MA40254 Differential and geometric analysis : Exercises 6

Hand in answers by 1:15pm on Wednesday 15 November for the Seminar of Thursday 16 November
Homepage: http://moodle.bath.ac.uk/course/view.php?id=57709

0 (Warmup)**.** Let $\gamma = \dfrac{-x_2\,dx_1 + x_1\,dx_2}{x_1^2 + x_2^2} \in \Omega^1(\mathbb{R}^2 \setminus \{0\})$. Show that $d\gamma = 0$.

[**Solution**: *By the product rule*

$$d\gamma = \frac{d(-x_2\,dx_1 + x_1\,dx_2)}{x_1^2 + x_2^2} - \frac{2x_1\,dx_1 + 2x_2\,dx_2}{(x_1^2 + x_2^2)^2} \wedge (-x_2\,dx_1 + x_1\,dx_2) = 0,$$

since $d(-x_2\,dx_1 + x_1\,dx_2) = 2dx_1 \wedge dx_2$ *and* $(x_1\,dx_1 + x_2\,dx_2) \wedge (-x_2\,dx_1 + x_1\,dx_2) = (x_1^2 + x_2^2) dx_1 \wedge dx_2$.]

1. Let $U = \{p \in \mathbb{R}^4 : x_2(p) \neq 0\}$, and $\varphi = (x_1,\, x_2^2 x_3,\, x_4/x_2) : U \to \mathbb{R}^3$, where x_1, x_2, x_3, x_4 are the coordinate functions on $U \subseteq \mathbb{R}^4$. Let $\alpha = y_1 dy_2 \wedge dy_3 \in \Omega^2(\mathbb{R}^3)$ where $y_1, y_2, y_3 : \mathbb{R}^3 \to \mathbb{R}$ denote the coordinate functions.

(i) Express $\varphi^*\alpha$ and $\varphi^* d\alpha$ in standard form, i.e., as a sum of terms $f\,dx_I$.

[**Hint**: *First expand* $\varphi^*\alpha$ *as* $(\varphi^* y_1) d(\varphi^* y_2) \wedge d(\varphi^* y_3)$ *and similarly* $\varphi^* d\alpha$.]

(ii) Compute directly $d(\varphi^*\alpha)$. What do you observe?

[**Hint**: *To save work, just compute the terms that don't appear in* $\varphi^* d\alpha$.]

2. For U open in $\mathbb{R}^n$, $\alpha \in \Omega^k(U)$, $p \in U$, and $v_1, \ldots, v_k \in \mathbb{R}^n$, show that

$$d\alpha_p(v_0, \ldots, v_k) = \sum_{i=0}^{k} (-1)^i D\alpha_p(v_i)(v_0, \ldots, \widehat{v_i}, \ldots, v_k)$$

where $v_0, \ldots, \widehat{v_i}, \ldots, v_k$ denotes the list obtained from $v_0, \ldots, v_k$ by omitting v_i. Equivalently

$$d\alpha_p = \sum_{i=0}^{k} \operatorname{sgn}(\sigma_i)\, \sigma_i \cdot D\alpha_p^{\vee},$$

where $\sigma_i = (0\ 1\ \cdots\ i)^{-1} \in G := \operatorname{Sym}(\{0, 1, \ldots k\}) \cong S_{k+1}$, and for $\sigma \in G$ and $\beta \in \operatorname{Alt}^{k+1}(\mathbb{R}^n)$, $(\sigma \cdot \beta)(v_0, \ldots, v_k) = \beta(v_{\sigma(0)}, \ldots, v_{\sigma(k)})$ (which is $\beta(v_i, v_0, \ldots, \widehat{v_i}, \ldots, v_k)$ when $\sigma = \sigma_i$).

