MA40254 Differential and geometric analysis : Exercises 10

Hand in answers by 18:10pm on Tuesday 12 December for the Seminar of Wednesday 13 December
Homepage: http://moodle.bath.ac.uk/course/view.php?id=57709

0 (Warmup)**.** Let M be an open subset of H^n with the standard orientation. Show that integration of differential n-forms on M (as in Section 4.3) defines an integration map $\int_M$ on M (as in Section 4.5).

[**Solution**: *Multiple integration is linear by basic results of analysis, hence integration of forms defines a linear map* $\Omega^n_c(M) \to \mathbb{R}$. *Now if* $\varphi \colon \widetilde{U} \to U \subseteq M$ *is an oriented parametrization (with* U *open in* M *and* $\widetilde{U}$ *open in* H^n*), and* $\operatorname{supp}(\alpha) \subseteq U$*, then* $\int_M \alpha = \int_U \alpha$ *because of the way integration of forms is defined, and* $\int_U \alpha = \int_{\widetilde{U}} \varphi^*\alpha$ *by the change of variables formula for integration of forms. Thus both properties of integration maps are satisfied.*]

1. Let $M \subseteq \mathbb{R}^s$ be an oriented compact n-dimensional SMWB. Let $\omega \in \Omega^n(M)$ be any orientation form on M compatible with the chosen orientation. Show that $\int_M \omega > 0$.

[**Hint**: *Use a partition of unity to write* ω *as the sum of forms, each of which has support contained in the image of a parametrisation, and show that the integral of each term has positive integral.*]

2. Let $M \subseteq \mathbb{R}^s$ and $N \subseteq \mathbb{R}^\ell$ be oriented n-dimensional SMWBs, and let $\varphi : M \to N$ be an orientation-preserving diffeomorphism. Show that for any $\alpha \in \Omega^n_c(N)$ we have

$$\int_M \varphi^*\alpha = \int_N \alpha.$$

[**Hint**: *Show that* $\Omega^n_c(N) \to \mathbb{R}$, $\alpha \mapsto \int_M \varphi^*\alpha$ *is an integration map.*]

3. Let $\omega \in \Omega^2(S^2)$ be the orientation form defined by $\omega_p = p \lrcorner \operatorname{Det}$ for $p \in S^2$. Consider the oriented manifold S^2 with orientation defined by ω. Define parametrisations of S^2 by

$$\varphi : \mathbb{R}^2 \to S^2 \setminus \{(0,0,1)\},\ x \mapsto \frac{1}{1+\|x\|^2}\left(2x_1, 2x_2, \|x\|^2 - 1\right)$$

and

$$\psi : \mathbb{R}^2 \to S^2 \setminus \{(0,0,-1)\},\ y \mapsto \frac{1}{1+\|y\|^2}\left(2y_1, 2y_2, -\|y\|^2 + 1\right).$$

(i) Express $\varphi^*\omega \in \Omega^2(\mathbb{R}^2)$ in terms of $dx_1 \wedge dx_2$, and $\psi^*\omega \in \Omega^2(\mathbb{R}^2)$ in terms of $dy_1 \wedge dy_2$.

[**Hint**: *The usual process would be to write* $\omega = z_1 dz_2 \wedge dz_3 - z_2 dz_1 \wedge dz_3 + z_3 dz_1 \wedge dz_2$ *and then work out* $\varphi^*\omega$ *as* $\varphi^*(z_1) d(\varphi^* z_2) \wedge d(\varphi^* z_3) + \cdots$. *However, in this case it may be more convenient to organise the calculation as follows: if* $\varphi^*\omega = f dx_1 \wedge dx_2$, *then* $f(x) = \operatorname{Det}\left(\varphi(x), \frac{\partial\varphi}{\partial x_1}, \frac{\partial\varphi}{\partial x_2}\right)$.]

(ii) Is φ an oriented parametrisation? Is ψ? Is $\varphi^{-1} \circ \psi|_{\mathbb{R}^2 \setminus \{0\}}$ orientation-preserving?

[**Hint**: *You only need to examine the signs of the coefficients of* $dx_1 \wedge dx_2$ *and* $dy_1 \wedge dy_2$ *in the results from (i).*]

(iii) Evaluate $\int_{S^2} \omega$.

[**Hint**: *It is enough to evaluate* $\int_{\mathbb{R}^2} \psi^*\omega$.]

4. Let M be any orientable n-dimensional closed manifold (i.e., M is a compact SMWB with $\partial M = \varnothing$). Show that M admits a differential n-form ω which is closed but not exact.

