MA40254 Differential and geometric analysis : Exercises 9

*Hand in answers by 1:15pm on Wednesday 6 December for the Seminar of Thursday 7 December*
*Homepage: http://moodle.bath.ac.uk/course/view.php?id=57709*

**0** (Warmup)**.** Let $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2; p \mapsto (x_1(p), x_2(p)^2)$. Compute $\varphi^*(dx_1 \wedge dx_2)$ directly, and find the largest open subset of $\mathbb{R}^2$ on which $\varphi$ is an orientation-preserving local diffeomorphism.

**1.** Let $M \subseteq \mathbb{R}^s$ be an orientable submanifold, and let $U \subseteq M$ be an open subset. Show that $U$ is also orientable.

**2** (Less essential)**.** Let $M \subseteq \mathbb{R}^{n+1}$ be a submanifold of dimension $n$. Show that $M$ is orientable if and only if there is a nowhere-vanishing normal vector field on $M$, i.e., a smooth function $\nu : M \to \mathbb{R}^{n+1}$ such that $\nu(p) \neq 0$ and $\nu(p)$ is orthogonal to $T_pM$ for all $p \in M$.

**3.** Let $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$, and let $a : S^n \to S^n$ be the *antipodal map*, i.e., the diffeomorphism $p \mapsto -p$. For which values of $n$ is $a$ orientation-preserving?

**4.** Let $U$ and $\tilde{U}$ be open subsets of $\mathbb{R}^n$, and $\alpha \in \Omega^n_{cpt}(U)$. Let $\varphi : \tilde{U} \to U$ be an orientation-reversing diffeomorphism, i.e., $\det(D\varphi_p) < 0$ for all $x \in \tilde{U}$. Show that

$$\int_{\tilde{U}} \varphi^*\alpha = -\int_U \alpha.$$

**5.** Plan an essay on one of the following topics.

(i) The inverse function theorem and its use in submanifold theory.

(ii) Alternating multilinear forms and their properties.

(iii) Using pullback to define the exterior derivative on submanifolds.

*DMJC 28 November*