[**Hint**: *One approach, using the second formula, is to let* $H \cong S_k$ *be the subgroup of* G *fixing* 0, *and observe that* $\sigma_i H : i = 0, \ldots k$ *is a left coset partition of* G. *Now split the sum into sums over each coset: what is* $\tau \cdot D\alpha_p^{\vee}$ *for* $\tau \in H$ *?*]

3 (Less essential)**.** Let $U \subseteq \mathbb{R}^3$ an open subset. Given a vector-valued function $v = (v_1, v_2, v_3) : U \to \mathbb{R}^3$, define $v^\flat \in \Omega^1(U)$ and $v \lrcorner \operatorname{Det} \in \Omega^2(U)$ by applying the corresponding operations on vectors pointwise.

(i) Let $\operatorname{div}(v) : U \to \mathbb{R}$ be defined by

$$\operatorname{div}(v) = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}.$$

Show that

$$d(v \lrcorner \operatorname{Det}) = \operatorname{div}(v) \operatorname{Det} \in \Omega^3(U).$$

(ii) Let $\mathrm{curl}(v) : U \to \mathbb{R}^3$ be defined by

$$\mathrm{curl}(v) = \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right).$$

Show that

$$d(v^\flat) = \mathrm{curl}(v) \lrcorner \mathrm{Det} \in \Omega^2(U).$$

[**Hint**: *Write all the forms in standard form, e.g.,* $v \lrcorner \mathrm{Det}$ *can be expressed as* $v_1 dx_2 \wedge dx_3 - v_2 dx_1 \wedge dx_3 + v_3 dx_1 \wedge dx_2$.]

4. Let $x_1, x_2 \colon \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ be the coordinate functions. Which of the following elements of $\Omega^1(\mathbb{R}^2 \setminus \{0\})$ are closed? Which are exact?

(i) $\alpha = -2x_1 x_2\, dx_1 + x_1^2 dx_2$

(ii) $\beta = x_2 dx_1 + x_1 dx_2$

(iii) $\gamma = \dfrac{-x_2 dx_1 + x_1 dx_2}{x_1^2 + x_2^2}$

[**Hint**: *For (iii), consider* $\varphi : \mathbb{R} \to \mathbb{R}^2 \setminus \{0\}$ *defined by* $t \mapsto (\cos t, \sin t)$. *Suppose that* $\gamma = df$ *for some* $f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$. *What can you say about* $\varphi^* f$ *and* $\varphi^* \gamma$? *Or about* $\int_0^{2\pi} \frac{d(f \circ \varphi)}{dt} dt$?]

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1. (i)

$$\begin{aligned}\varphi^*(y_1 dy_2\wedge dy_3) &= (\varphi^* y_1) d((\varphi^* y_2)\wedge d(\varphi^* y_3))) = x_1 d\left(x_2^2 x_3 \frac{x_2 dx_4 - x_4 dx_2}{x_2^2}\right)\\ &= x_1(d(x_3x_2)\wedge dx_4 - d(x_3x_4)\wedge dx_2)\\ &= 2x_1x_3dx_2\wedge dx_4 + x_1x_4dx_2\wedge dx_3 + x_1x_2dx_3\wedge dx_4\end{aligned}$$

Similarly, $\varphi^* d\alpha = d(\varphi^* y_1)\wedge d((\varphi^* y_2)\wedge d(\varphi^* y_3))) = 2x_3dx_1\wedge dx_2\wedge dx_4 + x_4dx_1\wedge dx_2\wedge dx_3 + x_2dx_1\wedge dx_3\wedge dx_4$

(ii) Observe that $2x_1dx_3\wedge dx_2\wedge dx_4 + x_1dx_4\wedge dx_2\wedge dx_3 + x_1dx_2\wedge dx_3\wedge dx_4 = 0$ so $d(\varphi^*\alpha) = \varphi^* d\alpha$.

