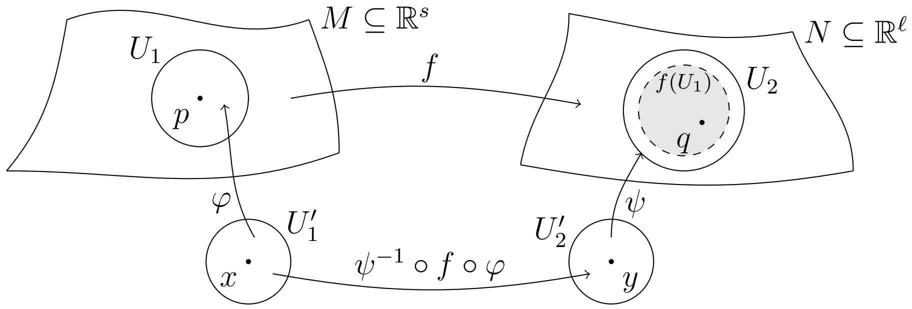
sending $v = \gamma'(0) \in T_pM$ to $Df_p(v) := (f \circ \gamma)'(0) \in T_{f(p)}N$.

Lemma 2.12. *Let $F : \widetilde{U} \to \mathbb{R}^\ell$ be a smooth function on an open neighbourhood $\widetilde{U}$ of p in $\mathbb{R}^s$ such that $F|_{\widetilde{U}\cap M} = f$. Then $Df_p(v) = DF_p(v)$, hence is well-defined and linear in v.*

Proof. If $v = \gamma'(0) \in T_pM$, then $Df_p(v) = (F \circ \gamma)'(0) = DF_p(v)$ by the chain rule. $\square$

Lemma 2.13. *Let $\varphi : U_1' \to U_1$ and $\psi : U_2' \to U_2$ be parametrisations of M and N respectively with $p = \varphi(x) \in U_1$ and $q = f(p) = \psi(y) \in U_2$. Suppose that $f(U_1) \subseteq U_2$, so that $\psi^{-1} \circ f \circ \varphi : U_1' \to U_2'$ is a well-defined smooth function. Then*

$$Df_p = D\psi_y \circ D(\psi^{-1} \circ f \circ \varphi)_x \circ (D\varphi_x)^{-1}.$$



Proof. Let $F\colon \widetilde{U} \to \mathbb{R}^\ell$ be a local extension of f near p as in Lemma 2.12. Then on $\varphi^{-1}(\widetilde{U} \cap M)$, $f \circ \varphi = F \circ \varphi = \psi \circ (\psi^{-1} \circ f \circ \varphi)$, so by the chain rule, $DF_p \circ D\varphi_x = D\psi_y \circ D(\psi^{-1} \circ f \circ \varphi)_x$, which rearranges to the stated formula by Lemma 2.12. $\square$

Remark 2.14. If $M \subseteq \mathbb{R}^s$ and $N \subseteq \mathbb{R}^\ell$ are open then by Lemma 2.12 the definition of $Df_p\colon T_pM \to T_{f(p)}N$ here coincides with its usual definition as a linear map $\mathbb{R}^s \to \mathbb{R}^\ell$.

Definition 2.15. A smooth function $f : M \to N$ between submanifolds is called

(1) a *local diffeomorphism* if $\forall\, p \in M$, $Df_p : T_pM \to T_{f(p)}N$ is an isomorphism;
(2) an *immersion* if $\forall\, p \in M$, Df_p is injective; and
(3) a *submersion* if $\forall\, p \in M$, Df_p is surjective.

3. Differential forms

3.1. **Motivation.** Suppose $U \subseteq \mathbb{R}^n$ is open and $\alpha\colon U \to \mathbb{R}^{n*} = \mathcal{M}^1(\mathbb{R}^n)$ is smooth. When does there exist a function $f\colon U \to \mathbb{R}$ such that $\alpha = Df$?

Partial answer. If $\alpha = Df$, then for all $p \in U$, $(D\alpha_p)^\vee = D^2 f_p \in \mathcal{M}^2(\mathbb{R}^n)$ is symmetric. Hence if we define, for any smooth $\alpha\colon U \to \mathbb{R}^{n*}$, any $p \in U$ and any $v_1, v_2 \in \mathbb{R}^n$, $\mathrm{d}\alpha_p(v_1, v_2) := (D\alpha_p)^\vee(v_1, v_2) - (D\alpha_p)^\vee(v_2, v_1)$, then $\alpha = Df$ implies $\forall\, p \in U$, $\mathrm{d}\alpha_p = 0$.

Because this is a natural question, we can expect it to behave well with respect to smooth changes of coordinates. Indeed if $\varphi\colon U' \to U$ is a (local) diffeomorphism, and we define $\varphi^* f := f \circ \varphi\colon U' \to \mathbb{R}$, then by the chain rule $D(\varphi^* f)_p = Df_{\varphi(p)} \circ D\varphi_p$. Hence if we define $(\varphi^*\alpha)_p(v) := \alpha_{\varphi(p)}(D\varphi_p(v))$ then $\alpha = Df$ if and only if $\varphi^*\alpha = D(\varphi^* f)$.

A further computation with the chain rule shows that $\mathrm{d}(\varphi^*\alpha)_p = (\varphi^*\mathrm{d}\alpha)_p$, where

$$(\varphi^*\mathrm{d}\alpha)_p(v_1, v_2) := d\alpha_{\varphi(p)}(D\varphi_p(v_1), D\varphi_p(v_2)).$$

Hence if $\mathrm{d}\alpha_{\varphi(p)} = 0$ then $\mathrm{d}(\varphi^*\alpha)_p = 0$.

Notice that $d\alpha_p(v, v) = 0$, so that $\mathrm{d}\alpha_p \in \mathcal{M}^2(\mathbb{R}^n)$ is *alternating.* This calculus extends to *differential forms*, which are functions with values in the vector space $\mathrm{Alt}^k(\mathbb{R}^n)$ of alternating k-linear forms on $\mathbb{R}^n$. In this chapter we define vector spaces $\Omega^k(U)$ of differential k-forms on U, linear operators $\mathrm{d}\colon \Omega^k(U) \to \Omega^{k+1}(U)$, called *exterior derivatives*, and, for any smooth $\varphi\colon U' \to U$, linear operators $\varphi^*\colon \Omega^k(U) \to \Omega^k(U')$ called *pullbacks* such that:

- $\Omega^0(U)$ is the space of smooth functions $f\colon U \to \mathbb{R}$ and $\mathrm{d}f = Df \in \Omega^1(U)$;
- For any $\alpha \in \Omega^k(U)$, $\mathrm{d}(\mathrm{d}\alpha) = 0$;
- $\mathrm{d}(\varphi^*\alpha) = \varphi^*\mathrm{d}\alpha$.

In addition, there is an associative multiplication on differential forms, and all of this structure can be extended from open subsets U of $\mathbb{R}^n$ to arbitrary submanifolds M.

3.2. **Alternating forms.** Recall that if V is a real vector space then $\mathcal{M}^k(V) = \mathcal{M}^k(V;\mathbb{R})$ is the vector space of maps $\alpha\colon V^k \to \mathbb{R}$, which are k-(multi)linear, *i.e.*, for all i (etc.),

$$\alpha(v_1, \ldots, v_{i-1}, \lambda v_i + \mu w_i, v_{i+1}, \ldots, v_k) = \lambda\alpha(v_1, \ldots, v_k) + \mu\alpha(v_1, \ldots, w_i, \ldots, v_k).$$

Definition 3.1. A multilinear form $\alpha \in \mathcal{M}^k(V)$ is *alternating* if $\alpha(v_1, \ldots, v_k) = 0$ whenever $v_i = v_j$ for some $i \neq j$. Denote the subspace of alternating forms by $\mathrm{Alt}^k(V) \subseteq \mathcal{M}^k(V)$. The *degree* of $\alpha \in \mathrm{Alt}^k(V)$ is k.

Remark 3.2. If $k = 1$, then $\mathrm{Alt}^1(V) = \mathcal{M}^1(V) = V^*$ (the alternating condition is vacuous in degree 1). Also, by definition, $\mathrm{Alt}^0(V) = \mathcal{M}^0(V) = \mathbb{R}$.

Example 3.3. For $v_1, \ldots, v_n \in \mathbb{R}^n$, let $\mathrm{Det}(v_1, \ldots, v_n) = \det(A) \in \mathbb{R}$, where A is the matrix whose columns are v_i. Then $\mathrm{Det} \in \mathrm{Alt}^n(\mathbb{R}^n)$.

Recall that for each $k \in \mathbb{N}$, there is a *symmetric group* S_k of permutations σ of $\{1, \ldots, k\}$, that any $\sigma \in S_k$ is a composite of transpositions, and that the sign homomorphism $\mathrm{sgn} : S_k \to \{\pm 1\}$ is characterised by $\mathrm{sgn}(\tau) = -1$ for all transpositions τ; let $A_k = \ker(\mathrm{sgn})$.

Definition 3.4. For $\alpha \in \mathcal{M}^k(V)$ and $\sigma \in S_k$, define $\sigma \cdot \alpha \in \mathcal{M}^k(V)$ by

$$(\sigma \cdot \alpha)(v_1, \ldots, v_k) = \alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$$

and

$$\mathrm{alt}(\alpha) = \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\, \sigma \cdot \alpha \in \mathcal{M}^k(V).$$

Clearly $\mathrm{Id} \cdot \alpha = \alpha$, and if $\sigma, \tau \in S_k$, consider $w_i := v_{\sigma(i)}$; then $w_{\tau(j)} = v_{\sigma(\tau(j))}$, so

$$\begin{aligned}(\sigma \cdot (\tau \cdot \alpha))(v_1, \ldots, v_k) &= (\tau \cdot \alpha)(w_1, \ldots, w_k) = \alpha(w_{\tau(1)}, \ldots, w_{\tau(k)}) \\ &= \alpha(v_{\sigma\circ\tau(1)}, \ldots, v_{\sigma\circ\tau(k)}) = ((\sigma \circ \tau) \cdot \alpha)(v_1, \ldots, v_k).\end{aligned}$$

Hence $\sigma \cdot (\tau \cdot \alpha) = (\sigma \circ \tau) \cdot \alpha$, *i.e.*, $S_k \times \mathcal{M}^k(V) \to \mathcal{M}^k(V)$; $(\sigma, \alpha) \mapsto \sigma \cdot \alpha$ defines a (left) action of S_k on $\mathcal{M}^k(V)$.

Lemma 3.5. *Let $\alpha \in \mathcal{M}^k(V)$.*

(1) $\operatorname{alt}(\alpha) \in \operatorname{Alt}^k(V)$.
(2) *If $\alpha \in \operatorname{Alt}^k(V)$ then for all $\sigma \in S_k$, $\sigma \cdot \alpha = \operatorname{sgn}(\sigma)\alpha$.*
(3) *If $\sigma \cdot \alpha = \operatorname{sgn}(\sigma)\alpha$ for all $\sigma \in S_k$, then $\operatorname{alt}(\alpha) = k!\,\alpha$.*

Proof. (1) Suppose $v_i = v_j$ for $i \neq j$, and consider the transposition $\tau = (i\ j)$. Then $S_k = A_k \cup \tau A_k$ (a disjoint union of left cosets) and so

$$\operatorname{alt}(\alpha)(v_1, \dots, v_k) = \sum_{\sigma \in A_k} (\sigma \cdot \alpha - (\tau \circ \sigma) \cdot \alpha)(v_1, \dots, v_k) = 0,$$

since $(\tau \circ \sigma) \cdot \alpha = \tau \cdot (\sigma \cdot \alpha)$ and $(\tau \cdot (\sigma \cdot \alpha))(v_1, \dots, v_k) = (\sigma \cdot \alpha)(v_1, \dots, v_k)$ because $v_i = v_j$.

(2) Since S_k is generated by transpositions, sgn is a homomorphism, and $(\sigma, \alpha) \mapsto \sigma \cdot \alpha$ is an action, it suffices to check that $\sigma \cdot \alpha = -\alpha$ for $\sigma = (i\ j)$ with $i < j$:

$$\begin{aligned} 0 &= \alpha(v_1, \dots, v_i + v_j, \dots, v_i + v_j, \dots, v_k) \\ &= \alpha(v_1, \dots, v_i, \dots, v_j, \dots, v_k) + \alpha(v_1, \dots, v_j, \dots, v_i, \dots, v_k) \\ &= \alpha(v_1, v_2, \dots, v_k) + (\sigma \cdot \alpha)(v_1, v_2, \dots, v_k) \end{aligned}$$

as required.

(3) Immediate because $|S_k| = k!$, and for all $\sigma \in S_k$, $\operatorname{sgn}(\sigma)^2 = 1$. □

Corollary 3.6. *For $\alpha \in \mathcal{M}^k(V)$, we have*

$$\alpha \in \operatorname{Alt}^k(V) \quad \Leftrightarrow \quad \forall\, \sigma \in S_k,\ \sigma \cdot \alpha = \operatorname{sgn}(\sigma)\alpha \quad \Leftrightarrow \quad \alpha = \frac{1}{k!} \operatorname{alt}(\alpha).$$

Lemma 3.7. *For any $\alpha \in \mathcal{M}^k(V)$ and any $\sigma \in S_k$, $\operatorname{alt}(\sigma \cdot \alpha) = \operatorname{sgn}(\sigma) \operatorname{alt}(\alpha)$.*

Proof. Since sgn is a homomorphism and $(\sigma, \alpha) \mapsto \sigma \cdot \alpha$ is an action, we have

$$\begin{aligned} \operatorname{alt}(\sigma \cdot \alpha) &= \sum_{\tau \in S_k} \operatorname{sgn}(\tau)\, \tau \cdot (\sigma \cdot \alpha) = \operatorname{sgn}(\sigma) \sum_{\tau \in S_k} \operatorname{sgn}(\tau \circ \sigma)\, (\tau \circ \sigma) \cdot \alpha \\ &= \operatorname{sgn}(\sigma) \sum_{\tau' \in S_k} \operatorname{sgn}(\tau')\, \tau' \cdot \alpha = \operatorname{sgn}(\sigma) \operatorname{alt}(\alpha), \end{aligned}$$

where the penultimate equality uses that $\tau \mapsto \tau \circ \sigma = \tau'$ is a bijection $S_k \to S_k$. □

Definition 3.8. For a list $\alpha_1, \alpha 2, \dots, \alpha_k \in V^*$, we define $\alpha_1 \alpha_2 \cdots \alpha_k \in \mathcal{M}^k(V)$ and $\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_k \in \operatorname{Alt}^k(V)$ by

$$\begin{aligned} (\alpha_1 \alpha_2 \cdots \alpha_k)(v_1, \dots, v_k) &:= \alpha_1(v_1)\alpha_2(v_2) \cdots \alpha_k(v_k), \\ \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_k &:= \operatorname{alt}(\alpha_1 \alpha_2 \cdots \alpha_k). \end{aligned}$$

It follows from Lemma 3.7 that for any $\sigma \in S_k$, $\alpha_{\sigma(1)} \wedge \cdots \wedge \alpha_{\sigma(k)} = \operatorname{sgn}(\sigma)\alpha_1 \wedge \cdots \wedge \alpha_k$.

