MA40254 Differential and geometric analysis : Exercises 5

Hand in answers by 1:15pm on Wednesday 8 November for the Seminar of Thursday 9 November
Homepage: http://moodle.bath.ac.uk/course/view.php?id=57709

0 (Warmup)**.** Let V be a real vector space of dimension n, $\alpha_1, \alpha_2 \dots \alpha_k, \beta_j \in \mathcal{M}^1(V)$ and $\lambda, \mu \in \mathbb{R}$. Show that $\alpha_1\alpha_2\cdots(\lambda\alpha_j + \mu\beta_j)\alpha_{j+1}\cdots\alpha_k = \lambda\,\alpha_1\alpha_2\cdots\alpha_j\alpha_{j+1}\cdots\alpha_k + \mu\,\alpha_1\alpha_2\cdots\beta_j\alpha_{j+1}\cdots\alpha_k$ in $\mathcal{M}^k(V)$ and deduce that

$$\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge (\lambda\alpha_j + \mu\beta_j) \wedge \alpha_{j+1} \wedge \cdots \wedge \alpha_k$$
$$= \lambda\,\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_j \wedge \alpha_{j+1} \wedge \cdots \wedge \alpha_k + \mu\,\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \beta_j \wedge \alpha_{j+1} \wedge \cdots \wedge \alpha_k.$$

[**Solution**: *Evaluating the left hand side of the first identity on $v_1, v_2, \dots v_k \in V$ (using the definition) gives*

$$\alpha_1(v_1)\alpha_2(v_2)\cdots(\lambda\alpha_j + \mu\beta_j)(v_j)\alpha_{j+1}(v_{j+1})\cdots\alpha_k(v_k) \in \mathbb{R}$$

Now $(\lambda\alpha_j + \mu\beta_j)(v_j) = \lambda\,\alpha_j(v_j) + \mu\,\beta_j(v_j)$ (pointwise operations) so the result follows by the distributive law. To obtain the second identity, apply alt *to both sides of the first identity, and use that* $\mathrm{alt}\colon \mathcal{M}^k(V) \to \mathrm{Alt}^k(V)$ *is linear:* $\mathrm{alt}(\lambda\alpha + \mu\beta) = \sum_{\sigma\in S_k} \mathrm{sgn}(\sigma)\sigma\cdot(\lambda\alpha + \mu\beta)$ *and it is easy to see that* $\sigma\cdot(\lambda\alpha + \mu\beta) = \lambda\,\sigma\cdot\alpha + \mu\,\sigma\cdot\beta$ *(evaluate both sides on $v_1, v_2, \dots v_k$).*]

1. Let V be a real vector space of dimension n.

(i) Let $v_1, \dots, v_k \in V$ and $\alpha_1, \dots, \alpha_k \in V^*$. Let $A \in M_{k,k}(\mathbb{R})$ be the matrix with $A_{ij} = \alpha_i(v_j)$. Show that $(\alpha_1 \wedge \cdots \wedge \alpha_k)(v_1, \dots, v_k) = \det A$.

[**Hint**: *Compare the result of evaluating the full antisymmetrisation of $\alpha_1 \cdots \alpha_k \in \mathcal{M}^k(V)$ on $v_1, \dots, v_k$ with the sum formula for the determinant of a matrix.*]

(ii) Let $\phi : V \to V$ be a linear operator. Show that for any $\alpha \in \mathrm{Alt}^n(V)$,

$$\phi^*\alpha = (\det\phi)\alpha \in \mathrm{Alt}^n(V)$$

[**Hint**: *Recall that* $\det\phi$ *means the determinant of the matrix A representing ϕ with respect to any basis $e_1, e_2, \dots e_n$ of V. Use the dual basis $\varepsilon_1, \varepsilon_2, \dots \varepsilon_n$ to write the entries A_{ij} in the form of part (i), and note that $\varepsilon_1 \wedge \varepsilon_2 \wedge \ldots \wedge \varepsilon_n$ is a basis for* $\mathrm{Alt}^n(V)$.]

