MA40254 DIFFERENTIAL AND GEOMETRIC ANALYSIS : EXERCISES 7

Hand in answers by 1:15pm on Wednesday 22 November for the Seminar of Thursday 23 November
Homepage: http://moodle.bath.ac.uk/course/view.php?id=57709

0 (Warmup)**.** Show that $\alpha := \sin(x_2)^3 dx_1 \wedge dx_2$ is exact.

[**Solution**: *There are three easy ways and one hard way. The hard way is to integrate* $\sin(x_2)^3$ *explicitly. The easy ways are:* $\alpha = d(x_1 \sin(x_2)^3 dx_2)$*;* $\alpha = d(-(\int^{x_2} \sin(t)^3 dt) dx_1)$ *by the Second Fundamental Theorem of Calculus; or* α *is exact by the Poincaré Lemma.*]

1. Let $U = \{p \in \mathbb{R}^4 : x_2(p) \neq 0\}$, and

$$\alpha = \frac{x_2 dx_1 - x_1 dx_2}{x_2^2} \wedge (x_1 dx_3 + x_2^2 dx_4) \in \Omega^2(U).$$

Compute $d\alpha \in \Omega^3(U)$ and express it in standard form.

[**Hint**: *One way to organise the calculation is to first show that* $d\left(\frac{x_2 dx_1 - x_1 dx_2}{x_2^2}\right) = 0$.]

2. For each of the following differential 3-forms α, find a differential 2-form β such that $d\beta = \alpha$. (Note we abbreviate multi-index notation as $dx_{ijk} = dx_i \wedge dx_j \wedge dx_k$.)

(i) $\alpha = x_3 x_4\, dx_{123} + x_3^2\, dx_{124} + 2x_2 x_3\, dx_{134} + x_1 x_3\, dx_{234} \;\in\; \Omega^3(\mathbb{R}^4)$.

[**Hint**: *First look for* $\gamma \in \Omega^2(\mathbb{R}^4)$ *of the form* $\gamma = f dx_{23} + g dx_{24} + h dx_{34}$ *(for* f, g *and* h *functions* $\mathbb{R}^4 \to \mathbb{R}$*) such that* $\alpha - d\gamma$ *has no* dx_{123}, dx_{124} *or* dx_{134} *component.*]

(ii) $\alpha = \log(x_1) \exp(x_2) \cos(x_3)^2 dx_{123} \;\in\; \Omega^3(\mathbb{R}^+ \times \mathbb{R}^2)$.

[**Hint**: *Which function is easiest to integrate?*]

3. (i) Show that any $\omega \in \mathrm{Alt}^2(\mathbb{R}^3)$ can be written as $\omega = \alpha \wedge \beta$ for some $\alpha, \beta \in \mathbb{R}^{3*}$.

[**Hint**: *If* $\omega = \alpha \wedge \beta$*, what can you say about* $\omega \wedge \alpha$*? What is the dimension of the subspace* $\{\gamma : \omega \wedge \gamma = 0\} \subseteq \mathbb{R}^{3*}$*?*]

(ii) Show that $\varepsilon_1 \wedge \varepsilon_2 + \varepsilon_3 \wedge \varepsilon_4 \in \mathrm{Alt}^2(\mathbb{R}^4)$ cannot be written in the form $\alpha \wedge \beta$ for any $\alpha, \beta \in \mathbb{R}^{4*}$. (Here ε_i is the standard dual basis of $\mathbb{R}^{4*}$ as usual.)

[**Hint**: *If* $\varepsilon_1 \wedge \varepsilon_2 + \varepsilon_3 \wedge \varepsilon_4 = \alpha \wedge \beta$*, consider the result of taking the wedge product of each side with itself.*]

4. Let $\alpha \in \Omega^k(U)$ and $\beta \in \Omega^\ell(U)$

(i) Show that if α and β are closed, then so is $\alpha \wedge \beta$.

(ii) Show that if α is closed and β is exact, then $\alpha \wedge \beta$ is exact.

5. Sketch a proof of the Inverse Function Theorem.
[Please also indicate if you are willing to have your sketch discussed in the seminar.]

[**Hint**: *See the guidance about sketch proofs on moodle.*]

DMJC 14 November