For a *multi-index* $I = \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ with $i_1 < \cdots < i_k$, let

$$\alpha_I := \alpha_{i_1} \wedge \cdots \wedge \alpha_{i_k} \in \operatorname{Alt}^k(V).$$

Theorem 3.9. *Let $e_1, \dots, e_n$ be a basis for V with dual basis $\varepsilon_1, \dots, \varepsilon_n \in V^*$. Then for $k > n$, $\operatorname{Alt}^k(V) = \{0\}$, while for $0 \leq k \leq n$, any $\alpha \in \operatorname{Alt}^k(V)$ may be written in the form*

$$\alpha = \sum_{|I|=k} \lambda_I \varepsilon_I \quad \textit{with} \quad \lambda_I \in \mathbb{R} \quad \textit{for} \quad I \subseteq \{1, \dots, n\},\ |I| = k,$$

and then for all $J = \{j_1, \dots, j_k\}$ *with* $j_1 < \cdots < j_k$*, we have* $\lambda_J = \alpha(e_{j_1}, \dots, e_{j_k})$*. In particular,* $\varepsilon_I : I \subseteq \{1, \dots, n\}$*,* $|I| = k$ *is a basis for* $\mathrm{Alt}^k(V)$ *and* $\dim \mathrm{Alt}^k(V) = \binom{n}{k}$*.*

Proof. Suppose first that $\alpha = \sum_{|I|=k} \lambda_I \varepsilon_I$. Since $e_1, \dots, e_n$ and $\varepsilon_1, \dots, \varepsilon_n$ are dual bases *i.e.*, $\varepsilon_i(e_j) = \delta_{ij}$, it follows that if $I = \{i_1, \dots, i_k\}$ with $i_1 < \cdots < i_k$ and $J = \{j_1, \dots, j_k\}$ with $j_1 < \cdots < j_k$, then $\varepsilon_I(e_{j_1}, \dots, e_{j_k}) = \delta_{IJ}$. Thus $\alpha(e_{j_1}, \dots, e_{j_k}) = \lambda_J$.

Now suppose $\alpha \in \mathrm{Alt}^k(V)$ with $k \in \mathbb{N}$ and set $\beta = \alpha - \sum_{|I|=k} \alpha(e_{i_1}, \dots, e_{i_k})\varepsilon_I$ where the sum is empty (hence zero) for $k > n$. By construction $\beta(e_{j_1}, \dots, e_{j_k}) = 0$ whenever $j_1 < \cdots < j_k$. Hence also $\beta(e_{j_1}, \dots, e_{j_k}) = 0$ for any $j_1, \dots, j_k \in \{1, \dots, n\}$ as β is alternating. Since β is multilinear, $\beta(v_1, \dots, v_k) = 0$ for all $v_1, \dots, v_k \in V$. □

3.3. **Differential forms and pullback.**

Definition 3.10. For $U \subseteq \mathbb{R}^n$ open, a (*smooth*) *differential* k*-form on* U is a smooth function $\alpha : U \to \mathrm{Alt}^k(\mathbb{R}^n)$, written $p \mapsto \alpha_p$. Thus if $p \in U$ and $v_1, \dots, v_k \in \mathbb{R}^n$, then $\alpha_p(v_1, \dots, v_k) \in \mathbb{R}$. Let $\Omega^k(U)$ the vector space of differential k-forms on U under pointwise operations, *i.e.*, $(\alpha + \beta)_p = \alpha_p + \beta_p$ and $(\lambda\alpha)_p = \lambda\alpha_p$.

Notation 3.11. Since $\mathrm{Alt}^0(\mathbb{R}^n) = \mathbb{R}$, $\Omega^0(U)$ is the vector space of smooth functions $f\colon U \to \mathbb{R}$. For any such f, we let $\mathrm{d}f \in \Omega^1(U)$ denote the differential 1-form defined by the derivative of f, *i.e.*, $\mathrm{d}f_p = Df_p \in \mathrm{Alt}^1(\mathbb{R}^n) = \mathbb{R}^{n*}$. For $f \in \Omega^0(U)$ and $\alpha \in \Omega^k(U)$ we define $f\alpha \in \Omega^k(U)$ by $(f\alpha)_p = f_p\alpha_p = f(p)\alpha_p$ for all $p \in U$.

Let $e_1, \dots, e_n$ be the standard basis of $\mathbb{R}^n$ with dual basis $\varepsilon_1, \dots, \varepsilon_n$, and let $x_1, \dots, x_n : U \to \mathbb{R}$ denote the coordinate functions on U, so that $p = (x_1(p), \dots, x_n(p))$ for all $p \in U$. Then for $i \in \{1, \dots n\}$, $x_i = \varepsilon_i|_U$ and hence $(\mathrm{d}x_i)_p = \varepsilon_i \in \mathbb{R}^{n*}$ for $p \in U$, *i.e.*, $\mathrm{d}x_i \in \Omega^1(U)$ is a constant differential 1-form on U with $(\mathrm{d}x_i)_p(e_j) = \delta_{ij}$.

We may extend the wedge and multi-index notation from Definition 3.8 to differential forms: for $\alpha_1, \dots, \alpha_k \in \Omega^1(U)$, we define $\alpha_1 \wedge \cdots \wedge \alpha_k \in \Omega^k(U)$ pointwise: for $p \in U$,

$$(\alpha_1 \wedge \cdots \wedge \alpha_k)_p = (\alpha_1)_p \wedge \cdots \wedge (\alpha_k)_p \in \mathrm{Alt}^k(\mathbb{R}^n);$$

also for $I = \{i_1, \dots, i_k\}$ with $i_1 < \cdots < i_k$, we let $\mathrm{d}x_I = \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k} \in \Omega^k(U)$, so that $(\mathrm{d}x_I)_p = \varepsilon_I \in \mathrm{Alt}^k(\mathbb{R}^n)$. Since $\{\varepsilon_I : |I| = k\}$ is a basis of $\mathrm{Alt}^k(\mathbb{R}^n)$ by Theorem 3.9, any $\alpha \in \Omega^k(U)$ can be written uniquely as

$$\alpha = \sum_{|I|=k} \alpha_I \, \mathrm{d}x_I \tag{3.1}$$

for $\binom{n}{k}$ smooth functions $\alpha_I : U \to \mathbb{R}$. In particular, for $f \in \Omega^0(U)$, we have

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \mathrm{d}x_i \in \Omega^1(U)$$

since $\mathrm{d}f_p(e_j) = Df_p(e_j) = \partial f / \partial x_j$. If $n = 1$, then $\mathrm{d}f = f' \, \mathrm{d}x$, where $f' = \mathrm{d}f/\mathrm{d}x$ (!).

Definition 3.12. For a linear map $\phi : V \to W$ and $\alpha \in \mathcal{M}^k(W)$, define $\phi^*\alpha \in \mathcal{M}^k(V)$ by

$$(\phi^*\alpha)(v_1, \dots, v_k) = \alpha(\phi(v_1), \dots, \phi(v_k)) \quad \forall v_1, \dots, v_k \in V.$$

Note that if $\alpha \in \mathrm{Alt}^k(W)$, then $\phi^*\alpha \in \mathrm{Alt}^k(V)$. Hence $\alpha \mapsto \phi^*\alpha$ defines a linear map $\phi^* : \mathrm{Alt}^k(W) \to \mathrm{Alt}^k(V)$.

Remark 3.13. If $\phi : V \to W$ and $\psi : W \to X$ are linear maps, then $(\psi \circ \phi)^* = \phi^* \circ \psi^*$. Hence if $\phi : V \to W$ is an isomorphism, then so is $\phi^* : \mathrm{Alt}^k(W) \to \mathrm{Alt}^k(V)$. For $p = 0$, $\phi^*\alpha = \alpha$, and for $p = 1$, $\phi^* : W^* \to V^*$ is the transpose of ϕ, and (exercise) for $V = W$ with $p = \dim V$, $\phi^*\alpha = \det(\phi)\alpha$.

Definition 3.14. Let $U \subseteq \mathbb{R}^n$ and $\widetilde{U} \subseteq \mathbb{R}^m$ be open and $\varphi : U \to \widetilde{U}$ a smooth function and $\alpha \in \Omega^k(\widetilde{U})$. Then the *pullback* $\varphi^*\alpha \in \Omega^k(U)$ is defined by

$$(\varphi^*\alpha)_p = (D\varphi_p)^*\alpha_{\varphi(p)} \in \mathrm{Alt}^k(\mathbb{R}^n)$$

—here $\alpha_{\varphi(p)} \in \mathrm{Alt}^k(\mathbb{R}^m)$, and $D\varphi_p : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map, so $(D\varphi_p)^* : \mathrm{Alt}^k(\mathbb{R}^m) \to \mathrm{Alt}^k(\mathbb{R}^n)$ is defined in Definition 3.12. In other words, for all $p \in U$ and $v_1, \ldots, v_k \in \mathbb{R}^n$,

$$(\varphi^*\alpha)_p(v_1, \ldots, v_k) = \alpha_{\varphi(p)}(D\varphi_p(v_1), \ldots, D\varphi_p(v_k)) \in \mathbb{R}.$$

Since operations on differential forms are defined pointwise, $\varphi^* : \Omega^k(\widetilde{U}) \to \Omega^k(U)$ is a linear map, and for any $f \in \Omega^0(\widetilde{U})$ and $\alpha \in \Omega^k(\widetilde{U})$, $\varphi^*(f\alpha) = (\varphi^* f)(\varphi^*\alpha)$.

Lemma 3.15. *Let $\varphi : U \to \widetilde{U}$ and $\psi : \widetilde{U} \to U'$ be smooth maps between open sets.*

(1) *For any $f \in \Omega^0(\widetilde{U})$, $\varphi^* f = f \circ \varphi$ and $\varphi^*\mathrm{d}f = \mathrm{d}(\varphi^* f)$.*
(2) *$(\psi \circ \varphi)^* = \varphi^* \circ \psi^* : \Omega^k(U') \to \Omega^k(U)$.*

Proof. (1) For all $p \in U$, $(\varphi^* f)_p = f_{\varphi(p)} = (f \circ \varphi)_p$; hence the chain rule gives $\mathrm{d}(\varphi^* f)_p = Df_{\varphi(p)} \circ D\varphi_p = (D\varphi_p)^*(\mathrm{d}f_{\varphi(p)}) = (\varphi^*\mathrm{d}f)_p$.

(2) For $\alpha \in \Omega^k(W)$ and $p \in U$, $((\psi \circ \varphi)^*\alpha)_p = (D(\psi \circ \varphi)_p)^*\alpha_{\psi(\varphi(p))}$, and $(D(\psi \circ \varphi)_p)^* = (D\psi_{\varphi(p)} \circ D\varphi_p)^* = (D\varphi_p)^* \circ (D\psi_{\varphi(p)})^*$, so this is $(\varphi^*(\psi^*\alpha))_p$. □

Example 3.16. Let $\widetilde{U} = \{v \in \mathbb{R}^2 \mid \|v\| < 1\}$, $U = (-1, 1) \times \mathbb{R} \subseteq \mathbb{R}^2$,

$$\alpha = \frac{\mathrm{d}x_2}{1 - x_1^2 - x_2^2} \in \Omega^1(\widetilde{U}),$$

and $\varphi : U \to \widetilde{U}; p \mapsto (r(p)\cos\theta(p), r(p)\sin\theta(p))$. Thus, as smooth functions from U to $\mathbb{R}$, $\varphi^* x_1 = r\cos\theta$ and $\varphi^* x_2 = r\sin\theta$ are the components of φ. Using Lemma 3.15, we have

$$\begin{aligned}\varphi^*\alpha &= \frac{\varphi^*(\mathrm{d}x_2)}{\varphi^*(1 - x_1^2 - x_2^2)} = \frac{\mathrm{d}(r\sin\theta)}{1 - (r\cos\theta)^2 - (r\sin\theta)^2} \\ &= \frac{\sin\theta\,\mathrm{d}r + r\mathrm{d}(\sin\theta)}{1 - r^2} = \frac{1}{1 - r^2}\left(\sin\theta\,\mathrm{d}r + r\cos\theta\,\mathrm{d}\theta\right).\end{aligned}$$

Note that φ is not even a local diffeomorphism, as $D\varphi_0$ is not invertible.

3.4. **The exterior derivative on open subsets.** For any $\alpha \in \Omega^k(U)$, the derivative of α (which is smooth, by definition of $\Omega^k(U)$) at $p \in U$ in the usual sense is $(D\alpha)_p : \mathbb{R}^n \to \mathrm{Alt}^k(\mathbb{R}^n)$. Using Notation 1.7, we then have $(D\alpha)_p^\vee \in \mathcal{M}^{k+1}(\mathbb{R}^n)$ defined by

$$(D\alpha)_p^\vee(v_1, v_2, \ldots, v_{k+1}) := (D\alpha)_p(v_1)(v_2, \ldots, v_{k+1}).$$

However, there is no reason to expect that $(D\alpha)_p^\vee$ is alternating (although it is alternating in the last k arguments).