2. Let $e_1, \dots, e_5 \in \mathbb{R}^5$ be the standard basis, and let $\varepsilon_1, \dots, \varepsilon_5 \in (\mathbb{R}^5)^*$ be the dual basis. Let

$$\alpha = 3\varepsilon_1 \wedge \varepsilon_3 + \varepsilon_2 \wedge (7\varepsilon_3 - 2\varepsilon_5) \in \mathrm{Alt}^2(\mathbb{R}^5).$$

(i) Evaluate $\alpha(e_1 + 2e_3, e_3 + e_4) \in \mathbb{R}$.

(ii) Express $\alpha \wedge (2\varepsilon_1 + \varepsilon_2 - 3\varepsilon_4) \in \mathrm{Alt}^3(\mathbb{R}^5)$ in terms of the standard basis $\varepsilon_1\wedge\varepsilon_2\wedge\varepsilon_3$, $\varepsilon_1\wedge\varepsilon_2\wedge\varepsilon_4$, $\dots$, $\varepsilon_3\wedge\varepsilon_4\wedge\varepsilon_5$.

[**Hint**: *When expanding, bear in mind that $\varepsilon_i \wedge \varepsilon_j = -\varepsilon_j \wedge \varepsilon_i$, and $\varepsilon_i \wedge \varepsilon_i = 0$.*]

3 (Less essential)**.** For a real inner product space V and $v \in V$, define $v^\flat \in V^*$ to be the linear map $V \to \mathbb{R}$, $w \mapsto v.w$. For $\alpha \in \mathrm{Alt}^{k+1}(V)$ and $v \in V$, define the 'contraction' $v \lrcorner \alpha \in \mathrm{Alt}^k(V)$ by

$$(v \lrcorner \alpha)(w_1, \dots, w_k) = \alpha(v, w_1, \dots, w_k)$$

for all $w_1, \dots, w_k \in V$. Show that the cross product on $\mathbb{R}^3$ is related to the wedge product on $(\mathbb{R}^3)^*$ by

$$(u \times v) \lrcorner \mathrm{Det} = u^\flat \wedge v^\flat \in \mathrm{Alt}^2(\mathbb{R}^3)$$

for any $u, v \in \mathbb{R}^3$.

[**Hint**: *Express u and v in terms of the standard basis, i.e., write* $u = u_1e_1 + u_2e_2 + u_3e_3$ *etc., and find the components of both sides of the equation with respect to the standard basis* $\varepsilon_1 \wedge \varepsilon_2, \varepsilon_1 \wedge \varepsilon_3, \varepsilon_2 \wedge \varepsilon_3$ *of* $\mathrm{Alt}^2\mathbb{R}^3$.]

4. Let $\phi : V \to W$ be a linear map between real vector spaces. Show that

(i) $\phi^* \operatorname{alt}(\alpha) = \operatorname{alt}(\phi^*\alpha) \in \mathrm{Alt}^k(V)$ for any $\alpha \in \mathcal{M}^k(W)$.

[**Hint**: *First check that* $\sigma \cdot (\phi^*\alpha) = \phi^*(\sigma \cdot \alpha) \in \mathcal{M}^k(V)$ *for any* $\alpha \in \mathcal{M}^k(W)$ *and* $\sigma \in S_k$.]

(ii) $\phi^*(\alpha_1 \wedge \cdots \wedge \alpha_k) = (\phi^*\alpha_1) \wedge \cdots \wedge (\phi^*\alpha_k) \in \mathrm{Alt}^k(V)$ for any $\alpha_1, \ldots, \alpha_k \in \mathrm{Alt}^1(W)$.

[**Hint**: *First show* $\phi^*(\alpha_1 \cdots \alpha_k) = (\phi^*\alpha_1) \cdots (\phi^*\alpha_k)$ *by evaluating both sides on* $v_1, \ldots v_k$.]

