MA40254 Differential and geometric analysis : Exercises 10

*Hand in answers by 18:10pm on Tuesday 12 December for the Seminar of Wednesday 13 December*
*Homepage: http://moodle.bath.ac.uk/course/view.php?id=57709*

**0** (Warmup)**.** Let $M$ be an open subset of $H^n$ with the standard orientation. Show that integration of differential $n$-forms on $M$ (as in Section 4.3) defines an integration map $\int_M$ on $M$ (as in Section 4.5).

**1.** Let $M \subseteq \mathbb{R}^s$ be an oriented compact $n$-dimensional SMWB. Let $\omega \in \Omega^n(M)$ be any orientation form on $M$ compatible with the chosen orientation. Show that $\int_M \omega > 0$.

**2.** Let $M \subseteq \mathbb{R}^s$ and $N \subseteq \mathbb{R}^\ell$ be oriented $n$-dimensional SMWBs, and let $\varphi : M \to N$ be an orientation-preserving diffeomorphism. Show that for any $\alpha \in \Omega^n_c(N)$ we have

$$\int_M \varphi^*\alpha = \int_N \alpha.$$

**3.** Let $\omega \in \Omega^2(S^2)$ be the orientation form defined by $\omega_p = p \lrcorner \mathrm{Det}$ for $p \in S^2$. Consider the oriented manifold $S^2$ with orientation defined by $\omega$. Define parametrisations of $S^2$ by

$$\varphi : \mathbb{R}^2 \to S^2 \setminus \{(0,0,1)\},\ x \mapsto \frac{1}{1+\|x\|^2}\Big(2x_1, 2x_2, \|x\|^2 - 1\Big)$$

and

$$\psi : \mathbb{R}^2 \to S^2 \setminus \{(0,0,-1)\},\ y \mapsto \frac{1}{1+\|y\|^2}\Big(2y_1, 2y_2, -\|y\|^2 + 1\Big).$$

(i) Express $\varphi^*\omega \in \Omega^2(\mathbb{R}^2)$ in terms of $dx_1 \wedge dx_2$, and $\psi^*\omega \in \Omega^2(\mathbb{R}^2)$ in terms of $dy_1 \wedge dy_2$.

(ii) Is $\varphi$ an oriented parametrisation? Is $\psi$? Is $\varphi^{-1} \circ \psi|_{\mathbb{R}^2\setminus\{0\}}$ orientation-preserving?

(iii) Evaluate $\int_{S^2} \omega$.

**4.** Let $M$ be any orientable $n$-dimensional closed manifold (i.e., $M$ is a compact SMWB with $\partial M = \varnothing$). Show that $M$ admits a differential $n$-form $\omega$ which is closed but not exact.

**5.** Sketch a proof of the Poincaré Lemma. [Other examples of sketch proof questions can be found in past papers.]

*DMJC 5 December*