Definition 3.17. For U open in $\mathbb{R}^n$, the *exterior derivative* $\mathrm{d}\alpha \in \Omega^{k+1}(U)$ of $\alpha \in \Omega^k(U)$ is given, at each $p \in U$, by

$$(\mathrm{d}\alpha)_p = \frac{1}{k!}\,\mathrm{alt}((D\alpha)_p^\vee) \in \mathrm{Alt}^{k+1}(\mathbb{R}^n).$$

This defines a linear map $\mathrm{d} : \Omega^k(U) \to \Omega^{k+1}(U)$.

Remark 3.18. Note that if α is constant (*i.e.*, $\alpha_p = \alpha_q$ for all $p, q \in U$) then $D\alpha = 0$, so α is closed. However the converse only holds (locally) when $k = 0$.

In order to compute $\mathrm{d}\alpha$ in coordinates, we need a bit more algebra.

Definition 3.19. For $\alpha \in \mathcal{M}^k(V)$ and $\beta \in \mathcal{M}^\ell(V)$, define $\alpha\beta \in \mathcal{M}^{k+\ell}(V)$ by

$$(\alpha\beta)(v_1, \ldots, v_{k+\ell}) := \alpha(v_1, \ldots, v_k)\beta(v_{k+1}, \ldots, v_{k+\ell}).$$

Lemma 3.20. *Let $\alpha \in \mathcal{M}^k(V)$ and $\beta \in \mathcal{M}^\ell(V)$. Then*

$$\mathrm{alt}(\mathrm{alt}(\alpha)\beta) = k!\;\mathrm{alt}(\alpha\beta) \quad \textit{and} \quad \mathrm{alt}(\alpha\,\mathrm{alt}(\beta)) = \ell!\;\mathrm{alt}(\alpha\beta)$$

Proof. We consider $\sigma \in S_k$ as an element of $S_{k+\ell}$ by letting σ fix each element of $\{k+1, \ldots, k+\ell\}$, so that $(\sigma\cdot\alpha)\beta = \sigma\cdot(\alpha\beta)$. Hence

$$\begin{aligned}\mathrm{alt}(\mathrm{alt}(\alpha)\beta) &= \sum_{\tau\in S_{k+\ell}} (\operatorname{sgn}\tau)\,\tau\cdot\left(\sum_{\sigma\in S_p}(\operatorname{sgn}\sigma)\sigma\cdot(\alpha\beta)\right)\\ &= \sum_{\tau\in S_{k+\ell}}\sum_{\sigma\in S_k}(\operatorname{sgn}(\tau\circ\sigma))\,(\tau\circ\sigma)\cdot(\alpha\beta).\end{aligned}$$

Now for each $\rho \in S_{k+\ell}$, there are precisely $k!$ ways to write $\rho = \tau\circ\sigma$ for $\tau \in S_{k+\ell}$ and $\sigma \in S_k$ (we can take σ to be any of the $k!$ elements of S_k and set $\tau = \rho\circ\sigma^{-1} \in S_{k+\ell}$). In other words, there are $k!$ terms in the double sum with $\rho = \tau\circ\sigma$, hence

$$\mathrm{alt}(\mathrm{alt}(\alpha)\beta) = k! \sum_{\rho\in S_{k+\ell}} (\operatorname{sgn}\rho)\,\rho\cdot(\alpha\beta) = k!\;\mathrm{alt}(\alpha\beta)$$

as required. The second equality follows by a similar argument. □

Proposition 3.21. *For $f : U \to \mathbb{R}$ and $i_1 < \cdots < i_k$, let $\alpha = f\,\mathrm{d}x_{i_1}\wedge\cdots\wedge\mathrm{d}x_{i_k} \in \Omega^k(U)$. Then*

$$\mathrm{d}\alpha = \mathrm{d}f\wedge\mathrm{d}x_{i_1}\wedge\cdots\wedge\mathrm{d}x_{i_k} = \sum_{j=1}^n \frac{\partial f}{\partial x_j}\mathrm{d}x_j\wedge\mathrm{d}x_{i_1}\wedge\cdots\wedge\mathrm{d}x_{i_k}$$

Proof. Since $\mathrm{d}x_I = \mathrm{d}x_{i_1}\wedge\cdots\wedge\mathrm{d}x_{i_k}$ is constant, $D(f dx_I)_p(v) = Df_p(v)(\mathrm{d}x_I)_p$ for any $p \in U$, and so $D(f\mathrm{d}x_I)_p^\vee \in \mathcal{M}^{k+1}(\mathbb{R}^n)$ is equal to the product $(Df_p)(\mathrm{d}x_I)_p$ of $Df_p \in \mathcal{M}^1(\mathbb{R}^n)$ with $(\mathrm{d}x_I)_p = \mathrm{alt}(\varepsilon_{i_1}\cdots\varepsilon_{i_k}) \in \mathcal{M}^k(\mathbb{R}^n)$. Hence

$$\mathrm{d}(f\mathrm{d}x_I)_p = \frac{1}{k!}\,\mathrm{alt}(D(f\mathrm{d}x_I)_p) = \frac{1}{k!}\,\mathrm{alt}((Df_p)\,\mathrm{alt}(\varepsilon_{i_1}\cdots\varepsilon_{i_k})) = \mathrm{alt}((Df_p)\varepsilon_{i_1}\cdots\varepsilon_{i_k})$$

by Lemma 3.20. Since $Df_p = \sum_j(\partial f/\partial x_j)(p)\varepsilon_j$, this is $\sum_j(\partial f/\partial x_j)(p)\varepsilon_j\wedge\varepsilon_{i_1}\wedge\cdots\wedge\varepsilon_{i_k}$, which is the value at p of the right hand side as required. □

Example 3.22. Let $U = \{p \in \mathbb{R}^2 : x_1(p) \neq 0\}$ and $\alpha = x_2 \mathrm{d}x_1 + \frac{\mathrm{d}x_2}{x_1} \in \Omega^1(U)$. Then

$$\begin{aligned}\mathrm{d}\alpha &= \mathrm{d}x_2 \wedge \mathrm{d}x_1 + \mathrm{d}(\tfrac{1}{x_1}) \wedge \mathrm{d}x_2 = -\mathrm{d}x_1 \wedge \mathrm{d}x_2 + \frac{-\mathrm{d}x_1}{x_1^2} \wedge \mathrm{d}x_2 \\ &= -(1 + \tfrac{1}{x_1^2})\mathrm{d}x_1 \wedge \mathrm{d}x_2\end{aligned}$$

For another example $\mathrm{d}(x_2 x_3 \mathrm{d}x_1) = x_3 \mathrm{d}x_2 \wedge \mathrm{d}x_1 + x_2 \mathrm{d}x_3 \wedge \mathrm{d}x_1$. Notice that applying d again gives $\mathrm{d}x_3 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_1 + \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_1 = 0$. This is a general fact.

Theorem 3.23. *If* $\alpha \in \Omega^k(U)$*, then* $\mathrm{d}(\mathrm{d}\alpha) = 0 \in \Omega^{k+2}(U)$*.*

Proof. By linearity of the exterior derivative, it suffices to check that the claim holds when $\alpha = f\, \mathrm{d}x_I$ for some $f \in \Omega^0(U)$, where Proposition 3.21 computes $d\alpha$. Using linearity and Proposition 3.21 once again, we obtain

$$\mathrm{d}(\mathrm{d}\alpha) = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} \mathrm{d}x_i \wedge \mathrm{d}x_j \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}$$

However $\mathrm{d}x_i \wedge \mathrm{d}x_j \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k} = -\mathrm{d}x_j \wedge \mathrm{d}x_i \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}$ by Lemma 3.7, whereas $\partial^2 f / \partial x_i \partial x_j$ is symmetric in i, j, so $\mathrm{d}(\mathrm{d}\alpha) = -\mathrm{d}(\mathrm{d}\alpha)$, hence is zero. □

Definition 3.24. We say α is *closed* if $\mathrm{d}\alpha = 0$, and *exact* if $\alpha = \mathrm{d}\beta$ for some $\beta \in \Omega^{k-1}(U)$, which for $k = 0$ is taken to mean $\alpha = 0$. Thus any exact α is closed.

The converse is false in general; however it does hold on $\mathbb{R}^n$ (if $k > 0$).

Theorem 3.25 (The Poincaré Lemma)**.** *Suppose* $k > 0$ *and* $\alpha \in \Omega^k(\mathbb{R}^n)$ *is closed,* i.e., $\mathrm{d}\alpha = 0$*. Then* α *is exact.*

We will prove this later.

Example 3.26. Let $\alpha = (x_3^2 - x_1^2)\mathrm{d}x_1 \wedge \mathrm{d}x_2 + x_2 \mathrm{d}x_2 \wedge \mathrm{d}x_3 + 2x_2x_3 \mathrm{d}x_1 \wedge \mathrm{d}x_3 + \in \Omega^2(\mathbb{R}^3)$, which is closed, as

$$\mathrm{d}\alpha = 2x_3 \mathrm{d}x_3 \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 + 2x_3 \mathrm{d}x_2 \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_3 = 0.$$

To find a β such that $\alpha = \mathrm{d}\beta$, we first find some γ of the form $\gamma = f\mathrm{d}x_1 + g\mathrm{d}x_2$ such that $\alpha - \mathrm{d}\gamma$ has no terms involving $\mathrm{d}x_3$. Thus we choose f such that $\frac{\partial f}{\partial x_3} = -2x_2x_3$, say $f = -x_2x_3^2$, and g such that $\frac{\partial g}{\partial x_3} = -x_2$, say $g = -x_2x_3$. Then

$$\begin{aligned}d\gamma &= -\mathrm{d}(x_2x_3^2) \wedge \mathrm{d}x_1 - \mathrm{d}(x_2x_3) \wedge \mathrm{d}x_2 \\ &= x_3^2 \mathrm{d}x_1 \wedge \mathrm{d}x_2 + 2x_2x_3 \mathrm{d}x_1 \wedge \mathrm{d}x_3 + x_2 \mathrm{d}x_2 \wedge \mathrm{d}x_3\end{aligned}$$

So $\alpha' = \alpha - \mathrm{d}\gamma = -x_1^2 \mathrm{d}x_1 \wedge \mathrm{d}x_2$. Note that α' is independent of x_3 as well as $\mathrm{d}x_3$, so we can iterate the process and eliminate the $\mathrm{d}x_2$ term to get $\alpha' = \mathrm{d}(x_1^2 x_2 \mathrm{d}x_1)$ (or alternatively, $\alpha' = -\frac{1}{3}\mathrm{d}(x_1^3 \mathrm{d}x_2)$). Hence $\alpha = \mathrm{d}\beta$ with $\beta = x_1^2x_2\mathrm{d}x_1 + \gamma = x_2(x_1^2 - x_3^2)\mathrm{d}x_1 - x_2x_3\mathrm{d}x_2$.

3.5. **The wedge product and Leibniz rule.**

Definition 3.27. For $\alpha \in \mathrm{Alt}^k(V)$ and $\beta \in \mathrm{Alt}^\ell(V)$, define

$$\alpha \wedge \beta = \frac{1}{k!\ell!} \mathrm{alt}(\alpha\beta) \in \mathrm{Alt}^{k+\ell}(V).$$

Lemma 3.28. *For $\alpha \in \mathrm{Alt}^k(V)$, $\beta \in \mathrm{Alt}^\ell(V)$ and $\gamma \in \mathrm{Alt}^m(V)$, we have $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$*

Proof. By Lemma 3.20,

$$\begin{aligned}(\alpha \wedge \beta) \wedge \gamma &= \frac{1}{(k+\ell)!m!\,k!\ell!}\,\mathrm{alt}(\mathrm{alt}(\alpha\beta)\gamma) = \frac{1}{k!\ell!m!}\,\mathrm{alt}(\alpha\beta\gamma) \\ &= \frac{1}{k!(\ell+m)!\,\ell!m!}\,\mathrm{alt}(\alpha\,\mathrm{alt}(\beta\gamma)) = \alpha \wedge (\beta \wedge \gamma)\end{aligned}$$ □

Remark 3.29. Since $\wedge$ is associative, we may omit brackets, and then for $\alpha_j \in \mathrm{Alt}^{\ell_j}(V)$ ($j \in \{1, \dots k\}$), we have $\alpha_1 \wedge \cdots \wedge \alpha_k = \mathrm{alt}(\alpha_1 \cdots \alpha_k)/(\ell_1! \cdots \ell_k!)$, which is consistent with Definition 3.8 when $\ell_j = 1$ for all j.

Lemma 3.30. *For $\alpha \in \mathrm{Alt}^k(V)$ and $\beta \in \mathrm{Alt}^\ell(V)$, we have $\alpha \wedge \beta = (-1)^{k\ell}\beta \wedge \alpha$.*

Proof. Since $\wedge$ is bilinear, it suffices to take $\alpha = \alpha_1 \wedge \cdots \wedge \alpha_k$ and $\beta = \beta_1 \wedge \cdots \wedge \beta_q$ with $\alpha_i, \beta_j \in \mathrm{Alt}^1(V)$. Since $\alpha_i \wedge \beta_j = -\beta_j \wedge \alpha_i$ and $\wedge$ is associative,

$$\alpha_1 \wedge \cdots \wedge \alpha_k \wedge \beta_1 \wedge \cdots \wedge \beta_\ell = (-1)^k \beta_1 \wedge \alpha_1 \wedge \cdots \wedge \alpha_k \wedge \beta_2 \wedge \cdots \wedge \beta_\ell$$

and iterating this process gives $(-1)^{k\ell}\beta_1 \wedge \cdots \wedge \beta_\ell \wedge \alpha_1 \wedge \cdots \wedge \alpha_k$ as required. □

Definition 3.31. Let $U \subseteq \mathbb{R}^n$ be open, $\alpha \in \Omega^k(U)$ and $\beta \in \Omega^\ell(U)$. Then the *wedge product* $\alpha \wedge \beta \in \Omega^{k+\ell}(U)$ is defined by $(\alpha \wedge \beta)_p = \alpha_p \wedge \beta_p$ for all $p \in U$.

The wedge product of differential forms is bilinear and associative. In particular, if $\alpha = f\mathrm{d}x_I$ and $\beta = g\mathrm{d}x_J$ then $\alpha \wedge \beta = fg\,\mathrm{d}x_I \wedge \mathrm{d}x_J$, hence also $f \wedge \beta = f\beta$ and $\alpha \wedge f = f\alpha$. Also $\alpha \wedge \beta = (-1)^{k\ell}\beta \wedge \alpha$ and there is the following Leibniz/product rule.

