MA40254 Differential and geometric analysis : Exercises 9

Hand in answers by 1:15pm on Wednesday 6 December for the Seminar of Thursday 7 December
Homepage: http://moodle.bath.ac.uk/course/view.php?id=57709

0 (Warmup)**.** Let $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2; p \mapsto (x_1(p), x_2(p)^2)$. Compute $\varphi^*(dx_1 \wedge dx_2)$ directly, and find the largest open subset of $\mathbb{R}^2$ on which φ is an orientation-preserving local diffeomorphism.

[**Solution**: $\varphi^*(dx_1 \wedge dx_2) = d(\varphi^* x_1) \wedge d(\varphi^* x_2) = dx_1 \wedge d(x_2^2) = 2x_2\, dx_1 \wedge dx_2$, *which is* $J_\varphi\, dx_1 \wedge dx_2$ *in accordance with the general result in lectures, since the matrix of* $D\varphi_p$ *is diagonal with entries* $1, 2x_2(p)$ *hence* $J_\varphi = 2x_2$*. Thus* φ *is a local diffeomorphism when* $x_2 \neq 0$ *and orientation preserving on* $\{p \in \mathbb{R}^2 : x_2(p) > 0\}$.]

1. Let $M \subseteq \mathbb{R}^s$ be an orientable submanifold, and let $U \subseteq M$ be an open subset. Show that U is also orientable.

[**Hint**: *If* $\omega \in \Omega^n(M)$ *is an orientation form, what can you say about its pullback to* U *?*]

2 (Less essential)**.** Let $M \subseteq \mathbb{R}^{n+1}$ be a submanifold of dimension n. Show that M is orientable if and only if there is a nowhere-vanishing normal vector field on M, i.e., a smooth function $\nu : M \to \mathbb{R}^{n+1}$ such that $\nu(p) \neq 0$ and $\nu(p)$ is orthogonal to T_pM for all $p \in M$.

[**Hint**: *Given such a* ν*, consider* $\omega \in \Omega^n(M)$ *defined by* $\omega_p = \nu(p) \lrcorner \operatorname{Det}$*. Show that* ν *is uniquely determined by* ω*—you may then assume that* ν *is smooth (this is not so easy to prove rigorously).*]

3. Let $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$, and let $a : S^n \to S^n$ be the *antipodal map*, i.e., the diffeomorphism $p \mapsto -p$. For which values of n is a orientation-preserving?

[**Hint**: *If* $\omega \in \Omega^n(S^n)$ *is an orientation form, then* $a^*\omega = f\omega$ *for some function* $f : S^n \to \mathbb{R} \setminus \{0\}$*. You need to decide whether* f *takes positive or negative values. Consider the orientation form* $\omega \in \Omega^n(S^n)$ *given by* $\omega_p = p \lrcorner \operatorname{Det}$.]

4. Let U and $\tilde{U}$ be open subsets of $\mathbb{R}^n$, and $\alpha \in \Omega^n_{cpt}(U)$. Let $\varphi : \tilde{U} \to U$ be an orientation-reversing diffeomorphism, i.e., $\det(D\varphi_p) < 0$ for all $x \in \tilde{U}$. Show that

$$\int_{\tilde{U}} \varphi^* \alpha = -\int_U \alpha.$$

[**Hint**: *Imitate the proof of that the integral is invariant under orientation-preserving diffeomorphisms.*]

5. Plan an essay on one of the following topics.

(i) The inverse function theorem and its use in submanifold theory.

(ii) Alternating multilinear forms and their properties.

(iii) Using pullback to define the exterior derivative on submanifolds.

DMJC 28 November