MA40254 Differential and geometric analysis : Exercises 7

Hand in answers by 1:15pm on Wednesday 22 November for the Seminar of Thursday 23 November
Homepage: http://moodle.bath.ac.uk/course/view.php?id=57709

0 (Warmup). Show that $\alpha := \sin(x_2)^3 dx_1 \wedge dx_2$ is exact.

[**Solution**: *There are three easy ways and one hard way. The hard way is to integrate* $\sin(x_2)^3$ *explicitly. The easy ways are:* $\alpha = d(x_1 \sin(x_2)^3 dx_2)$*;* $\alpha = d(-(\int^{x_2} \sin(t)^3 dt) dx_1)$ *by the Second Fundamental Theorem of Calculus; or* α *is exact by the Poincaré Lemma.*]

1. Let $U = \{p \in \mathbb{R}^4 : x_2(p) \neq 0\}$, and

$$\alpha = \frac{x_2 dx_1 - x_1 dx_2}{x_2^2} \wedge (x_1 dx_3 + x_2^2 dx_4) \in \Omega^2(U).$$

Compute $d\alpha \in \Omega^3(U)$ and express it in standard form.

[**Hint**: *One way to organise the calculation is to first show that* $d\left(\frac{x_2 dx_1 - x_1 dx_2}{x_2^2}\right) = 0$.]

2. For each of the following differential 3-forms α, find a differential 2-form β such that $d\beta = \alpha$. (Note we abbreviate multi-index notation as $dx_{ijk} = dx_i \wedge dx_j \wedge dx_k$.)

(i) $\alpha = x_3 x_4\, dx_{123} + x_3^2\, dx_{124} + 2x_2 x_3\, dx_{134} + x_1 x_3\, dx_{234} \in \Omega^3(\mathbb{R}^4)$.

[**Hint**: *First look for* $\gamma \in \Omega^2(\mathbb{R}^4)$ *of the form* $\gamma = f dx_{23} + g dx_{24} + h dx_{34}$ *(for* f, g *and* h *functions* $\mathbb{R}^4 \to \mathbb{R}$*) such that* $\alpha - d\gamma$ *has no* dx_{123}, dx_{124} *or* dx_{134} *component.*]

(ii) $\alpha = \log(x_1) \exp(x_2) \cos(x_3)^2 dx_{123} \in \Omega^3(\mathbb{R}^+ \times \mathbb{R}^2)$.

[**Hint**: *Which function is easiest to integrate?*]

3. (i) Show that any $\omega \in \mathrm{Alt}^2(\mathbb{R}^3)$ can be written as $\omega = \alpha \wedge \beta$ for some $\alpha, \beta \in \mathbb{R}^{3*}$.

[**Hint**: *If* $\omega = \alpha \wedge \beta$*, what can you say about* $\omega \wedge \alpha$*? What is the dimension of the subspace* $\{\gamma : \omega \wedge \gamma = 0\} \subseteq \mathbb{R}^{3*}$*?*]

(ii) Show that $\varepsilon_1 \wedge \varepsilon_2 + \varepsilon_3 \wedge \varepsilon_4 \in \mathrm{Alt}^2(\mathbb{R}^4)$ cannot be written in the form $\alpha \wedge \beta$ for any $\alpha, \beta \in \mathbb{R}^{4*}$. (Here ε_i is the standard dual basis of $\mathbb{R}^{4*}$ as usual.)

[**Hint**: *If* $\varepsilon_1 \wedge \varepsilon_2 + \varepsilon_3 \wedge \varepsilon_4 = \alpha \wedge \beta$*, consider the result of taking the wedge product of each side with itself.*]

4. Let $\alpha \in \Omega^k(U)$ and $\beta \in \Omega^\ell(U)$

(i) Show that if α and β are closed, then so is $\alpha \wedge \beta$.

(ii) Show that if α is closed and β is exact, then $\alpha \wedge \beta$ is exact.

5. Sketch a proof of the Inverse Function Theorem.
[Please also indicate if you are willing to have your sketch discussed in the seminar.]