Theorem 3.32. *For U open in $\mathbb{R}^n$, $\alpha \in \Omega^k(U)$ and $\beta \in \Omega^\ell(U)$, we have*

$$\mathrm{d}(\alpha \wedge \beta) = (\mathrm{d}\alpha) \wedge \beta + (-1)^k \alpha \wedge (\mathrm{d}\beta) \quad \in \Omega^{k+\ell+1}(U).$$

Proof. We first check the equation holds for $k = \ell = 0$. If $f, g \in \Omega^0(U)$, then

$$d(fg) = g(df) + f(dg) = df \wedge g + f \wedge dg$$

by the usual Leibniz rule for the derivative of a product of real-valued functions.

In general, since $\wedge$ is bilinear and d is linear, it suffices to consider $\alpha = f\,\mathrm{d}x_I$ and $\beta = g\,\mathrm{d}x_J$ for multi-indices I, J and $f, g \in \Omega^0(U)$. Then by Proposition 3.21,

$$\begin{aligned}\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}(fg\,\mathrm{d}x_I \wedge \mathrm{d}x_J) &= \mathrm{d}(fg) \wedge \mathrm{d}x_I \wedge \mathrm{d}x_J = ((\mathrm{d}f)g + f(\mathrm{d}g)) \wedge \mathrm{d}x_I \wedge \mathrm{d}x_J \\ &= g\,\mathrm{d}f \wedge \mathrm{d}x_I \wedge \mathrm{d}x_J + f\,\mathrm{d}g \wedge \mathrm{d}x_I \wedge \mathrm{d}x_J \\ &= g\,\mathrm{d}f \wedge \mathrm{d}x_I \wedge \mathrm{d}x_J + (-1)^k f\,\mathrm{d}x_I \wedge \mathrm{d}g \wedge \mathrm{d}x_J \\ &= \mathrm{d}(f\mathrm{d}x_I) \wedge (g\,\mathrm{d}x_J) + (-1)^k (f\,\mathrm{d}x_I) \wedge \mathrm{d}(g\,\mathrm{d}x_J) = \mathrm{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \mathrm{d}\beta.\end{aligned}$$ □

Remark 3.33. The exterior derivative d is characterised as a linear operator by:

(1) If $f \in \Omega^0(U)$ (*i.e.*, $f : U \to \mathbb{R}$ is smooth) then $\mathrm{d}f = Df$ as functions $U \to (\mathbb{R}^n)^*$;
(2) If $\alpha \in \Omega^k(U)$ and $\beta \in \Omega^\ell(U)$, then $\mathrm{d}(\alpha \wedge \beta) = (\mathrm{d}\alpha) \wedge \beta + (-1)^k \alpha \wedge (\mathrm{d}\beta) \in \Omega^{k+\ell+1}(U)$;
(3) If $\alpha \in \Omega^k(U)$, then $\mathrm{d}(\mathrm{d}\alpha) = 0 \in \Omega^{k+2}(U)$.

Indeed it follows straightforwardly from these properties that if $\alpha = f\,\mathrm{d}x_I$ then

$$\mathrm{d}\alpha = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\mathrm{d}x_j \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}.$$

3.6. **Pullbacks and the exterior derivative on submanifolds.**

Proposition 3.34. *For $U \subseteq \mathbb{R}^n$ and $\widetilde{U} \subseteq \mathbb{R}^m$ open, for $\alpha \in \Omega^k(\widetilde{U})$ and $\beta \in \Omega^\ell(\widetilde{U})$, and for $\varphi\colon U \to \widetilde{U}$ smooth, $\varphi^*(\alpha \wedge \beta) = \varphi^*\alpha \wedge \varphi^*\beta$.*

Proof. It is a straightforward exercise to check that if $\psi\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear map and $q \in \widetilde{U}$ then

$$\psi^*(\alpha_q\beta_q) = \psi^*(\alpha_q)\,\psi^*(\beta_q) \in \mathcal{M}^{k+\ell}(\mathbb{R}^n),$$

and for any $\gamma \in \mathcal{M}^m(\mathbb{R}^s)$ $\psi^*\operatorname{alt}(\gamma) = \operatorname{alt}(\psi^*\gamma) \in \operatorname{Alt}^m(\mathbb{R}^n)$, so that $\psi^*(\alpha_q \wedge \beta_q) = \psi^*(\alpha_q) \wedge \psi^*(\beta_q)$. Hence for any $p \in U$ (taking $q = \varphi(p)$ and $\psi = D\varphi_p$)

$$\varphi^*(\alpha \wedge \beta)_p = D\varphi_p^*(\alpha_{\varphi(p)} \wedge \beta_{\varphi(p)}) = D\varphi_p^*(\alpha_{\varphi(p)}) \wedge D\varphi_p^*(\alpha_{\varphi(p)}) = (\varphi^*\alpha \wedge \varphi^*\beta)_p. \qquad \square$$

Theorem 3.35. *Let $U \subseteq \mathbb{R}^n, \widetilde{U} \subseteq \mathbb{R}^m$ be open and $\varphi : U \to \widetilde{U}$ smooth. Then for any $\alpha \in \Omega^k(\widetilde{U})$,*

$$\mathrm{d}(\varphi^*\alpha) = \varphi^*(\mathrm{d}\alpha) \in \Omega^{k+1}(U).$$

Proof. By linearity of pullback and the exterior derivative, it suffices to check that the claim holds when $\alpha = f\,\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}$. Then by Proposition 3.34 and Lemma 3.15,

$$\varphi^*\alpha = (\varphi^*f)(\varphi^*\mathrm{d}x_{i_1} \wedge \cdots \wedge \varphi^*\mathrm{d}x_{i_k}) = (\varphi^*f)\mathrm{d}(\varphi^*x_{i_1}) \wedge \cdots \wedge \mathrm{d}(\varphi^*x_{i_k}),$$

so Theorem 3.23 ($\mathrm{d}^2 = 0$), Theorem 3.32 (Leibniz), Proposition 3.34 and Lemma 3.15 give

$$\begin{aligned}\mathrm{d}(\varphi^*\alpha) &= \mathrm{d}(\varphi^*f) \wedge \mathrm{d}(\varphi^*x_{i_1}) \wedge \cdots \wedge \mathrm{d}(\varphi^*x_{i_k}) = \varphi^*(\mathrm{d}f) \wedge (\varphi^*\mathrm{d}x_{i_1}) \wedge \cdots \wedge (\varphi^*\mathrm{d}x_{i_k})\\ &= \varphi^*(\mathrm{d}f \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}) = \varphi^*(\mathrm{d}(f\,\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k})) = \varphi^*(\mathrm{d}\alpha). \qquad \square\end{aligned}$$

Let $M \subseteq U \subseteq \mathbb{R}^s$ with U open and M an m-dimensional submanifold. Then the inclusion map $\iota = \mathrm{Id}_U|_M\colon M \to U$ is smooth, with derivative $D\iota_p\colon T_pM \to \mathbb{R}^s$ for any $p \in M$. Motivated by pullback, for any $\beta \in \Omega^k(U)$ we would like to define a "differential form" $\alpha = \iota^*\beta$ on M by

$$\alpha_p = (\iota^*\beta)_p = (D\iota_p)^*(\beta_{\iota(p)}) = \beta_p|_{T_pM^k} \in \operatorname{Alt}^k(T_pM)$$

for any $p \in M$. In order to provide α with a fixed codomain, we let

$$\mathcal{A}^k_m(\mathbb{R}^s) := \bigsqcup_W \operatorname{Alt}^k(W),$$

where the disjoint union is taken over all m-dimensional subspaces $W \subseteq \mathbb{R}^s$. This allows us to define differential forms on submanifolds as (local) pullbacks.

Definition 3.36. Let $M \subseteq \mathbb{R}^s$ be a submanifold of dimension m. A (*smooth*) *differential k-form* on M is a function $\alpha : M \to \mathcal{A}^k_m(\mathbb{R}^s); x \mapsto \alpha_p$ such that:

- $\alpha_p \in \operatorname{Alt}^k(T_pM)$ for all $p \in M$;
- for all $p \in M$ there is an open neighbourhood U of p in $\mathbb{R}^s$ and $\beta \in \Omega^k(U)$ such that for all $q \in U \cap M$, $\alpha_q = \beta_q|_{T_qM^k}$.

We let $\Omega^k(M)$ be the vector space of differential k-forms on M under pointwise operations.

If $\varphi : N \to M$ is smooth, where $N \subseteq \mathbb{R}^\ell$ is an n-dimensional submanifold then the *pullback* $\varphi^*\alpha$ of α by φ is defined by $(\varphi^*\alpha)_q = (D\varphi_q)^*\alpha_{\varphi(q)} \in \mathrm{Alt}^k(T_qN)$ for all $q \in N$.

Remarks 3.37.
(1) Recall that if M is an open subset of $\mathbb{R}^m$, then $T_pM = \mathbb{R}^m$ for all $p \in M$. Thus the two definitions of $\Omega^k(M)$ agree, as do the definitions of pullback.
(2) As in Lemma 3.15, the definition of pullback ensures that if $\varphi\colon N \to M$ and $\psi\colon P \to N$, then $(\varphi \circ \psi)^* = \psi^* \circ \varphi^*$ (with essentially the same proof).

Example 3.38. Let $S^1 = \{v \in \mathbb{R}^2 \mid \|v\|^2 = 1\}$ and let $i\colon S^1 \to \mathbb{R}^2$ be the inclusion. Then $\omega := i^*(-x_2\mathrm{d}x_1 + x_1\mathrm{d}x_2) \in \Omega^1(S^1)$. To see what ω looks like, use the parametrisation

$$\varphi : (0, 2\pi) \to S^1\backslash\{(1, 0)\}, \quad \theta \mapsto (\cos\theta, \sin\theta).$$

Then $\varphi^*\omega = (i \circ \varphi)^*(-x_2\mathrm{d}x_1 + x_1\mathrm{d}x_2) = -(\sin\theta)\,\mathrm{d}(\cos\theta) + (\cos\theta)\,\mathrm{d}(\sin\theta) = (\sin\theta)^2\,\mathrm{d}\theta + (\cos\theta)^2\,\mathrm{d}\theta = \mathrm{d}\theta$.

Lemma 3.39. *Let $M \subseteq U \subseteq \mathbb{R}^s$ and $N \subseteq \widetilde{U} \subseteq \mathbb{R}^\ell$, where U and $\widetilde{U}$ are open, while M and N are submanifolds of dimension m and n respectively. Let $i\colon M \to U$ and $j\colon N \to \widetilde{U}$ denote the inclusions and let $\varphi\colon N \to M$ be the restriction of a smooth map $\widetilde{\varphi}\colon \widetilde{U} \to U$.*

Suppose that $\beta \in \Omega^k(U)$ and $\alpha = i^*\beta \in \Omega^k(M)$. Then $\varphi^*\alpha = j^*\gamma \in \Omega^k(N)$ with $\gamma = \widetilde{\varphi}^*\beta \in \Omega^k(\widetilde{U})$, and $\varphi^*i^*\mathrm{d}\beta = j^*\mathrm{d}\gamma \in \Omega^{k+1}(N)$.

Proof. Since $\widetilde{\varphi} \circ j = i \circ \varphi$, we have $j^*\widetilde{\varphi}^*\beta = (\widetilde{\varphi} \circ j)^*\beta = (i \circ \varphi)^*\beta = \varphi^*i^*\beta = \varphi^*\alpha$. Furthermore, by Theorem 3.35, $\varphi^*i^*\mathrm{d}\beta = j^*\widetilde{\varphi}^*\mathrm{d}\beta = j^*\mathrm{d}(\widetilde{\varphi}^*\beta) = j^*\mathrm{d}\gamma$. □

For any smooth map $\varphi\colon N \to M$ between submanifolds M and N and any $\alpha \in \Omega^k(M)$, this lemma applies to $N \cap \widetilde{U}$ and $M \cap U$ for sufficiently small open neighbourhoods of any $q \in N$ and $\varphi(q) \in M$ such that $\alpha = i^*\beta$ on $U \cap M$ ($i\colon U \cap M \to U$) and φ has a smooth extension $\widetilde{\varphi}\colon \widetilde{U} \to U$. Hence $\varphi^*\alpha \in \Omega^k(N)$, *i.e.*, is smooth.

Secondly, suppose that $\varphi\colon \widetilde{U} \to U \cap M$ is a parametrisation of M (for $\widetilde{U} \subseteq \mathbb{R}^n$ open) and $\alpha \in \Omega^k(M)$ agrees with $i^*\beta$ on $U \cap M$ ($i\colon U \cap M \to U$). Then the lemma applies with $\widetilde{\varphi} = \varphi \circ i$ and $j = \mathrm{id}_{\widetilde{U}}$ to give $\varphi^*i^*\mathrm{d}\beta = \mathrm{d}(\varphi^*\alpha)$ on $\widetilde{U}$ and hence for any $p \in U \cap M$,

$$(i^*\mathrm{d}\beta)_p = ((\varphi^{-1})^*\mathrm{d}(\varphi^*\alpha))_p \in \mathrm{Alt}^{k+1}(T_pM).$$

Thus $(i^*\mathrm{d}\beta)_p$ depends only on α, not on the choice of local extension β.

Definition 3.40. Let M be an n-dimensional submanifold of $\mathbb{R}^s$. Define the *exterior derivative* $\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$ by requiring that whenever $\beta \in \Omega^k(U)$ is a local extension of $\alpha \in \Omega^k(M)$ on $U \cap M$ (for $U \subseteq \mathbb{R}^s$ open), then $\mathrm{d}\alpha = i^*\mathrm{d}\beta$ on $U \cap M$, where $i\colon U \cap M \to U$ is the inclusion.

Now for any smooth map $\varphi\colon N \to M$ between submanifolds M and N and any $\alpha \in \Omega^k(M)$, applying Lemma 3.39 on sufficiently small open neighbourhoods as above, we obtain that $\varphi^*\mathrm{d}\alpha = \mathrm{d}(\varphi^*\alpha) \in \Omega^k(N)$, generalizing Theorem 3.35.