5. Let V and W be vector spaces with bases v_1, v_2, v_3 and w_1, w_2, w_3, w_4 respectively. Let $\phi : V \to W$ be the linear map represented with respect to these bases by

$$\begin{pmatrix} 2 & 0 & -3 \\ 1 & 6 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 5 \end{pmatrix}$$

Let $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in V^*$ and $\delta_1, \delta_2, \delta_3, \delta_4 \in W^*$ denote the dual bases to the given bases. Compute

$$\phi^*(3\delta_1 \wedge \delta_3 + \delta_2 \wedge \delta_4) \in \mathrm{Alt}^2(V)$$

in terms of the standard basis $\varepsilon_i \wedge \varepsilon_j : i < j$ for $\mathrm{Alt}^2(V)$.

[**Hint**: *What matrix represents* $\phi^* : W^* \to V^*$ *with respect to the bases* δ_i *and* ε_j*? There is a reason why the dual map* ϕ^* *is sometimes called the "transpose" of* ϕ*! You should find, for instance, that* $\phi^*\delta_1 = 2\varepsilon_1 - 3\varepsilon_3$.]

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1. (i) Recall that $\alpha_1 \wedge \cdots \wedge \alpha_k = \operatorname{alt}(\alpha_1 \cdots \alpha_k)$ i.e.,

$$(\alpha_1 \wedge \cdots \wedge \alpha_k)(v_1, \ldots, v_k) = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\, \alpha_1(v_{\sigma(1)}) \cdots \alpha_k(v_{\sigma(k)}).$$

The RHS is precisely the sum formula for $\det(\alpha_i(v_j))$.

(ii) Pick a basis $e_1, \ldots, e_n \in V$, and let $\varepsilon_1, \ldots, \varepsilon_n$ be the dual basis. Let A be the matrix that represents ϕ with respect to $e_1, \ldots, e_n$. Then $\varepsilon_i(\phi(e_j)) = A_{ij}$, so by (i)

$$(\phi^*(\varepsilon_1 \wedge \cdots \wedge \varepsilon_n))(e_1, \ldots, e_n) = (\varepsilon_1 \wedge \cdots \wedge \varepsilon_n)(\phi(e_1), \ldots, \phi(e_n)) = \det A,$$
$$(\varepsilon_1 \wedge \cdots \wedge \varepsilon_n)(e_1, \ldots, e_n) = 1.$$

Since $\varepsilon_1 \wedge \cdots \wedge \varepsilon_n$ spans $\operatorname{Alt}^n(V)$, and $\det \phi = \det A$ by definition, the claim follows.

2. (i) By multilinearity and alternating property

$$\alpha(e_1 + 2e_3, e_3 + e_4) = \alpha(e_1, e_3) + \alpha(e_1, e_4) + 2\alpha(e_3, e_4).$$

The last two terms vanish, while $\alpha(e_1, e_3) = 3$.

(ii) Since $\alpha \wedge \varepsilon_1 = 7\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3 - 2\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_5$ and $\alpha \wedge \varepsilon_2 = -3\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3$ and

$$\alpha \wedge \varepsilon_4 = 3\varepsilon_1 \wedge \varepsilon_3 \wedge \varepsilon_4 + 7\varepsilon_2 \wedge \varepsilon_3 \wedge \varepsilon_4 + 2\varepsilon_2 \wedge \varepsilon_4 \wedge \varepsilon_5,$$

we have

$$\alpha \wedge (2\varepsilon_1 + \varepsilon_2 - 3\varepsilon_4) = 11\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3 - 4\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_5 - 9\varepsilon_1 \wedge \varepsilon_3 \wedge \varepsilon_4 - 21\varepsilon_2 \wedge \varepsilon_3 \wedge \varepsilon_4 - 6\varepsilon_2 \wedge \varepsilon_4 \wedge \varepsilon_5.$$

