MA40254 Differential and geometric analysis : Exercises 8

Hand in answers by 1:15pm on Wednesday 29 November for the Seminar of Thursday 30 November
Homepage: http://moodle.bath.ac.uk/course/view.php?id=57709

0 (Warmup)**.** Let M be an n-dimensional submanifold of $\mathbb{R}^s$. Prove (from the definition of d on submanifolds) that if $\alpha \in \Omega^k(M)$ is exact, i.e., $\alpha = d\beta$, then α is closed, i.e., $d\alpha = 0$.

[**Solution**: *Suppose* $\alpha = d\beta$. *We have seen in lectures that* $d^2 = 0$ *so* $d\alpha = d^2\beta = 0$, *but let us recall how to prove this. Note that it suffices to prove* $d\alpha = 0$ *locally on* M, *i.e., on sufficiently small open neighbourhoods* $U \cap M$ *of each* $p \in M$. *There are two ways to proceed.*

First by definition, we may assume that on $U \cap M$, $\beta = \iota^*\gamma$, *where* $\gamma \in \Omega^k(U)$ *and* $\iota\colon U \cap M \to U$ *is the inclusion. Thus* $d\alpha = d^2(\iota^*\gamma) = d(\iota^* d\gamma) = \iota^*(d^2\gamma) = 0$ *on* $U \cap M$. *Alternatively, we may assume that there is a parametrisation* $\varphi\colon \widetilde{U} \to U \cap M$. *Then* $\varphi^* d\alpha = \varphi^* d^2\beta = d\varphi^* d\beta = d^2\varphi^*\beta = 0$, *so* $d\alpha = 0$ *on* $U \cap M$ *because* $\varphi^*\colon \Omega^k(U \cap M) \to \Omega^k(\widetilde{U})$ *is a bijection with inverse* $(\varphi^{-1})^*$.]

1. Let $M \subseteq \mathbb{R}^s$ be an n-dimensional submanifold and $\alpha \in \Omega^k(M)$ for $k > 0$.

(i) Suppose that M is diffeomorphic to $\mathbb{R}^n$ and α is closed; prove that α is exact.

[**Hint**: *Let* $\varphi : \mathbb{R}^n \to M$ *be a diffeomorphism, and consider the pullback* $\varphi^*\alpha \in \Omega^k(\mathbb{R}^n)$.]

(ii) Suppose $k = n$; show that α is closed.

[**Hint**: *What is the dimension of* $\mathrm{Alt}^{n+1}(T_x M)$*?*]

(iii) If $k = n$, does α have to be exact?

[**Hint**: *Look for examples on* $M = S^1$. *A previous exercise may help.*]

2. Give an example of an n-dimensional submanifold $M \subseteq \mathbb{R}^s$ and an $\alpha \in \Omega^k(M)$ such that α is not the pullback by the inclusion of any $\beta \in \Omega^k(\mathbb{R}^s)$.

[**Hint**: *The simplest examples have* $n = s = 1$ *and* $k = 0$.]

3. Let $S^2 = \{z \in \mathbb{R}^3 : \|z\| = 1\}$, and $U = S^2 \setminus \{(0,0,1)\}$, and consider the parametrisation $\varphi\colon \mathbb{R}^2 \to U$ defined by

$$(x_1, x_2) \mapsto \frac{1}{1 + \|x\|^2}\Big(2x_1,\, 2x_2,\, \|x\|^2 - 1\Big).$$

Let $\alpha = i^* dz_3$ where $i\colon S^2 \to \mathbb{R}^3$ is the inclusion and z_1, z_2, z_3 denote the coordinate functions on $\mathbb{R}^3$. Compute $\varphi^*\alpha$.

[**Hint**: $\varphi^* i^* dz_3 = (i \circ \varphi)^* dz_3 = \cdots$.]

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1. (i) Suppose α is closed. Let $\varphi : \mathbb{R}^n \to M$ be a diffeomorphism. Then $\varphi^*\alpha \in \Omega^k(\mathbb{R}^n)$ is closed, so exact by the Poincaré lemma, i.e, there is some $\gamma \in \Omega^{k-1}(\mathbb{R}^n)$ such that $d\gamma = \varphi^*\alpha$. If we let $\beta = (\varphi^{-1})^*\gamma \in \Omega^{k-1}(M)$ then $d\beta = (\varphi^{-1})^* d\gamma = (\varphi^{-1})^*(\varphi^*\alpha) = (\varphi \circ \varphi^{-1})^*\alpha = \alpha$.

(ii) For each $p \in M$, $d\alpha_p$ is an element of $\mathrm{Alt}^{n+1}(T_pM)$. Since $n+1 > \dim T_pM$, we have $\mathrm{Alt}^{n+1}(T_pM) = \{0\}$, so $d\alpha$ is zero everywhere.

(iii) No. Consider $M = S^1 \subseteq \mathbb{R}^2$, and let $\alpha \in \Omega^1(S^1)$ be the pullback by $i \colon S^1 \to \mathbb{R}^2 \setminus \{0\}$ of the non-exact 1-form

$$\gamma = \frac{-x_2\, dx_1 + x_1\, dx_2}{x_1^2 + x_2^2} \in \Omega^1(\mathbb{R}^2 \setminus \{0\})$$

from Exercises 6 Q 4(iii). The same argument as there shows that α is not exact either.

2. We can for instance take $M = (0, \infty) \subset \mathbb{R}$, and $k = 0$. Then an element of $\Omega^k(M)$ is just a smooth function $(0, \infty) \to \mathbb{R}$ and pullback reduces to restriction. If we take it to be $x \mapsto \frac{1}{x}$ then that is not the restriction of any smooth function $\mathbb{R} \to \mathbb{R}$.

3. We have $\varphi^* i^* dz_3 = (i \circ \varphi)^* dz_3 = d((i \circ \varphi)^* z_3)$ and $(i \circ \varphi)^* z_3 = (x_1^2 + x_2^2 - 1)/(1 + x_1^2 + x_2^2)$. The exterior derivative of this is

$$\frac{4x_1 dx_1 + 4x_2 dx_2}{(1 + x_1^2 + x_2^2)^2}.$$