3.7. **Proof of the Poincaré Lemma.** We turn the method of Example 3.26 into an algorithm. Note first that the two lists

$$\varepsilon_I : I \subseteq \{1,\dots,n-1\}, |I| = k \quad \text{and} \quad \varepsilon_n \wedge \varepsilon_I : I \subseteq \{1,\dots,n-1\}, |I| = k-1.$$

combine to give a basis for $\mathrm{Alt}^k(\mathbb{R}^n)$. Therefore, if for any k we let $B_n^k \leq \mathrm{Alt}^k(\mathbb{R}^n)$ be the subspace spanned by the first list, then any $\psi \in \mathrm{Alt}^k(\mathbb{R}^n)$ can be written uniquely as

$$\psi = \nu + \varepsilon_n \wedge \eta$$

for $\nu \in B_n^k$ and $\eta \in B_n^{k-1}$. Hence for any $\alpha \in \Omega^k(\mathbb{R}^n)$, there exists a unique function $\mathcal{L}(\alpha) : \mathbb{R}^n \to B_n^{k-1}$ such that for all $p \in \mathbb{R}^n$, $(\alpha - \mathrm{d}x_n \wedge \mathcal{L}(\alpha))_p \in B_n^k$. We now observe that if α doesn't involve $\mathrm{d}x_n$, then $\mathrm{d}\alpha$ will be the sum of $\mathrm{d}x_n \wedge \frac{\partial \alpha}{\partial x_n}$ and some terms that do not involve $\mathrm{d}x_n$.

Lemma 3.41. *If $\alpha \in \Omega^{k-1}(\mathbb{R}^n)$ satisfies $\mathcal{L}(\alpha) = 0$, then $\mathcal{L}(\mathrm{d}\alpha) = \dfrac{\partial \alpha}{\partial x_n} : \mathbb{R}^n \to B_n^{k-1}$.*

Proof. If $\mathcal{L}(\alpha) = 0$, then we can write $\alpha = \sum_{|I|=k} f_I \mathrm{d}x_I$ for some real functions $f_I : \mathbb{R}^n \to \mathbb{R}$, where the sum is over $I \subseteq \{1,\dots,n-1\}$. Then since $\mathcal{L}$ is linear,

$$\mathcal{L}(\mathrm{d}\alpha) = \sum_{|I|=p} \mathcal{L}\left(\frac{\partial f_I}{\partial x_1}\mathrm{d}x_1 \wedge \mathrm{d}x_I + \cdots + \frac{\partial f_I}{\partial x_n}\mathrm{d}x_n \wedge \mathrm{d}x_I\right) = \sum_{|I|=k} \frac{\partial f_I}{\partial x_n}\mathrm{d}x_I = \frac{\partial \alpha}{\partial x_n}. \qquad \square$$

Proof of Theorem 3.25. We use induction on n. If $n = 0$ then the claim is trivial since $\Omega^k(\mathbb{R}^0) = \{0\}$ for $k > 0$, so suppose the claim holds for $n = m-1 \geq 0$, and let $\alpha \in \Omega^k(\mathbb{R}^m)$. Define $\gamma : \mathbb{R}^m \to B_m^{k-1}$ by

$$p \mapsto \int_0^{x_m(p)} \mathcal{L}(\alpha)_{(x_1(p),\dots,x_{m-1}(p),t)}\,\mathrm{d}t.$$

Then $\frac{\partial \gamma}{\partial x_m} = \mathcal{L}(\alpha)$, so Lemma 3.41 gives $\mathcal{L}(\mathrm{d}\gamma) = \mathcal{L}(\alpha)$. Hence $\alpha' := \alpha - \mathrm{d}\gamma \in \Omega^k(\mathbb{R}^m)$ is closed with $\mathcal{L}(\alpha') = 0$. Now Lemma 3.41 gives $\frac{\partial \alpha'}{\partial x_m} = \mathcal{L}(\mathrm{d}\alpha') = 0$, *i.e.*, the function $\alpha' : \mathbb{R}^m \to B_m^k$ does not depend on x_m. Now let $\pi : \mathbb{R}^m \to \mathbb{R}^{m-1}$ denote the projection map $p \mapsto (x_1(p),\dots,x_{m-1}(p))$, and let $\bar{x}_1,\dots \bar{x}_{m-1}$ denote the coordinate functions on $\mathbb{R}^{m-1}$. Then $\pi^*\bar{x}_j = x_j$ and hence, by Theorem 3.35, $\pi^*\mathrm{d}\bar{x}_j = \mathrm{d}x_j$ for $j \in \{0,\dots,m-1\}$. It follows from Proposition 3.34 that $\pi^*\mathrm{d}\bar{x}_I = \mathrm{d}x_I$ for $I \subseteq \{1,\dots m-1\}$, and hence $\pi^* \colon \Omega^\ell(\mathbb{R}^{m-1}) \to \Omega^\ell(\mathbb{R}^m)$ is injective for all $\ell \in \mathbb{N}$. Also observe that for $f \in \Omega^0(\mathbb{R}^{m-1})$ $\pi^*\mathrm{d}(f \wedge \mathrm{d}\bar{x}_I) = \pi^*(\mathrm{d}f \wedge \mathrm{d}\bar{x}_I) = (\pi^*\mathrm{d}f) \wedge \mathrm{d}x_I = \mathrm{d}((\pi^*f)\mathrm{d}\bar{x}_I)$, so $\pi^* \circ \mathrm{d} = \mathrm{d} \circ \pi^*$ by Theorem 3.35.

Since α' does not involve x_m or $\mathrm{d}x_m$, it follows that there exists $\bar{\alpha} \in \Omega^k(\mathbb{R}^{m-1})$ such that $\alpha' = \pi^*\bar{\alpha}$. So $0 = \mathrm{d}\alpha' = \mathrm{d}\pi^*\bar{\alpha} = \pi^*\mathrm{d}\bar{\alpha}$ and hence $\bar{\alpha}$ is closed. The inductive hypothesis thus gives $\bar{\beta} \in \Omega^{k-1}(\mathbb{R}^{m-1})$ such that $\mathrm{d}\bar{\beta} = \bar{\alpha}$, and therefore $\alpha' = \pi^*\bar{\alpha} = \pi^*\mathrm{d}\bar{\beta} = \mathrm{d}(\pi^*\bar{\beta})$. Hence $\alpha = \mathrm{d}(\pi^*\bar{\beta} + \gamma)$ is exact. $\square$

4. Integration and Stokes' Theorem

4.1. **Submanifolds with boundary.** Let H^n be the closed half-space $\{p \in \mathbb{R}^n \mid x_1(p) \leq 0\}$, and let $\partial H^n = \{0\} \times \mathbb{R}^{n-1} \subset H^n$. For U open in H^n, let $\partial U = U \cap \partial H^n$.

If $f \colon H^n \to \mathbb{R}^m$ is smooth, then Df_p is well-defined at all $p \in H^n$, including $p \in \partial H^n$, since $D\tilde{f}_p$ is independent of the choice of smooth local extension $\tilde{f} \colon \widetilde{U} \to \mathbb{R}^m$ of f to an open neighbourhood $\widetilde{U}$ of p in $\mathbb{R}^n$: observe that for $v \in H^n \setminus \partial H^n$, $\tilde{f}(p+tv) - \tilde{f}(p) = f(p+tv) - f(p)$ for $t > 0$, so $D\tilde{f}_p(v)$ is determined by f, and such v span $\mathbb{R}^n$.

Definition 4.1. $M \subseteq \mathbb{R}^s$ is an n-dimensional *submanifold-with-boundary* (SMWB) if for every $p \in M$ there is a diffeomorphism $\varphi\colon \widetilde{U} \to U$ (called a *parametrisation*) from an open subset $\widetilde{U} \subseteq H^n$ to an open neighbourhood $U \subseteq M$ of p. The *boundary* of M is

$$\partial M = \{p \in M \,|\, p \in \varphi(\partial\widetilde{U}) \text{ for some parametrisation } \varphi\colon \widetilde{U} \to U\},$$

while the *interior* is $\mathring{M} = M \setminus \partial M$. For $p \in \partial M$, we define T_pM to be the span of $\gamma'(0)$ over all smooth curves $\gamma\colon (-\varepsilon, \varepsilon) \to \mathbb{R}^s$ with $\gamma(0) = p$ and $\gamma(t) \in M$ for $t > 0$.

Remarks 4.2. If $U \subseteq H^n$ is open, then $p \in U$ is in ∂U if and only if p has no open neighbourhood U' in $\mathbb{R}^n$ such that $U' \subseteq U$. Hence for any diffeomorphism $\psi\colon \widetilde{U} \to U$ of open subsets of H^n, $\psi(\partial\widetilde{U}) = \partial U$. If $M \subseteq \mathbb{R}^s$ is a SMWB then:

(1) the condition that $p \in \varphi(\partial\widetilde{U}) \subseteq M$ is independent of the choice of parametrisation $\varphi : \widetilde{U} \to U$ with $p \in U$;
(2) the interior of M is an n-dimensional submanifold of $\mathbb{R}^s$;
(3) the boundary ∂M is an $(n-1)$-dimensional submanifold of $\mathbb{R}^s$—indeed, for any parametrisation $\varphi : \widetilde{U} \to U$ of M, the restriction of φ to $\partial\widetilde{U}$ gives a diffeomorphism $\partial\widetilde{U} \to \partial U$, from an open subset $\partial\widetilde{U}$ of $\partial H^n = \mathbb{R}^{n-1}$ to an open subset ∂U of ∂M.

On the other hand, if $N \subseteq \mathbb{R}^s$ is a submanifold, then N is also a SMWB, with $\partial N = \varnothing$.

For a SMWB M, we can define spaces of differential forms $\Omega^k(M)$, pullbacks and exterior derivatives in exactly the same way as for submanifolds.

4.2. Multiple integrals.

Theorem 4.3 (Heine–Borel). *A subset of a finite dimensional normed vector space is compact if and only if it is closed and bounded.*

Definition 4.4. For $S \subseteq \mathbb{R}^n$ and a function $f : S \to \mathbb{R}$, the *support* of f is

$$\operatorname{supp}(f) := \overline{\{p \in S : f(p) \neq 0\}} \subseteq S$$

(the closure in S of the set $\{p \in S : f(p) \neq 0\}$). In other words, $\operatorname{supp}(f)$ is the smallest closed subset of S that contains all $p \in S$ with $f(p) \neq 0$. We say that f has *compact support* if $\operatorname{supp}(f)$ is compact, *i.e.*, $\operatorname{supp}(f)$ is a closed and bounded subset of $\mathbb{R}^n$ (by Heine–Borel 4.3). Write $C^0_c(S) := \{f \in C^0(S) : \operatorname{supp}(f) \text{ is compact}\}$.

We impose compact support to ensure convergence of the integrals in the following. For $f \in C^0_c(H^n)$, we define $\tilde{f} \in C^0_c(H^{n-1})$ as follows: for $n \geq 2$ let

$$\tilde{f} : H^{n-1} \to \mathbb{R}, \qquad p \mapsto \int_{\mathbb{R}} f(x_1(p), \ldots, x_{n-1}(p), t)\,\mathrm{d}t;$$

for $n = 1$, $H^1 = (-\infty, 0]$, and $\tilde{f} \in \mathbb{R}$ is the integral of f over this interval.

Definition 4.5. For $f \in C^0_c(H^n)$, define the *multiple integral* of f inductively by

$$\int_{H^n} f(x_1, \ldots, x_n)\,\mathrm{d}x_1 \cdots \mathrm{d}x_n = \int_{H^{n-1}} \tilde{f}(x_1, \ldots, x_{n-1})\,\mathrm{d}x_1 \cdots \mathrm{d}x_{n-1} \in \mathbb{R}$$

If $U \subseteq H^n$ is open and $f \in C^0_c(U)$, $\int_U f\,\mathrm{d}x_1 \cdots \mathrm{d}x_n$ is defined in the same way after extending f by zero to H^n.

Given $\varphi : \widetilde{U} \to U$ a diffeomorphism of open subsets of H^n, define the *Jacobian* $J_\varphi : \widetilde{U} \to \mathbb{R}$ by $J_\varphi(p) = \det(D\varphi_p)$. Let $f \in C_c^0(U)$ and note that $\operatorname{supp}(f \circ \varphi) = \varphi^{-1}(\operatorname{supp}(f)) \subseteq \widetilde{U}$ is the continuous image of a compact set, and thus compact. Hence $f \circ \varphi$ has compact support, so $(f \circ \varphi)\,|J_\varphi|\colon p \mapsto f(\varphi(p))|J_\varphi(p)|$ is in $C_c^0(\widetilde{U})$

Theorem 4.6 (Change of variables formula for multiple integrals)**.** *Given $f \in C_c^0(U)$ and a diffeomorphism $\varphi : \widetilde{U} \to U$ and Jacobian $J_\varphi : \widetilde{U} \to \mathbb{R}$ defined as above, then*

$$\int_U f \,\mathrm{d}y_1 \cdots \mathrm{d}y_n = \int_{\widetilde{U}} (f \circ \varphi)\,|J_\varphi| \,\mathrm{d}x_1 \cdots \mathrm{d}x_n.$$

A proof of this theorem is given in Appendix B, in the case of open subsets of $\mathbb{R}^n$ rather than H^n, but the proof in the latter case is similar.

4.3. **Integration of forms.** Recall that $\dim \operatorname{Alt}^n(\mathbb{R}^n) = 1$ with basis Det, and that for any $\alpha \in \operatorname{Alt}^n(\mathbb{R}^n)$ and linear map $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$, $\phi^*\alpha = \det(\phi)\alpha$ (exercise). Now suppose $\alpha = f\,\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_n \in \Omega^n(U)$ where U is open in H^n with coordinates $y_1, \ldots, y_n$ and $f\colon U \to \mathbb{R}$ smooth. If $\varphi\colon \widetilde{U} \to U$ is a diffeomorphism for $\widetilde{U}$ open in H^n (with coordinates $x_1, \ldots, x_n$), then

$$\begin{aligned}(\varphi^*\alpha)_p = (\varphi^*(f\,\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_n))_p &= (D\varphi_p)^*\big(f(\varphi(p))\mathrm{Det}\big) = f(\varphi(p))J_\varphi(p)\mathrm{Det} \\ &= f(\varphi(p))J_\varphi(p)(\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n)_p.\end{aligned}$$

We say φ is *orientation-preserving* if $\forall p \in \widetilde{U}$, $J_\varphi(p) > 0$; then $J_\varphi(p) = |J_\varphi(p)|$, so the transformation rule for $\varphi^*\alpha$ resembles the change of variables formula of Theorem 4.6.

