MA40254 Differential and geometric analysis : Exercises 6

Hand in answers by 1:15pm on Wednesday 15 November for the Seminar of Thursday 16 November
Homepage: http://moodle.bath.ac.uk/course/view.php?id=57709

0 (Warmup)**.** Let $\gamma = \dfrac{-x_2\,dx_1 + x_1\,dx_2}{x_1^2+x_2^2} \in \Omega^1(\mathbb{R}^2 \setminus \{0\})$. Show that $d\gamma = 0$.

[**Solution**: *By the product rule*

$$d\gamma = \frac{d(-x_2\,dx_1 + x_1\,dx_2)}{x_1^2+x_2^2} - \frac{2x_1\,dx_1 + 2x_2\,dx_2}{(x_1^2+x_2^2)^2} \wedge (-x_2\,dx_1 + x_1\,dx_2) = 0,$$

since $d(-x_2\,dx_1+x_1\,dx_2) = 2dx_1\wedge dx_2$ *and* $(x_1\,dx_1+x_2\,dx_2)\wedge(-x_2\,dx_1+x_1\,dx_2) = (x_1^2+x_2^2)dx_1\wedge dx_2$.]

1. Let $U = \{p \in \mathbb{R}^4 : x_2(p) \neq 0\}$, and $\varphi = (x_1,\, x_2^2x_3,\, x_4/x_2) : U \to \mathbb{R}^3$, where x_1, x_2, x_3, x_4 are the coordinate functions on $U \subseteq \mathbb{R}^4$. Let $\alpha = y_1 dy_2 \wedge dy_3 \in \Omega^2(\mathbb{R}^3)$ where $y_1, y_2, y_3 : \mathbb{R}^3 \to \mathbb{R}$ denote the coordinate functions.

(i) Express $\varphi^*\alpha$ and $\varphi^*d\alpha$ in standard form, i.e., as a sum of terms $f\,dx_I$.

[**Hint**: *First expand* $\varphi^*\alpha$ *as* $(\varphi^*y_1)d(\varphi^*y_2)\wedge d(\varphi^*y_3)$ *and similarly* $\varphi^*d\alpha$.]

(ii) Compute directly $d(\varphi^*\alpha)$. What do you observe?

[**Hint**: *To save work, just compute the terms that don't appear in* $\varphi^*d\alpha$.]

2. For U open in $\mathbb{R}^n$, $\alpha \in \Omega^k(U)$, $p \in U$, and $v_1, \ldots, v_k \in \mathbb{R}^n$, show that

$$d\alpha_p(v_0, \ldots, v_k) = \sum_{i=0}^{k} (-1)^i D\alpha_p(v_i)(v_0, \ldots, \widehat{v}_i, \ldots, v_k)$$

where $v_0, \ldots, \widehat{v}_i, \ldots, v_k$ denotes the list obtained from $v_0, \ldots, v_k$ by omitting v_i. Equivalently

$$d\alpha_p = \sum_{i=0}^{k} \operatorname{sgn}(\sigma_i)\,\sigma_i \cdot D\alpha_p^{\vee},$$

where $\sigma_i = (0\ 1\ \cdots\ i)^{-1} \in G := \mathrm{Sym}(\{0, 1, \ldots k\}) \cong S_{k+1}$, and for $\sigma \in G$ and $\beta \in \mathrm{Alt}^{k+1}(\mathbb{R}^n)$, $(\sigma\cdot\beta)(v_0, \ldots, v_k) = \beta(v_{\sigma(0)}, \ldots, v_{\sigma(k)})$ (which is $\beta(v_i, v_0, \ldots, \widehat{v}_i, \ldots, v_k)$ when $\sigma = \sigma_i$).

[**Hint**: *One approach, using the second formula, is to let* $H \cong S_k$ *be the subgroup of* G *fixing* 0*, and observe that* $\sigma_i H : i = 0, \ldots k$ *is a left coset partition of* G*. Now split the sum into sums over each coset: what is* $\tau \cdot D\alpha_p^{\vee}$ *for* $\tau \in H$ *?*]

3 (Less essential)**.** Let $U \subseteq \mathbb{R}^3$ an open subset. Given a vector-valued function $v = (v_1, v_2, v_3) : U \to \mathbb{R}^3$, define $v^\flat \in \Omega^1(U)$ and $v \lrcorner \mathrm{Det} \in \Omega^2(U)$ by applying the corresponding operations on vectors pointwise.

(i) Let $\mathrm{div}(v) : U \to \mathbb{R}$ be defined by

$$\mathrm{div}(v) = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}.$$

Show that

$$d(v \lrcorner \mathrm{Det}) = \mathrm{div}(v)\mathrm{Det} \in \Omega^3(U).$$

(ii) Let $\mathrm{curl}(v) : U \to \mathbb{R}^3$ be defined by

$$\mathrm{curl}(v) = \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right).$$

Show that

$$d(v^\flat) = \mathrm{curl}(v) \lrcorner \mathrm{Det} \in \Omega^2(U).$$

[**Hint**: *Write all the forms in standard form, e.g.,* $v \lrcorner \mathrm{Det}$ *can be expressed as* $v_1 dx_2 \wedge dx_3 - v_2 dx_1 \wedge dx_3 + v_3 dx_1 \wedge dx_2$.]

4. Let $x_1, x_2 \colon \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ be the coordinate functions. Which of the following elements of $\Omega^1(\mathbb{R}^2 \setminus \{0\})$ are closed? Which are exact?

(i) $\alpha = -2x_1 x_2 \, dx_1 + x_1^2 dx_2$

(ii) $\beta = x_2 dx_1 + x_1 dx_2$

(iii) $\gamma = \dfrac{-x_2 dx_1 + x_1 dx_2}{x_1^2 + x_2^2}$

[**Hint**: *For (iii), consider* $\varphi : \mathbb{R} \to \mathbb{R}^2 \setminus \{0\}$ *defined by* $t \mapsto (\cos t, \sin t)$. *Suppose that* $\gamma = df$ *for some* $f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$. *What can you say about* $\varphi^* f$ *and* $\varphi^* \gamma$? *Or about* $\int_0^{2\pi} \frac{d(f \circ \varphi)}{dt} dt$?]

DMJC 7 November