MA40254 Differential and geometric analysis : Exercises 8

Hand in answers by 1:15pm on Wednesday 30 November for the Seminar of Thursday 1 December
Homepage: http://moodle.bath.ac.uk/course/view.php?id=57709

0 (Warmup)**.** Let M be an n-dimensional submanifold of $\mathbb{R}^s$. Prove (from the definition of d on submanifolds) that if $\alpha \in \Omega^k(M)$ is exact, i.e., $\alpha = d\beta$, then α is closed, i.e., $d\alpha = 0$.

1. Let $M \subseteq \mathbb{R}^s$ be an n-dimensional submanifold and $\alpha \in \Omega^k(M)$ for $k > 0$.

(i) Suppose that M is diffeomorphic to $\mathbb{R}^n$ and α is closed; prove that α is exact.

(ii) Suppose $k = n$; show that α is closed.

(iii) If $k = n$, does α have to be exact?

2. Give an example of an n-dimensional submanifold $M \subseteq \mathbb{R}^s$ and an $\alpha \in \Omega^k(M)$ such that α is not the pullback by the inclusion of any $\beta \in \Omega^k(\mathbb{R}^s)$.

3. Let $S^2 = \{z \in \mathbb{R}^3 : \|z\| = 1\}$, and $U = S^2 \setminus \{(0,0,1)\}$, and consider the parametrisation $\varphi\colon \mathbb{R}^2 \to U$ defined by

$$(x_1, x_2) \mapsto \frac{1}{1 + \|x\|^2}\left(2x_1,\, 2x_2,\, \|x\|^2 - 1\right).$$

Let $\alpha = i^* dz_3$ where $i\colon S^2 \to \mathbb{R}^3$ is the inclusion and z_1, z_2, z_3 denote the coordinate functions on $\mathbb{R}^3$. Compute $\varphi^*\alpha$.

DMJC 22 November