We write $\alpha \in \Omega_c^n(U)$ if $\alpha = f\,\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_n \in \Omega^n(U)$ with $f \in C_c^0(U)$.

Definition 4.7. For $U \subseteq H^n$ an open subset and $\alpha \in \Omega_c^n(U)$, we define

$$\int_U \alpha := \int_U f\,\mathrm{d}y_1 \cdots \mathrm{d}y_n \in \mathbb{R}$$

where $f \in C_c^0(U)$ is such that $\alpha = f\,\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_n$.

Theorem 4.8 (Change of variables for differential forms)**.** *Suppose $\varphi : \widetilde{U} \to U$ is an orientation-preserving diffeomorphism and $\alpha \in \Omega_c^n(U)$; then*

$$\int_{\widetilde{U}} \varphi^*\alpha = \int_U \alpha$$

Proof. If $\alpha = f\,\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_n$, we have seen that $\varphi^*\alpha = (f \circ \varphi)J_\varphi\,\mathrm{d}x_1 \wedge \cdots \mathrm{d}x_n$. Since φ is orientation-preserving, $|J_\varphi(p)| = J_\varphi(p) > 0$, so Theorem 4.6 gives

$$\int_{\widetilde{U}} \varphi^*\alpha = \int_{\widetilde{U}} (f \circ \varphi)\,J_\varphi\,\mathrm{d}x_1 \cdots \mathrm{d}x_n = \int_U f\,\mathrm{d}y_1 \cdots \mathrm{d}y_n = \int_U \alpha. \qquad \square$$

4.4. **Orientations.** An *orientation* of an n-dimensional real vector space V is an element of the 2 element set $(\operatorname{Alt}^n(V)\backslash\{0\})/\sim$, where $\alpha \sim \widetilde{\alpha}$ if $\widetilde{\alpha} = \lambda\alpha$ for some $\lambda \in \mathbb{R}^+$.

Definition 4.9. Let $M \subseteq \mathbb{R}^s$ be an n-dimensional SMWB.

(1) We call $\omega \in \Omega^n(M)$ an *orientation form* if ω never vanishes, *i.e.*, $\forall\, p \in M,\ \omega_p \neq 0$;
(2) M is *orientable* if an orientation form exists;
(3) An *orientation* on M an equivalence class $[\omega] \in \mathcal{N}/{\sim}$, where $\mathcal{N}$ is the set of orientation forms on M, and $\omega \sim \widetilde{\omega}$ if for all $p \in M$, $\omega_p \sim \widetilde{\omega}_p$.

An *oriented SMWB* is a SMWB M together with a choice of orientation $[\omega]$.

Remarks 4.10. Thus an orientation form ω on a SMWB M defines an orientation $[\omega_p]$ on T_pM for each $p \in M$, with equivalent orientation forms defining the same pointwise orientation. Smoothness of ω means that the orientations of T_pM are "consistent" (*i.e.*, they do not change discontinuously). The intermediate value theorem can be used to show that if M is connected and orientable, it has exactly 2 orientations $[\omega]$ and $[-\omega]$. (If ω and $\widetilde{\omega}$ are orientation forms then $\widetilde{\omega} = f\omega$ with $f(p) \neq 0$ for all $p \in M$ and f cannot change sign if M is connected.)

Example 4.11. If $U \subseteq H^n$ is an open subset, then $\omega = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \in \Omega^n(U)$ is an orientation form, called the *standard orientation* of U. The standard orientation of ∂U is $\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_{n-1}$ where the inclusion $\partial U \to U$ is defined by $i(y_1, \dots, y_{n-1}) = (0, y_1, \dots, y_{n-1})$.

Proposition 4.12. *If a SMWB M is oriented, then ∂M is oriented.*

Proof. For $p \in \partial M$ let $v(p) \in T_pM$ be the outward unit normal to ∂M; thus $\|v(p)\| = 1$ and $v(p) \cdot w = 0$ for all $w \in T_p(\partial M)$, which determines $v(p)$ up to sign, and the sign is fixed by $v(p)$ being "outward pointing". Then $v\colon \partial M \to \mathbb{R}^s$ is smooth: indeed, if $\varphi\colon \widetilde{U} \to U$ is a parametrisation with inverse $\psi\colon U \to \widetilde{U} \subseteq H^n \subseteq \mathbb{R}^n$ then on ∂U, $v = \operatorname{grad}(x_1 \circ \psi)/\|\operatorname{grad}(x_1 \circ \psi)\|$, so v has local smooth extensions (because ψ does).

Now suppose M is oriented by an orientation form ω, and, for $p \in \partial M$, define $\beta_p \in \mathrm{Alt}^{n-1}(T_p\partial M)$ by $\beta_p(v_1, \dots, v_{n-1}) = \omega_p(v(p), v_1, \dots, v_{n-1})$. Then $\beta \in \Omega^{n-1}(\partial M)$: if $\widetilde{v}$ and $\widetilde{\omega}$ are smooth local extensions of v and ω, then $\widetilde{v} \lrcorner \widetilde{\omega}$, with $(\widetilde{v} \lrcorner \widetilde{\omega})_p(v_1, \dots, v_{n-1}) = \widetilde{\omega}_p(\widetilde{v}(p), v_1, \dots, v_{n-1})$, locally extends β. Finally, for all $p \in \partial M$, $\beta_p \neq 0$, since if $v_1, \dots, v_{n-1}$ is a basis for $T_p(\partial M)$, it follows that $v(p), v_1, \dots, v_{n-1}$ is a basis for T_pM and so $\omega_p(v(p), v_1, \dots, v_{n-1})$ is nonzero. □

The outward normal convention ensures that for $U \subseteq H^n$ the standard orientation of U induces the standard orientation of ∂U.

Definition 4.13. Let $\varphi : N \to M$ be local diffeomorphism of oriented SMWBs. Then φ is *orientation-preserving* if for an orientation form $\omega \in \Omega^n(M)$ defining the chosen orientation of M, the pullback $\varphi^*\omega$ defines the chosen orientation on N.

In particular, a parametrisation $\varphi : \widetilde{U} \to U \subseteq M$ is orientation-preserving (or *oriented*) if $\varphi^*\omega = f\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \in \Omega^n(\widetilde{U})$ with $f : \widetilde{U} \to \mathbb{R}^+$.

Proposition 4.14. *If M is oriented, we can cover M by images U_i of oriented parametrisations $\varphi_i : U_i' \to U_i$.*

Proof. Cover M by images of some parametrisations $\widetilde{\varphi}_i : \widetilde{U}_i \to U_i$. Without loss of generality the $\widetilde{U}_i$ are connected. Now either $\widetilde{\varphi}_i$ is oriented (and we take $U_i' = \widetilde{U}_i$ and $\varphi_i = \tilde{\varphi}_i$), or $\tilde{\varphi}_i^*\omega = -f\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ with $f : U_i \to \mathbb{R}^+$. In the latter case let $\tau_n = (x_1, \dots, x_{n-1}, -x_n)$ and $U_i' = \tau_n^{-1}(\widetilde{U}_i)$. Then $\varphi_i = \tilde{\varphi}_i \circ \tau_n$ is an oriented parametrisation of M with the same image U_i. □

4.5. **The integration map.** For a SMWB M and $\alpha \in \Omega^k(M)$, we let $\operatorname{supp}(\alpha) = \overline{\{p \in M : \alpha_p \neq 0\}} \subseteq M$, and write $\alpha \in \Omega^k_c(M)$ if $\operatorname{supp}(\alpha)$ is compact.

Definition 4.15. Let $M \subseteq \mathbb{R}^s$ be an oriented SMWB of dimension n. Then an *integration map* on M is a linear map

$$\int_M : \Omega^n_c(M) \to \mathbb{R}$$

such that if $\varphi : \widetilde{U} \to U$ is an oriented parametrisation and $\alpha \in \Omega^n_c(M)$ with $\operatorname{supp}(\alpha) \subseteq U$, then

$$\int_M \alpha = \int_{\widetilde{U}} \varphi^* \alpha \in \mathbb{R} \tag{4.1}$$

To prove the existence and uniqueness of integration maps, we need a technical tool.

Definition 4.16. Let $U_i : i \in I$ be an open cover of $S \subseteq \mathbb{R}^s$. A *partition of unity on S subordinate to $U_i : i \in I$* is a indexed family $\rho_i : i \in I$ such that

(1) each ρ_i is a nonnegative smooth function $S \to \mathbb{R}$;
(2) $\operatorname{supp}(\rho_i) \subseteq U_i$ for all $i \in I$;
(3) each $p \in S$ has a neighbourhood $U \subseteq S$ such that $U \cap \operatorname{supp}(\rho_i) \neq \varnothing$ only for finitely many $i \in I$; and
(4) for each $p \in S$, $\sum_{i \in I} \rho_i(p) = 1$.

Remark 4.17. If I is finite, then (3) is vacuous. In general, (3) ensures that the sum in (4) is well-defined (since only finitely many terms are nonzero).

Theorem 4.18. *Let $M \subseteq \mathbb{R}^s$ be a SMWB. Then for any open cover of M, there exists a subordinate partition of unity.*

A proof is given in Appendix A.

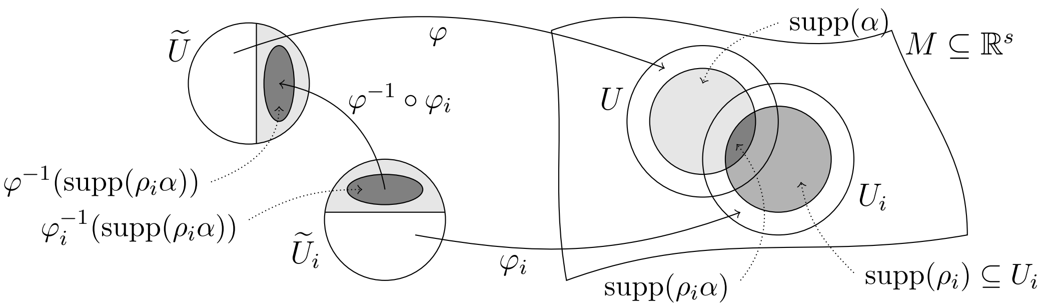
Theorem 4.19. *For any oriented SMWB M, there is a unique integration map.*

Proof. Let $\varphi_i : i \in I$ be oriented parametrisations $\varphi_i : \widetilde{U}_i \to U_i$ such that $U_i : i \in I$ cover M. Let $\rho_i : i \in I$ be a partition of unity on M subordinate to this cover. For $\alpha \in \Omega^n_c(M)$, Definition 4.16 (3) implies that each $p \in \operatorname{supp}(\alpha)$ has an open neighbourhood which meets $\operatorname{supp}(\rho_i)$ for only finitely many i; since $\operatorname{supp}(\alpha)$ is compact, this open cover has a finite subcover, so $\operatorname{supp}(\alpha)$ meets $\operatorname{supp}(\rho_i)$ for only finitely many i. Hence $\alpha = \sum_{i \in I} \rho_i \alpha$ is a finite sum with $\operatorname{supp}(\rho_i \alpha) \subseteq U_i$. Then linearity and (4.1) imply

$$\int_M \alpha = \sum_{i \in I} \int_{\widetilde{U}_i} \varphi_i^*(\rho_i \alpha) \in \mathbb{R} \tag{4.2}$$

So if there exists a map $\int_M$ satisfying (4.1), then it is unique.

It remains to prove that if we define $\int_M$ by (4.2)—which is clearly linear in α—then (4.1) holds. Suppose $\varphi : \widetilde{U} \to U$ is an oriented parametrisation with $\operatorname{supp}(\alpha) \subseteq U$.



Then (see above diagram) $\text{supp}(\rho_i\alpha) \subseteq U_i \cap U$. Note that

$$\text{supp}(\varphi_i^*(\rho_i\alpha)) = \varphi_i^{-1}(\text{supp}(\rho_i\alpha)) \subseteq \varphi_i^{-1}(U_i \cap U) \subseteq \widetilde{U}_i$$

and

$$\varphi_i^*(\rho_i\alpha) = (\varphi^{-1} \circ \varphi_i)^*(\varphi^*(\rho_i\alpha)) \in \Omega_c^n(\varphi_i^{-1}(U_i \cap U))$$

The function $\varphi^{-1} \circ \varphi_i : \varphi_i^{-1}(U_i \cap U) \to \varphi^{-1}(U_i \cap U)$ is orientation-preserving since φ and φ_i are both oriented. Hence

$$\int_{\widetilde{U}_i} \varphi_i^*(\rho_i\alpha) = \int_{\varphi_i^{-1}(U_i \cap U)} \varphi_i^*(\rho_i\alpha) = \int_{\varphi^{-1}(U_i \cap U)} \varphi^*(\rho_i\alpha) = \int_{\widetilde{U}} \varphi^*(\rho_i\alpha)$$

where the second equality follows by Theorem 4.8. Hence

$$\sum_{i\in I} \int_{\widetilde{U}_i} \varphi_i^*(\rho_i\alpha) = \sum_{i\in I} \int_{\widetilde{U}} \varphi^*(\rho_i\alpha) = \int_{\widetilde{U}} \varphi^*\left(\sum_{i\in I} \rho_i\alpha\right) = \int_{\widetilde{U}} \varphi^*\alpha,$$

as required. □

Remark 4.20. Note that the expression (4.2) for the integration map apparently depends upon the choice of parametrisations and partition of unity. By the second part of the proof, any other choice $\tilde{\varphi}_j \colon \widetilde{U}'_j \to U'_j$, $\tilde{\rho}_j : j \in J$ will also define an integration map

$$\alpha \mapsto \sum_{j\in J} \int_{\widetilde{U}'_j} \tilde{\varphi}_j^*(\tilde{\rho}_j\alpha).$$

However, by the first part of the proof, this integration map is equal to the one defined by (4.2). So in fact all such formula compute the same integrals.