3. If $e_1, e_2, e_3 \in \mathbb{R}^3$ is the standard basis and $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in (\mathbb{R}^3)^*$ is the dual basis, then $\operatorname{Det} = \varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3$. If we write $u = u_1e_1 + u_2e_2 + u_3e_3$ and $v = v_1e_1 + v_2e_2 + v_3e_3$, then $u^\flat \wedge v^\flat$ equals

$$(u_1v_2 - u_2v_1)\varepsilon_1 \wedge \varepsilon_2 + (u_3v_1 - u_1v_3)\varepsilon_3 \wedge \varepsilon_1 + (u_2v_3 - u_3v_2)\varepsilon_2 \wedge \varepsilon_3. \qquad (*)$$

This is equal to $(u \times v) \lrcorner (\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3)$ because evaluating the latter on e_i, e_j for $i \neq j$ gives

$$\sum_{\sigma \in S_3} \operatorname{sgn}(\sigma)\, (u \times v) \cdot e_{\sigma(1)}\, \varepsilon_{\sigma(2)}(e_i)\varepsilon_{\sigma(3)}(e_j) = \operatorname{sgn}(i,j)\operatorname{Det}(u, v, e_k)$$

where $k \neq i, j$ and $\operatorname{sgn}(i,j)$ is the sign of the permutation sending $(1,2,3)$ to (k,i,j). This agrees with $(*)$ on e_i, e_j. [One can also use vector algebra methods from MA10236.]

4. (i) Following the hint, observe that

$$(\sigma \cdot \phi^*\alpha)(v_1, \ldots, v_k) = (\phi^*\alpha)(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \alpha(\phi(v_{\sigma(1)}), \ldots, \phi(v_{\sigma(k)})) = \phi^*(\sigma \cdot \alpha)(v_1, \ldots, v_k)$$

for any $\alpha \in \mathcal{M}^k(W)$ and $\sigma \in S_k$. Now multiply both sides by $\operatorname{sgn}(\sigma)$ and sum over $\sigma \in S_k$.

(ii) Similarly for all $v_1, \ldots, v_k \in V$,

$$\begin{aligned}(\phi^*(\alpha_1 \cdots \alpha_k))(v_1, \ldots, v_k) &= (\alpha_1 \cdots \alpha_k)(\phi(v_1), \ldots, \phi(v_k)) = \alpha_1(\phi(v_1)) \cdots \alpha_k(\phi(v_k)) \\ &= (\phi^*\alpha_1)(v_1) \cdots (\phi^*\alpha_k)(v_k) = ((\phi^*\alpha_1) \cdots (\phi^*\alpha_k))(v_1, \ldots, v_k).\end{aligned}$$

Now apply (i).

5. With respect to the dual bases, $\phi^* : W^* \to V^*$ is represented by the transpose of the matrix defining ϕ, i.e., $\phi^*\delta_1 = 2\varepsilon_1 - 3\varepsilon_3$ etc. Thus (using that ϕ^* distributes over $\wedge$)

$$\begin{aligned}\phi^*(3\delta_1 \wedge \delta_3 + \delta_2 \wedge \delta_4) &= 3(2\varepsilon_1 - 3\varepsilon_3) \wedge (\varepsilon_2 - \varepsilon_3) + (\varepsilon_1 + 6\varepsilon_2) \wedge (\varepsilon_1 + 5\varepsilon_3)\\ &= (6\varepsilon_1 \wedge \varepsilon_2 - 6\varepsilon_1 \wedge \varepsilon_3 - 9\varepsilon_3 \wedge \varepsilon_2) + (5\varepsilon_1 \wedge \varepsilon_3 + 6\varepsilon_2 \wedge \varepsilon_1 + 30\varepsilon_2 \wedge \varepsilon_3)\\ &= -\varepsilon_1 \wedge \varepsilon_3 + 39\varepsilon_2 \wedge \varepsilon_3.\end{aligned}$$