[**Hint**: *Use Q1 and Stokes' Theorem.*]

5. Sketch a proof of the Poincaré Lemma. [Other examples of sketch proof questions can be found in past papers.]

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1. Let $\varphi_i : i \in I$ be a family of oriented parametrisations $\varphi_i : \widetilde{U}_i \to U_i$ of M, such that the set of images covers M. (Since M is compact, we can in fact take I to be finite.) Let $\rho_i : i \in I$ be a partition of unity on M subordinate to this cover. Then

$$\int_M \omega = \sum_i \int_M \rho_i \omega = \sum_i \int_{\widetilde{U}_i} \varphi^*(\rho_i \omega).$$

To evaluate each term on the RHS, we write

$$\varphi_i^* \omega = f dx_1 \wedge \cdots \wedge dx_n$$

for a function $f : \widetilde{U}_i \to \mathbb{R}$. By definition of φ_i being oriented f takes positive values, so

$$\int_{\widetilde{U}_i} \varphi^*(\rho_i \omega) = \int_{\widetilde{U}_i} \rho_i(\varphi(x)) f(x) dx_1 \cdots dx_n \in \mathbb{R}$$

is non-negative, and positive unless $\rho_i \equiv 0$. Since the ρ_i are certainly not all identically zero, the sum is positive.

2. Consider the linear map

$$L : \Omega^n_c(N) \to \mathbb{R}, \ \alpha \mapsto \int_M \varphi^* \alpha.$$

If $\psi : \widetilde{U} \to U$ is an oriented parametrisation of N and $U' := \varphi^{-1}(U) \subseteq M$, then $\varphi^{-1} \circ \psi : \widetilde{U} \to U'$ is an oriented parametrisation of M. If $\alpha \in \Omega^n_c(N)$ has $\operatorname{supp} \alpha \subseteq U$, then $\operatorname{supp} \varphi^* \alpha \subseteq U'$, so the characterising property of $\int_M$ gives

$$L(\alpha) = \int_{\widetilde{U}} (\varphi^{-1} \circ \psi)^*(\varphi^* \alpha) = \int_{\widetilde{U}} (\varphi \circ \varphi^{-1} \circ \psi)^* \alpha = \int_{\widetilde{U}} \psi^* \alpha.$$

Thus L is an integration map on N, and hence $L = \int_N$.

3. (i) To identify the function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $\varphi^* \omega = f dx_1 \wedge dx_2$, we can compute $f(x)$ as

$$\omega_{\varphi(x)} \left(\frac{\partial \varphi}{\partial x_1}, \frac{\partial \varphi}{\partial x_2} \right) = \operatorname{Det} \left(\varphi(x), \frac{\partial \varphi}{\partial x_1}, \frac{\partial \varphi}{\partial x_2} \right).$$

Now

$$\frac{\partial \varphi}{\partial x_1} = \frac{\partial \left(\frac{1}{1+\|x\|^2} \right)}{\partial x_1} (1 + \|x\|^2) \varphi(x) + \frac{2}{1 + \|x\|^2} (1, 0, x_1),$$

and $\frac{\partial \varphi}{\partial x_2}$ has an analogous expression. Because Det is alternating, the $\varphi(x)$ terms in $\frac{\partial \varphi}{\partial x_i}$ do not contribute to

$$\begin{aligned} \operatorname{Det} \left(\varphi(x), \frac{\partial \varphi}{\partial x_1}, \frac{\partial \varphi}{\partial x_1} \right) &= \frac{4}{(1 + \|x\|^2)^3} \det \begin{pmatrix} 2x_1 & 1 & 0 \\ 2x_2 & 0 & 1 \\ \|x\|^2 - 1 & x_1 & x_2 \end{pmatrix} \\ &= \frac{4}{(1 + \|x\|^2)^3} (\|x\|^2 - 1 - 2x_1^2 - 2x_2^2) = -\frac{4}{(1 + \|x\|^2)^2} \end{aligned}$$

Thus

$$\varphi^* \omega = -\frac{4}{(1 + \|x\|^2)^2} dx_1 \wedge dx_2.$$

A similar calculation shows

$$\psi^* \omega = \frac{4}{(1 + \|y\|^2)^2} dy_1 \wedge dy_2.$$

(ii) φ is not oriented, but ψ is, and therefore $\varphi^{-1} \circ \psi$ is orientation-reversing.

(iii) Since ψ is an oriented parametrisation and the complement of its image in S^2 is a lower-dimensional set,

$$\int_{S^2} \omega = \int_{\mathbb{R}^2} \psi^* \omega = \int_{\mathbb{R}^2} \frac{4dy_1 dy_2}{(1 + \|y\|^2)^2} = \int_0^{2\pi} \int_0^{\infty} \frac{4r dr d\theta}{(1+r^2)^2} = 4\pi,$$

since the integral over θ gives 2π and the integral over r gives 2 (e.g. the integrand is the derivative of $-2/(1+r^2)$).

4. Let ω be an orientation form. Then $d\omega = 0$ and (using the orientation defined by ω) $\int_M \omega > 0$ by previous exercises. We cannot have $\omega = d\beta$ since $\int_M d\beta = 0$ by Stokes' Theorem.

5. We do not give model sketch proofs. A good sketch, along the lines of the proof in the notes, would include the following main relevant points:

- the decomposition of $\mathrm{Alt}^k(\mathbb{R}^n)$, and hence $\alpha \in \Omega^k(U)$, with respect to one variable;
- the existence of γ with $\alpha' = \alpha - d\gamma$ independent of one variable (e.g. using the second fundamental theorem of calculus to define γ, and the formula for $\mathcal{L}(d\alpha)$ when $\mathcal{L}(\alpha) = 0$);
- an argument (e.g. pullback and induction) that α' is exact.