Example 4.21. Let $S^1 = \{v \in \mathbb{R}^2 \mid \|v\|^2 = 1\}$. Now equip S^1 with an orientation form $\omega \in \Omega^1(S^1)$ defined as the pullback of $-x_2 \mathrm{d}x_1 + x_1 \mathrm{d}x_2 \in \Omega^1(\mathbb{R}^2)$. The parametrisations

$$\varphi : (0, 2\pi) \to S^1 \backslash \{(1,0)\}, \qquad \theta \mapsto (\cos\theta, \sin\theta)$$
$$\psi : (-\pi, \pi) \to S^1 \backslash \{(-1,0)\}, \quad \mu \mapsto (\cos\mu, \sin\mu)$$

are both oriented and cover S^1. Now we claim that for any $\alpha \in \Omega^1(S^1)$, we can compute $\int_{S^1} \alpha$ as $\int_{(0,2\pi)} \varphi^*\alpha$. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function such that

$$f(t) = \begin{cases} 1, & t \geq 1 \\ 0, & t \leq 1/2. \end{cases}$$

For $\varepsilon > 0$, define $\rho_{1,\varepsilon}, \rho_{2,\varepsilon} : S^1 \to \mathbb{R}$ by $\rho_{1,\varepsilon}(q) = f(\varepsilon^{-2}\|q-(1,0)\|^2)$ and $\rho_{2,\varepsilon}(q) = 1-\rho_{1,\varepsilon}(q)$. Then for any $\varepsilon \in (0,1)$, we have that $\rho_{1,\varepsilon}, \rho_{2,\varepsilon}$ is a partition of unity subordinate to $U_1 := S^1\backslash\{(1,0)\}$ and $U_2 := S^1\backslash\{(-1,0)\}$, and so

$$\int_{S^1} \alpha = \int_{(0,2\pi)} \varphi^*(\rho_{1,\varepsilon}\, \alpha) + \int_{-\pi,\pi)} \psi^*(\rho_{2,\varepsilon}\, \alpha)$$

As $\varepsilon \to 0$, $\rho_{1,\varepsilon}$ tends to 1 except at $q = (1,0)$. Hence the first term converges to $\int_{(0,2\pi)} \varphi^*\alpha$ and the second term converges to 0, proving the claim.

Now let $\alpha \in \Omega^1(S^1)$ be the pullback of $x_1 \mathrm{d}x_2 \in \Omega^1(\mathbb{R}^2)$. Then $\varphi^*\alpha = \cos\theta\, \mathrm{d}(\sin\theta) = (\cos\theta)^2 \mathrm{d}\theta$ and so

$$\int_{S^1} \alpha = \int_{(0,2\pi)} \varphi^*\alpha = \int_0^{2\pi} (\cos\theta)^2\, \mathrm{d}\theta = \int_0^{2\pi} \tfrac{1}{2}(\cos(2\theta)+1)\, \mathrm{d}\theta = \pi.$$

Remark 4.22. The above example illustrates a general principle. When evaluating integrals in practice, we don't have to use partitions of unity: we can just find an oriented parametrisation $\varphi : \widetilde{U} \to U$ on M such that U is dense and evaluate $\int_{\widetilde{U}} \varphi^*\alpha$.

4.6. **Stokes' theorem.** Let $i : \partial M \to M$ denote the inclusion of the boundary of an oriented SMWB M; then any $\alpha \in \Omega^k(M)$ has a pullback $i^*\alpha \in \Omega^k(\partial M)$, and if $\alpha \in \Omega^k_c(M)$, then $i^*\alpha \in \Omega^k_c(\partial M)$. In particular, if $\beta \in \Omega^{n-1}_c(M)$, then $\mathrm{d}\beta \in \Omega^n_c(M)$ and $i^*\beta \in \Omega^{n-1}_c(\partial M)$ can be integrated on M and ∂M respectively.

Theorem 4.23. *Let $i \colon \partial M \to M$ be an oriented SMWB and $\beta \in \Omega^{n-1}_c(M)$, Then*

$$\int_M \mathrm{d}\beta = \int_{\partial M} i^*\beta.$$

Example 4.24. Let $\beta : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $\beta(p) = 0$ for $p \leq -1$ and $\beta(p) = 1$ for $p \geq 1$. Then $\mathrm{supp}(\mathrm{d}\beta) \subseteq [-1,1]$, so $\mathrm{d}\beta \in \Omega^1_c(\mathbb{R})$, and

$$\int_{\mathbb{R}} \mathrm{d}\beta = \int_{-1}^{1} \frac{\mathrm{d}\beta}{\mathrm{d}x} \mathrm{d}x = \beta(1) - \beta(-1) = 1.$$

Example 4.25. Let $M := \{v \in \mathbb{R}^2 : \|v\|^2 \leq 1\}$, and let

$$\beta = x_1\, \mathrm{d}x_2 \in \Omega^1(M).$$

Since M itself is compact, automatically $\beta \in \Omega^1_c(M)$. Now $\mathrm{d}\beta = \mathrm{d}x_1 \wedge \mathrm{d}x_2$, so $\int_M \mathrm{d}\beta$ is simply the double integral of the constant function 1 over the unit disc, which is π.

This agrees the integral of $i^*\beta$ on S^1, evaluated in Example 4.21.

To prove Stokes' theorem, we may as well assume that $\mathrm{supp}(\beta)$ is contained in U for some oriented parametrisation $\varphi : \widetilde{U} \to U$, since any β can be written (using a partition of unity) as a sum of such forms. Then

$$\int_M \mathrm{d}\beta = \int_{\widetilde{U}} \varphi^*(\mathrm{d}\beta) = \int_{\widetilde{U}} \mathrm{d}(\varphi^*\beta)$$

and (with $i_{\widetilde{U}} \colon \partial\widetilde{U} \to \widetilde{U}$ being the inclusion)

$$\int_{\partial M} i^*\beta = \int_{\partial\widetilde{U}} i^*_{\widetilde{U}}\varphi^*\beta.$$

The theorem now follows by applying the next lemma to $\gamma \in \Omega_c^{n-1}(H^n)$ defined by

$$\gamma_p = \begin{cases} (\varphi^*\beta)_p & \text{for } p \in \widetilde{U} \\ 0 & \text{for } p \in H^n \setminus \operatorname{supp}(\varphi^*\beta). \end{cases}$$

Lemma 4.26. *For any $\gamma \in \Omega_c^{n-1}(H^n)$ and $i \colon \partial H^n \to H^n$ the inclusion,*

$$\int_{H^n} \mathrm{d}\gamma = \int_{\partial H^n} i^*\gamma.$$

Proof. We may write γ as $v \lrcorner (\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n)$ with $v(p) = \sum_{i=1}^n f_i(p)e_i$. Then

$$\mathrm{d}\gamma = \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \in \Omega_c^n(H^n)$$

and

$$i^*\gamma = g\, \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_{n-1} \in \Omega_c^{n-1}(\partial H^n),$$

where $g(y_1, \ldots, y_{n-1}) = f_1(0, y_1, \ldots, y_{n-1})$. Thus it remains to prove

$$\sum_{i=1}^n \int_{H^n} \frac{\partial f_i}{\partial x_i} \mathrm{d}x_1 \cdots \mathrm{d}x_n = \int_{\mathbb{R}^{n-1}} g\, \mathrm{d}y_1 \cdots \mathrm{d}y_{n-1}.$$

By Theorem 4.6, we may evaluate the multiple integrals in any order. For $2 \le i \le n$, $\int_{-\infty}^{\infty} \frac{\partial f_i}{\partial x_i}(x_1, \ldots, x_n)\mathrm{d}x_i = 0$ for each fixed $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$, since f_i has compact support. Hence the sum reduces to the first term, which is

$$\begin{aligned} \int_{\mathbb{R}^{n-1}} \left(\int_{-\infty}^0 \frac{\partial f_1}{\partial x_1}(x_1, \ldots, x_n)\mathrm{d}x_1 \right) \mathrm{d}x_2 \cdots \mathrm{d}x_n &= \int_{\mathbb{R}^{n-1}} f_1(0, x_2, \ldots, x_n)\mathrm{d}x_2 \cdots \mathrm{d}x_n \\ &= \int_{\mathbb{R}^{n-1}} g \mathrm{d}y_1 \cdots \mathrm{d}y_{n-1}. \end{aligned}$$

□

Corollary 4.27 (Boundaryless case of Stokes' theorem)**.** *Let M be an oriented n-manifold and $\beta \in \Omega_c^{n-1}(M)$. Then*

$$\int_M \mathrm{d}\beta = 0.$$

Remarks 4.28. Stokes' theorem provides further justification for the importance of the exterior derivative. Notice that its proof reduces to a fairly straightforward application of the fundamental theorem of calculus. The hard work is really in setting up the definition of the integral in a diffeomorphism-invariant way, and the diffeomorphism invariance of the exterior derivative plays a crucial role.

If M is compact, then $\Omega_c^n(M) = \Omega^n(M)$, so (if M is oriented) $\int_M \alpha$ is defined for any $\alpha \in \Omega^n(M)$. Compact submanifolds (without boundary) are also called *closed manifolds* (although, as subsets of $\mathbb{R}^s$, they are not only closed, but also bounded). If M is not compact and $\beta \in \Omega^{n-1}(M)$, then it could be that $\mathrm{d}\beta$ has compact support even if β does not. In that case $\int_M \mathrm{d}\beta$ could be non-zero, even if M has no boundary.

Appendix A. Existence of partitions of unity

Let $U_i : i \in \mathcal{I}$ be an open cover of M. For each U_i, there is by definition an open subset $\widetilde{U}_i \subseteq \mathbb{R}^s$ such that $U_i = M \cap \widetilde{U}_i$. Let $\tilde{M} = \bigcup_{i\in\mathcal{I}} \widetilde{U}_i$ (an open subset of $\mathbb{R}^s$). Then any partition of unity on $\tilde{M}$ subordinate to $\widetilde{U}_i : i \in \mathcal{I}$ induces a partition of unity on M subordinate to $U_i : i \in \mathcal{I}$, so without loss, we can assume that M is open in $\mathbb{R}^s$.

Step 1: cover by a countable set of balls. Let $\mathcal{V}$ be the set of subsets $V \subseteq M$ such that:

- there exist $r, x_1, \dots x_k \in \mathbb{Q}$ such that $V = B_r(x)$ where $x = (x_1, \dots x_k)$;
- the closure $\overline{V}$ in $\mathbb{R}^s$ is contained in U_j for some $j \in \mathcal{I}$.

$\mathcal{V}$ is a countable set, so we may enumerate its elements as $V_j : j \in \mathbb{Z}^+$.

Claim 1. *For any open subset W with $\overline{W} \subseteq M$ and any $p \in M \setminus \overline{W}$, there is some $V \in \mathcal{V}$ such that $\overline{V} \cap W = \varnothing$ and $p \in V$.*

Proof. Pick some $i \in \mathcal{I}$ such that $p \in U_i$. Then $(M \setminus \overline{W}) \cap U_i$ is an open subset of $\mathbb{R}^s$ containing p, so it contains some open ball $B_R(p)$. Choose $x \in B_{R/2}(p)$ with rational coordinates and $r \in \mathbb{Q}$ with $|x - p| < r < R/2$. Then $p \in B_r(x)$ and $\overline{B_r(x)} \subseteq B_R(p)$ so we may take $V = B_r(x) \in \mathcal{V}$. □

Set $W_0 = \varnothing$, $\mathcal{A}_0 = \mathcal{V}$ and, for $m \in \mathbb{Z}^+$,

$$W_m = V_1 \cup \cdots \cup V_m \quad \text{and} \quad \mathcal{A}_m = \{V \in \mathcal{V} : \overline{V} \cap W_m = \varnothing\}.$$

Then Claim 1, with $W = W_m$, shows that $\mathcal{A}_m$ covers $M \setminus \overline{W_m}$: indeed $\bigcup \mathcal{A}_m = M \setminus \overline{W_m}$.

Step 2: making the cover locally finite. We now define inductively for $m \in \mathbb{N}$, a finite subset $\mathcal{B}_m \subseteq \mathcal{V}$, such that $\mathcal{B}_0 = \varnothing$, and for $m \in \mathbb{Z}^+$, $\mathcal{B}_m$ covers $\overline{W}_m$, so that $\mathcal{A}_m \cup \mathcal{B}_m$ covers M. To do this, observe that $\overline{W_m} \subseteq M$ is a closed and bounded subset of $\mathbb{R}^s$, hence compact by Heine–Borel 4.3. Since (inductively) $\mathcal{A}_{m-1} \cup \mathcal{B}_{m-1}$ covers M, it has a finite subset $\mathcal{B}_m$ which covers $\overline{W_m}$.

We now set $\mathcal{B} = \bigcup_{m\in\mathbb{N}} \mathcal{B}_m$, which is an open cover of $\bigcup_{m\in\mathbb{N}} \overline{W_m} = M$. However, it is also "locally finite": any $p \in M$ belongs to W_m for some $m \in \mathbb{N}$ and so if $V \in \mathcal{B}$ with $\overline{V} \cap W_m \neq \varnothing$, then $V \notin \mathcal{A}_m$, and so $V \in \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_{m-1}$, which is finite.

Step 3: defining the partition of unity. For each $V = B_r(x) \in \mathcal{B}$, choose $j(V) \in \mathcal{I}$ with $\overline{V} \subseteq U_{j(V)}$, and define $\rho_V : M \to \mathbb{R}$ by

$$\rho_V(y) = \exp\left(-\frac{1}{r^2 - |x - y|^2}\right)$$

for $y \in B_r(x)$, and $\rho_V(y) = 0$ for $y \notin B_r(x)$. Then ρ_V is smooth (exercise) with $\operatorname{supp}(\rho_V) = \overline{V}$ and ρ_V positive on V. Since $\mathcal{B}$ is locally finite, each $p \in M$ has an open neighbourhood $W \subseteq M$ with $\overline{V} \cap W \neq \emptyset$ for only finitely many $V \in \mathcal{B}$. Hence the functions

$$\sigma(x) = \sum_{V\in\mathcal{B}} \rho_V(x) \quad \text{and} \quad \sigma_i(x) = \sum_{V\in\mathcal{B}:j(V)=i} \rho_V(x)$$

(for $i \in \mathcal{I}$) are well-defined and smooth because only finitely many ρ_V are nonzero on an open neighbourhood of any point. Furthermore σ is nonvanishing, $\operatorname{supp}(\sigma_i) \subseteq U_i$, and any point has an open neighbourhood which meets $\operatorname{supp}(\sigma_i)$ for only finitely many $i \in \mathcal{I}$. Hence $\rho_i(x) := \sigma_i(x)/\sigma(x)$ defines a partition of unity subordinate to $U_i : i \in \mathcal{I}$.