[**Hint**: *See the guidance about sketch proofs on moodle.*]

MA40254 Differential and geometric analysis : Solutions 7

1. The first factor equals $d(\frac{x_1}{x_2})$, so its exterior derivative vanishes. Hence, using the Leibniz rule,

$$\begin{aligned} d\alpha &= -\frac{x_2 dx_1 - x_1 dx_2}{x_2^2} \wedge d(x_1 dx_3 + x_2^2 dx_4) \\ &= -\frac{x_2 dx_1 - x_1 dx_2}{x_2^2} \wedge (dx_1 \wedge dx_3 + 2x_2 dx_2 \wedge dx_4) \\ &= -\frac{2x_2^2 dx_1 \wedge dx_2 \wedge dx_4 + x_1 dx_1 \wedge dx_2 \wedge dx_3}{x_2^2} \end{aligned}$$

2. (i) We can first look for, for instance, a $\gamma \in \Omega^2(\mathbb{R}^4)$ of the form $\gamma = f dx_{23} + g dx_{24} + h dx_{34}$, for f, g and h functions $\mathbb{R}^4 \to \mathbb{R}$, such that $\alpha - d\gamma$ has no dx_{123}, dx_{124} or dx_{134} component. That means we should take f such that $\frac{\partial f}{\partial x_1} = x_3 x_4$, e.g., $f = x_1 x_3 x_4$. Similarly we can take $g = x_1 x_3^2$ and $h = 2x_1 x_2 x_3$. Evaluating $\alpha - d\gamma$ we find that actually the dx_{234} terms cancel too. Thus we can take

$$\beta = \gamma = x_1 x_3 x_4 dx_{23} + x_1 x_3^2 dx_{24} + 2x_1 x_2 x_3 dx_{34}.$$

(ii) $(-x_1 + x_1 \log(x_1)) \exp(x_2) \cos(x_3)^2 dx_{23}$ and $-\log(x_1) \exp(x_2) \cos(x_3)^2 dx_{13}$ and

$$\tfrac{1}{4} \log(x_1) \exp(x_2)(\sin(2x_3) + 2x_3) dx_{12}$$

are three possible choices for β, the middle one being the easiest!

3. (i) The linear map $(\mathbb{R}^3)^* \to \mathrm{Alt}^3(\mathbb{R}^3)$, $\alpha \mapsto \omega \wedge \alpha$ has kernel of dimension at least 2. Pick linearly independent elements α_1, α_2 in the kernel, and extend to a basis $\alpha_1, \alpha_2, \alpha_3$ of $(\mathbb{R}^3)^*$. We can then write

$$\omega = \lambda_1 \alpha_2 \wedge \alpha_3 + \lambda_2 \alpha_3 \wedge \alpha_1 + \lambda_3 \alpha_1 \wedge \alpha_2$$

for some coefficients $\lambda_i \in \mathbb{R}$. Now

$$0 = \omega \wedge \alpha_1 = \lambda_1 \alpha_1 \wedge \alpha_2 \wedge \alpha_3$$

implies that $\lambda_1 = 0$, and similarly $\lambda_2 = 0$. We can thus take $\alpha = \lambda_3 \alpha_1$ and $\beta = \alpha_2$.

(ii) Writing $(\gamma)^2$ for $\gamma \wedge \gamma$, we have

$$(\varepsilon_1 \wedge \varepsilon_2 + \varepsilon_3 \wedge \varepsilon_4)^2 = 2\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3 \wedge \varepsilon_4 \in \mathrm{Alt}^4 \mathbb{R}^4,$$

which is non-zero. On the other hand,

$$(\alpha \wedge \beta)^2 = 0$$

for any $\alpha, \beta \in (\mathbb{R}^4)^*$.

4. (i) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta = 0$ by assumption and the Leibniz rule.

(ii) Suppose $\beta = d\gamma$. Then $d((-1)^k \alpha \wedge \gamma) = (-1)^k d\alpha \wedge \gamma + \alpha \wedge d\gamma = \alpha \wedge \beta$, by assumption and the Leibniz rule.

5. I do not provide model sketch proofs as there is no single right answer, and I want to encourage you to develop your own. Instead, I indicate the most crucial ideas. For the proof in lectures of the IFT for $f \colon U \to \mathbb{R}^n$ with Df_x invertible, I would highlight the following.

- Using the continuity of Df to find a domain U' on which Df is close to Df_x (hence f is a local diffeomorphism).
- Using the Mean Value Inequality to establish the contraction mapping property on U'.
- Using the Contraction Mapping Theorem to show $f(U')$ is open.
- The estimate $||y - z|| \leq 2||f(y) - f(z)||$, which is used to establish both injectivity of f and differentiability of the inverse.