MA40254 Differential and geometric analysis : Exercises 10

Hand in answers by 18:10pm on Tuesday 12 December for the Seminar of Wednesday 13 December
Homepage: http://moodle.bath.ac.uk/course/view.php?id=57709

0 (Warmup). Let M be an open subset of H^n with the standard orientation. Show that integration of differential n-forms on M (as in Section 4.3) defines an integration map $\int_M$ on M (as in Section 4.5).

[**Solution**: *Multiple integration is linear by basic results of analysis, hence integration of forms defines a linear map* $\Omega^n_c(M) \to \mathbb{R}$. *Now if* $\varphi\colon \widetilde{U} \to U \subseteq M$ *is an oriented parametrization (with* U *open in* M *and* $\widetilde{U}$ *open in* H^n*), and* $\operatorname{supp}(\alpha) \subseteq U$*, then* $\int_M \alpha = \int_U \alpha$ *because of the way integration of forms is defined, and* $\int_U \alpha = \int_{\widetilde{U}} \varphi^*\alpha$ *by the change of variables formula for integration of forms. Thus both properties of integration maps are satisfied.*]

1. Let $M \subseteq \mathbb{R}^s$ be an oriented compact n-dimensional SMWB. Let $\omega \in \Omega^n(M)$ be any orientation form on M compatible with the chosen orientation. Show that $\int_M \omega > 0$.

[**Hint**: *Use a partition of unity to write* ω *as the sum of forms, each of which has support contained in the image of a parametrisation, and show that the integral of each term has positive integral.*]

2. Let $M \subseteq \mathbb{R}^s$ and $N \subseteq \mathbb{R}^\ell$ be oriented n-dimensional SMWBs, and let $\varphi : M \to N$ be an orientation-preserving diffeomorphism. Show that for any $\alpha \in \Omega^n_c(N)$ we have

$$\int_M \varphi^*\alpha = \int_N \alpha.$$

[**Hint**: *Show that* $\Omega^n_c(N) \to \mathbb{R},\ \alpha \mapsto \int_M \varphi^*\alpha$ *is an integration map.*]

3. Let $\omega \in \Omega^2(S^2)$ be the orientation form defined by $\omega_p = p \lrcorner \operatorname{Det}$ for $p \in S^2$. Consider the oriented manifold S^2 with orientation defined by ω. Define parametrisations of S^2 by

$$\varphi : \mathbb{R}^2 \to S^2 \setminus \{(0,0,1)\},\ x \mapsto \frac{1}{1+\|x\|^2}\left(2x_1, 2x_2, \|x\|^2 - 1\right)$$

and

$$\psi : \mathbb{R}^2 \to S^2 \setminus \{(0,0,-1)\},\ y \mapsto \frac{1}{1+\|y\|^2}\left(2y_1, 2y_2, -\|y\|^2 + 1\right).$$

(i) Express $\varphi^*\omega \in \Omega^2(\mathbb{R}^2)$ in terms of $dx_1 \wedge dx_2$, and $\psi^*\omega \in \Omega^2(\mathbb{R}^2)$ in terms of $dy_1 \wedge dy_2$.

[**Hint**: *The usual process would be to write* $\omega = z_1 dz_2 \wedge dz_3 - z_2 dz_1 \wedge dz_3 + z_3 dz_1 \wedge dz_2$ *and then work out* $\varphi^*\omega$ *as* $\varphi^*(z_1) d(\varphi^* z_2) \wedge d(\varphi^* z_3) + \cdots$. *However, in this case it may be more convenient to organise the calculation as follows: if* $\varphi^*\omega = f dx_1 \wedge dx_2$*, then* $f(x) = \operatorname{Det}\left(\varphi(x), \frac{\partial\varphi}{\partial x_1}, \frac{\partial\varphi}{\partial x_2}\right)$.]

(ii) Is φ an oriented parametrisation? Is ψ? Is $\varphi^{-1} \circ \psi|_{\mathbb{R}^2\setminus\{0\}}$ orientation-preserving?

[**Hint**: *You only need to examine the signs of the coefficients of* $dx_1 \wedge dx_2$ *and* $dy_1 \wedge dy_2$ *in the results from (i).*]

(iii) Evaluate $\int_{S^2} \omega$.

[**Hint**: *It is enough to evaluate* $\int_{\mathbb{R}^2} \psi^*\omega$.]

4. Let M be any orientable n-dimensional closed manifold (i.e., M is a compact SMWB with $\partial M = \varnothing$). Show that M admits a differential n-form ω which is closed but not exact.

[**Hint**: *Use Q1 and Stokes' Theorem.*]

5. Sketch a proof of the Poincaré Lemma. [Other examples of sketch proof questions can be found in past papers.]

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