Appendix B. Proof of the change of variables formula

For a diffeomorphism $\varphi : V \to U$ of open subset of $\mathbb{R}^n$, let $C(\varphi)$ be the statement:

$$\int_U f(y)\,\mathrm{d}y_1 \cdots \mathrm{d}y_n = \int_V f(\varphi(x))|J_\varphi(x)|\,\mathrm{d}x_1 \cdots \mathrm{d}x_n \tag{B.1}$$

for all $f \in C_c^0(V)$. We wish to prove that $C(\varphi)$ holds for any φ. Main steps:

(1) $C(\varphi)$ and $C(\psi) \quad \Rightarrow \quad C(\varphi \circ \psi)$
(2) φ a permutation of coordinates $\quad \Rightarrow \quad C(\varphi)$
(3) $n = 1 \quad \Rightarrow \quad C(\varphi)$
(4) φ of the form $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1}, h(x_1, \ldots, x_n)) \quad \Rightarrow \quad C(\varphi)$.
(5) for any φ, any $x \in V$ has an open neighbourhood $V' \subseteq V$ such that $\varphi|_{V'}$ is a composite of maps of the form (2) and (4).
(6) using a partition of unity $C(\varphi)$ holds for any φ.

Step 1. Let $U, V, W \subseteq \mathbb{R}^n$ be open subsets, and let $\varphi : V \to U$ and $\psi : W \to V$ be diffeomorphisms. Then

$$J_{\varphi\circ\psi}(x) = \det(D(\varphi\circ\psi)_x) = \det(D\varphi_{\psi(x)}\circ D\psi_x) = \det(D\varphi_{\psi(x)})\det(D\psi_x) = J_\varphi(\psi(x))J_\psi(x).$$

Now suppose $C(\psi)$, *i.e.*,

$$\int_W g(\psi(x))|J_\psi(x)|\,\mathrm{d}x_1 \cdots \mathrm{d}x_n = \int_V g(y)\,\mathrm{d}y_1 \cdots \mathrm{d}y_n$$

for any $g \in C_c^0(V)$. Now if $f \in C_c^0(U)$, then assuming $C(\varphi)$ and applying $C(\psi)$ with $g = (f \circ \varphi)|J_\varphi| \in C_c^0(V)$, we obtain that

$$\begin{aligned}\int_U f(z)\,\mathrm{d}z_1 \cdots \mathrm{d}z_n &= \int_V f(\varphi(y))|J_\varphi(y)|\,\mathrm{d}y_1 \cdots \mathrm{d}y_n \\ &= \int_W f(\varphi(\psi(x)))|J_\varphi(\psi(x))||J_\psi(x)|\,\mathrm{d}x_1 \cdots \mathrm{d}x_n \\ &= \int_W f((\varphi\circ\psi)(x))|J_{\varphi\circ\psi}(x)|\,\mathrm{d}x_1 \cdots \mathrm{d}x_n,\end{aligned}$$

which gives $C(\varphi \circ \psi)$.

Step 2. We want to show $C(\varphi)$ when $\exists \sigma \in S_n$ such that $\varphi(x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. Since this map is the restriction of a diffeomorphism $s_\sigma : \mathbb{R}^n \to \mathbb{R}^n$, we can assume without loss that $U = V = \mathbb{R}^n$. Since $|J_{s_\sigma}(x)| = 1$ for all x, to establish $C(s_\sigma)$, we need to show that we can change the order of the multiple integrals.

Let $P \subseteq C_c^0(\mathbb{R}^n)$ be the set of functions $f : \mathbb{R}^n \to \mathbb{R}$ such that $f(x_1, \ldots, x_n) = f_1(x_1) \cdots f_n(x_n)$ for some $f_1, \ldots, f_n \in C_c^0(\mathbb{R})$. Observe that for any such f,

$$\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x_1 \cdots \mathrm{d}x_n = \int_{-\infty}^{\infty} f_1(t)\,\mathrm{d}t \cdots \int_{-\infty}^{\infty} f_n(t)\,\mathrm{d}t = \int_{\mathbb{R}^n} (f \circ s_\sigma)\,\mathrm{d}x_1 \cdots \mathrm{d}x_n,$$

since each one-variable integral above is a real number, and multiplication is commutative. Hence (B.1) holds for all $f \in P$. By linearity of integration, it follows that (B.1) holds for all $f \in \mathrm{span}(P)$, the linear span of P.

To establish $C(s_\sigma)$, *i.e.*, that (B.1) holds for all $f \in C_c^0(\mathbb{R}^n)$, we suppose $\varepsilon > 0$ and apply the following special case of the Stone–Weierstrass theorem.

Theorem. The span of P is uniformly dense in $C_c^0(\mathbb{R}^n)$, *i.e.*, for any $f \in C_c^0(\mathbb{R}^n)$ and $\varepsilon > 0$, there exists $g \in \operatorname{span}(P)$ such that $\forall\, x \in \mathbb{R}^n$ we have $|f(x) - g(x)| < \varepsilon$.

Evidently this also implies that $\forall\, x \in \mathbb{R}^n$, $|f(s_\sigma(x)) - g(s_\sigma(x))| < \varepsilon$. Since $g \in \operatorname{span}(\mathrm{P})$ we now have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x_1 \cdots \mathrm{d}x_n - \int_{\mathbb{R}^n} f(s_\sigma(x))\,\mathrm{d}x_1 \cdots \mathrm{d}x_n\right| \\
&\quad = \left|\int_{\mathbb{R}^n} (f(x) - g(x))\,\mathrm{d}x_1 \cdots \mathrm{d}x_n - \int_{\mathbb{R}^n} (f(s_\sigma(x)) - g(s_\sigma(x)))\,\mathrm{d}x_1 \cdots \mathrm{d}x_n\right| \\
&\quad \le \left|\int_{\mathbb{R}^n} (f(x) - g(x))\,\mathrm{d}x_1 \cdots \mathrm{d}x_n\right| + \left|\int_{\mathbb{R}^n} (f(s_\sigma(x)) - g(s_\sigma(x)))\,\mathrm{d}x_1 \cdots \mathrm{d}x_n\right| \le 2\varepsilon V,
\end{aligned}$$

where V is the volume of a ball containing the supports of $f, g, f \circ s_\sigma$ and $g \circ s_\sigma$ (which exists as these functions all have compact support). Since $\varepsilon > 0$ is arbitrary, the left-hand side is zero. In other words $C(s_\sigma)$ holds.

Step 3. Suppose $n = 1$, so $\varphi : V \to U$ for $V, U \subseteq \mathbb{R}$ disjoint unions of open intervals, and $J_\varphi(x) = \frac{\mathrm{d}\varphi}{\mathrm{d}x}$. Now given $f \in C_c^0(U)$, then there exists a finite union of bounded open intervals $V' \subseteq V$ such that $\operatorname{supp}(f \circ \varphi) \subseteq V'$. Therefore without loss of generality, V is a single bounded interval $(a, b) \subseteq \mathbb{R}$ and $U = \varphi((a, b))$. Then by the change of variables formula for functions of one variable

$$\int_U f(y)\,\mathrm{d}y = \pm \int_{\varphi(a)}^{\varphi(b)} f(y)\,\mathrm{d}y = \pm \int_a^b f(\varphi(x)) \frac{\mathrm{d}\varphi}{\mathrm{d}x}\,\mathrm{d}x,$$

where the sign is positive if $\varphi(a) < \varphi(b)$ and negative if $\varphi(b) < \varphi(a)$. However, this sign is also the sign of $\frac{\mathrm{d}\varphi}{\mathrm{d}x}$ at all $x \in (a, b)$, so the right hand side is

$$\int_V f(\varphi(x)) \left|\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right| \mathrm{d}x$$

as required.

Step 4. Suppose $\varphi : V \to U$ has the form $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1}, h(x_1, \ldots, x_n))$ for some function h. Fix $(x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$ and set

$$U' = \{s \in \mathbb{R} \,|\, (x_1, \ldots, x_{n-1}, s) \in U\}, \quad \text{and} \quad V' = \{t \in \mathbb{R} \,|\, (x_1, \ldots, x_{n-1}, t) \in V\}$$

and $\varphi' : V' \to U'$, $t \mapsto h(x_1, \ldots, x_{n-1}, t)$. Then

$$J_{\varphi'}(t) = \frac{\mathrm{d}\varphi'}{\mathrm{d}t} = \frac{\partial h}{\partial t} = J_\varphi(x_1, \ldots, x_{n-1}, t)$$

and so for any $f \in C_c^0(U)$, $C(\varphi')$ (Step 3) implies

$$\tilde{f}(x_1, \ldots, x_{n-1}) := \int_{U'} f(x_1, \ldots, x_{n-1}, s)\,\mathrm{d}s = \int_{V'} g(x_1, \ldots, x_{n-1}, t)\,\mathrm{d}t$$

where $g = (f \circ \varphi)|J_\varphi| \in C_c^0(V)$. However, extending both integrands by zero to $\mathbb{R}$, the multiple integral of f over $\mathbb{R}^n$ is the multiple integral of $\tilde{f}$ over $\mathbb{R}^{n-1}$, which therefore equals the multiple integral of g over $\mathbb{R}^n$, proving $C(\varphi)$.

Step 5. A diffeomorphism $\varphi : V \to U$ is called a *k-graph* if it is of the form

$$\varphi(x) = \varphi(x_1, \dots, x_n) = (x_1, \dots, x_{n-k}, \varphi_{n-k+1}(x), \dots, \varphi_n(x)).$$

So

- φ is a 0-graph $\Leftrightarrow \varphi = \mathrm{Id}$;
- φ is a 1-graph $\Leftrightarrow \varphi$ is a diffeomorphism of the form (4);
- any φ is an n-graph.

We are interested in the $k = n$ case of the following claim.

Lemma B.1. *For any $k \leq n$, if φ is a k-graph, then any $x \in V$ has a neighbourhood $V' \subseteq V$ such that $\varphi|_{V'}$ is a composite of permutation maps and 1-graphs.*

Proof. The cases $k = 0$ and $k = 1$ are trivial. Now suppose the claim holds for $k - 1$. Let φ be a k-graph and $x \in V$. Then $\det(D\varphi_x) \neq 0$ since φ is a diffeomorphism, so there is an integer $i \in [n-k+1, n]$ such that $\frac{\mathrm{d}\varphi_n}{\mathrm{d}x_i} \neq 0$ at x. Let σ be the transposition $(n\ i) \in S_n$ and $V' = s_\sigma^{-1}(V)$. Then $\varphi' = \varphi \circ s_\sigma \colon V' \to U$ is a k-graph and has $\frac{\partial \varphi'_n}{\partial x_n} \neq 0$ at $y = s_\sigma^{-1}(x)$.

Now define $g : V' \to \mathbb{R}^n$ by $(y_1, \dots, y_n) \mapsto (y_1, \dots, y_{n-1}, \varphi'_n(y_1, \dots, y_n))$. Then g is a 1-graph, and $\det(Dg_y) = \frac{\partial \varphi'_n}{\partial x_n}(y) \neq 0$. So by the inverse function theorem, y has a neighbourhood $W \subseteq V'$ such that $g|_W$ defines a diffeomorphism $W \to g(W)$, with $g(W)$ open in $\mathbb{R}^n$. Let $\psi = \varphi' \circ g^{-1} : g(W) \to U$. Then ψ has the form

$$\psi(z_1, \dots, z_n) = (z_1, \dots, z_{n-k}, \psi_{n-k+1}(z_1, \dots, z_n), \dots, \psi_{n-1}(z_1, \dots, z_n), z_n)$$

where $z_1, \dots, z_{n-k}$ are fixed since φ and g both fix the first $n - k$ coordinates. If $\tau \in S_n$ is the transposition $(n\ k+1)$, then $\psi' = s_\tau \circ \psi \colon g(W) \to s_\tau(U)$ is $(k-1)$-fixed. So by the inductive hypothesis, $g(y)$ has a neighbourhood $W' \subseteq g(W)$ such that $\psi'|_{W'}$ is a composite of permutation maps and 1-graphs. Hence so is the restriction of $\varphi = \psi \circ g \circ s_\sigma = s_\tau \circ \psi' \circ g \circ s_\sigma$ to $s_\sigma^{-1}(g^{-1}(W'))$. □

Step 6. Let $\varphi : V \to U$ be any diffeomorphism. The preceding steps show that any $x \in V$ has a neighbourhood $V' \subseteq V$ such that $C(\varphi|_{V'})$. Equivalently, for any $f \in C_c^0(U)$ such that $\mathrm{supp}(f) \subseteq \mathrm{im}(\varphi(V'))$ we have

$$\int_V (f \circ \varphi)(x) |J_\varphi(x)| \, \mathrm{d}x_1 \cdots \mathrm{d}x_n = \int_U f(y) \, \mathrm{d}y_1 \cdots \mathrm{d}y_n$$

Let $U_i : i \in \mathcal{I}$ be the family of images of such V'. Then U_i is an open cover of M. Let $\rho_i : i \in \mathcal{I}$ be a partition of unity subordinate this cover. Then any $f \in C_c^0(U)$ can be written as $f = \sum_i \rho_i f$. Since $\mathrm{supp}(\rho_i f) \subseteq U_i$, then

$$\int_U f(y) \, \mathrm{d}y_1 \cdots \mathrm{d}y_n = \sum_i \int_U (\rho_i f)(y) \, \mathrm{d}y_1 \cdots \mathrm{d}y_n = \sum_i \int_V ((\rho_i f) \circ \varphi)(x) |J_\varphi(x)| \, \mathrm{d}x_1 \cdots \mathrm{d}x_n$$
$$= \int_V (f \circ \varphi)(x) |J_\varphi(x)| \, \mathrm{d}x_1 \cdots \mathrm{d}x_n.$$

This concludes the proof of the change of variables formula.