2. Let $H\cong S_k$ be the subgroup of G fixing 0; then $\sigma_i H$ consists of elements of G which send 0 to i, so these cosets are disjoint, and there are $k+1$ of them, which is the index of H in G. Hence

$$\operatorname{alt}(D\alpha_p^\vee) = \sum_{\sigma\in G}\operatorname{sgn}(\sigma)\sigma\cdot D\alpha_p^\vee = \sum_{i=0}^{k}\sum_{\tau\in H}\operatorname{sgn}(\sigma_i)\operatorname{sgn}(\tau)\sigma_i\cdot\tau\cdot D\alpha_p^\vee = k!\sum_{i=0}^{k}\operatorname{sgn}(\sigma_i)\,\sigma_i\cdot D\alpha_p^\vee,$$

since $\tau\cdot D\alpha_p^\vee = \operatorname{sgn}(\tau)D\alpha_p^\vee$ and $|H| = k!$. Dividing by $k!$ proves the result.

3. (i) $v\lrcorner\operatorname{Det} = v_1dx_2\wedge dx_3 + v_2dx_3\wedge dx_1 + v_3dx_1\wedge dx_2$, so

$$\begin{aligned}d(v\lrcorner\operatorname{Det}) &= dv_1\wedge dx_2\wedge dx_3 + dv_2\wedge dx_3\wedge dx_1 + dv_3\wedge dx_1\wedge dx_2\\ &= \left(\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}+\frac{\partial v_3}{\partial x_3}\right)dx_1\wedge dx_2\wedge dx_3.\end{aligned}$$

(ii) $v^\flat = v_1dx_1 + v_2dx_2 + v_3dx_3$, so

$$\begin{aligned}d(v^\flat) &= dv_1\wedge dx_1 + dv_2\wedge dx_2 + dv_3\wedge dx_3\\ &= \left(\frac{\partial v_1}{\partial x_2}dx_2+\frac{\partial v_1}{\partial x_3}dx_3\right)\wedge dx_1 + \left(\frac{\partial v_2}{\partial x_1}dx_1+\frac{\partial v_2}{\partial x_3}dx_3\right)\wedge dx_2 + \left(\frac{\partial v_3}{\partial x_1}dx_1+\frac{\partial v_3}{\partial x_2}dx_2\right)\wedge dx_3\\ &= \left(\frac{\partial v_3}{\partial x_2}-\frac{\partial v_2}{\partial x_3}\right)dx_2\wedge dx_3 + \left(\frac{\partial v_1}{\partial x_3}-\frac{\partial v_3}{\partial x_1}\right)dx_3\wedge dx_1 + \left(\frac{\partial v_2}{\partial x_1}-\frac{\partial v_1}{\partial x_2}\right)dx_1\wedge dx_2 = \operatorname{curl} v\lrcorner\operatorname{Det}.\end{aligned}$$

4. (i) $d(-2x_1x_2dx_1 + x_1^2dx_2) = -2x_1dx_2\wedge dx_1 + 2x_1dx_1\wedge dx_2 = 4x_1dx_1\wedge dx_2$ is not zero everywhere, so not closed. Hence also not exact.

(ii) $x_2dx_1 + x_1dx_2 = d(x_1x_2)$, so exact, hence also closed.

(iii) We've seen in the warmup that γ is closed. However, suppose that $\gamma = df$ for some $f\in\Omega^0(U)$. Let

$$\varphi:\mathbb{R}\to\mathbb{R}^2,\ \theta\mapsto(\cos\theta,\sin\theta).$$

Then $\varphi^*\gamma = d\theta$. So $\varphi^* f = f\circ\varphi:\mathbb{R}\to\mathbb{R}$ is a function such that $d(\varphi^* f) = d\theta$. Then $\varphi^* f - \theta$ is a constant function on $\mathbb{R}$. In particular, $(\varphi^* f)(2\pi) = (\varphi^* f)(0) + 2\pi$. But that contradicts $(\varphi^* f)(0) = f(\varphi(0)) = f(\varphi(2\pi)) = (\varphi^* f)(2\pi)$.