MA40254 Differential and geometric analysis : Exercises 8

Hand in answers by 1:15pm on Wednesday 29 November for the Seminar of Thursday 30 November
Homepage: http://moodle.bath.ac.uk/course/view.php?id=57709

0 (Warmup)**.** Let M be an n-dimensional submanifold of $\mathbb{R}^s$. Prove (from the definition of d on submanifolds) that if $\alpha \in \Omega^k(M)$ is exact, i.e., $\alpha = d\beta$, then α is closed, i.e., $d\alpha = 0$.

[**Solution**: *Suppose $\alpha = d\beta$. We have seen in lectures that $d^2 = 0$ so $d\alpha = d^2\beta = 0$, but let us recall how to prove this. Note that it suffices to prove $d\alpha = 0$ locally on M, i.e., on sufficiently small open neighbourhoods $U \cap M$ of each $p \in M$. There are two ways to proceed.*

First by definition, we may assume that on $U \cap M$, $\beta = \iota^*\gamma$, where $\gamma \in \Omega^k(U)$ and $\iota \colon U \cap M \to U$ is the inclusion. Thus $d\alpha = d^2(\iota^*\gamma) = d(\iota^* d\gamma) = \iota^*(d^2\gamma) = 0$ on $U \cap M$. Alternatively, we may assume that there is a parametrisation $\varphi \colon \widetilde{U} \to U \cap M$. Then $\varphi^* d\alpha = \varphi^* d^2\beta = d\varphi^* d\beta = d^2\varphi^*\beta = 0$, so $d\alpha = 0$ on $U \cap M$ because $\varphi^* \colon \Omega^k(U \cap M) \to \Omega^k(\widetilde{U})$ is a bijection with inverse $(\varphi^{-1})^*$.]

1. Let $M \subseteq \mathbb{R}^s$ be an n-dimensional submanifold and $\alpha \in \Omega^k(M)$ for $k > 0$.

(i) Suppose that M is diffeomorphic to $\mathbb{R}^n$ and α is closed; prove that α is exact.

[**Hint**: *Let $\varphi : \mathbb{R}^n \to M$ be a diffeomorphism, and consider the pullback $\varphi^*\alpha \in \Omega^k(\mathbb{R}^n)$.*]

(ii) Suppose $k = n$; show that α is closed.

[**Hint**: *What is the dimension of* $\mathrm{Alt}^{n+1}(T_xM)$*?*]

(iii) If $k = n$, does α have to be exact?

[**Hint**: *Look for examples on $M = S^1$. A previous exercise may help.*]

2. Give an example of an n-dimensional submanifold $M \subseteq \mathbb{R}^s$ and an $\alpha \in \Omega^k(M)$ such that α is not the pullback by the inclusion of any $\beta \in \Omega^k(\mathbb{R}^s)$.

[**Hint**: *The simplest examples have $n = s = 1$ and $k = 0$.*]

3. Let $S^2 = \{z \in \mathbb{R}^3 : \|z\| = 1\}$, and $U = S^2 \setminus \{(0,0,1)\}$, and consider the parametrisation $\varphi \colon \mathbb{R}^2 \to U$ defined by

$$(x_1, x_2) \mapsto \frac{1}{1 + \|x\|^2}\left(2x_1,\, 2x_2,\, \|x\|^2 - 1\right).$$

Let $\alpha = i^* dz_3$ where $i \colon S^2 \to \mathbb{R}^3$ is the inclusion and z_1, z_2, z_3 denote the coordinate functions on $\mathbb{R}^3$. Compute $\varphi^*\alpha$.

[**Hint**: $\varphi^* i^* dz_3 = (i \circ \varphi)^* dz_3 = \cdots$.]

DMJC 21 November