MA40254 Differential and geometric analysis : Exercises 9

Hand in answers by 1:15pm on Wednesday 6 December for the Seminar of Thursday 7 December
Homepage: http://moodle.bath.ac.uk/course/view.php?id=57709

0 (Warmup)**.** Let $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2; p \mapsto (x_1(p), x_2(p)^2)$. Compute $\varphi^*(dx_1 \wedge dx_2)$ directly, and find the largest open subset of $\mathbb{R}^2$ on which φ is an orientation-preserving local diffeomorphism.

[**Solution**: $\varphi^*(dx_1 \wedge dx_2) = d(\varphi^* x_1) \wedge d(\varphi^* x_2) = dx_1 \wedge d(x_2^2) = 2x_2\, dx_1 \wedge dx_2$, *which is* $J_\varphi\, dx_1 \wedge dx_2$ *in accordance with the general result in lectures, since the matrix of* $D\varphi_p$ *is diagonal with entries* $1, 2x_2(p)$ *hence* $J_\varphi = 2x_2$. *Thus* φ *is a local diffeomorphism when* $x_2 \neq 0$ *and orientation preserving on* $\{p \in \mathbb{R}^2 : x_2(p) > 0\}$.]

1. Let $M \subseteq \mathbb{R}^s$ be an orientable submanifold, and let $U \subseteq M$ be an open subset. Show that U is also orientable.

[**Hint**: *If* $\omega \in \Omega^n(M)$ *is an orientation form, what can you say about its pullback to* U *?*]

2 (Less essential)**.** Let $M \subseteq \mathbb{R}^{n+1}$ be a submanifold of dimension n. Show that M is orientable if and only if there is a nowhere-vanishing normal vector field on M, i.e., a smooth function $\nu : M \to \mathbb{R}^{n+1}$ such that $\nu(p) \neq 0$ and $\nu(p)$ is orthogonal to T_pM for all $p \in M$.

[**Hint**: *Given such a* ν, *consider* $\omega \in \Omega^n(M)$ *defined by* $\omega_p = \nu(p) \lrcorner \operatorname{Det}$. *Show that* ν *is uniquely determined by* ω*—you may then assume that* ν *is smooth (this is not so easy to prove rigorously).*]

3. Let $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$, and let $a : S^n \to S^n$ be the *antipodal map*, i.e., the diffeomorphism $p \mapsto -p$. For which values of n is a orientation-preserving?

[**Hint**: *If* $\omega \in \Omega^n(S^n)$ *is an orientation form, then* $a^*\omega = f\omega$ *for some function* $f : S^n \to \mathbb{R} \setminus \{0\}$. *You need to decide whether* f *takes positive or negative values. Consider the orientation form* $\omega \in \Omega^n(S^n)$ *given by* $\omega_p = p \lrcorner \operatorname{Det}$.]

4. Let U and $\tilde{U}$ be open subsets of $\mathbb{R}^n$, and $\alpha \in \Omega^n_{cpt}(U)$. Let $\varphi : \tilde{U} \to U$ be an orientation-reversing diffeomorphism, i.e., $\det(D\varphi_p) < 0$ for all $x \in \tilde{U}$. Show that

$$\int_{\tilde{U}} \varphi^* \alpha = -\int_U \alpha.$$

[**Hint**: *Imitate the proof of that the integral is invariant under orientation-preserving diffeomorphisms.*]

5. Plan an essay on one of the following topics.

(i) The inverse function theorem and its use in submanifold theory.

(ii) Alternating multilinear forms and their properties.

(iii) Using pullback to define the exterior derivative on submanifolds.

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1. Let $\omega \in \Omega^n(M)$ be an orientation form, so that $i^*\omega \in \Omega^n(U)$, with $i\colon U \to M$ the inclusion. Then for any $p \in U$, $(i^*\omega)_p = \omega_p \in \mathrm{Alt}^n(T_pU)$ is non-zero, so $i^*\omega$ is an orientation form too.

2. For a fixed $p \in M$, consider $N_p \subseteq \mathbb{R}^{k+1}$ the vector space orthogonal to T_pM. Then N_p has dimension 1, and the map

$$N_p \to \mathrm{Alt}^n(T_pM),\ u \mapsto (u \lrcorner \mathrm{Det})|_{T_pM^n}$$

is an isomorphism. Thus for each $p \in M$ any $\omega_p \in \mathrm{Alt}^n(T_pM)$ can be written uniquely as $\omega_p = \nu(p) \lrcorner \mathrm{Det}$ for some $\nu(p) \in N_p$. Clearly if $\nu\colon M \to \mathbb{R}^{n+1}$ has smooth local extensions, so does ω (using the same formula) and in fact the converse is also true (proof omitted). Now $\omega_p \neq 0$ if and only $\nu(p) \neq 0$, so ω is an orientation form if and only if ν never vanishes.

3. Define $\omega \in \Omega^n(S^n)$ by $\omega_p = p \lrcorner \mathrm{Det}$. Then x is a normal vector field to S^n and so ω is an orientation form.

Now compare ω and $a^*\omega$. First note that $\omega_{-p} = (-p) \lrcorner \mathrm{Det} = -(p \lrcorner \mathrm{Det}) = -\omega_p$; that we can at all compare ω_p and ω_{-p} like this relies on $T_pS^n = T_{-p}S^n$. Meanwhile $Da_p = -\mathrm{Id}_{T_pS^n}$ for any $p \in S^n$. Therefore

$$(a^*\omega)_p = (Da_p)^*(\omega_{-p}) = (-\mathrm{Id})^*(-\omega_p) = -(\det(-\mathrm{Id}))\omega_p = (-1)^{n+1}\omega_p \in \mathrm{alt}^n(T_pS^n).$$

So $a^*\omega = (-1)^{n+1}\omega$, implying that a is orientation-preserving if and only if n is odd.

4. Write $\alpha = f dy_1 \wedge \cdots \wedge dy_n$. Since $|J(x)| = -J(x)$,

$$\begin{aligned}\int_{\tilde U} \varphi^*\alpha &= \int_{\tilde U} f(\varphi(x))J(x)dx_1 \cdots dx_n = -\int_{\tilde U} f(\varphi(x))|J(x)|\,dx_1 \cdots dx_n \\ &= -\int_U f(y)dy_1 \cdots dy_n = -\int_U \alpha.\end{aligned}$$

5. We do not give model essays, and there is some flexibility on what you cover (in particular, what you prove and in what level of detail, and what examples you give). Here are some comments and suggestions.

(i) You certainly need to state the inverse function theorem and the regular value theorem, which means defining what is a submanifold and what is a regular value. A sketch proof of the inverse function theorem would be too long, but you might explain "why" the inverse function theorem is true at a higher level, and sketch how the regular value theorem follows from it. You should also give an example of a submanifold defined as the inverse image of a regular value.

(ii) Define alternation and wedge, state what is a basis, prove something, and give an example computation (e.g., you could discuss decomposability).

(iii) To define the exterior derivative on submanifolds, you will need the result that exterior derivative commutes with pullback, and to define what is a differential form on a submanifold. A proof and an example should be easy to come by.